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Anisotropic elliptic problem involving a singularity and a Radon measure

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Abstract. In this study, we demonstrate the existence of solutions to an anisotropic elliptic problem featuring a singularity, where the non-homogeneous term is characterized by a non-negative Radon measure μ. The model problem is

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2}\partial_i u\right) = \frac{f}{(e^u-1)^\gamma} + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\gamma > 0$, $f \in L^1(\Omega)$ and $2 < p_1 \leq p_2 \leq \ldots \leq p_N$. The primary goal of this work is to establish the existence of solutions based on the values of γ.

1. Introduction and some preliminaries

Anisotropic equations hold a pivotal position in a broad spectrum of mathematical models. A notable illustration is their utilization in the examination of fluid dynamics, where they capture the behavior of fluids with diverse conductivities in different orientations (refer to [2]). Additionally, these equations bear significance in the realm of biology, specifically in the modeling of epidemic disease propagation in heterogeneous environments, as investigated by Bendahmane, Langlais, and Saad in their research (see [3]). These instances underscore the adaptability and significance of anisotropic equations across various scientific disciplines.

The objective of this paper is to concentrate on the investigation of an anisotropic elliptic problem described by the following equations

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2}\partial_i u\right) = f(x)g(u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1}$$

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where Ω is a bounded open set in $\mathbb{R}^N$ (with $N > 2$), μ is a non-negative, bounded, Radon measure on Ω, and f belongs to the space $L^1(\Omega)$ with non-negative values, that could also be considered as a measure. The vector $\overrightarrow{p} = (p_1, ..., p_N) \in \mathbb{R}^N$ satisfies the following conditions

$$2 < p_1 \leq p_2 \leq ... \leq p_N \quad \text{and} \quad 2 < \overline{p} < N, \tag{2}$$

here, $\overline{p}$ represents the harmonic mean of p_i and is defined as

$$\overline{p} = N\left(\sum_{i=1}^{N} \frac{1}{p_i}\right)^{-1}.$$

The function $g : (0, +\infty) \to (0, +\infty)$ is a nonlinear function, such that

$$g \text{ is non-increasing, continuous and } \lim_{s \to 0^+} g(s) = +\infty, \tag{3}$$

and it also has the following growth conditions near zero and infinity

$$\exists \gamma > 0,\ \underline{M} > 0,\ \underline{t} > 0 \ \text{ such that } \ g(t) \leq \frac{\underline{M}}{t^{\gamma}} \ \text{ for all } \ t \leq \underline{t}, \tag{4}$$

$$\exists \theta > 0,\ \overline{M} > 0,\ \bar{t} > \underline{t} \ \text{ such that } \ g(t) \leq \frac{\overline{M}}{t^{\theta}} \ \text{ for all } \ t \geq \bar{t}. \tag{5}$$

It is worth mentioning that the singularity appearing in problem (1) can be controlled by the conditions (4).

The anisotropic Sobolev spaces naturally serve as the functional framework for problem (1) are $W^{1,\overrightarrow{p}}(\Omega)$ and $W_0^{1,\overrightarrow{p}}(\Omega)$, which are defined as follows

$$W^{1,\overrightarrow{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega) : \ \partial_i u \in L^{p_i}(\Omega), \forall i = 1, \ldots, N\right\},$$

and

$$W_0^{1,\overrightarrow{p}}(\Omega) = \left\{u \in W_0^{1,1}(\Omega) : \ \partial_i u \in L^{p_i}(\Omega), \forall i = 1, \ldots, N\right\}.$$

The space $W_0^{1,\overrightarrow{p}}(\Omega)$ can also be defined as the closure of $C_c^{\infty}(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}} = \sum_{i=1}^{N} \left(\int_{\Omega} |\partial_i u|^{p_i} dx\right)^{\frac{1}{p_i}},$$

endowed with this norm, $W_0^{1,\overrightarrow{p}}(\Omega)$ is a separable and reflexive Banach space.

The theory concerning such spaces was developed in [11, 20, 21, 23]. In particular, it has been demonstrated in [23] that when $\overline{p} < N$, the following continuous embedding holds

$$W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^r(\Omega), \quad \forall r \in [1, \overline{p}^{\star}], \text{ where } \overline{p}^{\star} = \frac{N\overline{p}}{N - \overline{p}},$$

additionally, this embedding is compact for $r < \overline{p}^{\star}$. Furthermore, in reference [23], positive constants C and $\overline{C}$, which depend solely on Ω, exist such that

$$\left(\int_{\Omega} |u|^r dx\right)^{\frac{1}{r}} \leq C \prod_{i=1}^{N} \left(\int_{\Omega} |\partial_i u|^{p_i} dx\right)^{\frac{1}{Np_i}}, \quad \forall r \in [1, \overline{p}^{\star}],\ \forall u \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{6}$$

$$\frac{1}{\overline{C}} \left(\int_{\Omega} |u|^{\overline{p}^{\star}}\right)^{\frac{p_N}{\overline{p}^{\star}}} \leq \sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i} dx, \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{7}$$

Furthermore, for each $i = 1, \ldots, N$, there exists a constant $C_i > 0$ (see [18, Lemma 1.1]) such that the following inequality holds

$$\left(\int_\Omega |u|^{p_i} dx\right)^{\frac{1}{p_i}} \leq C_i \left(\int_\Omega |\partial_i u|^{p_i} dx\right)^{\frac{1}{p_i}}, \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega). \tag{8}$$

We represent the collection of finite Radon measures on Ω as $\mathcal{M}(\Omega)$. This space is equipped with the "total variation norm," which is defined as

$$\|\mu\|_{\mathcal{M}(\Omega)} = \int_\Omega d|\mu|.$$

Recall that the Marcinkiewicz space $M^s(\Omega)$ (also known as the weak $L^s(\Omega)$ space), as defined in [4]. This space is defined for all $s > 0$ and is given by

$$M^s(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable} \Big/ \exists C > 0, \forall t > 0 : \text{meas}\left\{x \in \Omega : |u(x)| \geq t\right\} \leq \frac{C}{t^s}\right\},$$

where $\text{meas}\left\{x \in \Omega : |u(x)| \geq t\right\}$ denotes the Lebesgue measure of the set $\left\{x \in \Omega : |u(x)| \geq t\right\}$. Moreover, it holds that $M^s(\Omega) \subset M^{\bar{s}}(\Omega)$ if $s \geq \bar{s}$. Additionally, the following continuous embeddings hold for every $1 < s < \infty$ and $0 < \varepsilon \leq s - 1$,

$$L^s(\Omega) \hookrightarrow M^s(\Omega) \hookrightarrow L^{s-\varepsilon}(\Omega), \tag{9}$$

Furthermore, we have

$$\forall u \in M^s(\Omega),\ \exists C_u > 0,\ \forall E \subset \Omega \text{ (measurable)} : \int_E |u| dx \leq C_u \text{meas}\{E\}^{1-\frac{1}{s}}.$$

The simplest anisotropic problem has been studied in detail in [8], the author established the existence and regularity of solutions for the specific case of problem (1), where $g(u) \equiv 1$, $f \in L^m(\Omega)$ with $m \geq 1$, and $\mu \equiv 0$. The paper extensively discussed various cases by considering different values of m.

Numerous studies in the literature [13, 17, 19, 24] have addressed anisotropic problems involving singularities. One such problem was considered in [17], where the authors investigated the anisotropic problem

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \dfrac{f}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{10}$$

here, the anisotropic Laplace operator $\Delta_{\overrightarrow{p}} u$ is defined as follows

$$\Delta_{\overrightarrow{p}} u = \sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u\right) \text{ where } \partial_i u = \frac{\partial u}{\partial x_i}, \ \forall i = 1, \ldots, N, \tag{11}$$

where $\gamma > 0$, $1 \leq p_1 \leq p_2 \leq \ldots \leq p_N$, and f is a non-negative function in $L^m(\Omega)$. The authors obtained the existence and regularity results for problem (10), which are summarized in the following table.

