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Milne-type inequalities for differentiable s-preinvex functions via Riemann-Liouville fractional integrals

Badreddine Meftah[a], Abdelghani Lakhdari[b,*], Wedad Saleh[c]

[a]*Laboratory of Analysis and Control of Differential Equations "ACED", Faculty MISM, Department of Mathematics, University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria*
[b]*Laboratory of Energy Systems Technology, Department CPST, National Higher School of Technology and Engineering, Annaba 23005, Algeria*
[c]*Department of Mathematics, Taibah University, Al Medinah 42353, Saudi Arabia*

Abstract. The objective of this study is to introduce novel fractional Milne-type inequalities for s-preinvex first derivatives. These findings are derived from a fresh fractional identity and offer enhancements to existing outcomes. The study concludes by applying these findings to special means.

1. Introduction

Convexity theory is a powerful tool to deal with a variety of problems in mathematics as well as other sciences.This concept has a closed relationship in the development of the theory of inequalities which are important tool to predict upper and lower bounds in various applied sciences e.g in probability theory, functional inequalities and information theory.

We recall that a function $\Upsilon : I \to \mathbb{R}$ is said to be convex on the interval I, if

$$\Upsilon\left(\phi u+\left(1-\phi\right)v\right)\leq \phi\Upsilon\left(u\right)+\left(1-\phi\right)\Upsilon\left(v\right)$$

holds for all $\phi \in [0,1]$ and $u, v \in I$ (see [41]).

The fundamental inequality for convex functions is known as Hermite-Hadamard inequality (see [20, 24]), which can be stated as follows: For any convex function Υ on the interval $[\xi, \Im]$ with $\xi < \Im$, we have

$$\Upsilon\left(\frac{\xi+\Im}{2}\right)\leq \frac{1}{\Im-\xi}\int_{\xi}^{\Im}\Upsilon\left(u\right)du\leq \frac{\Upsilon(\xi)+\Upsilon(\Im)}{2}. \tag{1}$$

Since this discovery, several papers dealing with convex inequalities have been published, see [17, 18, 22, 26, 28, 29, 32, 35, 36, 40, 43, 47, 53, 54, 56].

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* Corresponding author: Abdelghani Lakhdari
Email addresses: badrimeftah@yahoo.fr (Badreddine Meftah), a.lakhdari@ensti-annaba.dz (Abdelghani Lakhdari), wlehabi@taibahu.edu.sa (Wedad Saleh)

Many researchers have devoted a lot of effort to generalizing classical convexity, the most significant generalization is that introduced by Hanson [23], called invex set and functions. Pini [42], Noor [38, 39], Weir et al. [52], and Yang et al. [55] have studied the basic properties of preinvex functions and their roles in optimization, variational inequalities and equilibrium problems.

Fractional calculus is a branch of mathematical analysis that grows out of the classical definitions of integral and derivative operators of noninteger order and provides an excellent tool for the description of the memory and hereditary properties of various materials and processes. It has been successfully used in various fields of science and engineering see [19, 30]. The most used operator is that of Riemann-Liouville defined as follow:

$$I_{\xi^{+}}^{\alpha}(u)=\frac{1}{\Gamma(\alpha)}\int_{\xi}^{u}\left(u-\phi\right)^{\alpha-1}\Upsilon\left(\phi\right)d\phi,\quad u>\xi$$

and

$$I_{\Im^{-}}^{\alpha}(u)=\frac{1}{\Gamma(\alpha)}\int_{u}^{\Im}\left(\phi-u\right)^{\alpha-1}\Upsilon\left(\phi\right)d\phi,\quad \Im>u,$$

where $\alpha>0$, $\Upsilon\in L^{1}[\xi,\Im]$, $\Gamma(\alpha)=\int_{0}^{\infty}e^{-\phi}\phi^{\alpha-1}d\phi$ is the gamma function and $I_{\xi^{+}}^{0}\Upsilon(u)=I_{\Im^{-}}^{0}\Upsilon(u)=\Upsilon(u)$ (see [30]).

Fractional convex inequalities have gained significant attention as a burgeoning area of research. The literature showcases a multitude of generalizations, enhancements, extensions, and novel integral inequalities. For a more comprehensive understanding, the interested reader is directed to references [5–7, 13, 33, 34, 44–46, 48, 49, 51] and the related citations therein.

In recent years, several papers have investigated Milne quadrature (see [1–4, 8–10, 14, 15, 21, 25, 27, 37, 50, 57]). The conventional formulation of this quadrature rule is as follows [3]:

$$\frac{1}{\Im-\xi}\int_{\xi}^{\Im}\Upsilon(u)\,du\simeq\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right).$$

In [16], Djenaoui and Meftah established some Milne-type inequalities for s-convex, bounded, and Lipschitzian functions as follow:

Theorem 1.1 ([16]). *Let $\Upsilon:[\xi,\Im]\to\mathbb{R}$ be a differentiable function on $[\xi,\Im]$ such that $\Upsilon'\in L^{1}[\xi,\Im]$ with $0\le\xi<\Im$. If $|\Upsilon'|$ is s-convex in the second sense for some fixed $s\in(0,1]$, then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{1}{\Im-\xi}\int_{\xi}^{\Im}\Upsilon(u)\,du\right|$$
$$\le\frac{\Im-\xi}{4}\left(\frac{4s+5}{3(s+1)(s+2)}\left|\Upsilon'(\xi)\right|+\frac{2s+10}{3(s+1)(s+2)}\left|\Upsilon'\left(\frac{\xi+\Im}{2}\right)\right|+\frac{4s+5}{3(s+1)(s+2)}\left|\Upsilon'(\Im)\right|\right).$$

Theorem 1.2 ([16]). *Under the assumptions of Theorem 1.1, if $|\Upsilon'|^{q}$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{1}{\Im-\xi}\int_{\xi}^{\Im}\Upsilon(u)\,du\right|$$
$$\le\frac{\Im-\xi}{4(p+1)^{\frac{1}{p}}}\left(\frac{4^{p+1}-1}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{|\Upsilon'(\xi)|^{q}+\left|\Upsilon'\left(\frac{\xi+\Im}{2}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\Upsilon'\left(\frac{\xi+\Im}{2}\right)\right|^{q}+|\Upsilon'(\Im)|^{q}}{s+1}\right)^{\frac{1}{q}}\right).$$

Theorem 1.3 ([16]). *Under the assumptions of Theorem 1.1, if $|\Upsilon'|^q$ is s-convex in the second sense for some fixed $s \in (0,1]$ where $q \geq 1$, then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{1}{\Im-\xi}\int_{\xi}^{\Im}\Upsilon(u)\,du\right|$$
$$\leq\frac{\Im-\xi}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\left(\frac{4s+5}{3(s+1)(s+2)}|\Upsilon'(\xi)|^q+\frac{s+5}{3(s+1)(s+2)}\left|\Upsilon'\left(\frac{\xi+\Im}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{s+5}{3(s+1)(s+2)}\left|\Upsilon'\left(\frac{\xi+\Im}{2}\right)\right|^q+\frac{4s+5}{3(s+1)(s+2)}\left|\Upsilon'(\Im)\right|^q\right)^{\frac{1}{q}}\right).$$

Theorem 1.4 ([16]). *Under the assumptions of Theorem 1.1, if there exist constants $-\infty < m < M < +\infty$ such that $m \leq \Upsilon'(x) \leq M$ for all $x \in [\xi, \Im]$, then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{1}{\Im-\xi}\int_{\xi}^{\Im}\Upsilon(u)\,du\right|\leq\frac{5(\Im-\xi)(M-m)}{24}.$$

Theorem 1.5 ([16]). *Under the assumptions of Theorem 1.1, if Υ' is L-Lipschitzian function on $[\xi, \Im]$, then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{1}{\Im-\xi}\int_{\xi}^{\Im}\Upsilon(u)\,du\right|\leq\frac{7(\Im-\xi)^2}{24}L.$$

Recently, Budak et al. [11], investigated some fractional Milne-type integral inequalities for functions whose first derivatives are convex as follow:

Theorem 1.6 ([11]). *Under the assumptions of Theorem 1.1, if $|\Upsilon'|^q$ is convex on $[\xi, \Im]$, then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Im-\xi)^{\alpha}}\left(I^{\alpha}_{\Im^-}\Upsilon\left(\frac{\xi+\Im}{2}\right)+I^{\alpha}_{\xi^+}\Upsilon\left(\frac{\xi+\Im}{2}\right)\right)\right|$$
$$\leq\frac{\Im-\xi}{12}\left(\frac{\alpha+4}{\alpha+1}\right)\left(|\Upsilon'(\xi)|+\left|\Upsilon'(\Im)\right|\right).$$

