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On the zeros of regular polynomial of a quaternionic variable

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Abstract. In this paper, we present some outcomes concerning the zero distributions of regular polynomials of a quaternionic variable. By invoking the maximum modulus theorem and zero sets of a regular product established in the recently developed theory of regular functions and polynomials of a quaternionic variable, we find new bounds of Eneström-Kakeya type for the zeros of these polynomials with restricted coefficients. Additionally, our findings extend some classical results from the complex domain to the realm of quaternionic variables.

1. Introduction

The Eneström-Kakeya theorem occupies a significant position within the realm of complex analysis and has got extensive attention since beginning of the 20th century. This theorem provides profound insights regarding the distribution of zeros of complex polynomials and finds valuable applications in the domain of geometric function theory [13].

Theorem 1.1. *(Eneström-Kakeya Theorem) If* $T(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $T(z)$ *lie in*

$$|z| \leq 1.$$

Numerous extensions of the Eneström-Kakeya theorem have been documented in existing literature, as evidenced by works such as [9] and [11]. For a comprehensive understanding of the theorem and its diverse generalizations, one can turn to the exhaustive surveys conducted by Marden [13] and Milovanović et al. [17]. Noteworthy contributions have also emerged from Joyal, Labelle and Rahman [11], who extended Theorem 1.1 to encompass polynomials with coefficients demonstrating monotonicity, without necessarily adhering to non-negativity. This extension is exemplified by the subsequent result.

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Theorem 1.2. *If* $T(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of $T(z)$ *lie in*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

The study of zero distribution in complex polynomials, particularly under constraints imposed on their coefficients, has been a subject of extensive investigation. The Eneström-Kakeya theorem and its various generalizations are considered classic and noteworthy examples of this kind. The goal of this paper is to present extensions to the quaternionic setting of some classical results of Eneström-Kakeya type as discussed above. There is now a very ample literature on regular functions and, in particular on polynomials of a quaternionic variable, which is the first example of a regular function. For this reason, the study of fine properties of such functions would be very useful and worth pursuing for specialists in approximation theory.

2. Preliminary knowledge

To introduce the theory within which we will operate, let us first provide some preliminary information about quaternions and regular functions of a quaternionic variable, which will be beneficial for our subsequent discussions. The non-commutative division ring $\mathbb{H}$ of quaternions comprises elements of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$, and the imaginary units i, j, and k satisfy certain properties: $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. Any quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ can be divided into its real part $\mathrm{Re}(q) = x_0$ and its imaginary part $\mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$. The conjugate of q is denoted by $\overline{q}$ and defined as $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$. The norm of q is given by $|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. Additionally, the inverse of each nonzero element q of $\mathbb{H}$ is expressed as $q^{-1} = |q|^{-2}\overline{q}$. For $r > 0$, the ball $B(0, r)$ is defined as the set of quaternions $\{q \in \mathbb{H}; |q| < r\}$. Furthermore, we denote the open unit ball in $\mathbb{H}$ centered at the origin as $\mathbb{B}$, i.e.,

$$\mathbb{B} = \{q \in \mathbb{H}; |q| < 1\} = \{q = x_0 + x_1 i + x_2 j + x_3 k : x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1\}.$$

Let $\mathbb{S}$ represent the unit sphere composed of purely imaginary quaternions, defined as:

$$\mathbb{S} = \{q = x_1 i + x_2 j + x_3 k : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

It is noteworthy that when $I \in \mathbb{S}$, the quaternion I satisfies $I^2 = -1$. Consequently, for any fixed $I \in \mathbb{S}$, we introduce the set $\mathbb{C}_I$ as follows:

$$\mathbb{C}_I = \{x + Iy : x, y \in \mathbb{R}\},$$

which can be identified with a complex plane. The real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$, implying that a real quaternion $q = x_0$ also belongs to $\mathbb{C}_I$ for any $I \in \mathbb{S}$. Moreover, for any non-real quaternion $q \in \mathbb{H} \setminus \mathbb{R}$, there exist unique real numbers x and y, where $y > 0$, and a unit quaternion $I \in \mathbb{S}$ such that $q = x + Iy$. A comprehensive background on these hyper-complex numbers can be found in reference [23]. In this paper, we will focus on a specific class of functions known as slice regular functions (as polynomials) of a quaternionic variable. These regular functions, defined over quaternionic variables, have been introduced and extensively studied in the past decade. They have emerged as a fertile subject in the field of analysis and have seen rapid development, particularly due to their applications in operator theory. For a more comprehensive understanding of these functions and their practical implications, interested readers can refer to works such as [2], [4]-[8], and the references provided therein. Inspired by Cullen's investigation of analytic intrinsic functions of quaternions [3], Gentili and Struppa, in their work [6], put forward the following definition of regularity for functions of a quaternionic variable.