Values of γ	Assumptions on f	Regularity of u
$\gamma = 1$	$f \in L^1(\Omega)$	$u \in W_0^{1,\vec{p}}(\Omega)$
	$f \in M^m(\Omega)$ where $m > \frac{N}{\overline{p}}$	$u \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$
	$f \in L^m(\Omega)$ where $(\overline{p}^*)' < m < \frac{N}{\overline{p}}$	$u \in L^s(\Omega)$ with $s = \frac{mN\overline{p}}{N-m\overline{p}}$
$0 < \gamma < 1$	$f \in L^1(\Omega)$	$u \in W_0^{1,\vec{s}}(\Omega) \cap L^{\overline{s}^*}(\Omega)$ for all $s_i < p_i \frac{N(\overline{p}-(1-\gamma)N)}{\overline{p}(N-(1-\gamma))}$
$1 < \gamma$	$f \in L^1(\Omega)$	$u \in L^s(\Omega)$ with $s = \frac{N(\gamma-1+\overline{p})}{N-\overline{p}}$

Table. Regularity results for different values of γ.

The problem (10) has been studied in [19] with a singular nonlinearity having a variable exponent, i.e., $\gamma \equiv \gamma(x) \in C(\overline{\Omega})$. In [24], the problem (10) is investigated with the substitution of the operator (11) by the degenerate operator

$$Au = \sum_{i=1}^{N} \partial_i \left(\frac{|\partial_i u|^{p_i-2} \partial_i u}{(1+|u|)^\theta} \right),$$

we also recommend, for instance, referring to [14] for addressing anisotropic degenerate problems.

The problems involving singularities in the isotropic case, where $p_i = 2$ for all i, have been extensively studied in the literature. In work [5], the authors obtained existence and regularity of solutions to the problem

$$\begin{cases} -\mathrm{div}\,(M(x)\nabla u) = \frac{f}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{12}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with $N \geq 2$, $\gamma > 0$, $f \in L^m(\Omega)$ or $f \in \mathcal{M}(\Omega)$, and M is a bounded elliptic matrix.

The existence and stability of solutions to a problem more general than (12) were explored in [12], it is as follows

$$\begin{cases} -\mathrm{div}\,(A(x)\nabla u) = F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{13}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 1$, A is a coercive matrix with coefficients in $L^\infty(\Omega)$, and $F : (x,s) \in \Omega \times [0,+\infty[\to F(x,s) \in [0,+\infty]$ is a Carathéodory function which satisfies $0 \leq F(x,s) \leq \frac{h(x)}{\Gamma(s)}$ a.e. $x \in \Omega, \forall s > 0$, with $h \in L^m(\Omega), m \geq 1$ and $\Gamma \in C^1([0,+\infty[)$ strictly increasing function such that $\Gamma(0) = 0$.

Regarding problems involving singularity and measure data, we recommend, for instance [1, 7, 10]. In [1] the authors studied the following model

$$\begin{cases} -\mathrm{div}\,(\nabla u) = f(x)g(u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{14}$$

here, μ represents a nonnegative, bounded, Radon measure on Ω, while f stands for a nonnegative function in $L^m(\Omega)$ with $m \geq 1$, which can be seen as a measure. The function $g : (0,+\infty) \to (0,+\infty)$ is a nonlinear, nonincreasing, continuous function satisfying the conditions (3), (4) and (5). Under these assumptions, the authors have proved the following results

(R1) For $0 < \gamma < 1$, a weak solution u of (14) exists in $W_0^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$.

(R2) For $\gamma \geq 1$, problem (14) has a weak solution u in the space $W_{loc}^{1,q}(\Omega)$, where $q < \frac{N}{N-1}$.

They have also obtained several other noteworthy results, which can be explored in detail within the referenced publication. The reference [10] addresses a specific case of (14) where $g(u) = u^{-\gamma}$ with $\gamma > 0$, and $f \in L^1(\Omega)$.

Drawing inspiration from the aforementioned facts and motivated by the previous results, we seek to generalize the problem (14) by introducing an anisotropic Laplace operator denoted as $\Delta_{\overrightarrow{p}} u$, which is defined in (11). By incorporating this anisotropic Laplace operator, we formulate the generalized problem presented in (1).

Regarding subsequences, we will need the following useful topological trick of uniqueness.

Lemma 1.1. *[16, Lemma 1.1] Let X be a topological space, and consider a sequence (x_n) in X with the property that for any subsequence (x_{n_k}), there exists a convergent subsequence $(x_{n_{k_j}})$ with a limit of x. In such cases, the sequence (x_n) converges to x.*

2. Main results

Currently, we present two essential definitions that play a crucial role in our analysis of the matter introduced in equation (1).

Definition 2.1. *Let (μ_n) be a sequence of measures in $\mathcal{M}(\Omega)$. We say (μ_n) weakly converges to $\mu \in \mathcal{M}(\Omega)$, denoted $\mu_n \rightharpoonup \mu$ in $\mathcal{M}(\Omega)$, if for any continuous function $f \in C_c(\Omega)$*

$$\int_\Omega f d\mu_n \to \int_\Omega f d\mu.$$

Definition 2.2. *(i) For $0 < \gamma < 1$, a weak solution to problem (1) is a function $u \in W_0^{1,1}(\Omega)$ that fulfills the equality*

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi dx = \int_\Omega f g(u) \varphi dx + \int_\Omega \varphi d\mu, \quad \forall \varphi \in C_c^1(\overline{\Omega}), \tag{15}$$

and the condition

$$\forall \omega \subset\subset \Omega, \ \exists C_\omega > 0 : u \geq C_\omega > 0, \ a.e. \ in \ \omega. \tag{16}$$

(ii) For $\gamma \geq 1$, a weak solution to problem (1) is a function $u \in W_{loc}^{1,1}(\Omega)$ satisfying (15) and (16), with $T_k^{\frac{\gamma+p_i-1}{p_i}}(u) \in W_0^{1,\overrightarrow{p}}(\Omega)$ for each fixed $k > 0$.

Our first result is presented in the following theorem.

Theorem 2.3. *Let $f \in L^1(\Omega)$ be a positive function, and $0 < \gamma < 1$. Assume that (2)–(5) hold true. Then, the problem (1) has at least one weak solution $u \in W_0^{1,\overrightarrow{q}}(\Omega)$ (As defined in Definition 2.2, under case (i)), where*

$$1 < q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i, \quad \forall i = 1, \dots, N. \tag{17}$$

The next result deals with the existence of solutions belonging to a space larger than the one mentioned in the previous theorem.

Theorem 2.4. *Let $f \in L^1(\Omega)$ and $\gamma \geq 1$. Assuming that (2) through (5) hold. Then, the problem (1) possesses at least one weak solution $u \in W_{loc}^{1,\overrightarrow{q}}(\Omega)$ (in the sense of Definition 2.2, under case (ii)), where q_i satisfies (17).*

Remark 2.5. *The assumption* (2) *ensures that*

$$\left(1, \frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i\right) \neq \emptyset \quad \textit{and} \quad q_i < p_i, \quad \textit{for all} \quad i = 1, \ldots, N.$$

Indeed,

$$\begin{aligned}
2 > 1 + \frac{N-1}{N+1} &\Longrightarrow \overline{p} > 1 + \frac{N-1}{N+1} \quad (\textit{because } \overline{p} > 2)\\
&\Longrightarrow \overline{p} > \frac{2N}{N+1}\\
&\Longrightarrow \overline{p}N + \overline{p} > 2N\\
&\Longrightarrow 2\overline{p}N - 2N > \overline{p}N - \overline{p}\\
&\Longrightarrow 2 > \frac{\overline{p}(N-1)}{N(\overline{p}-1)} \quad (\overline{p} - 1 > 0)\\
&\Longrightarrow p_i > \frac{\overline{p}(N-1)}{N(\overline{p}-1)} \quad (\textit{because } p_i > 2)\\
&\Longrightarrow \frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i > 1.
\end{aligned}$$

To show that $q_i < p_i$, it is enough to prove that $\frac{N(\overline{p}-1)}{\overline{p}(N-1)} < 1$, we have

$$\frac{N(\overline{p}-1)}{\overline{p}(N-1)} < 1 \Longleftrightarrow N(\overline{p}-1) < \overline{p}(N-1) \Longleftrightarrow N > \overline{p} \quad (\textit{true}).$$

Remark 2.6. *In the isotropic case, i.e., $p_i = 2$, the results of Theorem 2.3 and Theorem 2.4 coincide with regularity results for elliptic equation problems involving a singular term and a Radon measure (see Theorem 2.6 and Theorem 2.9 in [1]).*

3. Approximate solutions

We will use the following truncation functions: $T_k(s) = \min\{k, \max\{-k, s\}\}$ and $G_k(s) = s - T_k(s)$, where $s \in \mathbb{R}$ and $k > 0$. For any $s \in \mathbb{R}$ and $k > 0$, the equality $T_k(s) + G_k(s) = s$ is apparent.