Theorem 1.7 ([11]). *Under the assumptions of Theorem 1.1, if $|\Upsilon'|^q$ is convex, where $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have the following inequality:*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Im-\xi)^{\alpha}}\left(I^{\alpha}_{\Im^-}\Upsilon\left(\frac{\xi+\Im}{2}\right)+I^{\alpha}_{\xi^+}\Upsilon\left(\frac{\xi+\Im}{2}\right)\right)\right|$$
$$\leq\frac{\Im-\xi}{4}\left(\int_{\xi}^{\Im}\left(\upsilon^{\alpha}+\frac{1}{3}\right)^p\right)^{\frac{1}{p}}\left(\left(\frac{3|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(\xi)|^q+3\left|\Upsilon'(\Im)\right|^q}{4}\right)^{\frac{1}{q}}\right)$$
$$\leq\frac{\Im-\xi}{4}\left(4\int_{\xi}^{\Im}\left(\upsilon^{\alpha}+\frac{1}{3}\right)^p\right)^{\frac{1}{p}}\left(|\Upsilon'(\xi)|+\left|\Upsilon'(\Im)\right|\right).$$

Theorem 1.8 ([11]). *Under the assumptions of Theorem 1.1, if $|\Upsilon'|^q$ is convex, where $q \geq 1$, then we have the following inequality:*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Im-\xi)^{\alpha}}\left(I^{\alpha}_{\Im^-}\Upsilon\left(\frac{\xi+\Im}{2}\right)+I^{\alpha}_{\xi^+}\Upsilon\left(\frac{\xi+\Im}{2}\right)\right)\right|$$
$$\leq\frac{\Im-\xi}{4}\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left(\frac{\alpha^2+7\alpha+8}{4(\alpha+1)(\alpha+2)}|\Upsilon'(\xi)|^q+\frac{\alpha^2+3\alpha+8}{12(\alpha+1)(\alpha+2)}\left|\Upsilon'(\Im)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\alpha^2+3\alpha+8}{12(\alpha+1)(\alpha+2)}|\Upsilon'(\xi)|^q+\frac{\alpha^2+7\alpha+8}{4(\alpha+1)(\alpha+2)}\left|\Upsilon'(\Im)\right|^q\right)^{\frac{1}{q}}\right).$$

Theorem 1.9 ([11]). *Under the assumptions of Theorem 1.1, if there exist constants* $-\infty < m < M < +\infty$ *such that* $m \le \Upsilon'(x) \le M$ *for all* $x \in [\xi, \Im]$, *then we have*

$$\left|\tfrac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\tfrac{\xi+\Im}{2}\right) + 2\Upsilon(\Im)\right) - \tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Im-\xi)^{\alpha}}\left(I^{\alpha}_{\Im^-}\Upsilon(\tfrac{\xi+\Im}{2}) + I^{\alpha}_{\xi^+}\Upsilon(\tfrac{\xi+\Im}{2})\right)\right|$$
$$\le \tfrac{\Im-\xi}{12}\left(\tfrac{\alpha+4}{\alpha+1}\right)(M-m).$$

Theorem 1.10 ([11]). *Under the assumptions of Theorem 1.1, if* Υ' *is L-Lipschitzian function on* $[\xi, \Im]$, *then we have*

$$\left|\tfrac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\tfrac{\xi+\Im}{2}\right) + 2\Upsilon(\Im)\right) - \tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Im-\xi)^{\alpha}}\left(I^{\alpha}_{\Im^-}\Upsilon(\tfrac{\xi+\Im}{2}) + I^{\alpha}_{\xi^+}\Upsilon(\tfrac{\xi+\Im}{2})\right)\right|$$
$$\le \tfrac{(\Im-\xi)^2}{24}\left(\tfrac{\alpha+8}{\alpha+2}\right)L.$$

The aim of this study is to establish some new Milne-type integral inequalities for functions whose first derivatives are s-preinvex in the second sense via Riemann-Liouville fractional integral operators. Additionally, when classical convexity is taken into consideration, representing a specific scenario, the outcomes obtained from this analysis signify advancements or enhancements over previously established results, especially for $0 < \alpha \le 1$.

The paper is organized as follows: Section 2 is dedicated to recalling several fundamental definitions related to fractional calculus. In Section 3, we show a new identity as a partial result. Based on this equality, we derive some new fractional and classical Milne-type inequalities for various classes of functions. Certain specific cases are discussed. In Section 4, some applications to special means for different positive real numbers involving arithmetic, logarithmic, p-logarithmic and harmonic means are given.

The originality of the results established in our research has links with the results proven in previous works concerning the three-point Newton-Cotes inequalities. Our results open the horizons and stimulate other research in different fields of applied sciences, in particular the field of integral inequalities.

2. Preliminaries

In this section, we revisit a number of definitions essential for our study.

Definition 2.1 ([30]). *The beta function is defined for any* x, y *such that* $Re(x), Re(y) > 0$ *by*

$$B(x,y) = \int_0^1 \phi^{x-1}\left(1-\phi\right)^{y-1} d\phi.$$

Similarly, the incomplete beta function is given for $0 < a < 1$ *by*

$$B_a(x,y) = \int_0^a \phi^{x-1}\left(1-\phi\right)^{y-1} d\phi.$$

Definition 2.2 ([30]). *The hypergeometric function is defined as follows*

$${}_2F_1(a,b,c;z) = \tfrac{1}{B(b,c-b)}\int_0^1 \phi^{b-1}\left(1-\phi\right)^{c-b-1}\left(1-z\phi\right)^{-a} d\phi,$$

where $Re(c) > Re(b) > 0$, $|z| < 1$ *and* $B(.,.)$ *is the beta function.*

Definition 2.3 ([31]). *A nonnegative function* $\Upsilon : K \subset [0,\infty) \to \mathbb{R}$ *is said to be s-preinvex in the second sense with respect to* θ *for some fixed* $s \in (0,1]$, *if*

$$\Upsilon\left(u + \phi\theta(v,u)\right) \le (1-\phi)^s\Upsilon(u) + \phi^s\Upsilon(v)$$

holds for all $u, v \in K$ *and* $\phi \in [0,1]$.

3. Main results

Throughout, we assume that $\theta(\Im,\xi)>0, \alpha>0$ and $s\in(0,1]$.

Lemma 3.1. *For any differentiable function $\Upsilon:[\xi,\xi+\theta(\Im,\xi)]\subset\mathbb{R}\to\mathbb{R}$ of integrable first derivative Υ' on $[\xi,\Im]$, we have*

$$\begin{aligned}&\tfrac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\\&=\tfrac{\theta(\Im,\xi)}{4}\left(\int_0^1\left(\left(1-\phi\right)^{\alpha}-\tfrac{4}{3}\right)\Upsilon'\left(\xi+\tfrac{1-\phi}{2}\theta(\Im,\xi)\right)d\phi-\int_0^1\left(\phi^{\alpha}-\tfrac{4}{3}\right)\Upsilon'\left(\xi+\left(1-\tfrac{\phi}{2}\right)\theta(\Im,\xi)\right)d\phi\right),\end{aligned}\tag{2}$$

where

$$\mathcal{I}(\Upsilon)=\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\theta(\Im,\xi))^{\alpha}}\left(I^{\alpha}_{\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)^-}\Upsilon(\xi)+I^{\alpha}_{\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)^+}\Upsilon(\xi+\theta(\Im,\xi))\right).\tag{3}$$

Proof. Let

$$A=A_1-A_2,\tag{4}$$

where

$$A_1=\int_0^1\left(\left(1-\phi\right)^{\alpha}-\tfrac{4}{3}\right)\Upsilon'\left(\xi+\tfrac{1-\phi}{2}\theta(\Im,\xi)\right)d\phi$$

and

$$A_2=\int_0^1\left(\phi^{\alpha}-\tfrac{4}{3}\right)\Upsilon'\left(\xi+\left(1-\tfrac{\phi}{2}\right)\theta(\Im,\xi)\right)d\phi.$$

