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Blending type approximation by λ-Bernstein-Beta type operators

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Abstract. We propose a blending type generalized Bernstein-Beta operators associated with Bézier bases $\tilde{q}_{k,l}(\lambda; y)$ and a shape parameter λ. First, we study the convergence results for the proposed operators and then establish their rate of convergence with the help of the modulus of continuity and Peetre's K-functional. Next, we present quantitative Voronovskaja-type results to study their approximation speed. In addition, we estimate the error for absolutely continuous mappings possessing derivatives of bounded variation.

1. Introduction

The key to the popularity of the theory of Approximation is undoubtedly the famous Weierstrass approximation theorem [21] and then the constructive proof given by the Russian mathematician S. Bernstein [7]. He constructed the following sequence of positive linear operators

$$P_k(u;y) = \sum_{l=0}^{k} \binom{k}{l} y^l (1-y)^{k-l} u\left(\frac{l}{k}\right), \quad k \geq 1, \tag{1}$$

where $y \in [0,1]$ and u is a continuous function on $[0,1]$. These sequences of polynomials are known as Bernstein polynomials. These polynomials became more popular after the Bohman-Korovkin theorem, which states that if a positive linear operator defined in [0,1] preserves constant, linear, and quadratic polynomials, then it will preserve all the continuous functions on [0,1]. After that, many modifications and generalizations were made for the Bernstein operators, and it became the most extensively studied linear positive operators. Some of the works can be seen in ([11, 14, 19]) and the references cited therein.

In 2010, Ye et al. [22] defined a new Bézier bases $\tilde{p}_{k,l}(\lambda; y)$, $l = 0, 1, ..., k$ with the shape parameter $\lambda \in [-1, 1]$ by

$$\begin{cases} \tilde{p}_{k,0}(\lambda; y) = p_{k,0}(y) - \frac{\lambda}{k+1} p_{k+1,1}(y), \\ \tilde{p}_{k,l}(\lambda; y) = p_{k,l}(y) + \lambda \left(\frac{k-2l+1}{k^2-1} p_{k+1,l}(y) - \frac{k-2l-1}{k^2-1} p_{k+1,l+1}(y) \right), \quad (1 \leq l \leq k-1), \\ \tilde{p}_{k,k}(\lambda; y) = p_{k,k}(y) - \frac{\lambda}{k+1} p_{k+1,k}(y), \end{cases} \tag{2}$$

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where $p_{k,l}(y) = \binom{k}{l} y^l (1-y)^{k-l}$.

Referring to equation (2), it's essential to highlight that incorporating the shape parameter λ provides us with increased modeling flexibility for the positive linear operators. Due to the above property, Cai et al.[10], explored a modification of the Bernstein polynomials as follows

$$P_{k,\lambda}(u;y) = \sum_{l=0}^{k} \tilde{p}_{k,l}(\lambda;y) u\left(\frac{l}{k}\right), \tag{3}$$

where $\tilde{p}_{k,l}(\lambda;y)$ defined in (2). In particular, when $\lambda = 0$ the operators (3) boils down to the Bernstein polynomials (1). Further, some analysis of important approximation properties of (3) has been conducted and results regarding their rate of convergence has been established. Also, to ensure that some continuous functions and different values of λ lead to a better convergence speed compared to the classical Bernstein polynomials.

Further, Cai [9] introduced generalized λ-Bernstein operators by developing Kantorovich-type λ-Bernstein operators and their Bézier variant, and analyzed these operators in terms of several approximation properties. Later, Acu et al. [2] introduced λ-Bernstein-Kantorovich operators and discussed various approximation properties and asymptotic type results. In this context, we highlight the works of authors who introduced modifications to the λ-Bernstein operators and established their convergence in the following papers, see [4, 5, 8, 18, 20].

It is worth noting that, many Beta-type generalizations like the Stancu-Beta operator, Beta operator of the first kind, q-Stancu-Beta operator, (p,q)-Bernstein-Beta operator, etc. were contributed to this field of Approximation theory, see [3, 6, 16, 17].

In recent years, the Beta-type generalization of many operators has became quite a popular area of research. Motivated by the above-mentioned works, we introduce a Beta-type generalization of the operators (3) as follows

$$\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y) = \sum_{l=0}^{k} \tilde{p}_{k,l}(\lambda;y) \int_0^1 \frac{t^{l\nu+\rho(y)-1}(1-t)^{(k-l)\nu+\rho(y)-1}}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))} u(t)dt, \tag{4}$$

where the maps u and ρ are continuous on $[0,1]$, $\lambda \in [-1,1]$, $y \in [0,1]$ and $\tilde{p}_{k,l}(\lambda;y)$ is given in (2). Also, $\nu > 0$ and $\beta(q,r)$ is the beta function defined by

$$\beta(q,r) = \int_0^1 t^{q-1}(1-t)^{r-1}dt, \quad q,r > 0.$$

One can easily observe that the operators $\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y)$ are linear and positive.

Remark 1.1. *Some Special cases:*

1. *When $\rho(y) = \nu = 1$, we obtain the λ-Bernstein-Durrmeyer operators*

$$D_{n,\lambda}(u,x) = (n+1) \sum_{k=0}^{n} \tilde{p}_{k,l}(\lambda;y) \int_0^1 p_{k,l}(t)u(t)dt.$$

In addition, if $\lambda = 0$, then we get the classical Bernstein-Durrmeyer operators

$$D_n(u,x) = (n+1) \sum_{k=0}^{n} p_{k,l}(y) \int_0^1 p_{k,l}(t)u(t)dt.$$

2. *When $\rho(y) = 1$, the above defined operators get reduced to the operators defined in [1].*

During this discussion, our primary concern is to analyze some of the essential approximation properties of the operators (4). Also, we examine their speed of convergence by establishing the Voronovskaja-type results. At last, we propose a direct estimation result for absolutely continuous maps on $[0,1]$, whose derivatives are equivalent to some function of bounded variation.

Throughout the paper, we consider the real valued functions $e_j(y) = y^j$ and $e_{j,y}(t) = (t-y)^j, \forall y \in [0,1]$ and $j \in \mathbb{N} \cup \{0\}$. Also, by $\mathfrak{C}[0,1]$, we denote the vector space of all bounded continuous maps on $[0,1]$ associated with the sup norm: $\|u\| = \sup\{|u(y)| : y \in [0,1]\}$.

2. Preliminary Results

This section discusses some basic results that will be used to establish the main theorems.