Let's begin by looking at the following approximation problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i\left(|\partial_i u_n|^{p_i-2}\partial_i u_n\right) = f_n g_n\left(u_n + \frac{1}{n}\right) + \mu_n & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{18}$$

here, $f_n = T_n(f)$, $g_n = T_n(g)$, and (μ_n) is a sequence of smooth non-negative functions bounded in $L^1(\Omega)$, converging to μ as per Definition 2.1. The weak formulation of (18) reads

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i \varphi dx = \int_\Omega f_n g_n\left(u_n + \frac{1}{n}\right)\varphi dx + \int_\Omega \mu_n \varphi dx, \tag{19}$$

for all $\varphi \in C_c^1(\overline{\Omega})$.

By means of the following lemma, we will demonstrate the existence of a solution to problem (18).

Lemma 3.1. *Assuming* (2) *to* (5) *are satisfied. Then, the problem* (18) *has a non-negative weak solution $u_n \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$.*

Proof. The lemma's proof will be carried out by employing Schauder's fixed point argument.

Let $n \in \mathbb{N}^\star$ be fixed. We introduce a mapping $S : L^{\overline{p}}(\Omega) \to L^{\overline{p}}(\Omega)$, where for all $v \in L^{\overline{p}}(\Omega)$, $w = S(v)$ is the unique weak solution to the problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i w|^{p_i - 2} \partial_i w \right) = f_n g_n \left(|v| + \frac{1}{n} \right) + \mu_n & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

To demonstrate the existence of a unique solution $w \in W_0^{1,\overrightarrow{p}}(\Omega)$, to (20), please consult Appendix A.

Let us consider w as a test function in the weak formulation of (20). Using (4), (5), and the fact that $f_n \leq n$, we obtain

$$\begin{aligned} \sum_{i=1}^{N} \|\partial_i u_n\|_{L^{p_i}(\Omega)}^{p_i} &= \sum_{i=1}^{N} \int_\Omega |\partial_i w|^{p_i} dx \\ &= \int_\Omega f_n g_n \left(|v| + \frac{1}{n} \right) w dx + \int_\Omega \mu_n w dx \\ &\leq \underline{M} \int_{\left\{|v| + \frac{1}{n} < \underline{t}\right\}} \frac{f_n w}{\left(|v| + \frac{1}{n} \right)^\gamma} dx + \overline{M} \int_{\left\{|v| + \frac{1}{n} > \overline{t}\right\}} \frac{f_n w}{\left(|v| + \frac{1}{n} \right)^\theta} dx + \max_{t \in [\underline{t}, \overline{t}]} h(t) \int_{\left\{\underline{t} \leq |v| + \frac{1}{n} \leq \overline{t}\right\}} f_n w dx \\ &\quad + C(n) \int_\Omega |w| dx \\ &\leq \underline{M} n^{\gamma+1} \int_{\left\{|v| + \frac{1}{n} < \underline{t}\right\}} |w| dx + \overline{M} n^{\theta+1} \int_{\left\{|v| + \frac{1}{n} > \overline{t}\right\}} |w| dx + n \max_{t \in [\underline{t}, \overline{t}]} h(t) \int_{\left\{\underline{t} \leq |v| + \frac{1}{n} \leq \overline{t}\right\}} |w| dx \\ &\quad + C(n) \int_\Omega |w| dx \\ &\leq C(n, \gamma, \theta) \int_\Omega |w| dx. \end{aligned}$$

By utilizing (7) on the left-hand side and Hölder's inequality with exponent $\overline{p}^\star$ on the right-hand side, we obtain

$$\frac{1}{C} \|w\|_{L^{\overline{p}^\star}(\Omega)}^{p_N} \leq \sum_{i=1}^{N} \|\partial_i u_n\|_{L^{p_i}(\Omega)}^{p_i} \leq C(n, \gamma, \theta) |\Omega|^{\frac{1}{\overline{p}^{\star\prime}}} \|w\|_{L^{\overline{p}^\star}(\Omega)}. \tag{21}$$

As $p_N > 1$, a positive constant $R(n, |\Omega|)$, independent of v and w, exists such that

$$\|w\|_{L^{\overline{p}^\star}(\Omega)} \leq \overline{C} C(n, \gamma, \theta) = R(n, |\Omega|). \tag{22}$$

As $\overline{p} < \overline{p}^\star$, then

$$\|w\|_{L^{\overline{p}}(\Omega)} \leq \overline{C} C(n, \gamma, \theta) = R(n, |\Omega|). \tag{23}$$

Thus, equation (23) implies that the ball B in $L^{\overline{p}}(\Omega)$, with radius $R(n, |Q|)$, is invariant under the map S.
Claim: S is continuous on $L^{\overline{p}}(\Omega)$.
Let $v \in L^{\overline{p}}(\Omega)$ and let (v_k) be a sequence of functions converges to v in $L^{\overline{p}}(\Omega)$. We denote $w_k = S(v_k)$ and $w = S(v)$. To prove that $w_k \longrightarrow w$ in $L^{\overline{p}}(\Omega)$, it suffices to demonstrate that $w_k \longrightarrow w$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ because $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{\overline{p}}(\Omega)$. According to Lemma 1.1, to verify that $w_k \longrightarrow w$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, it is sufficient to show that for any subsequence of (w_k), it is possible to extract a further subsequence that converges to w.

Let $(w_{\sigma(k)})$ be a subsequence of (w_k). Firstly, since $v_{\sigma(k)} \longrightarrow v$ in $L^{\overline{p}}(\Omega)$ as $\sigma(k) \to \infty$, we can extract a subsequence $(v_{\sigma_1(k)})$ of $(v_{\sigma(k)})$ such that

$$v_{\sigma_1(k)} \xrightarrow{\sigma_1(k)\to\infty} v \text{ a.e. in } \Omega. \tag{24}$$

Secondly, for every integer $\sigma_1(k)$, one has

$$\left| f_n g_n\left(v_{\sigma_1(k)} + \frac{1}{n}\right) + \mu_n \right| \leq n^2 + C(n). \tag{25}$$

From (24) and (25), we can apply the dominated convergence theorem to deduce that

$$\left\| \left[f_n g_n\left(v_{\sigma_1(k)} + \frac{1}{n}\right) + \mu_n \right] - \left[f_n g_n\left(v + \frac{1}{n}\right) + \mu_n \right] \right\|_{L^{\alpha}(\Omega)} \xrightarrow{\sigma_1(k)\to\infty} 0, \quad \forall \alpha \geq 1.$$

Hence

$$\left\| f_n g_n\left(v_{\sigma_1(k)} + \frac{1}{n}\right) - f_n g_n\left(v + \frac{1}{n}\right) \right\|_{L^{\alpha}(\Omega)} \xrightarrow{\sigma_1(k)\to\infty} 0, \quad \forall \alpha \geq 1. \tag{26}$$