Integrating by parts A_1, we get

$$\begin{aligned}A_1&=-\tfrac{2}{\theta(\Im,\xi)}\left(\left(1-\phi\right)^{\alpha}-\tfrac{4}{3}\right)\Upsilon\left(\xi+\tfrac{1-\phi}{2}\theta(\Im,\xi)\right)\Big|_{t=0}^{t=1}-\tfrac{2}{\theta(\Im,\xi)}\int_0^1\left(1-\phi\right)^{\alpha-1}\Upsilon\left(\xi+\tfrac{1-\phi}{2}\theta(\Im,\xi)\right)d\phi\\&=\tfrac{8}{3\theta(\Im,\xi)}\Upsilon(\xi)-\tfrac{2}{3\theta(\Im,\xi)}\Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right)-\tfrac{2}{\theta(\Im,\xi)}\int_0^1\left(1-\phi\right)^{\alpha-1}\Upsilon\left(\xi+\tfrac{1-\phi}{2}\theta(\Im,\xi)\right)d\phi\\&=\tfrac{8}{3\theta(\Im,\xi)}\Upsilon(\xi)-\tfrac{2}{3\theta(\Im,\xi)}\Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right)-\tfrac{2^{\alpha+1}}{(\theta(\Im,\xi))^{\alpha+1}}\int_{\xi}^{\frac{2\xi+\theta(\Im,\xi)}{2}}(u-\xi)^{\alpha-1}\Upsilon(u)\,du\\&=\tfrac{8}{3\theta(\Im,\xi)}\Upsilon(\xi)-\tfrac{2}{3\theta(\Im,\xi)}\Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right)-\tfrac{2^{\alpha+1}\Gamma(\alpha+1)}{(\theta(\Im,\xi))^{\alpha+1}}I^{\alpha}_{\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)^-}\Upsilon(\xi).\end{aligned}\tag{5}$$

Likewise, we get

$$A_2=-\tfrac{2}{\theta(\Im,\xi)}\left(\phi^{\alpha}-\tfrac{4}{3}\right)\Upsilon\left(\xi+\left(1-\tfrac{\phi}{2}\right)\theta(\Im,\xi)\right)\Big|_{t=0}^{t=1}+\tfrac{2\alpha}{\theta(\Im,\xi)}\int_0^1\phi^{\alpha-1}\Upsilon\left(\xi+\left(1-\tfrac{\phi}{2}\right)\theta(\Im,\xi)\right)d\phi$$

$$=\frac{2}{3\theta(\Im,\xi)}\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)-\frac{8}{3\theta(\Im,\xi)}\Upsilon(\xi+\theta(\Im,\xi))+\frac{2\alpha}{\theta(\Im,\xi)}\int_0^1\phi^{\alpha-1}\Upsilon\left(\xi+\left(1-\frac{\phi}{2}\right)\theta(\Im,\xi)\right)d\phi$$

$$=\frac{2}{3\theta(\Im,\xi)}\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)-\frac{8}{3\theta(\Im,\xi)}\Upsilon(\xi+\theta(\Im,\xi))+\frac{2^{\alpha+1}\alpha}{(\theta(\Im,\xi))^{\alpha+1}}\int_{\frac{2\xi+\theta(\Im,\xi)}{2}}^{\xi+\theta(\Im,\xi)}(\xi+\theta(\Im,\xi)-u)^{\alpha-1}\Upsilon(u)\,du$$

$$=\frac{2}{3\theta(\Im,\xi)}\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)-\frac{8}{3\theta(\Im,\xi)}\Upsilon(\xi+\theta(\Im,\xi))+\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\theta(\Im,\xi))^{\alpha+1}}I^{\alpha}_{\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)^+}\Upsilon(\xi+\theta(\Im,\xi)). \tag{6}$$

Substituting (5) and (6) into (4), then multiplying the resulting equality by $\frac{\theta(\Im,\xi)}{4}$, we get the required result. □

Theorem 3.2. *Under the assumptions of Lemma 3.1, if $|\Upsilon'|$ is s-preinvex, then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\right|$$
$$\leq\frac{\theta(\Im,\xi)}{4}\left(\frac{2^{s+3}(\alpha+s+1)-3(s+1)}{2^s\times3(s+1)(\alpha+s+1)}-2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)\left(|\Upsilon'(\xi)|+|\Upsilon'(\Im)|\right),$$

where $B_{\frac{1}{2}}(.,.)$ is the incomplete beta function.

Proof. Applying the absolute value to both sides of (2), then using the s-preinvexity of $|\Upsilon'|$, we get

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\right|$$
$$\leq\frac{\theta(\Im,\xi)}{4}\left(\int_0^1\left|(1-\phi)^{\alpha}-\frac{4}{3}\right|\left|\Upsilon'\left(\xi+\frac{1-\phi}{2}\theta(\Im,\xi)\right)\right|d\phi+\int_0^1\left|\phi^{\alpha}-\frac{4}{3}\right|\left|\Upsilon'\left(\xi+\left(1-\frac{\phi}{2}\right)\theta(\Im,\xi)\right)\right|d\phi\right)$$
$$\leq\frac{\theta(\Im,\xi)}{4}\left(\int_0^1\left(\frac{4}{3}-(1-\phi)^{\alpha}\right)\left(\left(1-\frac{1-\phi}{2}\right)^s|\Upsilon'(\xi)|+\left(\frac{1-\phi}{2}\right)^s|\Upsilon'(\Im)|\right)d\phi\right.$$
$$\left.+\int_0^1\left(\frac{4}{3}-\phi^{\alpha}\right)\left(\left(1-\left(1-\frac{\phi}{2}\right)\right)^s|\Upsilon'(\xi)|+\left(1-\frac{\phi}{2}\right)^s|\Upsilon'(\Im)|\right)d\phi\right)$$
$$=\frac{\theta(\Im,\xi)}{4}\left(\int_0^1\left(\frac{4}{3}-(1-\phi)^{\alpha}\right)\left(\left(1-\frac{1-\phi}{2}\right)^s|\Upsilon'(\xi)|+\left(\frac{1-\phi}{2}\right)^s|\Upsilon'(\Im)|\right)d\phi\right.$$
$$\left.+\int_0^1\left(\frac{4}{3}-\phi^{\alpha}\right)\left(\left(1-\left(1-\frac{\phi}{2}\right)\right)^s|\Upsilon'(\xi)|+\left(1-\frac{\phi}{2}\right)^s|\Upsilon'(\Im)|\right)d\phi\right)$$
$$=\frac{\theta(\Im,\xi)}{4}\left(|\Upsilon'(\xi)|\left(\int_0^1\left(\frac{4}{3}-(1-\phi)^{\alpha}\right)\left(1-\frac{1-\phi}{2}\right)^s d\phi+\int_0^1\left(\frac{4}{3}-\phi^{\alpha}\right)\left(1-\left(1-\frac{\phi}{2}\right)\right)^s d\phi\right)\right.$$
$$\left.+|\Upsilon'(\Im)|\left(\int_0^1\left(\frac{4}{3}-(1-\phi)^{\alpha}\right)\left(\frac{1-\phi}{2}\right)^s d\phi+\int_0^1\left(\frac{4}{3}-\phi^{\alpha}\right)\left(1-\frac{\phi}{2}\right)^s d\phi\right)\right)$$
$$=\frac{\theta(\Im,\xi)}{4}\left(\frac{2^{s+3}(\alpha+s+1)-3(s+1)}{2^s\times3(s+1)(\alpha+s+1)}-2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)\left(|\Upsilon'(\xi)|+|\Upsilon'(\Im)|\right),$$

where we have used the fact that

$$\begin{aligned}\int_0^1 \left(\tfrac{4}{3} - (1-\phi)^{\alpha}\right)\left(1 - \tfrac{1-\phi}{2}\right)^s d\phi &= \int_0^1 \left(\tfrac{4}{3} - \phi^{\alpha}\right)\left(1 - \tfrac{\phi}{2}\right)^s d\phi \\ &= 2\int_0^{\frac{1}{2}} \left(\tfrac{4}{3}(1-\tau)^s - 2^{\alpha}\tau^{\alpha}(1-\tau)^s\right) d\tau \\ &= \tfrac{8}{3(s+1)}\left(1 - \left(\tfrac{1}{2}\right)^{s+1}\right) - 2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1, s+1) \end{aligned} \tag{7}$$

and

$$\begin{aligned}\int_0^1 \left(\tfrac{4}{3} - \phi^{\alpha}\right)\left(1 - \left(1 - \tfrac{\phi}{2}\right)\right)^s d\phi &= \int_0^1 \left(\tfrac{4}{3} - (1-\phi)^{\alpha}\right)\left(\tfrac{1-\phi}{2}\right)^s d\phi \\ &= \tfrac{1}{2^s}\int_0^1 \left(\tfrac{4}{3}\tau^s - \tau^{\alpha+s}\right) d\tau = \tfrac{4\alpha+(s+1)}{2^s 3(s+1)(\alpha+s+1)}. \end{aligned} \tag{8}$$