Lemma 2.1. *[10] The operators* (3) *satisfy the following relations*

$$\begin{aligned} P_{k,\lambda}(e_0;y) =& 1,\\ P_{k,\lambda}(e_1;y) =& y + \lambda\left\{\frac{1-2y+y^{k+1}-(1-y)^{k+1}}{k(k-1)}\right\},\\ P_{k,\lambda}(e_2;y) =& y^2 + \frac{y(1-y)}{k} + \lambda\left\{\frac{2y-4y^2+2y^{k+1}}{k(k-1)} + \frac{y^{k+1}+(1-y)^{k+1}-1}{k^2(k-1)}\right\},\\ P_{k,\lambda}(e_3;y) =& y^3 + \frac{3y^2(1-y)}{k} + \frac{2y^3-3y^2+y}{k^2} + \lambda\left[\frac{-6y^3+6y^{k+1}}{k^2} + \frac{3y^2-3y^{k+1}}{k(k-1)}\right.\\ &\left. + \frac{-9y^2+9y^{k+1}}{k^2(k-1)} + \frac{-4y+4y^{k+1}}{k^3(k-1)} + \frac{(1-y^{k+1}-(1-y)^{k+1})(k+3)}{k^3(k^2-1)}\right]. \end{aligned}$$

Lemma 2.2. *For the newly defined operators* (4), *we calculate the usual moments as*

$$\begin{aligned} \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_0;y) =& 1,\\ \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_1;y) =& \frac{1}{k\nu+2\rho(y)}\left[\{k\nu y+\rho(y)\} + \lambda\nu\left\{\frac{1-2y+y^{k+1}-(1-y)^{k+1}}{k-1}\right\}\right],\\ \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_2;y) =& \frac{1}{\{k\nu+2\rho(y)\}(k\nu+2\rho(y)+1)}\Big[\{k^2\nu^2y^2 + k\nu^2 y(1-y) + ky\nu(2\rho(y)+1) + \rho(y)(\rho(y)+1)\}\\ &+\frac{\lambda\nu}{k-1}\Big\{k\nu(2y-4y^2+2y^{k+1}) + \nu\left(y^{k+1}+(1-y)^{k+1}-1\right)\\ &+(2\rho(y)+1)(1-2y+y^{k+1}-(1-y)^{k+1})\Big\}\Big]. \end{aligned}$$

Proof. From the operators (4), we have the following relations

$$\begin{aligned} \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_0;y) =& P_{k,\lambda}(e_0;y),\\ \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_1;y) =& \sum_{l=0}^{k}\tilde{p}_{k,l}(\lambda;y)\int_0^1 \frac{t^{l\nu+\rho(y)}(1-t)^{(k-l)\nu+\rho(y)-1}}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))}dt\\ =& \sum_{l=0}^{k}\tilde{p}_{k,l}(\lambda;y)\frac{\beta(l\nu+\rho(y)+1,(k-l)\nu+\rho(y))}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))}\\ =& \sum_{l=0}^{k}\tilde{p}_{k,l}(\lambda;y)\left(\frac{l\nu+\rho(y)}{k\nu+2\rho(y)}\right) \end{aligned} \tag{5}$$

$$=\frac{k\nu}{k\nu+2\rho(y)}P_{k,\lambda}(e_1;y)+\frac{\rho(y)}{k\nu+2\rho(y)}P_{k,\lambda}(e_0;y), \tag{6}$$

$$\begin{aligned}\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_2;y)&=\sum_{l=0}^{k}\tilde{p}_{k,l}(\lambda;y)\int_0^1\frac{t^{l\nu+\rho(y)+1}(1-t)^{(k-l)\nu+\rho(y)-1}}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))}dt\\&=\sum_{l=0}^{k}\tilde{p}_{k,l}(\lambda;y)\frac{\beta(l\nu+\rho(y)+2,(k-l)\nu+\rho(y))}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))}\\&=\sum_{l=0}^{k}\tilde{p}_{k,l}(\lambda;y)\left(\frac{l\nu+\rho(y)}{k\nu+2\rho(y)}\right)\left(\frac{l\nu+\rho(y)+1}{k\nu+2\rho(y)+1}\right)\\&=\frac{k^2\nu^2P_{k,\lambda}(e_2;y)+k\nu(2\rho(y)+1)P_{k,\lambda}(e_1;y)+\rho(y)(1+\rho(y))P_{k,\lambda}(e_0;y)}{(k\nu+2\rho(y))(k\nu+2\rho(y)+1)}.\end{aligned} \tag{7}$$

To conclude our desired assertions, we use the Lemma 2.1 in (5), (6) and (7). □

Lemma 2.3. *The operators* (4) *have the following central moments*

$$\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_{0,y};y)=1,$$

$$\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_{1,y};y)=\frac{1}{k\nu+2\rho(y)}\left[\rho(y)(1-2y)+\lambda\nu\left\{\frac{1-2y+y^{k+1}-(1-y)^{k+1}}{k-1}\right\}\right]=\alpha_{k,\lambda}^{\rho,\nu}(y), \tag{8}$$

$$\begin{aligned}\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_{2,y};y)=&\frac{k(y\nu^2-y^2\nu^2+y\nu-y^2\nu)+2(2y^2\rho^2(y)+y^2\rho(y)-2y\rho^2(y)-y\rho(y))+\rho^2(y)+\rho(y)}{(k\nu+2\rho(y))(k\nu+2\rho(y)+1)}\\&+\frac{\lambda\nu\left\{2k\nu y(1-y)(y^k+(1-y)^k)+\nu(y^{k+1}+(1-y)^{k+1}-1)\right\}}{(k-1)(k\nu+2\rho(y))(k\nu+2\rho(y)+1)}\\&+\frac{\lambda\nu\left\{(2\rho(y)+1)(1-4y+4y^2-2y^{k+2}+2y(1-y)^{k+1}+y^{k+1}-(1-y)^{k+1})\right\}}{(k-1)(k\nu+2\rho(y))(k\nu+2\rho(y)+1)}\\&=\beta_{k,\lambda}^{\rho,\nu}(y).\end{aligned} \tag{9}$$

Remark 2.4. *The calculated central moments in Lemma 2.3 satisfy the following limiting conditions*

$$\lim_{k\to\infty}k\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_{1,y};y)=\frac{(1-2y)\rho(y)}{\nu},$$

$$\lim_{k\to\infty}k\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_{2,y};y)=y(1-y)\left(1+\frac{1}{\nu}\right).$$

Lemma 2.5. *The inequality* $\|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)\|\le\|u\|$ *holds for any* $u\in\mathfrak{C}[0,1]$.

Proof. In view of the operator (4), Lemma 2.2, and the norm defined for $\mathfrak{C}[0,1]$, we immediately conclude the result. □

Theorem 2.6. *For any* $u\in\mathfrak{C}[0,1]$ *and* $\lambda\in[-1,1]$,

$$\lim_{m\to\infty}\|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u)-u\|=0.$$

Proof. From Lemma 2.2, one can easily derive that

$$\lim_{k\to\infty}\|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(e_j)-e_j\|=0,\quad\text{for j=0,1,2}.$$

Hence the result follows from the Korovkin's Theorem [15]. □

3. Local and Global Approximation Results

For $\delta > 0$ and $u \in \mathfrak{C}[0,1]$, the first order modulus of continuity can be defined by

$$\Omega(u;\delta) = \sup_{0\leq h\leq \delta} \sup_{y\in[0,1]} |u(y+h) - u(y)|,$$

The second order modulus of continuity is defined by

$$\Omega_2(u;\delta) = \sup_{0\leq h\leq \delta} \sup_{y\in[0,1]} |u(y+2h) - 2u(y+h) + u(y)|.$$

The Peetre's K-functional of a function $u \in \mathfrak{C}[0,1]$ is defined by

$$K(u,\delta) := \inf_{v\in\mathfrak{C}^2[0,1]} \{\|u - v\| + \delta\|v''\|\},$$

where

$$\mathfrak{C}^2[0,1] := \{v \in \mathfrak{C}[0,1] : v', v'' \in \mathfrak{C}[0,1]\}.$$

The relationship between $K(u,\delta)$ and $\Omega_2(u,\delta)$ is described in [12] as

$$K(u,\delta) \leq C\Omega_2(u, \sqrt{\delta}). \tag{10}$$

Here δ is a positive real number and the constant $C > 0$ doesn't depend on u or δ.