Thirdly, we have $w_{\sigma_1(k)}$ and w satisfying the equation

$$-\sum_{i=1}^{N} \partial_i \left(|\partial_i w_{\sigma_1(k)}|^{p_i-2} \partial_i w_{\sigma_1(k)} \right) + \sum_{i=1}^{N} \partial_i \left(|\partial_i w|^{p_i-2} \partial_i w \right) = f_n g_n\left(v_{\sigma_1(k)} + \frac{1}{n}\right) - f_n g_n\left(v + \frac{1}{n}\right). \tag{27}$$

By selecting $w_{\sigma_1(k)} - w$ as a test function in (27), we get

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \left[|\partial_i w_{\sigma_1(k)}|^{p_i-2} \partial_i w_{\sigma_1(k)} - |\partial_i w|^{p_i-2} \partial_i w \right] (\partial_i w_{\sigma_1(k)} - \partial_i w) dx \\ &\quad = \int_\Omega \left[f_n g_n\left(v_{\sigma_1(k)} + \frac{1}{n}\right) - f_n g_n\left(v + \frac{1}{n}\right) \right] (w_{\sigma_1(k)} - w) dx. \end{aligned}$$

By Hölder's inequality, (8), (21) and (23), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \left[|\partial_i w_{\sigma_1(k)}|^{p_i-2} \partial_i w_{\sigma_1(k)} - |\partial_i w|^{p_i-2} \partial_i w \right] (\partial_i w_{\sigma_1(k)} - \partial_i w) dx \\ &\quad \leq \left\| f_n g_n\left(v_{\sigma_1(k)} + \frac{1}{n}\right) - f_n g_n\left(v + \frac{1}{n}\right) \right\|_{L^{p_i'}(\Omega)} \|w_{\sigma_1(k)} - w\|_{L^{p_i}(\Omega)} \\ &\quad \leq \left\| f_n g_n\left(v_{\sigma_1(k)} + \frac{1}{n}\right) - f_n g_n\left(v + \frac{1}{n}\right) \right\|_{L^{p_i'}(\Omega)} \|\partial_i (w_{\sigma_1(k)} - w)\|_{L^{p_i}(\Omega)} \\ &\quad \leq C_n \left\| f_n g_n\left(v_{\sigma_1(k)} + \frac{1}{n}\right) - f_n g_n\left(v + \frac{1}{n}\right) \right\|_{L^{p_i'}(\Omega)}. \end{aligned}$$

Consequently, from (26), we obtain

$$\lim_{\sigma_1(k)\to\infty} \sum_{i=1}^{N} \int_\Omega \left[|\partial_i w_{\sigma_1(k)}|^{p_i-2} \partial_i w_{\sigma_1(k)} - |\partial_i w|^{p_i-2} \partial_i w \right] \partial_i (w_{\sigma_1(k)} - w) dx = 0.$$

Finally, by following the same line of reasoning as in Lemma 2.4 of [15], we can extract a subsequence $(w_{\sigma_2(k)})$ from $(w_{\sigma_1(k)})$ such that $(w_{\sigma_2(k)})$ converges to w in $W_0^{1,\overrightarrow{p}}(\Omega)$. This establishes the continuity of S.

Claim: $S(L^{\overline{p}}(\Omega))$ is relatively compact in $L^{\overline{p}}(\Omega)$.
Using equations (21) and (22), we can deduce that

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i w|^{p_i}dx=\sum_{i=1}^{N}\int_{\Omega}|\partial_i S(v)|^{p_i}dx\le R(n,|\Omega|),\quad \forall v\in L^{\overline{p}}(\Omega).$$

By Sobolev embedding, $S(L^{\overline{p}}(\Omega))$ can be shown to be compact in $L^{\overline{p}}(\Omega)$.

As a result, through the utilization of the Schauder fixed point theorem on S, we establish the existence of a fixed point $u_n\in L^{\overline{p}}(\Omega)$. This fixed point is identified as a weak solution to (20) in $W_0^{1,\overrightarrow{p}}(\Omega)$.

Moreover, taking $\varphi=u_n^-=\min\{u_n,0\}$ in (19). Using the fact that $f_ng_n\left(u_n+\frac{1}{n}\right)+\mu_n\ge 0$, we obtain

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i u_n^-|^{p_i}dx\le\int_{\Omega}\left(f_ng_n\left(u_n+\frac{1}{n}\right)+\mu_n\right)u_n^-dx\le 0.$$

This leads to $u_n^-=0$ almost everywhere in Ω, implying $u_n\ge 0$.

Furthermore, for a fixed n, we have u_n belongs to $L^\infty(\Omega)$ (by [22, Théorème 4.2, page 215]) because the right-hand side of (18) is in $L^\infty(\Omega)$ and this concludes the proof. □

Lemma 3.2. *Let u_n be a solution to problem* (18). *Then, for every $\omega\subset\subset\Omega$, there exists a constant $C_\omega>0$ independent of n such that*

$$u_n(x)\ge C_\omega>0\quad a.e.\ x\in\omega. \tag{28}$$

Proof. Let v_n be the unique weak solution of

$$\begin{cases}-\sum_{i=1}^{N}\partial_i\left(|\partial_i v_n|^{p_i-2}\partial_i v_n\right)=f_ng_n\left(v_n+\frac{1}{n}\right) & \text{in }\ \Omega,\\ v_n=0 & \text{on }\ \partial\Omega.\end{cases} \tag{29}$$

By the strong maximum principle (see [9, Theorem 3.18]), since $f_ng_n\left(v_n+\frac{1}{n}\right)\ge 0$ and not identically zero, we have $v_n\ge C_\omega>0$ in any $\omega\subset\subset\Omega$ for some constant C_ω (see [17, Lemma 2.3 and Lemma 2.6]).

Demonstrating that $u_n\ge v_n$ holds almost everywhere in Ω is straightforward. Suppose this is not the case, meaning that $u_n<v_n$ in Ω. In such a scenario, we subtract the weak formulations (18) and (29) with the test function $v_n-u_n>0$, we obtain

$$\sum_{i=1}^{N}\int_{\Omega}\left(|\partial_i v_n|^{p_i-2}\partial_i v_n-|\partial_i u_n|^{p_i-2}\partial_i u_n\right)\partial_i(v_n-u_n)dx=\int_{\Omega}f_n\left[g_n\left(v_n+\frac{1}{n}\right)-g_n\left(u_n+\frac{1}{n}\right)\right](v_n-u_n)dx-\int_{\Omega}\mu_n(v_n-u_n)dx. \tag{30}$$

Thanks to the well-known inequalities (see Appendix B)

$$\left(|\partial_i v_n|^{p_i-2}\partial_i v_n-|\partial_i u_n|^{p_i-2}\partial_i u_n\right)(\partial_i v_n-\partial_i u_n)\ge|\partial_i v_n-\partial_i u_n|^{p_i},\ \forall i=1,\dots,N,$$

we get

$$\sum_{i=1}^{N}\int_{\Omega}\left(|\partial_i v_n|^{p_i-2}\partial_i v_n-|\partial_i u_n|^{p_i-2}\partial_i u_n\right)\partial_i(v_n-u_n)dx\ge\sum_{i=1}^{N}\int_{\Omega}|\partial_i(v_n-u_n)|^{p_i}dx. \tag{31}$$

Thus, (30) and (31) provide

$$\int_{\Omega}f_n\left[g_n\left(v_n+\frac{1}{n}\right)-g_n\left(u_n+\frac{1}{n}\right)\right](v_n-u_n)dx-\int_{\Omega}\mu_n(v_n-u_n)dx\ge\sum_{i=1}^{N}\int_{\Omega}|\partial_i(v_n-u_n)|^{p_i}dx. \tag{32}$$

Since $u_n < v_n$ and g_n is non-increasing function, then $g_n\left(v_n + \frac{1}{n}\right) - g_n\left(u_n + \frac{1}{n}\right) \leq 0$, and given that f_n and μ_n are non-negative functions, we have

$$\int_\Omega f_n \left[g_n\left(v_n + \frac{1}{n}\right) - g_n\left(u_n + \frac{1}{n}\right)\right](v_n - u_n)dx - \int_\Omega \mu_n(v_n - u_n)dx \leq 0, \tag{33}$$

Combining (32) and (33), we conclude

$$\sum_{i=1}^{N} \int_\Omega |\partial_i(v_n - u_n)|^{p_i} dx \leq 0.$$

This leads to the conclusion that $u_n \geq v_n$ almost everywhere in Ω, and consequently, this inequality holds true within ω as well. Hence, we've shown that for any $\omega \subset\subset \Omega$, there's a constant C_ω such that $u_n \geq v_n > C_\omega > 0$ almost everywhere in ω. □

Throughout the ensuing discussion, let $u_n \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ represent a solution to problem (18).