The proof is completed. □

Corollary 3.3. *In Theorem 3.2, taking $s = 1$, we obtain*

$$\begin{aligned}&\left|\tfrac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi + \theta(\Im,\xi))\right) - \mathcal{I}(\Upsilon)\right| \\ &\leq \tfrac{\theta(\Im,\xi)}{12}\left(\tfrac{4\alpha+1}{\alpha+1}\right)\left(|\Upsilon'(\xi)| + |\Upsilon'(\Im)|\right).\end{aligned}$$

Corollary 3.4. *In Theorem 3.2, taking $\alpha = 1$, we obtain*

$$\begin{aligned}&\left|\tfrac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi + \theta(\Im,\xi))\right) - \tfrac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)} \Upsilon(u)\,du\right| \\ &\leq \tfrac{\theta(\Im,\xi)}{12}\left(\tfrac{2^{s+1}(2s+1)+3}{2^{s-1}(s+1)(s+2)}\right)\left(|\Upsilon'(\xi)| + |\Upsilon'(\Im)|\right).\end{aligned}$$

Corollary 3.5. *In Theorem 3.2, taking $\alpha = s = 1$, we obtain*

$$\begin{aligned}&\left|\tfrac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi + \theta(\Im,\xi))\right) - \tfrac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)} \Upsilon(u)\,du\right| \\ &\leq \tfrac{5\theta(\Im,\xi)}{24}\left(|\Upsilon'(\xi)| + |\Upsilon'(\Im)|\right).\end{aligned}$$

Corollary 3.6. *In Theorem 3.2, choosing $\theta(\Im,\xi) = \Im - \xi$, we obtain*

$$\begin{aligned}&\left|\tfrac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\tfrac{\xi+\Im}{2}\right) + 2\Upsilon(\Im)\right) - \widehat{\mathcal{I}}(\Upsilon)\right| \\ &\leq \tfrac{\Im-\xi}{4}\left(\tfrac{2^{s+3}(\alpha+s+1)-3(s+1)}{2^s\times 3(s+1)(\alpha+s+1)} - 2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)\left(|\Upsilon'(\xi)| + |\Upsilon'(\Im)|\right),\end{aligned}$$

where

$$\widehat{\mathcal{I}}(\Upsilon) = \tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\Im-\xi)^{\alpha}}\left(I^{\alpha}_{\left(\frac{\xi+\Im}{2}\right)^-}\Upsilon(\xi) + I^{\alpha}_{\left(\frac{\xi+\Im}{2}\right)^+}\Upsilon(\Im)\right). \tag{9}$$

Remark 3.7. *For $s=1$, Corollary 3.6 is the same as the second inequality of Theorem 2.2 from [12].*

Corollary 3.8. *In Corollary 3.6, taking $\alpha=1$, we obtain*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{1}{\Im-\xi}\int_{\xi}^{\Im}\Upsilon(u)\,du\right| \leq \frac{\Im-\xi}{12}\left(\frac{2^{s+1}(2s+1)+3}{2^{s-1}(s+1)(s+2)}\right)\left(|\Upsilon'(\xi)|+|\Upsilon'(\Im)|\right).$$

Remark 3.9. *The outcome derived in Corollary 3.8 refines the second inequality of Corollary 2.3 in reference [16], as depicted by the curves in Figure 1, where the red curve represents the coefficient of $\frac{\Im-\xi}{12}\left(|\Upsilon'(\xi)|+|\Upsilon'(\Im)|\right)$ from Corollary 3.8, and the blue curve represents the coefficient of the same term established in [16], given repectively by $\frac{2^{s+1}(2s+1)+3}{2^{s-1}(s+1)(s+2)}$ and $\frac{2^{2-s}(s+5)+(4s+5)(s+1)}{(s+1)^2(s+2)}$.*

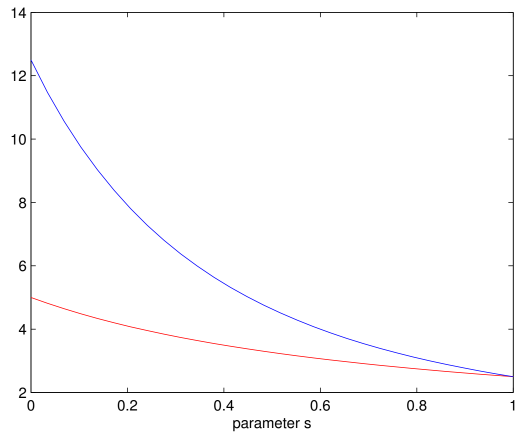


Figure 1: $s\in(0,1]$

Remark 3.10. *By setting $\alpha=s=1$, Corollary 3.6 will be reduced to Corollary 2.4. from [16] and Remark 1 from [11].*

Example 3.11. *Let us consider the function $\Upsilon:[0,1]\to\mathbb{R}$ defined by $\Upsilon(u)=\frac{u^{s+1}}{s+1}$ and $\theta(\Im,\xi)=\Im-\xi$, so that $\Upsilon'(u)=u^s$ which is s-convex on $[0,1]$ for $s\in(0,1]$.*

By the definition of the Riemann-Liouville integrals, we have

$$I^{\alpha}_{\left(\frac{1}{2}\right)^-}\Upsilon(0)=\frac{1}{(s+1)(s+\alpha+1)\Gamma(\alpha)}\left(\frac{1}{2}\right)^{s+\alpha+1} \tag{10}$$

and

$$I^{\alpha}_{\left(\frac{1}{2}\right)^+}\Upsilon(1)=\frac{1}{(s+1)\Gamma(\alpha)}B_{\frac{1}{2}}(\alpha,s+2), \tag{11}$$

where $B(.,.)$ is the incomplete beta function. From (10) and (11), we obtain

$$\widehat{\mathcal{I}}(\Upsilon)=\frac{2^{\alpha-1}\alpha}{s+1}\left(\frac{1}{2^{s+\alpha+1}(s+\alpha+1)}+B_{\frac{1}{2}}(\alpha,s+2)\right).$$

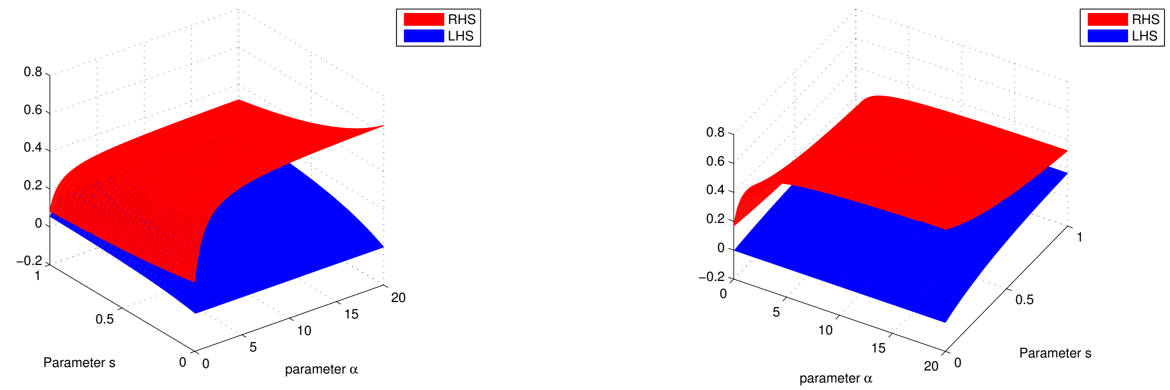
Thus, the left side (LHS) of Corollary 3.6 is reduced to

$$LHS := \left|\frac{2^{s+2}-1}{3(s+1)2^{s+1}} - \frac{2^{\alpha-1}\alpha}{s+1}\left(\frac{1}{2^{s+\alpha+1}(s+\alpha+1)} + B_{\frac{1}{2}}(\alpha, s+2)\right)\right|.$$

The right hand side (RHS) of Corollary 3.6 is reduced to

$$RHS := \frac{1}{4}\left(\frac{2^{s+3}(\alpha+s+1)-3(s+1)}{2^s\times 3(s+1)(\alpha+s+1)} - 2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1, s+1)\right).$$

The outcomes from Example 3.11 are illustrated in Figure 2.