Theorem 3.1. *Let $u \in \mathfrak{C}[0,1]$ and $y \in [0,1]$. Then for any $k \in \mathbb{N}$, the operators* (4) *satisfy*

$$|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq 2\Omega\left(u, \sqrt{\beta_{k,\lambda}^{\rho,\nu}(y)}\right),$$

where $\beta_{k,\lambda}^{\rho,\nu}(y)$ is defined in (9).

Proof. We note the following relation associated with the modulus of continuity.

$$|u(t) - u(y)| \leq \Omega(u,\delta)\left(\frac{(t-y)^2}{\delta^2} + 1\right). \tag{11}$$

Now, apply the operators $\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}$ on (11) both sides to obtain

$$|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(|u(t) - u(y)|;y) \leq \Omega(u,\delta)\left(1 + \frac{1}{\delta^2}\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2 : y)\right).$$

Next, we select $\delta = \sqrt{\beta_{k,\lambda}^{\rho,\nu}(y)}$, to obtain our desired result. □

Theorem 3.2. *For $y \in [0,1]$ and $u \in \mathfrak{C}^1[0,1]$, we have the following relation*

$$|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq \left|\alpha_{k,\lambda}^{\rho,\nu}(y)\right||u'(y)| + 2\sqrt{\beta_{k,\lambda}^{\rho,\nu}(y)}\,\Omega\left(u', \sqrt{\beta_{k,\lambda}^{\rho,\nu}(y)}\right),$$

where $\alpha_{k,\lambda}^{\rho,\nu}(y)$ is defined in (8). *Also, $\mathfrak{C}^1[0,1]$ denotes all such $u \in \mathfrak{C}[0,1]$ such that $u' \in \mathfrak{C}[0,1]$.*

Proof. Taylor series expansion ensures us that

$$u(t)-u(y)=u'(y)(t-y)+\int_y^t (u'(z)-u'(y))dz,$$

for any $y,t\in[0,1]$. This yields

$$\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((u(t)-u(y);y)=u'(y)\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y);y)+\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}\left(\int_y^t (u'(z)-u'(y))dz;y\right). \tag{12}$$

Next, for $y\in[0,1]$, $u\in\mathfrak{C}[0,1]$ and $\delta>0$, we know the following relation holds:

$$|u(t)-u(y)|\leq\Omega(u,\delta)\left(\frac{|t-y|}{\delta}+1\right).$$

Hence as $u'\in\mathfrak{C}[0,1]$, the above relation yields

$$\left|\int_y^t (u'(z)-u'(y))dz\right|\leq\Omega(u',\delta)\left(\frac{(t-y)^2}{\delta}+|t-y|\right).$$

So, by (12), we must have

$$|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)-u(y)|\leq|\alpha_{k,\lambda}^{\rho,\nu}(y)||u'(y)|+\Omega(u',\delta)\left\{\frac{1}{\delta}\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)+\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(t-y;y)\right\}.$$

Next, we apply the Cauchy-Schwarz inequality to obtain

$$|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)-u(y)|\leq|u'(y)||\alpha_{k,\lambda}^{\rho,\nu}(y)|+\Omega(u',\delta)\left\{\frac{1}{\delta}\sqrt{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)}+1\right\}\sqrt{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)}.$$

Lastly, we select $\delta=\sqrt{\beta_{k,\lambda}^{\rho,\nu}(y)}$, to prove our claim. □

Theorem 3.3. *Let $u\in\mathfrak{C}[0,1]$ and $v\in\mathfrak{C}^2[0,1]$. Then $\exists$ a constant $C>0$ obeying the following relation for each $k\in\mathbb{N}$*

$$\left|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)-u(y)-\alpha_{k,\lambda}^{\rho,\nu}(y)v'(y)\right|\leq C\Omega_2\left(u,\sqrt{\beta_{k,\lambda}^{\rho,\nu}(y)}\right).$$

Proof. By Taylor's expansion, for $v\in\mathfrak{C}^2[0,1]$, we must have

$$v(t)=v(y)+v'(y)(t-y)+\int_y^t (t-z)v''(z)dz. \tag{13}$$

Next, we apply the operators $\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}$ on (13), to obtain

$$\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(v;y)=v(y)+v'(y)\alpha_{k,\lambda}^{\rho,\nu}(y)+\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}\left(\int_y^t (t-z)v''(z)dz;y\right).$$

By taking modulus on both sides, we have

$$\begin{aligned}\left|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(v;y)-v(y)-\alpha_{k,\lambda}^{\rho,\nu}(y)v'(y)\right| &\leq\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)\frac{\|v''\|}{2}\\ &\leq\beta_{k,\lambda}^{\rho,\nu}(y)\|v''\|.\end{aligned} \tag{14}$$

Now, it is easy to observe that, for $u \in \mathfrak{C}[0,1]$,

$$\left|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y) - \alpha_{k,\lambda}^{\rho,\nu}(y)v'(y)\right| \leq \left|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u-v;y) - (u-v)(y)\right| + \left|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(v;y) - v(y) - \alpha_{k,\lambda}^{\rho,\nu}(y)v'(y)\right|. \quad (15)$$

So, using (14) and Lemma 2.5 in (15), we can deduce

$$\left|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y) - \alpha_{k,\lambda}^{\rho,\nu}(y)v'(y)\right| \leq \|u-v\| + \beta_{k,\lambda}^{\rho,\nu}(y)\|v''\|.$$

Now, taking infimum over all $v \in \mathfrak{C}^2[0,1]$ on the right side of above inequality, yields

$$\left|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y) - \alpha_{k,\lambda}^{\rho,\nu}(y)v'(y)\right| \leq K\left(u, \alpha_{k,\lambda}^{\rho,\nu}(y)\right),$$

Finally, we use the relation (10) to get our desired assertion. □

Next, we present a local approximation result for the Lipschitz type class of functions. In this context we have the following definition.
The Lipschitz class of functions with two parameters $\gamma_1 > 0$ and $\gamma_2 > 0$ is defined as

$$Lip_M^{\gamma_1,\gamma_2}(\eta) := \left\{u \in \mathfrak{C}[0,1] : |u(t) - u(y)| \leq M\frac{|t-y|^\eta}{(\gamma_1 y^2 + \gamma_2 y + t)}; y,t \in [0,1]\right\},$$

where $\eta \in (0,1]$ and $M > 0$ depends on u only. (M will not be the same for forthcoming results.)