4. Proof of main results

4.1. Proof of Theorem 2.3

Step 1: A priori estimates

In the following lemma, we shall provide $W_0^{1,\overrightarrow{p}}(\Omega)$−estimates for the solutions u_n of problem (18).

Lemma 4.1. *Under the assumptions of Theorem 2.3, there's a fixed positive constant C not dependent on n, such that*

$$\|T_k(u_n)\|_{\overrightarrow{q}} \leq C, \quad \forall k \geq 1, \tag{34}$$

$$\|\partial_i u_n\|_{M^{\frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i}(\Omega)} \leq C, \tag{35}$$

$$\|u_n\|_{\overrightarrow{q}} \leq C, \tag{36}$$

where $q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i$, $\forall i = 1, \ldots, N$.

Proof. We adopt the reasoning presented in [4] to prove this lemma. Let $\varphi = T_k(u_n)$ ($\forall k \geq 1$) and use it as a test function in (19), thus

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n)|^{p_i} dx \leq \int_\Omega f_n g_n\left(u_n + \frac{1}{n}\right) T_k(u_n)dx + \int_\Omega \mu_n T_k(u_n)dx, \tag{37}$$

Using (4) and (5) in the right hand side of (37) and the fact that

$$\frac{T_k(u_n)}{\left(u_n + \frac{1}{n}\right)^\gamma} \leq \frac{u_n}{\left(u_n + \frac{1}{n}\right)^\gamma} = \frac{u_n^\gamma u_n^{1-\gamma}}{\left(u_n + \frac{1}{n}\right)^\gamma} \leq u_n^{1-\gamma},$$

we have

$$\begin{aligned}\int_\Omega f_n g_n\left(u_n+\frac{1}{n}\right)T_k(u_n)dx+\int_\Omega \mu_n T_k(u_n)dx &\le \underline{M}\int_{\left\{u_n+\frac{1}{n}<\underline{t}\right\}}\frac{f_nT_k(u_n)}{\left(u_n+\frac{1}{n}\right)^\gamma}dx+\overline{M}\int_{\left\{u_n+\frac{1}{n}>\bar{t}\right\}}\frac{f_nT_k(u_n)}{\left(u_n+\frac{1}{n}\right)^\theta}dx\\ &\quad+\max_{t\in[\underline{t},\bar{t}]}h(t)\int_{\left\{\underline{t}\le u_n+\frac{1}{n}\le\bar{t}\right\}}f_nT_k(u_n)dx+k\|\mu_n\|_{L^1(\Omega)}\\ &\le \underline{M}\int_{\left\{u_n+\frac{1}{n}<\underline{t}\right\}}f\left(u_n+\frac{1}{n}\right)^{1-\gamma}dx+\overline{M}\int_{\left\{u_n+\frac{1}{n}>\bar{t}\right\}}\frac{kf}{\left(u_n+\frac{1}{n}\right)^\theta}dx\\ &\quad+k\max_{t\in[\underline{t},\bar{t}]}h(t)\int_{\left\{\underline{t}\le u_n+\frac{1}{n}\le\bar{t}\right\}}fdx+Ck\\ &\le\left(\underline{M}k\underline{t}^{1-\gamma}+\frac{k\overline{M}}{\bar{t}^\theta}+k\max_{t\in[\underline{t},\bar{t}]}h(t)\right)\|f\|_{L^1(\Omega)}+Ck\\ &\le Ck. \end{aligned}\tag{38}$$

By (37) and (38) we obtain

$$\int_\Omega|\partial_iT_k(u_n)|^{p_i}dx\le Ck,\quad \forall i=1,\dots,N.\tag{39}$$

Hence, from (39), we obtain (34).

Now, we prove that (∂_iu_n) is bounded in $M^{\frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i}(\Omega)$. For $\delta\ge1$ and any $k\ge1$, we get

$$\begin{aligned}\text{meas}\{|\partial_iu_n|>\delta\}&\le\text{meas}\{|\partial_iu_n|>\delta,\ u_n\le k\}+\text{meas}\{|\partial_iu_n|>\delta,\ u_n>k\}\\ &\le\text{meas}\{|\partial_iu_n|>\delta,\ u_n\le k\}+\text{meas}\{u_n>k\}.\end{aligned}$$

Now, using (39) on the right-hand side of the previous inequality, and the anisotropy inequality (6) we get

$$\begin{aligned}\text{meas}\{|\partial_iu_n|>\delta\}&\le\int_{\{u_n\le k\}}\left(\frac{|\partial_iu_n|}{\delta}\right)^{p_i}dx+\frac{1}{k^{\overline{p}^\star}}\int_{\{u_n>k\}}T_k(u_n)^{\overline{p}^\star}dx\\ &\le\frac{1}{\delta^{p_i}}\int_\Omega|\partial_iT_k(u_n)|^{p_i}dx+\frac{1}{k^{\overline{p}^\star}}\int_\Omega T_k(u_n)^{\overline{p}^\star}dx\\ &\le\frac{C}{\delta^{p_i}}k+\frac{1}{k^{\overline{p}^\star}}\left[\prod_{i=1}^N\left(\int_\Omega|\partial_iT_k(u_n)|^{p_i}dx\right)^{\frac{1}{Np_i}}\right]^{\overline{p}^\star}\\ &\le\frac{C}{\delta^{p_i}}k+C\frac{1}{k^{\overline{p}^\star}}\left[\prod_{i=1}^N k^{\frac{1}{Np_i}}\right]^{\overline{p}^\star}\\ &\le\frac{C}{\delta^{p_i}}k+Ck^{-\overline{p}^\star\left(1-\frac{1}{\overline{p}}\right)}.\end{aligned}\tag{40}$$

On choosing $k=\delta^{\frac{N-\overline{p}}{N\overline{p}-\overline{p}}p_i}$ in (40), we obtain

$$\text{meas}\{|\partial_iu_n|>\delta\}\le C\delta^{-\frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i},\quad\forall\delta\ge1,\ \ \forall i=1,\dots,N.$$

Consequently, we have established the boundedness of (∂_iu_n) in $M^{\frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i}(\Omega)$, which, in turn, leads by the property stated in (9) to the conclusion that (u_n) is bounded in $W_0^{1,\overrightarrow{q}}(\Omega)$ with $q_i<\frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i$, $\forall i=1,\dots,N$. This finishes the proof of Lemma 4.1. □

Step 2: Passage to the limit
Referencing Lemma 4.1, we ascertain the existence of a subsequence (u_n), also denoted as (u_n), and a measurable function u belonging to $W_0^{1,\overrightarrow{q}}(\Omega)$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,\overrightarrow{q}}(\Omega) \quad \text{and a.e. in } \Omega, \tag{41}$$

for every $q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)} p_i$. Now, adapting the approach of the proof of Lemma 5.1 in [24], we can show that there exists a subsequence (still denoted (u_n)) such that for all $i = 1, \ldots, N$

$$\partial_i u_n \to \partial_i u \quad \text{a.e. in } \Omega. \tag{42}$$

Via (41), (42), and the Lebesgue dominated convergence theorem, we achieve the following for all $\varphi \in C_c^1(\overline{\Omega})$