(a) View.1 (b) View.2

Figure 2: $\alpha \in (0, 20]$ and $s \in (0, 1]$

Theorem 3.12. *Under the assumptions of Lemma 3.1, if $|\Upsilon'|^q$ is convex, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\mathfrak{I},\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\mathfrak{I},\xi))\right) - \mathcal{I}(\Upsilon)\right|$$
$$\leq \frac{\theta(\mathfrak{I},\xi)}{3}\left({}_2F_1\left(-p, \frac{1}{\alpha}, \frac{1}{\alpha}+1; \frac{3}{4}\right)\right)^{\frac{1}{p}}\left(\left(\frac{(2^{s+1}-1)|\Upsilon'(\xi)|^q + |\Upsilon'(\mathfrak{I})|^q}{2(s+1)}\right)^{\frac{1}{q}} + \left(\frac{|\Upsilon'(\xi)|^q + (2^{s+1}-1)|\Upsilon'(\mathfrak{I})|^q}{2(s+1)}\right)^{\frac{1}{q}}\right),$$

where ${}_2F_1(.,.,.;.)$ is the hypergeometric function.

Proof. Applying the absolute value to both sides of (2), Hölder's inequality and s-preinvexity of $|\Upsilon'|^q$, we get

$$\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\mathfrak{I},\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\mathfrak{I},\xi))\right) - \mathcal{I}(\Upsilon)\right|$$
$$\leq \frac{\theta(\mathfrak{I},\xi)}{4}\left(\left(\int_0^1\left(\frac{4}{3} - (1-\phi)^\alpha\right)^p d\phi\right)^{\frac{1}{p}}\left(\int_0^1\left|\Upsilon'\left(\xi + \frac{1-\phi}{2}\theta(\mathfrak{I},\xi)\right)\right|^q d\phi\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1\left(\frac{4}{3} - \phi^\alpha\right)^p d\phi\right)^{\frac{1}{p}}\left(\int_0^1\left|\Upsilon'\left(\xi + \left(1-\frac{\phi}{2}\right)\theta(\mathfrak{I},\xi)\right)\right|^q d\phi\right)^{\frac{1}{q}}\right)$$
$$\leq \frac{\theta(\mathfrak{I},\xi)}{4}\left(\int_0^1\left(\frac{4}{3} - \phi^\alpha\right)^p d\phi\right)^{\frac{1}{p}}\left(\left(\int_0^1\left(\left(1-\frac{1-\phi}{2}\right)^s|\Upsilon'(\xi)|^q + \left(\frac{1-\phi}{2}\right)^s|\Upsilon'(\mathfrak{I})|^q\right)d\phi\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left(\left(1-\left(1-\frac{\phi}{2}\right)\right)^s|\Upsilon'(\xi)|^q+\left(1-\frac{\phi}{2}\right)^s\left|\Upsilon'(\Im)\right|^q\right)d\phi\right)^{\frac{1}{q}}\Bigg)$$

$$=\frac{\theta(\Im,\xi)}{3}\left({}_2F_1\left(-p,\frac{1}{\alpha},\frac{1}{\alpha}+1;\frac{3}{4}\right)\right)^{\frac{1}{p}}\left(\left(\frac{\left(2^{s+1}-1\right)|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{2(s+1)}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(\xi)|^q+\left(2^{s+1}-1\right)\left|\Upsilon'(\Im)\right|^q}{2(s+1)}\right)^{\frac{1}{q}}\right),$$

where we have used

$$\int_0^1\left(\frac{4}{3}-\phi^\alpha\right)^p d\phi=\frac{1}{\alpha}\left(\frac{4}{3}\right)^p\int_0^1\left(1-\frac{3}{4}u\right)^p u^{\frac{1}{\alpha}-1}du$$
$$=\left(\frac{4}{3}\right)^p {}_2F_1\left(-p,\frac{1}{\alpha},\frac{1}{\alpha}+1;\frac{3}{4}\right).$$

The proof is completed. □

Corollary 3.13. *In Theorem 3.12, taking $s=1$, we obtain*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\right|$$
$$\leq\frac{\theta(\Im,\xi)}{3}\left({}_2F_1\left(-p,\frac{1}{\alpha},\frac{1}{\alpha}+1;\frac{3}{4}\right)\right)^{\frac{1}{p}}\left(\left(\frac{3|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(\xi)|^q+3\left|\Upsilon'(\Im)\right|^q}{4}\right)^{\frac{1}{q}}\right).$$

Corollary 3.14. *In Theorem 3.12, taking $\alpha=1$, we obtain*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\frac{1}{\theta(\Im,\xi)}\int_\xi^{\xi+\theta(\Im,\xi)}\Upsilon(u)\,du\right|$$
$$\leq\frac{\theta(\Im,\xi)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{\left(2^{s+1}-1\right)|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{2(s+1)}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(\xi)|^q+\left(2^{s+1}-1\right)\left|\Upsilon'(\Im)\right|^q}{2(s+1)}\right)^{\frac{1}{q}}\right).$$

Corollary 3.15. *In Theorem 3.12, taking $\alpha=s=1$, we obtain*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\frac{1}{\theta(\Im,\xi)}\int_\xi^{\xi+\theta(\Im,\xi)}\Upsilon(u)\,du\right|$$
$$\leq\frac{\theta(\Im,\xi)}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{3|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(\xi)|^q+3\left|\Upsilon'(\Im)\right|^q}{4}\right)^{\frac{1}{q}}\right).$$

Remark 3.16. *Corollary 3.15 will be reduced to Corollary 1 from [11], and second inequality of Corollary 2.8 from [16], if we choose $\theta(\Im,\xi)=\Im-\xi$.*

Corollary 3.17. *In Theorem 3.12, using the discrete power mean inequality i.e. $\xi^\lambda+\Im^\lambda\leq 2^{1-\lambda}(\xi+\Im)^\lambda$ for $\xi,\Im>0$ and $0\leq\lambda\leq1$, we get*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\right|$$
$$\leq\frac{2\theta(\Im,\xi)}{3}\left({}_2F_1\left(-p,\frac{1}{\alpha},\frac{1}{\alpha}+1;\frac{3}{4}\right)\right)^{\frac{1}{p}}\left(\frac{2^s}{s+1}\right)^{\frac{1}{q}}\left(\frac{|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.18. *In Corollary 3.17, taking $s=1$, we obtain*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\right|$$
$$\leq\frac{2\theta(\Im,\xi)}{3}\left({}_2F_1\left(-p,\frac{1}{\alpha},\frac{1}{\alpha}+1;\frac{3}{4}\right)\right)^{\frac{1}{p}}\left(\frac{|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.19. *In Corollary 3.17, taking $\alpha = 1$, we obtain*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\Im,\xi))\right) - \frac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)}\Upsilon(u)\,du\right|$$
$$\leq \frac{\theta(\Im,\xi)}{6}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{2^s}{s+1}\right)^{\frac{1}{q}}\left(\frac{|\Upsilon'(\xi)|^q+|\Upsilon'(\Im)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.20. *In Corollary 3.17, taking $\alpha = s = 1$, we obtain*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\Im,\xi))\right) - \frac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)}\Upsilon(u)\,du\right|$$
$$\leq \frac{\theta(\Im,\xi)}{6}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Upsilon'(\xi)|^q+|\Upsilon'(\Im)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 3.21. *Under the assumptions of Lemma 3.1, if $|\Upsilon'|^q$ is convex where $q \geq 1$, then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\Im,\xi))\right) - \mathcal{I}(\Upsilon)\right|$$
$$\leq \frac{\theta(\Im,\xi)}{4}\left(\frac{4\alpha+1}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{8(2^{s+1}-1)}{2^{s+1}3(s+1)} - 2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)|\Upsilon'(\xi)|^q + \frac{4\alpha+(s+1)}{2^s3(s+1)(\alpha+s+1)}|\Upsilon'(\Im)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{4\alpha+(s+1)}{2^s3(s+1)(\alpha+s+1)}|\Upsilon'(\xi)|^q + \left(\frac{8(2^{s+1}-1)}{2^{s+1}3(s+1)} - 2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)|\Upsilon'(\Im)|^q\right)^{\frac{1}{q}}\right],$$

where $B_{\frac{1}{2}}(.,.)$ is the incomplete beta function.