Theorem 3.4. *For any $\eta \in (0,1]$ and $u \in Lip_M^{\gamma_1,\gamma_2}(\eta)$, we must have*

$$|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq M\left(\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{\gamma_1 y^2 + \gamma_2 y}\right)^{\eta/2},$$

where $\beta_{k,\lambda}^{\rho,\nu}(y)$ is given in (9) *and $M > 0$ depends on u.*

Proof. It is easily seen that for $\eta = 1$,

$$|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(|u(t) - u(y)|;y) \leq M\left\{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}\left(\frac{|t-y|}{\sqrt{t + \gamma_1 y^2 + \gamma_2 y}};y\right)\right\}.$$

Using the fact that $\frac{1}{\sqrt{t+\gamma_1 y^2+\gamma_2 y}} \leq \frac{1}{\sqrt{\gamma_1 y^2+\gamma_2 y}}$ and Cauchy-Schwarz inequality, the above inequality yields

$$\begin{aligned}|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| &\leq \frac{M}{\sqrt{\gamma_1 y^2 + \gamma_2 y}}\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(|t-y|;y) \\ &\leq \frac{M}{\sqrt{\gamma_1 y^2 + \gamma_2 y}}\left\{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)\right\}^{1/2} \leq M\left(\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{\gamma_1 y^2 + \gamma_2 y}\right)^{1/2}.\end{aligned}$$

Hence the required inequality is true for $\eta = 1$. Next, we consider the case when $\eta \in (0,1)$.
Using the fact that $\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u(y);y) = u(y)$, we have

$$\begin{aligned}|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| &\leq \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(|u(t) - u(y)|;y) \\ &= \sum_{l=0}^{k} \tilde{p}_{k,l}(\lambda;y) \int_0^1 \frac{t^{l\nu+\rho(y)-1}(1-t)^{(k-l)\nu+\rho(y)-1}}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))}|u(t) - u(y)|dt. \quad (16)\end{aligned}$$

Setting $p = \frac{2}{\eta}$ and $q = \frac{2}{2-\eta}$ and then applying the Hölder inequality to (16), we obtain

$$
\begin{aligned}
|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq & \left[\sum_{l=0}^{k} \tilde{p}_{k,l}(\lambda;y) \int_0^1 \frac{t^{l\nu+\rho(y)-1}(1-t)^{(k-l)\nu+\rho(y)-1}}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))} |u(t)-u(y)|^{2/\eta} dt\right]^{\eta/2} \\
& \times \left[\sum_{l=0}^{k} \tilde{p}_{k,l}(\lambda;y) \int_0^1 \frac{t^{l\nu+\rho(y)-1}(1-t)^{(k-l)\nu+\rho(y)-1}}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))} dt\right]^{(2-\eta)/2} \\
\leq & M\left[\sum_{l=0}^{k} \tilde{p}_{k,l}(\lambda;y) \int_0^1 \frac{t^{l\nu+\rho(y)-1}(1-t)^{(k-l)\nu+\rho(y)-1}}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))} \frac{|t-y|^2}{(t+\gamma_1 y^2+\gamma_2 y)} dt\right]^{\eta/2} \\
\leq & \frac{M}{(\gamma_1 y^2+\gamma_2 y)^{\eta/2}} \left[\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)\right]^{\eta/2} \\
\leq & M\left(\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{\gamma_1 y^2+\gamma_2 y}\right)^{\eta/2}.
\end{aligned}
$$

This completes the proof. □

We recall the 1st and 2nd-order Ditzian-Totik uniform modulus of smoothness

$$\Omega_\xi(u,\delta) := \sup_{0\leq h\leq\delta} \sup_{y,y+h\xi(y)\in[0,1]} \{|u(y+h\xi(y)) - u(y)|\}$$

and

$$\Omega_2^\xi(u,\delta) := \sup_{0\leq h\leq\delta} \sup_{y,y+h\xi(y)\in[0,1]} \{|u(y+h\xi(y)) - 2u(y) + u(y-h\xi(y))|\},$$

respectively, where $\xi(y) = [y(1-y)]^{1/2}$ and $\delta > 0$.
Let the corresponding K-functional be

$$K_{2,\xi}(u,\delta) = \inf_{v\in\mathcal{W}^2(\xi)} \left\{\|u-v\| + \delta\|\xi^2 v''\| : v \in \mathfrak{C}^2[0,1]\right\},$$

where

$$\mathcal{W}^2(\xi) = \{v \in \mathfrak{C}[0,1] : v' \in AC^{loc}[0,1], \|\xi^2 v''\| \leq \infty\}, \tag{17}$$

where by $AC^{loc}[0,1]$, we denote the set of all locally absolute continuous maps on $[0,1]$. Also, it is evident from [12] that

$$M^{-1}\Omega_2^\xi(u,\sqrt{\delta}) \leq K_{2,\xi}(u,\delta) \leq M\Omega_2^\xi(u,\sqrt{\delta}), \tag{18}$$

where $M > 0$ is a constant.
Now we are in the state to establish a global approximation result for the operators (4).

Theorem 3.5. *Suppose that ξ^2 ($\xi \neq 0$) is concave on $[0,1]$. Then for any $u \in \mathfrak{C}[0,1]$ and $y \in [0,1]$, $\exists\, M > 0$, such that*

$$|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq M\Omega_2^\xi\left(u, \frac{\tau_{k,\lambda}^{\rho,\nu}(y)}{2\xi(y)}\right) + \Omega_\xi\left(u, \frac{\alpha_{k,\lambda}^{\rho,\nu}(y)}{\xi(y)}\right), \tag{19}$$

where $\tau_{k,\lambda}^{\rho,\nu}(y) = \left\{\beta_{k,\lambda}^{\rho,\nu}(y) + \left(\alpha_{k,\lambda}^{\rho,\nu}(y)\right)^2\right\}^{1/2}$.

Proof. For $u \in \mathfrak{C}[0,1]$, we define an auxiliary operator

$$\mathfrak{P}_{k,\lambda}^{\rho,\nu}(u;y) = \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) + u(y) - u\left(y + \alpha_{k,\lambda}^{\rho,\nu}(y)\right). \tag{20}$$

Clearly, the above defined auxiliary operator preserves linear as well as constant functions.
Let $z = \vartheta y + (1-\vartheta)t$, $\vartheta \in [0,1]$. As ξ^2 is concave on $[0,1]$, we must have $\xi^2(z) \geq \vartheta\xi^2(y) + (1-\vartheta)\xi^2(t)$ and hence we have

$$\frac{|t-z|}{\xi^2(z)} \leq \frac{\vartheta|y-t|}{\vartheta\xi^2(y) + (1-\vartheta)\xi^2(t)} \leq \frac{|t-y|}{\xi^2(y)}.$$

Also, using Lemma 2.5, for the operators (20), we obtain

$$\begin{aligned}
|\mathfrak{P}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| &\leq |\mathfrak{P}_{k,\lambda}^{\rho,\nu}(u-v;y)| + |\mathfrak{P}_{k,\lambda}^{\rho,\nu}(v;y) - v(y)| + |u(y) - v(y)| \\
&\leq 4\|u - v\| + |\mathfrak{P}_{k,\lambda}^{\rho,\nu}(v;y) - v(y)|.
\end{aligned} \tag{21}$$

By applying Taylor's formula, we get the following relation:

$$\begin{aligned}
|\mathfrak{P}_{k,\lambda}^{\rho,\nu}(v;y) - v(y)| \leq& \widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}\left(\int_y^t |t-z||v''(z)|dz;y\right) + \left|\int_y^{y+\alpha_{k,\lambda}^{\rho,\nu}(y)} |y + \alpha_{k,\lambda}^{\rho,\nu}(y) - z||v''(z)|dz\right| \\
\leq& \|\xi^2 v''\|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}\left(\left|\int_y^t \frac{|t-z|}{\xi^2(z)}dz\right|;y\right) + \|\xi^2 v''\|\left|\int_y^{y+\alpha_{k,\lambda}^{\rho,\nu}(y)} \frac{|y+\alpha_{k,\lambda}^{\rho,\nu}(y) - z|}{\xi^2(z)}dz\right| \\
\leq& \xi^{-2}(y)\|\xi^2 v''\|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y) + \xi^{-2}(y)\|\xi^2 v''\|\left(\alpha_{k,\lambda}^{\rho,\nu}(y)\right)^2 \\
\leq& \xi^{-2}(y)\|\xi^2 v''\|\left[\beta_{k,\lambda}^{\rho,\nu}(y) + \left(\alpha_{k,\lambda}^{\rho,\nu}(y)\right)^2\right].
\end{aligned}$$