Definition 2.1. *Consider an open set U in the quaternionic space $\mathbb{H}$. A real differentiable function $f : U \to \mathbb{H}$ is termed "left slice regular" or simply "slice regular" if, for each unit quaternion $I \in \mathbb{S}$ (belonging to the unit sphere of purely imaginary quaternions), its restriction f_I to the complex plane $\mathbb{C}_I$ satisfies the following condition:*

$$\overline{\partial}_I f(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0.$$

Since for all $n \geq 1$ and for all $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)(x+Iy)^n = 0.$$

By definition, the monomial function $P(q) = q^n$ is considered regular. Furthermore, due to the properties that addition and right multiplication by a constant preserve regularity, all polynomials of the form

$$T(q) = \sum_{v=0}^{n} q^v a_v, \ a_v \in \mathbb{H} \text{ for } v = 0, 1, 2, ..., n, \tag{2.1}$$

with coefficients on the right and the indeterminate on the left are also regular. In the context of polynomials of this nature over skew-fields, a distinct multiplication operation (denoted by $*$) is defined to ensure that the product of regular functions remains regular. For quaternionic polynomials, this product is established using the convolution product, which follows the Cauchy multiplication rule. To elaborate, given two quaternionic polynomials of this kind, $T_1(q) = \sum\limits_{v=0}^{n} q^v a_v$ and $T_2(q) = \sum\limits_{t=0}^{m} q^t b_t$, we define their product as:

$$(T_1 * T_2)(q) := \sum_{\substack{v=0,1,...,n \\ t=0,1,...,m}} q^{v+t} a_v b_t.$$

When T_1 possesses real coefficients, the so-called $*$ multiplication coincides with the usual point-wise multiplication. Notice that the $*$ product is associative and not, in general, commutative. The absence of commutativity leads to a behavior of polynomials rather unlike their behavior in the real or complex setting. It is observed (see [4, 22]) that the zeros of the polynomial of type (2.1) of a quaternionic variable are either isolated or spherical. In the quaternionic setting, for example, the second degree polynomial $q^2 + 1$ vanishes for every $q \in \mathbb{S}$. The following result, which comprehensively describes the zero sets of a regular product of two polynomials in terms of the zero sets of the two factors, is derived from [12] (also see [5] and [7]):

Theorem 2.2. *Let f and g be quaternionic polynomials. Then, $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or if $f(q_0) \neq 0$, it implies that $g\left(f(q_0)^{-1} q_0 f(q_0)\right) = 0$.*

Gentili and Struppa [6] introduced a maximum modulus theorem for regular functions, encompassing convergent power series and polynomials, as stated in the following result:

Theorem 2.3. *(Maximum Modulus Theorem): Let $B = B(0, r)$ be a ball in $\mathbb{H}$ centered at 0 with radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ attains a relative maximum at a point $a \in B$, then f must be a constant over the entire ball B.*

It is worth noting that an algebraic proof of the Fundamental Theorem of Algebra for regular polynomials with coefficients in $\mathbb{H}$ can be found, for instance, in references such as [20] and [21]. Alternatively, a topological proof is presented in [8]. Consequently, this complete identification of the zeros of polynomials in terms of their factorization has led to an interesting perspective on regions containing all the zeros of a regular polynomial in a quaternionic variable. More recently, Carney et al. [1] extended the Eneström-Kakeya theorem and its various generalizations from complex polynomials to quaternionic polynomials, leveraging Theorems 2.2 and 2.3. They first established the quaternionic analogue of Theorem 1.1 as follows:

Theorem 2.4. *If* $T(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $T(q)$ *are located within the region defined by* $|q| \leq 1$.