Proof. Applying the absolute value to both sides of (2), power mean inequality and s-preinvexity of $|\Upsilon'|^q$, we get

$$\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\Im,\xi))\right) - \mathcal{I}(\Upsilon)\right|$$
$$\leq \frac{\theta(\Im,\xi)}{4}\left[\left(\int_0^1\left(\frac{4}{3}-(1-\phi)^\alpha\right)d\phi\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{4}{3}-(1-\phi)^\alpha\right)\left|\Upsilon'\left(\xi+\frac{1-\phi}{2}\theta(\Im,\xi)\right)\right|^q d\phi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left(\frac{4}{3}-\phi^\alpha\right)d\phi\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{4}{3}-\phi^\alpha\right)\left|\Upsilon'\left(\xi+\left(1-\frac{\phi}{2}\right)\theta(\Im,\xi)\right)\right|^q d\phi\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\theta(\Im,\xi)}{4}\left(\int_0^1\left(\frac{4}{3}-\phi^\alpha\right)d\phi\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left(\frac{4}{3}-(1-\phi)^\alpha\right)\left(\left(1-\frac{1-\phi}{2}\right)^s|\Upsilon'(\xi)|^q + \left(\frac{1-\phi}{2}\right)^s|\Upsilon'(\Im)|^q\right)d\phi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left(\frac{4}{3}-\phi^\alpha\right)\left(\left(1-\left(1-\frac{\phi}{2}\right)\right)^s|\Upsilon'(\xi)|^q + \left(1-\frac{\phi}{2}\right)^s|\Upsilon'(\Im)|^q\right)d\phi\right)^{\frac{1}{q}}\right]$$
$$= \frac{\theta(\Im,\xi)}{4}\left(\frac{4\alpha+1}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\left(|\Upsilon'(\xi)|^q\int_0^1\left(\frac{4}{3}-(1-\phi)^\alpha\right)\left(1-\frac{1-\phi}{2}\right)^s d\phi + |\Upsilon'(\Im)|^q\int_0^1\left(\frac{4}{3}-(1-\phi)^\alpha\right)\left(\frac{1-\phi}{2}\right)^s d\phi\right)^{\frac{1}{q}}\right.$$

$$+\left(|\Upsilon'(\xi)|^q\int_0^1\left(\tfrac{4}{3}-\phi^\alpha\right)\left(\tfrac{\phi}{2}\right)^s d\phi+\left|\Upsilon'(\Im)\right|^q\int_0^1\left(\tfrac{4}{3}-\phi^\alpha\right)\left(1-\tfrac{\phi}{2}\right)^s d\phi\right)^{\frac{1}{q}}\Bigg]$$

$$=\tfrac{\theta(\Im,\xi)}{4}\left(\tfrac{4\alpha+1}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\left(\left(\tfrac{8\left(2^{s+1}-1\right)}{2^{s+1}3(s+1)}-2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)|\Upsilon'(\xi)|^q+\tfrac{4\alpha+(s+1)}{2^{s}3(s+1)(\alpha+s+1)}\left|\Upsilon'(\Im)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\tfrac{4\alpha+(s+1)}{2^{s}3(s+1)(\alpha+s+1)}|\Upsilon'(\xi)|^q+\left(\tfrac{8\left(2^{s+1}-1\right)}{2^{s+1}3(s+1)}-2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)\left|\Upsilon'(\Im)\right|^q\right)^{\frac{1}{q}}\right],$$

where we have used (7) and (8). The proof is finished. □

Corollary 3.22. *In Theorem 3.21, taking $s = 1$, we obtain*

$$\left|\tfrac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\right|$$

$$\leq\tfrac{(4\alpha+1)\theta(\Im,\xi)}{12(\alpha+1)}\left(\left(\tfrac{\left(6\alpha^2+15\alpha+3\right)|\Upsilon'(\xi)|^q+\left(2\alpha^2+3\alpha+1\right)\left|\Upsilon'(\Im)\right|^q}{8\alpha^2+18\alpha+4}\right)^{\frac{1}{q}}+\left(\tfrac{\left(2\alpha^2+3\alpha+1\right)|\Upsilon'(\xi)|^q+\left(6\alpha^2+15\alpha+3\right)\left|\Upsilon'(\Im)\right|^q}{8\alpha^2+18\alpha+4}\right)^{\frac{1}{q}}\right).$$

Corollary 3.23. *In Theorem 3.21, taking $\alpha = 1$, we obtain*

$$\left|\tfrac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\tfrac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)}\Upsilon(u)\,du\right|$$

$$\leq\tfrac{5\theta(\Im,\xi)}{24}\left(\left(\tfrac{\left(2^{1-s}(1-s)+16s+8\right)|\Upsilon'(\xi)|^q+2^{1-s}(s+5)\left|\Upsilon'(\Im)\right|^q}{5(s+1)(s+2)}\right)^{\frac{1}{q}}+\left(\tfrac{2^{1-s}(s+5)|\Upsilon'(\xi)|^q+\left(2^{1-s}(1-s)+16s+8\right)\left|\Upsilon'(\Im)\right|^q}{5(s+1)(s+2)}\right)^{\frac{1}{q}}\right).$$

Corollary 3.24. *In Theorem 3.21, taking $\alpha = s = 1$, we obtain*

$$\left|\tfrac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\tfrac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)}\Upsilon(u)\,du\right|$$

$$\leq\tfrac{5\theta(\Im,\xi)}{24}\left(\left(\tfrac{4|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{5}\right)^{\frac{1}{q}}+\left(\tfrac{|\Upsilon'(\xi)|^q+4\left|\Upsilon'(\Im)\right|^q}{5}\right)^{\frac{1}{q}}\right).$$

Remark 3.25. *Corollary 3.24 will be reduced to Corollary 2.11. from [16] and that of Remark 2 from [11], if we choose $\theta(\Im,\xi)=\Im-\xi$.*

Corollary 3.26. *In Theorem 3.21, using the discrete power mean inequality, we get*

$$\left|\tfrac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\right|\leq\tfrac{(4\alpha+1)\theta(\Im,\xi)}{6(\alpha+1)}$$

$$\times\left(\tfrac{(\alpha+1)\left(2^{s+3}(\alpha+s+1)-3(s+1)\right)}{2^{s}(s+1)(\alpha+s+1)(4\alpha+1)}-2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)^{\frac{1}{q}}\left(\tfrac{|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.27. *In Corollary 3.26, taking $s = 1$, we obtain*

$$\left|\tfrac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\tfrac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\right|$$

$$\leq\tfrac{(4\alpha+1)\theta(\Im,\xi)}{6(\alpha+1)}\left(\tfrac{|\Upsilon'(\xi)|^q+\left|\Upsilon'(\Im)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.28. *In Corollary 3.26, taking $\alpha = 1$, we obtain*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\Im,\xi))\right) - \frac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)} \Upsilon(u)\,du\right|$$
$$\leq \frac{5\theta(\Im,\xi)}{12}\left(\frac{\left(2^{1-s}6+8(2s+1)\right)}{5(s+1)(s+2)}\right)^{\frac{1}{q}}\left(\frac{|\Upsilon'(\xi)|^{q}+\left|\Upsilon'(\Im)\right|^{q}}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.29. *In Corollary 3.26, taking $\alpha = s = 1$, we obtain*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\Im,\xi))\right) - \frac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)} \Upsilon(u)\,du\right|$$
$$\leq \frac{5\theta(\Im,\xi)}{12}\left(\frac{|\Upsilon'(\xi)|^{q}+\left|\Upsilon'(\Im)\right|^{q}}{2}\right)^{\frac{1}{q}}.$$

Theorem 3.30. *Under the assumptions of Lemma 3.1, if Υ' is r-L-Hölderian function on $[\xi, \xi+\theta(\Im,\xi)]$ (i.e. there exist $L > 0$ and $0 < r \leq 1$ such that $\left|\Upsilon'(x) - \Upsilon'(y)\right| \leq L\left|x-y\right|^{r}$), then we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\Im,\xi))\right) - \mathcal{I}(\Upsilon)\right|$$
$$\leq \frac{(\theta(\Im,\xi))^{r+1}}{4}L\left(\frac{4}{3(r+1)} - B(r+1,\alpha+1)\right).$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\Im,\xi))\right) - \mathcal{I}(\Upsilon)\\
&= \frac{\theta(\Im,\xi)}{4}\left(\int_0^1 \left(\left(1-\phi\right)^{\alpha} - \frac{4}{3}\right)\Upsilon'\left(\xi + \frac{1-\phi}{2}\theta(\Im,\xi)\right)d\phi - \int_0^1 \left(\phi^{\alpha} - \frac{4}{3}\right)\Upsilon'\left(\xi + \left(1-\frac{\phi}{2}\right)\theta(\Im,\xi)\right)d\phi\right)\\
&= \frac{\theta(\Im,\xi)}{4}\left(\int_0^1 \left(\left(1-\phi\right)^{\alpha} - \frac{4}{3}\right)\Upsilon'\left(\xi + \frac{1-\phi}{2}\theta(\Im,\xi)\right)d\phi - \int_0^1 \left(\left(1-\phi\right)^{\alpha} - \frac{4}{3}\right)\Upsilon'\left(\xi + \frac{1+\phi}{2}\theta(\Im,\xi)\right)d\phi\right)\\
&= \frac{\theta(\Im,\xi)}{4}\int_0^1 \left(\left(1-\phi\right)^{\alpha} - \frac{4}{3}\right)\left(\Upsilon'\left(\xi + \frac{1-\phi}{2}\theta(\Im,\xi)\right) - \Upsilon'\left(\xi + \frac{1+\phi}{2}\theta(\Im,\xi)\right)\right)d\phi. \qquad (12)
\end{aligned}$$