Using the above inequality, (21) yields

$$|\mathfrak{P}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq 4\|u-v\| + \xi^{-2}(y)\|\xi^2 v''\|\left[\beta_{k,\lambda}^{\rho,\nu}(y) + \left(\alpha_{k,\lambda}^{\rho,\nu}(y)\right)^2\right].$$

Now, by taking the infimum over all $v \in \mathcal{W}^2(\xi)$ and then using (18), we get there exists $C > 0$, such that

$$|\mathfrak{P}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq M\Omega_2^{\xi}\left(u, \frac{\sqrt{\beta_{k,\lambda}^{\rho,\nu}(y) + \left(\alpha_{k,\lambda}^{\rho,\nu}(y)\right)^2}}{2\xi(y)}\right).$$

But, by the definition of first order Ditzian-Totik uniform modulus of smoothness, we must have

$$\begin{aligned}
|u(y + \alpha_{k,\lambda}^{\rho,\nu}(y)) - u(y)| =& \left|u\left(y + \xi(y)\frac{\alpha_{k,\lambda}^{\rho,\nu}(y)}{\xi(y)}\right) - u(y)\right| \\
\leq& \Omega_\xi\left(u, \frac{\alpha_{k,\lambda}^{\rho,\nu}(y)}{\xi(y)}\right).
\end{aligned}$$

Finally, we obtain

$$|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| \leq |\mathfrak{P}_{k,\lambda}^{\rho,\nu}(u;y) - u(y)| + |u(y + \alpha_{k,\lambda}(y)) - u(y)|$$

$$\leq M\Omega_2^{\xi}\left(u, \frac{\sqrt{\beta_{k,\lambda}^{\rho,\nu}(y)+\left(\alpha_{k,\lambda}^{\rho,\nu}(y)\right)^2}}{2\xi(y)}\right)+\Omega_{\xi}\left(u, \frac{\alpha_{k,\lambda}^{\rho,\nu}(y)}{\xi(y)}\right),$$

which is the required result. □

4. Voronovskaja-type Asymptotic Results

This section presents Voronovskaja-type asymptotic results to study the speed of convergence of the operators (4).

Theorem 4.1. *Assume that an integrable function u on $[0,1]$ with u' and u'' exists at some point $y \in [0,1]$. Then we have the following relation*

$$\lim_{k\to\infty} k\{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)-u(y)\} = \frac{(1-2y)\rho(y)}{\nu}u'(y)+y(1-y)\left(1+\frac{1}{\nu}\right)u''(y).$$

Proof. By the well-known Taylor's expansion,

$$u(t) = u(y)+u'(y)(t-y)+\frac{1}{2}u''(y)(t-y)^2+\Theta(t,y)(t-y)^2, \tag{22}$$

where $\Theta(t,y) \in \mathfrak{C}[0,1]$ and satisfies $\lim_{t\to y}\Theta(t,y) = 0$.
For $k \in \mathbb{N}$, we apply the operators $\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}$ on (22) to obtain

$$\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y) = u(y)+u'(y)\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y);y)+\frac{1}{2}u''(y)\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)+\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(\Theta(t,y)(t-y)^2;y),$$

and of course applying limit $k \to \infty$ on both sides, we get

$$\begin{aligned}\lim_{k\to\infty} k\{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)-u(y)\} =& u'(y)\lim_{k\to\infty} k\,\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y);y)+\frac{u''(y)}{2}\lim_{k\to\infty} k\,\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)\\ &+\lim_{k\to\infty} k\,\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(\Theta(t,y)(t-y)^2;y).\end{aligned}$$

Next, using Remark 2.4, we can easily obtain

$$\begin{aligned}\lim_{k\to\infty} k\{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)-u(y)\} =& \frac{(1-2y)\rho(y)}{\nu}u'(y)+y(1-y)\left(1+\frac{1}{\nu}\right)u''(y)\\ &+\lim_{k\to\infty} k\,\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(\Theta(t,y)(t-y)^2;y).\end{aligned} \tag{23}$$

Hence, from the Cauchy-Schwarz inequality, we get

$$\lim_{k\to\infty} k\,\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(\Theta(t,y)(t-y)^2;y) \leq \sqrt{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(\Theta^2(t,y);y)}\sqrt{k^2\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^4;y)}. \tag{24}$$

Also, Theorem 2.6 ensures that

$$\lim_{k\to\infty}\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(\Theta^2(t,y);y) = \Theta^2(y,y) = 0. \tag{25}$$

Lastly, in view of the fact that $\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^4;y)$ is of order k^{-2} and then applying (24) and (25) in (23), we conclude the proposed result. □

Next, we proceed for a quantitative Voronovskaja type estimation for the newly defined operators. For this context, we have the following notations:
For $u \in \mathfrak{C}[0,1]$ and $\delta > 0$, the Ditzian-Totik uniform modulus of smoothness can also be defined as

$$\Omega_\xi(u,\delta) := \sup_{0\le h\le\delta}\left\{\left|u\left(y+\frac{h\xi(y)}{2}\right)-u\left(y-\frac{h\xi(y)}{2}\right)\right|, y\pm\frac{h\xi(y)}{2}\in[0,1]\right\},$$

and the corresponding Peetre's K-functional is given by

$$K_\xi(u,\delta) = \inf_{v\in\mathcal{W}^2(\xi)}\left\{\|u-v\|+\delta\|\xi v'\| : v\in\mathfrak{C}^1[0,1]\right\},$$

where $\xi(y) = \{y(1-y)\}^{1/2}$ and $\mathcal{W}^2(\xi)$ is given in (17).
Also, it is clear from [12] that $\exists$ a constant $M > 0$, with the property that $K_\xi(u,\delta) \le M\Omega_\xi(u,\delta)$.

Now, we are at the stage of establishing a quantitative Voronovskaja type result.

Theorem 4.2. *For $u \in \mathfrak{C}^2[0,1]$, $y \in [0,1]$ and sufficiently large k, there exists $M > 0$ satisfying the following relation*

$$\left|\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y)u(y)-u(y)-\alpha^{\rho,\nu}_{k,\lambda}(y)u'(y)-\frac{\beta^{\rho,\nu}_{k,\lambda}(y)}{2}u''(y)\right| \le \frac{M}{k}\xi^2(y)\Omega_\xi\left(u'',\frac{1}{\sqrt{k}}\right),$$

where $\alpha^{\rho,\nu}_{k,\lambda}(y)$ and $\beta^{\rho,\nu}_{k,\lambda}(y)$ are given in (8) *and* (9) *respectively.*

Proof. For $u \in \mathfrak{C}^2[0,1]$ and $y \in [0,1]$. Taylor's series ensures us the following

$$u(t)-u(y)-(t-y)u'(y) = \int_y^t (t-z)u''(z)dz.$$

This leads to the following relation

$$u(t)-u(y)-(t-y)u'(y)-\frac{u''(y)}{2}(t-y)^2 = \int_y^t (t-z)[u''(z)-u''(y)]dz.$$