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2} \partial_i u_n \partial_i \varphi dx = \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi dx. \tag{43}$$

On the other hand by Lemma 3.2, we have

$$\begin{aligned}\int_\Omega \left| f_n g_n\left(u_n + \frac{1}{n}\right)\varphi \right| dx &\le \underline{M} \int_{\left\{u_n + \frac{1}{n} < \underline{t}\right\}} \frac{f_n|\varphi|}{\left(u_n + \frac{1}{n}\right)^\gamma} dx + \overline{M} \int_{\left\{u_n + \frac{1}{n} > \bar{t}\right\}} \frac{f_n|\varphi|}{\left(u_n + \frac{1}{n}\right)^\theta} dx \\ &\quad + \max_{t\in[\underline{t},\bar{t}]} h(t) \int_{\left\{\underline{t} \le u_n + \frac{1}{n} \le \bar{t}\right\}} f_n |\varphi| dx \\ &\le \left(\frac{\underline{M}}{C_\omega^\gamma} + \frac{\overline{M}}{C_\omega^\theta} + C \right) \|\varphi\|_{L^\infty(\Omega)} \|f\|_{L^1(\Omega)}\end{aligned}$$

where, $C > 0$ and $\omega = \{x \in \Omega : \ \varphi(x) \neq 0\}$. Consequently,

$$\text{the sequence } \left(f_n g_n\left(u_n + \frac{1}{n}\right)\varphi \right) \text{ is bounded in } L^1(\Omega). \tag{44}$$

By (41), (44) and the Lebesgue's theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega f_n g_n\left(u_n + \frac{1}{n}\right)\varphi dx = \int_\Omega f g(u) \varphi dx, \quad \forall \varphi \in C_c^1(\Omega). \tag{45}$$

Using the convergence results (43), (45) and $\mu_n \rightharpoonup \mu$ in $\mathcal{M}(\Omega)$, we can then take the limit as $n \to +\infty$ in the identities (19) for all $\varphi \in C_c^1(\Omega)$. This yields (15). So, the proof of Theorem 2.3 has now been completed.

4.2. *Proof of Theorem 2.4*

Step 1: A priori estimates:
In view of the extraordinary singularity of this case, it's possible to obtain local estimates of u_n. We intend to provide a global estimation of $\left(T_k^{\frac{\gamma+p_i-1}{p_i}}(u_n) \right)$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, aiming to attribute meaning to the boundary values of u, albeit in a weaker manner than in the context of the trace sense.

Lemma 4.2. *Under the assumptions of Theorem 2.4, there exists a positive constant C independent of n, such that*

$$\left\| T_k^{\frac{\gamma+p_i-1}{p_i}}(u_n) \right\|_{\overrightarrow{p}} \le C, \quad \forall k \ge 1, \tag{46}$$

$$\|u_n\|_{W_{loc}^{1,\overrightarrow{q}}(\Omega)} \le C, \tag{47}$$

where q_i give as in (17).

Proof. Choosing $\varphi = T_k^{\gamma}(u_n)$ for all $k \geq 1$ as the test function in (19), we get

$$\mathcal{I}_1 = \gamma \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i T_k(u_n) T_k^{\gamma-1}(u_n)dx = \int_\Omega f_n g_n\left(u_n + \frac{1}{n}\right) T_k^{\gamma}(u_n)dx + \int_\Omega \mu_n T_k^{\gamma}(u_n)dx = \mathcal{I}_2. \quad (48)$$

By the definition of $T_k(u_n)$ and since $\gamma \geq 1$, we estimate the term $\mathcal{I}_1$ of (48) as

$$\begin{aligned}\mathcal{I}_1 &= \gamma \sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n)|^{p_i} T_k^{\gamma-1}(u_n)dx \\ &= \gamma \sum_{i=1}^{N} \left(\frac{p_i}{\gamma + p_i - 1}\right)^{p_i} \int_\Omega \left|\partial_i T_k^{\frac{\gamma+p_i-1}{p_i}}(u_n)\right|^{p_i} dx \\ &\geq \frac{\gamma p_1}{\gamma + p_1 - 1} \sum_{i=1}^{N} \int_\Omega \left|\partial_i T_k^{\frac{\gamma+p_i-1}{p_i}}(u_n)\right|^{p_i} dx. \end{aligned} \quad (49)$$

For $\mathcal{I}_2$, using (4), (5) and the fact that

$$\frac{T_k^{\gamma}(u_n)}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \leq \frac{u_n^{\gamma}}{\left(u_n + \frac{1}{n}\right)^{\gamma}} \leq 1,$$

we have

$$\begin{aligned}\mathcal{I}_2 &\leq \underline{M} \int_{\left\{u_n+\frac{1}{n}<\underline{t}\right\}} \frac{f_n T_k^{\gamma}(u_n)}{\left(u_n + \frac{1}{n}\right)^{\gamma}}dx + \overline{M}\int_{\left\{u_n+\frac{1}{n}>\bar{t}\right\}} \frac{f_n T_k^{\gamma}(u_n)}{\left(u_n + \frac{1}{n}\right)^{\theta}}dx + \max_{t\in[\underline{t},\bar{t}]} h(t) \int_{\left\{\underline{t}\leq u_n+\frac{1}{n}\leq \bar{t}\right\}} f_n T_k^{\gamma}(u_n)dx + k^{\gamma}\int_\Omega \mu_n dx \\ &\leq \underline{M}\int_{\left\{u_n+\frac{1}{n}<\underline{t}\right\}} f dx + \frac{\overline{M}k^{\gamma}}{\bar{t}^{\theta}} \int_{\left\{u_n+\frac{1}{n}>\bar{t}\right\}} f dx + k^{\gamma} \max_{t\in[\underline{t},\bar{t}]} h(t) \int_{\left\{\underline{t}\leq u_n+\frac{1}{n}\leq \bar{t}\right\}} f dx + k^{\gamma}\|\mu_n\|_{L^1(\Omega)} \\ &\leq \left[\left(\underline{M} + \frac{\overline{M}}{\bar{t}^{\theta}} + C\right)\|f\|_{L^1(\Omega)} + \|\mu_n\|_{L^1(\Omega)}\right]k^{\gamma}. \end{aligned} \quad (50)$$

Upon merging the inequalities presented in (49) and (50), we arrive at

$$\int_\Omega \left|\partial_i T_k^{\frac{\gamma+p_i-1}{p_i}}(u_n)\right|^{p_i} dx \leq Ck^{\gamma}, \quad \forall i = 1,\ldots,N. \quad (51)$$

Hence (51) yields (46).

We prove the estimate (47) through two stages.

Stage 1: We prove that $G_1(u_n)$ is bounded in $W_0^{1,\overrightarrow{q}}(\Omega)$ for every $q_i < \frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i$.