Applying the absolute value in both sides of (12), and by using the fact that Υ' is r-L-Hölderian

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\Upsilon(\xi) - \Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right) + 2\Upsilon(\xi+\theta(\Im,\xi))\right) - \mathcal{I}(\Upsilon)\right|\\
&\leq \frac{\theta(\Im,\xi)}{4}\int_0^1 \left(\frac{4}{3} - \left(1-\phi\right)^{\alpha}\right)\left|\Upsilon'\left(\xi + \frac{1-\phi}{2}\theta(\Im,\xi)\right) - \Upsilon'\left(\xi + \frac{1+\phi}{2}\theta(\Im,\xi)\right)\right|d\phi\\
&\leq \frac{(\theta(\Im,\xi))^{r+1}}{4}L\int_0^1 \left(\frac{4}{3} - \left(1-\phi\right)^{\alpha}\right)\phi^{r}d\phi\\
&= \frac{(\theta(\Im,\xi))^{r+1}}{4}L\left(\int_0^1 \left(\frac{4}{3}\phi^{r} - \phi^{r}\left(1-\phi\right)^{\alpha}\right)d\phi\right)
\end{aligned}$$

$$=\frac{(\theta(\Im,\xi))^{r+1}}{4}L\left(\frac{4}{3(r+1)}-B(r+1,\alpha+1)\right).$$

The proof is over. □

Corollary 3.31. *In Theorem 3.30, taking $r=1$, we get*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\mathcal{I}(\Upsilon)\right|\leq\frac{\left(2\alpha^2+6\alpha+1\right)(\theta(\Im,\xi))^2}{12(\alpha+1)(\alpha+2)}L.$$

Moreover, if we choose $\theta(\Im,\xi)=\Im-\xi$, we get

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\widehat{\mathcal{I}}(\Upsilon)\right|\leq\frac{\left(2\alpha^2+6\alpha+1\right)(\Im-\xi)^2}{12(\alpha+1)(\alpha+2)}L,$$

where $\widehat{\mathcal{I}}(\Upsilon)$ is defined as in (9)

Corollary 3.32. *In Theorem 3.30, taking $\alpha=1$, we have*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\frac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)}\Upsilon(u)\,du\right|\leq\frac{(4r+5)(\theta(\Im,\xi))^{r+1}}{12(r+1)(r+2)}L.$$

Moreover, if we choose $\theta(\Im,\xi)=\Im-\xi$, we get

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{\xi+\Im}{2}\right)+2\Upsilon(\Im)\right)-\frac{1}{\Im-\xi}\int_{\xi}^{\Im}\Upsilon(u)\,du\right|\leq\frac{(4r+5)(\Im-\xi)^{r+1}}{12(r+1)(r+2)}L.$$

Corollary 3.33. *In Theorem 3.30, taking $\alpha=r=1$, we get*

$$\left|\frac{1}{3}\left(2\Upsilon(\xi)-\Upsilon\left(\frac{2\xi+\theta(\Im,\xi)}{2}\right)+2\Upsilon(\xi+\theta(\Im,\xi))\right)-\frac{1}{\theta(\Im,\xi)}\int_{\xi}^{\xi+\theta(\Im,\xi)}\Upsilon(u)\,du\right|\leq\frac{(\theta(\Im,\xi))^2}{8}L.$$

Moreover, if we choose $\theta(\Im,\xi)=\Im-\xi$, we get Corollary 4 from [11].

4. Applications

Let $\xi,\Im$ be two arbitrary real numbers. Then, we have:
Arithmetic mean: $A(\xi,\Im)=\frac{\xi+\Im}{2}$.
Logarithmic mean: $L(\xi,\Im)=\frac{\Im-\xi}{\ln\Im-\ln\xi}$, $\xi,\Im>0$ and $\xi\neq\Im$.
p-Logarithmic mean: $L_p(\xi,\Im)=\left(\frac{\Im^{p+1}-\xi^{p+1}}{(p+1)(\Im-\xi)}\right)^{\frac{1}{p}}$, $\xi,\Im>0,\xi\neq\Im$ and $p\in\mathbb{R}\setminus\{-1,0\}$.
Harmonic mean: $H(\xi,\Im)=\frac{2\xi\Im}{\xi+\Im}$, $\xi,\Im>0$.

Proposition 4.1. *Let $0<\xi<\Im$. Then, we have*

$$\left|4H(\xi,\xi+A(\xi,\Im))-A^{-1}(\xi,\xi+A(\xi,\Im))-3L^{-1}(\xi,\xi+A(\xi,\Im))\right|$$
$$\leq\frac{5}{4}A(\xi,\Im)H^{-1}\left(\xi^2,\Im^2\right).$$

Proof. Applying Corollary 3.5 with $\theta(\Im,\xi)=A(\xi,\Im)$, to the function $\Upsilon(u)=\frac{1}{u}$. □

Proposition 4.2. *Let $0<\xi<\Im, q,p>1$ and $s\in(0.1]$. Then, we have*

$$\left|4A\left(\xi^{1+\frac{s}{q}},\Im^{1+\frac{s}{q}}\right)-A^{1+\frac{s}{q}}(\xi,\Im)-3L_{1+\frac{s}{q}}^{1+\frac{s}{q}}(\xi,\Im)\right|$$
$$\leq\frac{\Im-\xi}{2}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{2^s}{s+1}\right)^{\frac{1}{q}}\left(1+\frac{s}{q}\right)\left(\frac{\xi^s+\Im^s}{2}\right)^{\frac{1}{q}}.$$

Proof. Using Corollary 3.19 with $\theta(\Im,\xi)=\Im-\xi$, to the function $\Upsilon(u)=u^{1+\frac{s}{q}}$. □

5. Conclusion

In this study, we have explored fractional Milne-type integral inequalities for functions with s-preinvex first derivatives. We began by demonstrating a novel integral identity and subsequently established a series of fresh Milne-type inequalities that incorporate the Riemann-Liouville integral operator. Furthermore, we addressed specific scenarios and offered practical applications of our findings. We trust that the concepts presented in this paper will inspire researchers engaged in inequality studies to extend our findings to various forms of classical and generalized convexity, as well as to broaden this research into different realms of calculus.