Now, applying the operators (4) on both sides, we obtain

$$\begin{aligned}&\left|\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y)-u(y)-\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(t-y;y)u'(y)-\frac{u''(y)}{2}\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}((t-y)^2;y)\right|\\ &\le \widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}\left(\left|\int_y^t |t-z||u''(z)-u''(y)|dz\right|;y\right).\end{aligned} \tag{26}$$

It is easy to notice from [13, p.337] that the following inequality is true for any $v \in \mathcal{W}^2(\xi)$

$$\left|\int_y^t |t-z||u''(z)-u''(y)|dz\right| \le 2\|u''-v\|(t-y)^2 + 2\|\xi v'\|\xi^{-1}(y)|t-y|^3. \tag{27}$$

Also, using the facts that $\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}((t-y)^2;y)$ is of order k^{-1} and $\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}((t-y)^4;y)$ is of order k^{-2}, for sufficiently large k, we can get a constant $M > 0$, such that

$$\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}((t-y)^2;y) \le \frac{M}{2k}\xi^2(y) \quad \text{and} \quad \widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}((t-y)^4;y) \le \frac{M}{2k^2}\xi^4(y). \tag{28}$$

In view of (27), (28) and the well-known Cauchy-Schwarz inequality, (26) yields

$$\begin{aligned}
&\left|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)u(y)-u(y)-\alpha_{k,\lambda}^{\rho,\nu}(y)u'(y)-\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{2}u''(y)\right|\\
&\le 2\|u''-v\|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)+2\|\xi v'\|\xi^{-1}(y)\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(|t-y|^3;y)\\
&\le \frac{M}{k}\xi^2(y)\|u''-v\|+2\|\xi v'\|\xi^{-1}(y)\sqrt{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^2;y)}\sqrt{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}((t-y)^4;y)}\\
&\le \frac{M}{k}\xi^2(y)\left(\|u''-v\|+k^{-1/2}\|\xi v'\|\right).
\end{aligned}$$

Lastly, taking infimum over all $v\in\mathcal{W}^2(\xi)$, we get the desired assertion. □

In light of the above theorem, one can draw the following conclusion.

Corollary 4.3. *For $u\in\mathfrak{C}^2[0,1]$*

$$\lim_{k\to\infty}k\left|\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)u(y)-u(y)-\alpha_{k,\lambda}^{\rho,\nu}u'(y)-\frac{\beta_{k,\lambda}^{\rho,\nu}}{2}u''(y)\right|=0.$$

We complete this section by showing a Grüss-Voronovskaja type result for a particular class of the newly defined sequence of operators.

Theorem 4.4. *For $u,v\in\mathfrak{C}^2[0,1]$ and $y\in[0,1]$,*

$$\lim_{k\to\infty}k\{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(uv;y)-\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(v;y)\}=3y(1-y)u'(y)v'(y).$$

Proof. For $u,v\in\mathfrak{C}^2[0,1]$, we notice the following relation

$$\begin{aligned}
\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(uv;y)-\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(v;y)=&\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(uv;y)-u(y)v(y)-(uv)'(y)\alpha_{k,\lambda}^{\rho,\nu}(y)-(uv)''(y)\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{2}\\
&-v(y)\left[\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)-u(y)-u'(y)\alpha_{k,\lambda}^{\rho,\nu}(y)-u''(y)\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{2}\right]\\
&-\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)\left[\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(v;y)-v(y)-v'(y)\alpha_{k,\lambda}^{\rho,\nu}(y)-v''(y)\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{2}\right]\\
&+\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{2}[u(y)v''(y)+2u'(y)v'(y)-v''(y)\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)]\\
&+\alpha_{k,\lambda}^{\rho,\nu}(y)[u(y)v'(y)-v'(y)\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(v;y)].
\end{aligned}$$

Multiplying both sides by k and then taking limit $k\to\infty$, we have

$$\begin{aligned}
&\lim_{k\to\infty}k\{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(uv;y)-\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(v;y)\}\\
&\qquad=\lim_{k\to\infty}k\left\{\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(uv;y)-u(y)v(y)-(uv)'(y)\alpha_{k,\lambda}^{\rho,\nu}(y)-(uv)''(y)\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{2}\right\}\\
&\qquad-v(y)\lim_{k\to\infty}k\left[\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)-u(y)-u'(y)\alpha_{k,\lambda}^{\rho,\nu}(y)-u''(y)\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{2}\right]\\
&\qquad-\lim_{k\to\infty}\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(u;y)k\left[\widetilde{\mathfrak{P}}_{k,\lambda}^{\rho,\nu}(v;y)-v(y)-v'(y)\alpha_{k,\lambda}^{\rho,\nu}(y)-v''(y)\frac{\beta_{k,\lambda}^{\rho,\nu}(y)}{2}\right]
\end{aligned}$$

$$+\lim_{k\to\infty}[k\beta^{\rho,\nu}_{k,\lambda}(y)]u'(y)v'(y)+\lim_{k\to\infty}\frac{k\beta^{\rho,\nu}_{k,\lambda}(y)}{2}[u(y)v''(y)-v''(y)\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y)]$$
$$+\lim_{k\to\infty}[k\alpha^{\rho,\nu}_{k,\lambda}(y)][u(y)v'(y)-v'(y)\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(v;y)].$$

Lastly, using Theorem 2.6, Corollary 4.3 and Remark 2.4, we get the desired assertion. □

5. A direct Estimation

The final step of our discussion is to calculate the error while approximating the functions from a special class, namely $BV'[0,1]$, which consists of all continuous maps with derivatives of bounded variation. One can easily represent any function $u\in BV'[0,1]$ by

$$u(y)=\int_0^y v(t)dt+u(0),$$

where $v\in BV[0,1]$ i.e., the mapping v is of bounded variation on $[0,1]$.

By setting $\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)=\sum_{l=0}^{k}\tilde{p}_{k,l}(\lambda;y)\frac{t^{l\nu+\rho(y)-1}(1-t)^{(k-l)\nu+\rho(y)-1}}{\beta(l\nu+\rho(y),(k-l)\nu+\rho(y))}$ as the kernel, we can easily rewrite our operator (4) as

$$\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y)=\int_0^1\tilde{\mathcal{K}}_{k,\lambda}(y,t)u(t)dt. \tag{29}$$

Lemma 5.1. *For $y\in(0,1]$ and large positive integer k, we obtain the following inequalities*

1. *If $0\le x<y$, then*

$$\zeta^{\rho,\nu}_{k,\lambda}(y,x)=\int_0^x\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt\le\frac{\beta^{\rho,\nu}_{k,\lambda}(y)}{(y-x)^2}.$$

2. *If $y<z<1$, then*

$$1-\zeta^{\rho,\nu}_{k,\lambda}(y,z)=\int_z^1\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt\le\frac{\beta^{\rho,\nu}_{k,\lambda}(y)}{(z-y)^2}.$$

Proof. Using (29) and (9) we observe that, for $0\le x<y$,

$$\zeta^{\rho,\nu}_{k,\lambda}(y,x)=\int_0^x\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt\le\int_0^x\left(\frac{y-t}{y-x}\right)^2\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt$$
$$=\frac{\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}((t-y)^2;y)}{(y-x)^2}=\frac{\beta^{\rho,\nu}_{k,\lambda}(y)}{(y-x)^2}.$$