To establish this, it is sufficient to demonstrate that $\partial_i G_1(u_n)$ is bounded in the Marcinkiewicz space $M^{\frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i}(\Omega)$. Let $h > 0$, we have

$$\begin{aligned}\{|\partial_i u_n| > h, u_n > 1\} &= \{|\partial_i u_n| > h, 1 < u_n \leq k+1\} \cup \{|\partial_i u_n| > h, u_n > k+1\} \\ &\subset \{|\partial_i u_n| > h, 1 < u_n \leq k+1\} \cup \{u_n > k+1\},\end{aligned}$$

which implies that

$$\text{meas}\{|\partial_i u_n| > h, u_n > 1\} \leq \text{meas}\{|\partial_i u_n| > h, 1 < u_n \leq k+1\} + \text{meas}\{u_n > k+1\}. \quad (52)$$

To estimate (52), we use $\varphi = T_k(G_1(u_n))$ as the test function in (19), where $k \geq 1$. Upon examination, it becomes evident that $\partial_i T_k(G_1(u_n)) = \partial_i u_n$ only holds true for $1 < u_n \leq k+1$; otherwise, this value is zero. Additionally, $T_k(G_1(u_n)) = 0$ when $u_n \leq 1$. With these considerations, we arrive at

$$\begin{aligned}
\int_\Omega |\partial_i T_k(G_1(u_n))|^{p_i} dx &\leq \int_\Omega f_n g_n\left(u_n + \frac{1}{n}\right) T_k(G_1(u_n)) dx + \int_\Omega \mu_n T_k(G_1(u_n)) dx \\
&\leq \underline{M} \int_{\left\{u_n+\frac{1}{n}<\underline{t}\right\}} \frac{f_n T_k(G_1(u_n))}{\left(u_n+\frac{1}{n}\right)^\gamma} dx + \overline{M} \int_{\left\{u_n+\frac{1}{n}>\bar{t}\right\}} \frac{f_n T_k(G_1(u_n))}{\left(u_n+\frac{1}{n}\right)^\theta} dx \\
&\quad + \max_{t\in[\underline{t},\bar{t}]} h(t) \int_{\left\{\underline{t}\leq u_n+\frac{1}{n}\leq \bar{t}\right\}} f_n T_k(G_1(u_n)) dx + \int_\Omega \mu_n T_k(G_1(u_n)) dx \\
&\leq \underline{M} k \int_{\left\{u_n+\frac{1}{n}<\underline{t}\right\}} \frac{f}{\left(1+\frac{1}{n}\right)^\gamma} dx + \frac{\overline{M} k}{\bar{t}^\theta} \int_{\left\{u_n+\frac{1}{n}>\bar{t}\right\}} f dx \\
&\quad + k \max_{t\in[\underline{t},\bar{t}]} h(t) \int_{\left\{\underline{t}\leq u_n+\frac{1}{n}\leq \bar{t}\right\}} f dx + k\|\mu_n\|_{L^1(\Omega)} \\
&\leq \left[\left(\underline{M} + \frac{\overline{M}}{\bar{t}^\theta} + C\right)\|f\|_{L^1(\Omega)} + \|\mu_n\|_{L^1(\Omega)}\right] k \\
&\leq Ck, \quad \forall i = 1, \dots, N. \qquad (53)
\end{aligned}$$

By (53), we have

$$\begin{aligned}
\text{meas}\{|\partial_i u_n| > h, 1 < u_n \leq k+1\} &\leq \int_{\{1<u_n<k+1\}} \left(\frac{|\partial_i u_n|}{h}\right)^{p_i} dx \\
&= \frac{1}{h^{p_i}} \int_{\{1<u_n<k+1\}} |\partial_i T_k(G_1(u_n))|^{p_i} dx \\
&\leq \frac{Ck}{h^{p_i}}, \quad \forall i = 1, \dots, N. \qquad (54)
\end{aligned}$$

On the other hand, by computing the $p_i N$-th root of each side of the inequality (51), we deduce

$$\prod_{i-1}^{N} \left(\int_\Omega \left|\partial_i T_k^{\frac{\gamma+p_i-1}{p_i}}(u_n)\right|^{p_i} dx\right)^{\frac{1}{Np_i}} \leq Ck^{\gamma \sum_{i=1}^{N} \frac{1}{Np_i}} = Ck^{\frac{\gamma}{\overline{p}}}.$$

We invoke (6) with $r = \overline{p}^\star$, to derive

$$\left(\int_\Omega \left|T_k^{\frac{\gamma+p_i-1}{p_i}}(u_n)\right|^{\overline{p}^\star}\right)^{\frac{\overline{p}}{\overline{p}^\star}} \leq Ck^\gamma. \qquad (55)$$

By limiting the integral on the left-hand side of (55) to the set where $\{u_n > k+1\}$, we get

$$k^{\frac{\overline{p}(\gamma+p_i-1)}{p_i}} \text{meas}\{u_n > k+1\} \leq \left(\int_{\{u_n>k\}} k^{\frac{\gamma+p_i-1}{p_i}\overline{p}^\star}\right)^{\frac{\overline{p}}{\overline{p}^\star}} \leq \left(\int_\Omega \left|T_k^{\frac{\gamma+p_i-1}{p_i}}(u_n)\right|^{\overline{p}^\star}\right)^{\frac{\overline{p}}{\overline{p}^\star}} \leq Ck^\gamma,$$

hence

$$\text{meas}\{u_n > k+1\}^N = \prod_{i=1}^{N} \text{meas}\{u_n > k+1\} \leq C \prod_{i=1}^{N} k^{-\left(\gamma+\gamma\frac{\overline{p}}{p_i}+\overline{p}-\frac{\overline{p}}{p_i}\right)} \leq Ck^{-\left(N\gamma+N\gamma\overline{p}\sum_{i=1}^{N}\frac{1}{Np_i}+N\overline{p}-N\overline{p}\sum_{i=1}^{N}\frac{1}{Np_i}\right)} \leq k^{-\frac{N^2(\overline{p}-1)}{N-\overline{p}}},$$

this inequality implies that

$$\text{meas}\{u_n > k+1\} \le Ck^{-\frac{N(\overline{p}-1)}{N-\overline{p}}}. \tag{56}$$

Consequently, (u_n) is bounded in $M^{\frac{N(\overline{p}-1)}{N-\overline{p}}}$, leading to the conclusion that $(G_1(u_n))$ is similarly bounded in $M^{\frac{N(\overline{p}-1)}{N-\overline{p}}}$. Now from (52), (54) and (56), we have

$$\text{meas}\{|\partial_i u_n| > h, u_n > 1\} \le \frac{Ck}{h^{p_i}} + Ck^{-\frac{N(\overline{p}-1)}{N-\overline{p}}}, \quad \forall k \ge 1, \ \forall i = 1,\dots,N. \tag{57}$$

Minimizing the right-hand side of (57) with respect to k, we obtain

$$\text{meas}\{|\partial_i u_n| > h, u_n > 1\} \le Ch^{-\frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i}, \quad \forall k \ge 1, \ \forall i = 1,\dots,N.$$

We thus proved that $(\partial_i u_n) = (\partial_i G_1(u_n))$ is bounded in $M^{\frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i}$. Hence, by the property in (9), we can infer that $(G_1(u_n))$ is bounded in $W_0^{1,\overrightarrow{q}}(\Omega)$ where $q_i \le \frac{N(\overline{p}-1)}{\overline{p}(N-1)}p_i, \forall i = 1,\dots,N$.

Stage 2: We show that $(T_1(u_n))$ is bounded in $W_{loc}^{1,\overrightarrow{p}}(\Omega)$.

Demonstrating this claim requires a thorough examination of how u_n behaves for small values across different values of n. We have already proved in Lemma 3.2 that $u_n \ge C_\omega > 0$ on $\omega \subset\subset \Omega$. Upon utilizing $\varphi = T_1^\gamma(u_n)$ as a test function in equation (19), the outcome is

$$\gamma \sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i T_1(u_n)\partial_i T_1^{\gamma-1}(u_n)dx = \int_\Omega f_n g_n \left(u_n + \frac{1}{n}\right) T_1^\gamma(u_n)dx + \int_\Omega \mu_n T_1^\gamma(u_n)dx.$$

In the same way as the proof followed in (50), we find

$$\int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i T_1(u_n) T_1^{\gamma-1}(u_n)dx \le C, \quad \forall i = 1,\dots,N. \tag{58}$$

We observe that

$$\begin{aligned}\int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i T_1(u_n) T_1^{\gamma-1}(u_n)dx &\ge \int_\omega |\partial_i T_1(u_n)|^{p_i} T_1^{\gamma-1}(u_n)dx \\ &\ge C_\omega^{\gamma-1}\int_\omega |\partial_i T_1(u_n)|^{p_i}dx, \quad \forall i = 1,\dots,N. \end{aligned} \tag{59}$$

Therefore, from (58) and (59), we obtain

$$\int_\omega |\partial_i T_1(u_n)|^{p_i}dx \le C, \quad \forall i = 1,\dots,N.$$

The previous estimate implies that $(T_1(u_n))$ is bounded in $W_{loc}^{1,\overrightarrow{p}}(\Omega)$. Since $u_n = G_1(u_n) + T_1(u_n)$ we obtain the estimate (47). This completes the proof of Lemma 4.2. □

Step 2: Passage to the limit.