References

[1] A. A. Almoneef, A.-A. Hyder and H. Budak, *Weighted Milne-type inequalities through Riemann-Liouville fractional integrals and diverse function classes*, AIMS Math. **9** (2024), no. 7, 18417–18439.
[2] A. A. Almoneef, A. A. Hyder, H. Budak, and M. A. Barakat, *Fractional Milne-type inequalities for twice differentiable functions*, AIMS Math. **9** (2024), no. 7, 19771–19785.
[3] M. Alomari and Z. Liu, *New error estimations for the Milne's quadrature formula in terms of at most first derivatives*, Konuralp J. Math., **1** (2013), no.1, 17-23.
[4] S. U. Al-sa'di, M. Bibi, Y. Seol and M. Muddassar, *Milne-type fractional integral inequalities for generalized m-convex mapping*, Fractals, (2023), 2350081.
[5] S. U. Al-sa'di, M. Bibi and M. Muddassar, *Some Hermite-Hadamard's type local fractional integral inequalities for generalized γ-preinvex function with applications*, Math. Methods Appl. Sci. **46** (2023), no. 2, 2941–2954.
[6] D. Baleanu, A. Kashuri, P.O. Mohammed, and B. Meftah, *General Raina fractional integral inequalities on coordinates of convex functions*, Adv. Difference Equ. 2021, Paper no. 82, 23 pp.
[7] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, JIPAM. J. Inequal. Pure Appl. Math. **10** (2009), no. 3, Article 86, 5 pp.
[8] B. Bin-Mohsin, M. Z. Javed, M. U. Awan, A. G. Khan, C. Cesarano and M. A. Noor, *Exploration of Quantum Milne–Mercer-Type Inequalities with Applications*, Symmetry, **15** (2023), no.5, 1096.
[9] P. Bosch, J. M. Rodríguez and J. M. Sigarreta, *On new Milne-type inequalities and applications*, J. Inequal. Appl. 2023, Paper No. 3, 18 pp.
[10] H. Budak and A. A. Hyder, *Enhanced bounds for Riemann-Liouville fractional integrals: Novel variations of Milne inequalities*, AIMS Mathematics, **8**(12) (2023), 30760-30776.
[11] H. Budak, P. Kösem and H. Kara, *On new Milne-type inequalities for fractional integrals*, J. Inequal. Appl. 2023, Paper No. 10, 15 pp.
[12] H. Budak and P. Kösem, *Fractional Milne type inequalities*, Acta Math. Univ. Comenian. **93** (2024), no. 1, 1–15.
[13] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. **1** (2010), no. 1, 51–58.
[14] İ. Demir, *A new approach of Milne-type inequalities based on proportional Caputo-Hybrid operator*, Journal of Advances in Applied & Computational Mathematics, **10** (2023), 102-119.
[15] H. D. Desta, H. Budak and H. Kara, *New perspectives on fractional Milne-type inequalities: insights from twice-differentiable functions*, Universal Journal of Mathematics and Applications, **7**(1) (2024), 30-37.
[16] M. Djenaoui and B. Meftah, *Milne-type inequalities for differentiable s-convex functions*, Honam Math. J. **44** (2022), no. 3, 325–338.
[17] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), no. 5, 91–95.
[18] S. S. Dragomir and B. Mond, *Integral inequalities of Hadamard type for* log-*convex functions*, Demonstratio Math. **31** (1998), no. 2, 355–364.
[19] R. Gorenflo and F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*. Fractals and fractional calculus in continuum mechanics (Udine, 1996), 223–276, CISM Courses and Lect., 378, Springer, Vienna, 1997.
[20] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58** (1893), 171-215.
[21] W. Haider, H. Budak, A. Shehzadi, F. Hezenci and H. Chen, *A comprehensive study on Milne-type inequalities with tempered fractional integrals*, Bound. Value Probl. **2024**, Paper No. 53, 16 pp.
[22] S. Hamida and B. Meftah, *Some Simpson type inequalities for differentiable h-preinvex functions*, Indian J. Math. **62** (2020), no. 3, 299–319.
[23] M.A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80** (1981), no. 2, 545–550.
[24] C. Hermite, *Sur deux limites d'une intégrale définie*, Mathesis **3** (1883), n0.1, 1-82.
[25] F. Hezenci, H. Budak, H. Kara and U. Baş, *Novel results of Milne-type inequalities involving tempered fractional integrals*, Bound. Value Probl. 2024, Paper No. 12, 15 pp.
[26] S. Hussain and S. Qaisar, *Generalizations of Simpson's type inequalities through preinvexity and prequasiinvexity*, Punjab Univ. J. Math. (Lahore) **46** (2014), no. 2, 1–9.
[27] A.-A. Hyder, H. Budak and M. A. Barakat, *Milne-type inequalities via expanded fractional operators: a comparative study with different types of functions*, AIMS Math. **9** (2024), no. 5, 11228–11246.
[28] İ. İşcan, *A new generalization of some integral inequalities for (α, m)-convex functions*, Math. Sci. (Springer) **7** (2013), Art. 22, 8 pp.

[29] A. Kashuri, B. Meftah and P. O. Mohammed, *Some weighted Simpson type inequalities for differentiable s-convex functions and their applications*, Journal of Fractional Calculus and Nonlinear Systems, **1** (2020), no. 1, 75-94.
[30] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.
[31] J.-Y. Li, *On Hadamard-type inequalities for s-preinvex functions*, Journal of Chongqing Normal University (Natural Science) 27(2010), no. 4, p. 003.
[32] B. Meftah, *Ostrowski's inequality for functions whose first derivatives are s-preinvex in the second sense*, Khayyam J. Math. **3** (2017), no. 1, 61–80.
[33] B. Meftah, *Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of derivatives are co-ordinated* log*-preinvex*, Punjab Univ. J. Math. (Lahore). **51** (2019), no. 2, 21-37.
[34] B. Meftah, *Ostrowski inequalities for functions whose first derivatives are logarithmically preinvex*, Chin. J. Math. (N.Y.) 2016, Art. ID 5292603, 10 pp.
[35] B. Meftah, *Maclaurin type inequalities for multiplicatively convex functions*, Proc. Amer. Math. Soc. **151** (2023), no. 5, 2115–2125.
[36] B. Meftah, A. Lakhdari and D. C. Benchettah, *Some new Hermite-Hadamard type integral inequalities for twice differentiable s-convex functions*, Comput. Math. Model. **33** (2022), no. 3, 330–353.
[37] B. Meftah, A. Lakhdari, W. Saleh and A. Kiliçman, *Some new fractal Milne-type integral inequalities via generalized convexity with applications*, Fractal and Fractional, **7** (2023), no.2, 166.
[38] M. A. Noor, *Variational-like inequalities*, Optimization **30** (1994), no. 4, 323–330.
[39] M. A. Noor, *Invex equilibrium problems*, J. Math. Anal. Appl. **302** (2005), no. 2, 463–475.
[40] B. G. Pachpatte, *A note on integral inequalities involving two log-convex functions*, Math. Inequal. Appl. **7** (2004), no. 4, 511–515.
[41] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.
[42] R. Pini, *Invexity and generalized convexity*, Optimization **22** (1991), no. 4, 513–525.
[43] W. Saleh, B. Meftah and A. Lakhdari, *Quantum dual Simpson type inequalities for q-differentiable convex functions*, Int. J. Nonlinear Anal. Appl. **14**(4) (2023), 63-76.
[44] W. Saleh, A. Lakhdari, T. Abdeljawad, and B. Meftah, *On fractional biparameterized Newton-type inequalities*, J. Inequal. Appl. 2023, Paper No. 122, 18 pp.
[45] W. Saleh, A. Lakhdari, O. Almutairi and A. Kiliçman, *Some remarks on local fractional integral inequalities involving Mittag–Leffler kernel using generalized* (E, h)*-convexity*, Mathematics, **11**(6) (2023), 1373.
[46] W. Saleh, A. Lakhdari, A. Kiliçman, A. Frioui and B. Meftah, *Some new fractional Hermite-Hadamard type inequalities for functions with co-ordinated extended* (s, m)*-prequasiinvex mixed partial derivatives*, Alexandria Engineering Journal, **72** (2023), 261-267.
[47] M. Z. Sarikaya and N. Aktan, *On the generalization of some integral inequalities and their applications*, Math. Comput. Modeling **54** (2011), no. 9-10, 2175–2182.
[48] M. Z. Sarikaya and H. Yaldiz, *On generalized Hermite-Hadamard type integral inequalities involving Riemann-Liouville fractional integrals*, Nihonkai Math. J. **25** (2014), no. 2, 93–104.
[49] M. Z. Sarikaya and H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Math. Notes **17** (2016), no. 2, 1049–1059.
[50] I. B. Sial, H. Budak and M. A. Ali, *Some Milne's rule type inequalities in quantum calculus*, Filomat **37**(27) (2023): 9119-9134
[51] M. Vivas-Cortez, M. Bibi, M. Muddassar and S. U. Al-sa'di, *On local fractional integral inequalities via generalized* (h_1, h_2)*-preinvexity involving local fractional integral operators with Mittag-Leffler kernel*, Demonstr. Math. **56** (2023), no. 1, Paper No. 20220216, 14 pp.
[52] T. Weir and B. Mond, *Pre-invex functions in multiple objective optimization*, J. Math. Anal. Appl. **136** (1988), no. 1, 29–38.
[53] B.-Y. Xi, and F. Qi, *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means*, J. Funct. Spaces Appl. 2012, Art. ID 980438, 14 pp.
[54] G. S. Yang and K. L. Tseng, *On certain integral inequalities related to Hermite-Hadamard inequalities*, J. Math. Anal. Appl. **239** (1999), no. 1, 180–187.
[55] X. M. Yang and D. Li, *On properties of preinvex functions*, J. Math. Anal. Appl. **256** (2001), no. 1, 229–241.
[56] Z. Yang, Y. Li and T. Du, *A generalization of Simpson type inequality via differentiable functions using* (s, m)*-convex functions*, Ital. J. Pure Appl. Math. No. **35** (2015), 327–338.
[57] R. Ying, A. Lakhdari, H. Xu, W. Saleh and B. Meftah, *On Conformable Fractional Milne-Type Inequalities*, Symmetry, **16**(2) (2024), p.196.