For the second part, the proof is simailar. □

Theorem 5.2. *Let $u\in BV'(0,1)$ and $y\in(0,1)$. If $k\in\mathbb{N}$ is sufficiently large, then the following inequality is estimated:*

$$\left|\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y)-u(y)\right|\le\frac{1}{2}\{u'(y+)+u'(y-)\}\alpha^{\rho,\nu}_{k,\lambda}(y)+\frac{1}{2}\sqrt{\beta^{\rho,\nu}_{k,\lambda}(y)}\{u'(y+)-u'(y-)\}$$
$$+y^{-1}\beta^{\rho,\nu}_{k,\lambda}(y)\sum_{l=1}^{[\sqrt{k}]}\mathfrak{T}^{y}_{y-\frac{y}{l}}(u'_y)+\frac{y}{\sqrt{k}}\mathfrak{T}^{y}_{y-\frac{y}{\sqrt{k}}}(u'_y)$$

$$+(1-y)^{-1}\beta^{\rho,\nu}_{k,\lambda}(y)\sum_{l=1}^{[\sqrt{k}]}\mathfrak{T}_y^{y+\frac{(1-y)}{l}}(u'_y)+\frac{(1-y)}{\sqrt{k}}\mathfrak{T}_y^{y+\frac{(1-y)}{\sqrt{k}}}(u'_y).$$

By $\mathfrak{T}_c^d(u)$*, we denote the total variation of the function* u *on* $[c,d]\subset[0,1]$ *and the function* u'_y *is defined as*

$$u'_y(t)=\begin{cases}u'(t)-u'(y-), & 0\le t<y,\\ u'(t)-u'(y+), & y<t\le 1,\\ 0, & t=y.\end{cases}$$

Proof. We know that the operator (4) preserves the constant functions and hence

$$\begin{aligned}\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y)-u(y)&=\int_0^1(u(t)-u(y))\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt\\&=\int_0^1\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)\left(\int_y^t u'(z)dz\right)dt.\end{aligned}\tag{30}$$

Also, for $u\in BV'(0,1)$, we can write

$$u'(z)=\frac{1}{2}\{u'(y+)+u'(y-)\}+u'_y(z)+\frac{1}{2}\{u'(y+)-u'(y-)\}sgn(z-y)+[u'(z)-\frac{1}{2}\{u'(y+)+u'(y-)\}]\delta_y^z,\tag{31}$$

where δ_y^z is the Kronecker delta function and defined as

$$\delta_y^z=\begin{cases}1, & z=y\\ 0, & z\neq y\end{cases}$$

and $sgn(x)$ is the signum function defined by

$$sgn(x)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 0, & x=0.\end{cases}$$

It is easily observed that

$$\int_0^1\left(\int_y^t\left[u'(z)-\frac{1}{2}\{u'(y+)+u'(y-)\}\right]\delta_y^z dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt=0.\tag{32}$$

Now, using relation (29), we have

$$\int_0^1\left(\int_y^t\frac{1}{2}\{u'(y+)+u'(y-)\}dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt=\frac{1}{2}\{u'(y+)+u'(y-)\}\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(t-y;y).\tag{33}$$

Also, by some manipulations and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\int_0^1\left(\int_y^t\frac{1}{2}\{u'(y+)-u'(y-)\}sgn(z-y)dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt\\&\quad\le\frac{1}{2}\{u'(y+)-u'(y-)\}\int_0^1|t-y|\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt\\&\quad=\frac{1}{2}\{u'(y+)-u'(y-)\}\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(|t-y|;y)\\&\quad\le\frac{1}{2}\{u'(y+)-u'(y-)\}\left[\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}((t-y)^2;y)\right]^{1/2}.\end{aligned}\tag{34}$$

Now, using (31)-(34) in (30), we have

$$\begin{aligned}\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y)-u(y)\leq&\frac{1}{2}\{u'(y+)+u'(y-)\}\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(t-y;y)+\frac{1}{2}\{u'(y+)-u'(y-)\}\left[\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}((t-y)^2;y)\right]^{1/2}\\&+\int_0^1\left(\int_y^t u'_y(z)dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt\\=&\frac{1}{2}\{u'(y+)+u'(y-)\}\alpha^{\rho,\nu}_{k,\lambda}(y)+\frac{1}{2}\sqrt{\beta^{\rho,\nu}_{k,\lambda}(y)}\{u'(y+)-u'(y-)\}\\&+\int_0^y\left(\int_y^t u'_y(z)dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt+\int_y^1\left(\int_y^t u'_y(z)dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt.\end{aligned}$$

This yields

$$\left|\widetilde{\mathfrak{P}}^{\rho,\nu}_{k,\lambda}(u;y)-u(y)\right|\leq\frac{1}{2}\{u'(y+)+u'(y-)\}\alpha^{\rho,\nu}_{k,\lambda}(y)+\frac{1}{2}\sqrt{\beta^{\rho,\nu}_{k,\lambda}(y)}\{u'(y+)-u'(y-)\}+I_1+I_2,\tag{35}$$

where $I_1=\left|\int_0^y\left(\int_y^t u'_y(z)dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt\right|$, and $I_2=\left|\int_y^1\left(\int_y^t u'_y(z)dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt\right|$.
To establish our claim, we need to estimate the integrals I_1 and I_2. It is noticed that for $t<y$, we have $\tilde{\mathcal{K}}^{\rho,\nu}_{k,\lambda}(y,t)dt=d_t\zeta^{\rho,\nu}_{k,\lambda}(y,t)$, where d_t denotes the differential operator with respect to t.
Now, applying integration by parts, we get

$$\begin{aligned}I_1=&\left|\int_0^y\left(\int_y^t u'_y(z)dz\right)d_t\zeta^{\rho,\nu}_{k,\lambda}(y,t)\right|=\left|\left[\left(\int_y^t u'_y(z)dz\right)\zeta^{\rho,\nu}_{k,\lambda}(y,t)\right]_0^y-\int_0^y u'_y(t)\zeta^{\rho,\nu}_{k,\lambda}(y,t)dt\right|\\=&\left|\int_0^y\zeta^{\rho,\nu}_{k,\lambda}(y,t)u'_y(t)dt\right|\leq\int_0^x|\zeta^{\rho,\nu}_{k,\lambda}(y,t)||u'_y(t)|dt+\int_x^y|\zeta^{\rho,\nu}_{k,\lambda}(y,t)||u'_y(t)|dt.\end{aligned}$$

Substituting $x=y-\frac{y}{\sqrt{k}}$,

$$I_1\leq\int_0^{y-\frac{y}{\sqrt{k}}}|\zeta^{\rho,\nu}_{k,\lambda}(y,t)||u'_y(t)|dt+\int_{y-\frac{y}{\sqrt{k}}}^y|\zeta^{\rho,\nu}_{k,\lambda}(y,t)||u'_y(t)|dt.$$

Now, by noting the facts that $u'_y(y)=0$ and $|\zeta^{\rho,\nu}_{k,\lambda}(y,t)|\leq 1$, it follows

$$\begin{aligned}\int_{y-\frac{y}{\sqrt{k}}}^y|\zeta^{\rho,\nu}_{k,\lambda}(y,t)||u'_y(t)|dt=&\int_{y-\frac{y}{\sqrt{k}}}^y|\zeta^{\rho,\nu}_{k,\lambda}(y,t)||u'_y(t)-u'_y(y)|dt\leq\int_{y-\frac{y}{\sqrt{k}}}^y\mathfrak{T}^y_t(u'_y)dt\\\leq&\frac{y}{\sqrt{k}}\mathfrak{T}^y_{y-\frac{y}{\sqrt{k}}}(u'_y).\end{aligned}\tag{36}$$