Thanks to Lemmata 4.2, the sequence $\left(T^{\frac{\gamma+p_i-1}{p_i}}(u_n)\right)$ is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ and (u_n) is bounded in $W_{loc}^{1,\overrightarrow{q}}(\Omega)$. By applying the same proof methodology as used in Theorem 2.3, we can pass to the limit as $n \to +\infty$ in the identities (19) for all $\varphi \in C_c^1(\Omega)$ to establish (15).

Appendix A (Proof of the existence and uniqueness of the problem (20)**)**

Step 1: Existence.

As (μ_n) is a sequence of smooth functions, i.e., $\mu_n \in L^\infty(\Omega)$, and since $f_n g_n\left(|v| + \frac{1}{n}\right) \in L^\infty(\Omega)$, the right-hand side of (20) is in $L^\infty(\Omega)$. Therefore, by the well-known existence results stated in [8, Theorem 2.1 (i)], there exists a weak solution $w \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ to problem (20).

Step 2: Uniqueness.

Suppose that the problem (20) has two solutions w_1 and w_2. Then, we have

$$\sum_{i=1}^{N} \int_\Omega |\partial_i w_1|^{p_i-2} \partial_i w_1 \partial_i \varphi dx = \int_\Omega \left(f_n g_n\left(|v| + \frac{1}{n}\right) + \mu_n\right) \varphi dx, \ \forall \varphi \in W_0^{1,\overrightarrow{p}}(\Omega),$$
$$\sum_{i=1}^{N} \int_\Omega |\partial_i w_2|^{p_i-2} \partial_i w_2 \partial_i \varphi dx = \int_\Omega \left(f_n g_n\left(|v| + \frac{1}{n}\right) + \mu_n\right) \varphi dx, \ \forall \varphi \in W_0^{1,\overrightarrow{p}}(\Omega).$$

For the test function $\varphi = w_1 - w_2$, we obtain

$$\sum_{i=1}^{N} \int_\Omega \left(|\partial_i w_1|^{p_i-2} \partial_i w_1 - |\partial_i w_2|^{p_i-2} \partial_i w_2\right) (\partial_i w_1 - \partial_i w_2)\, dx = 0.$$

Thanks to the inequalities (see Appendix B)

$$\left(|\partial_i w_1|^{p_i-2} \partial_i w_1 - |\partial_i w_2|^{p_i-2} \partial_i w_2\right) (\partial_i w_1 - \partial_i w_2) \ge |\partial_i w_1 - \partial_i w_2|^{p_i}, \ \forall i = 1, \dots, N,$$

we get

$$\sum_{i=1}^{N} \int_\Omega |\partial_i (w_1 - w_2)|^{p_i} dx \le 0,$$

which implies that $w_1 = w_2$.

Appendix B

If $p > 2$, thanks to the symmetry, we prove the inequality

$$\left(|x|^{p-2}x - |y|^{p-2}y\right)(x - y) \ge |x - y|^p,$$

in the case

$$\left(x^{p-1} - y^{p-1}\right)(x - y) \ge (x - y)^p, \quad x > y > 0,$$

which is equivalent to ensuring the positivity of

$$\psi(x) = \left(x^{p-1} - y^{p-1}\right) - (x - y)^{p-1} \ge 0, \quad x > y > 0.$$

The function $\psi(x)$ is positive because it is increasing and $\psi(y) = 0$.

References

[1] P. Akasmika, G. Sekhar and C. Deajyoti, *Elliptic partial differential equation involving a singularity and a Radon measure*, Journal of the Indian Math. Soc. **86** (2019), no. 1-2, 95–117.

[2] J. Bear, *Dynamics of Fluids in Porous Media*, American Elsevier. New York, 1972.

[3] M. Bendahmane, M. Langlais and M. Saad, *On some anisotropic reaction-diffusion systems with L^1−data modeling the propagation of an epidemic disease*, Nonlinear. Anal. **54** (2003), no. 4, 617–636.

[4] P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, *An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa. **22** (1995), 241–273.

[5] L. Boccardo and L. Orsina, *Semilinear elliptic equations with singular nonlinearities*, Calc. Var. **37** (2010), 363–380.

[6] Y.O. Boukarabila and S.E. Miri, *Anisotropic system with singular and regular nonlinearities*, Complex Var. Elliptic Equ. **65** (2019), no. 4, 621–631.

[7] L.M. De Cave and O. Francescantonio, *Elliptic equations with general singular lower order term and measure data*, Nonlinear Analysis. **128** (2015), 391–411.

[8] A. Di Castro, *Existence and regularity results for anisotropic elliptic problems*, Adv. Nonlin. Stud. **9** (2009), no. 2, 367–393.

[9] A. Di Castro, *Elliptic problems for some anisotropic operators*, Ph.D. Thesis. University of Rome "Sapienza". a. y, 2008/2009.

[10] O. Francescantonio and P. Francesco, *On singular elliptic equations with measure sources*, ESAIM: COCV. **22** (2016), no. 1, 289–308.

[11] I. Fragalà and F. Gazzola, *Existence and nonexistence results for anisotropic quasilinear equations*, Ann. I. H. Poincaré-AN. **21** (2004), 715–734.

[12] D. Giachetti, J.M. Pedro and M. François, *Definition, existence, stability and uniqueness of the solution to a semilinear elliptic problem with a strong singularity at $u = 0$*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **5** (2018), 1395–1442.

[13] H. Khelifi, *Anisotropic degenerate elliptic problem with singular radient lower order term*, Boll. Unione. Mat. Ital. (2023), https://doi.org/10.1007/s40574-023-00395-3

[14] H. Khelifi, *Application of the Stampacchia lemma to anisotropic degenerate elliptic equations*, J. Innov. Appl. Math. Comput. Sci. **3** (2023), no. 1, 75–82.

[15] H. Khelifi, F. Mokhtari, *Nonlinear degenerate anisotropic elliptic equations with variable exponents and L^1 data*, J. Part. Diff. Eq. **33** (2020), no. 1, 1–16.

[16] H. Le Dret, *Équations aux dérivées partielles elliptiques non linéaires*, Springer Berlin Heidelberg, 2013.

[17] A.R. Leggat and S.E. Miri, *Anisotropic problem with singular nonlinearity*, Complex Var. Elliptic Equ. **61** (2015), no. 4, 496-509.

[18] F. Mokhtari, *Anisotropic parabolic problems with measures data*, Differential Equations and Applications. **2** (2010), no. 1, 123–150.

[19] S.E. Miri, *On an anisotropic problem with singular nonlinearity having variable exponent*, Ricerche mat. **66** (2017), 415-424.

[20] J. Ràkosník, *Some remarks to anisotropic Sobolev spaces I*, Beiträge zur Anal. **13** (1979), 55–68.

[21] J. Ràkosník, *Some remarks to anisotropic Sobolev spaces II*, Beiträge zur Anal. **15** (1981), 127–140.

[22] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficientes discontinus*, Annales de l'institut Fourier, **15**, (1965), no. 1, 189–257.

[23] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. **18** (1969), 3–24.

[24] M.A. Zouatini, H. Khelifi and F. Mokhtari, *Anisotropic degenerate elliptic problem with a singular nonlinearity*, Advances in Operator Theory. **8** (2023), no. 1, https://doi.org/10.1007/s43036-022-00240-y.