Also, by using Lemma 5.1 and the substitution $t=y-\frac{y}{z}$, we have

$$\begin{aligned}\int_0^{y-\frac{y}{\sqrt{k}}}|\zeta^{\rho,\nu}_{k,\lambda}(y,t)||u'_y(t)|dt\leq&\beta^{\rho,\nu}_{k,\lambda}(y)\int_0^{y-\frac{y}{\sqrt{k}}}\frac{|u'_y(t)|}{(y-t)^2}dt\\=&\beta^{\rho,\nu}_{k,\lambda}(y)\int_0^{y-\frac{y}{\sqrt{k}}}\frac{|u'_y(t)-u'_y(y)|}{(y-t)^2}dt\\\leq&y^{-1}\beta^{\rho,\nu}_{k,\lambda}(y)\int_1^{\sqrt{k}}\mathfrak{T}^y_{y-\frac{y}{z}}(u'_y)dz\\\leq&y^{-1}\beta^{\rho,\nu}_{k,\lambda}(y)\sum_{l=1}^{[\sqrt{k}]}\int_l^{l+1}\mathfrak{T}^y_{y-\frac{y}{z}}(u'_y)dz\end{aligned}$$

$$\leq y^{-1}\beta_{k,\lambda}^{\rho,\nu}(y)\sum_{l=1}^{[\sqrt{k}]}\mathfrak{T}_{y-\frac{y}{l}}^{y}(u_y'). \tag{37}$$

Combining (36) and (37), we get

$$I_1 \leq y^{-1}\beta_{k,\lambda}^{\rho,\nu}(y)\sum_{l=1}^{[\sqrt{k}]}\mathfrak{T}_{y-\frac{y}{l}}^{y}(u_y') + \frac{y}{\sqrt{k}}\mathfrak{T}_{y-\frac{y}{\sqrt{k}}}^{y}(u_y'). \tag{38}$$

Again, we note the fact that for $t > y$, the relation $\tilde{\mathcal{K}}_{k,\lambda}^{\rho,\nu}(y,t)dt = d_t(1-\zeta_{k,\lambda}^{\rho,\nu}(y,t))$ holds and hence by applying the by parts rule of integration on I_2, we yield

$$\begin{aligned}
I_2 =& \left|\int_y^1 \left(\int_y^t u_y'(z)dz\right) d_t(1-\zeta_{k,\lambda}^{\rho,\nu}(y,t))\right| \\
=& \left|\left[\left(\int_y^t u_y'(z)dz\right)(1-\zeta_{k,\lambda}^{\rho,\nu}(y,t))\right]_y^1 - \int_y^1 u_y'(t)(1-\zeta_{k,\lambda}^{\rho,\nu}(y,t))dt\right| \\
=& \left|\int_y^1 (1-\zeta_{k,\lambda}^{\rho,\nu}(y,t))u_y'(t)dt\right| \\
\leq& \int_y^w |1-\zeta_{k,\lambda}^{\rho,\nu}(y,t)||u_y'(t)|dt + \int_w^1 |1-\zeta_{k,\lambda}^{\rho,\nu}(y,t)||u_y'(t)|dt.
\end{aligned}$$

Substituting $w = y + \frac{(1-y)}{\sqrt{k}}$, in the above inequality

$$I_2 \leq \int_y^{y+\frac{(1-y)}{\sqrt{k}}} |1-\zeta_{k,\lambda}^{\rho,\nu}(y,t)||u_y'(t)|dt + \int_{y+\frac{(1-y)}{\sqrt{k}}}^1 |1-\zeta_{k,\lambda}^{\rho,\nu}(y,t)||u_y'(t)|dt.$$

By using Lemma 5.1 and the substitution $t = y + \frac{(1-y)}{z}$, we have

$$\begin{aligned}
\int_{y+\frac{(1-y)}{\sqrt{k}}}^1 |1-\zeta_{k,\lambda}^{\rho,\nu}(y,t)||u_y'(t)|dt \leq& \beta_{k,\lambda}^{\rho,\nu}(y)\int_{y+\frac{(1-y)}{\sqrt{k}}}^1 \frac{|u_y'(t)|}{(t-y)^2}dt \\
=& \beta_{k,\lambda}^{\rho,\nu}(y)\int_{y+\frac{(1-y)}{\sqrt{k}}}^1 \frac{|u_y'(t)-u_y'(y)|}{(t-y)^2}dt \\
\leq& (y-1)^{-1}\beta_{k,\lambda}^{\rho,\nu}(y)\int_{\sqrt{k}}^1 \mathfrak{T}_y^{y+\frac{(1-y)}{z}}(u_y')dz \\
=& (1-y)^{-1}\beta_{k,\lambda}^{\rho,\nu}(y)\int_1^{\sqrt{k}} \mathfrak{T}_y^{y+\frac{(1-y)}{z}}(u_y')dz \\
\leq& (1-y)^{-1}\beta_{k,\lambda}^{\rho,\nu}(y)\sum_{l=1}^{[\sqrt{k}]}\int_l^{l+1} \mathfrak{T}_y^{y+\frac{(1-y)}{z}}(u_y')dz \\
\leq& (1-y)^{-1}\beta_{k,\lambda}^{\rho,\nu}(y)\sum_{l=1}^{[\sqrt{k}]} \mathfrak{T}_y^{y+\frac{(1-y)}{l}}(u_y').
\end{aligned}$$

Using the fact $|1-\zeta_{k,\lambda}^{\rho,\nu}(y,t)| \leq 1$, we can deduce

$$\int_y^{y+\frac{(1-y)}{\sqrt{k}}} |1-\zeta_{k,\lambda}^{\rho,\nu}(y,t)||u_y'(t)|dt = \int_y^{y+\frac{(1-y)}{\sqrt{k}}} |1-\zeta_{k,\lambda}^{\rho,\nu}(y,t)||u_y'(t)-u_y'(y)|dt$$

$$\leq \int_{y}^{y+\frac{(1-y)}{\sqrt{k}}} \mathfrak{T}_y^t(u_y')dt \leq \frac{(1-y)}{\sqrt{k}} \mathfrak{T}_y^{y+\frac{(1-y)}{\sqrt{k}}}(u_y').$$

This leaves us with the following inequality

$$I_2 \leq (1-y)^{-1} \beta_{k,\lambda}^{\rho,\nu}(y) \sum_{l=1}^{[\sqrt{k}]} \mathfrak{T}_y^{y+\frac{(1-y)}{l}}(u_y') + \frac{(1-y)}{\sqrt{k}} \mathfrak{T}_y^{y+\frac{(1-y)}{\sqrt{k}}}(u_y'). \tag{39}$$

Finally, substituting (38) and (39) in (35), we get the desired estimation. □

6. Concluding Remarks

As a consequence, in this paper, a Beta-type generalization of a modified Bernstein operators has been introduced which have a better modeling flexibility than the Bernstein-Durrmryer operators. We established the important convergence properties of newly defined operators, such as degree of local and global approximation, quantitative Voronovskaja type and Gruss-Voronovskaja type results. Finally, we estimate the error for absolutely continuous functions with derivatives of bounded variation.

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