Besides proving some interesting results on quaternionic polynomials, Tripathi [24] (see also [10], Corollary 3.2) also established the following generalization of Theorem 2.4.

Theorem 2.5. *If* $T(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $T(q)$ *lie in*

$$|q| \leq \frac{|a_0| - a_0 + a_n}{|a_n|}.$$

Recently, Milovanović et al. [14] further generalized Theorem 2.5 in the form of the following result:

Theorem 2.6. *If* $T(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \leq a_{n-1} \leq \cdots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \cdots \geq a_1 \geq a_0,$$

where $0 \leq \lambda \leq n$. *Then all the zeros of* $T(q)$ *lie in*

$$|q| \leq \frac{2a_\lambda - a_n + |a_0| - a_0}{|a_n|}.$$

In the same paper, they also established the following result:

Theorem 2.7. *If* $T(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$0 < a_n \leq a_{n-1} \leq \cdots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \cdots \geq a_1 \geq a_0 \geq 0,$$

where $0 \leq \lambda \leq n-1$. *Then all the zeros of* $T(q)$ *lie in*

$$\left| q + \frac{a_{n-1}}{a_n} - 1 \right| \leq 2\frac{a_\lambda}{a_n} - \frac{a_{n-1}}{a_n}.$$

Recently, several works appeared in the literature, including generalizations and refinements of the above results; see, e.g., [15], [16], [18], [19]. The primary objective of this paper is to further explore the extensions of various Eneström-Kakeya type results from the complex domain to the quaternionic domain. These extensions are achieved by utilizing a recently established maximum modulus theorem (Theorem 2.3) and the knowledge of the structure of the zero sets of a regular product of two polynomials (Theorem 2.2) defined over a quaternionic variable. As a result of these investigations, the paper presents diverse generalizations of Theorems 2.4 - 2.7.

3. Main results

We start by introducing an Eneström-Kakeya type result concerning the distribution of zeros of a polynomial with a quaternionic variable. This result not only extends Theorem 2.5 but also offers various generalizations of related results.

Theorem 3.1. *Let $T(q) = \sum_{v=0}^{n} q^v a_v$ be a polynomial of degree n in the quaternionic variable q and with real coefficients. If for some non negative real numbers μ and σ, we have*

$$a_n - \mu \leq a_{n-1} \leq \cdots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \cdots \geq a_1 \geq a_0 - \sigma,$$

where λ is an integer such that $0 \leq \lambda \leq n$, then all the zeros of $T(q)$ lie in

$$|q| \leq \frac{2a_\lambda - a_n + |a_0| - a_0 + 2\sigma + 2\mu}{|a_n|}. \tag{1}$$

Remark 3.2. Theorem 3.1 simplifies to Theorem 2.6 for $\mu = 0 = \sigma$.

Next, we present a refinement of Theorem 3.1 for $0 \leq \lambda \leq n-1$.

Theorem 3.3. *Let $T(q) = \sum_{v=0}^{n} q^v a_v$ be a polynomial of degree n in the quaternionic variable q and with real coefficients. If for some non negative real numbers μ and σ, we have*

$$a_n - \mu \leq a_{n-1} \leq \cdots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \cdots \geq a_1 \geq a_0 - \sigma,$$

where λ is an integer such that $0 \leq \lambda \leq n-1$, then all the zeros of $T(q)$ lie in

$$\left| q + \frac{a_{n-1}}{a_n} - \left(1 - \frac{\mu}{a_n}\right) \right| \leq \frac{|a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu}{|a_n|}. \tag{2}$$

Remark 3.4. Note that Theorem 3.3 gives much better estimate than Theorem 3.1 for $0 \leq \lambda \leq n-1$. To verify this, it is sufficient to show that the region given by (2) is contained in the region given by (1).
Let q be any point belonging to the region given by (2), then

$$\left| q + \frac{a_{n-1}}{a_n} - \left(1 - \frac{\mu}{a_n}\right) \right| \leq \frac{|a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu}{|a_n|}.$$

Now,

$$\begin{aligned} |q| &\leq \left| q + \frac{a_{n-1}}{a_n} - \left(1 - \frac{\mu}{a_n}\right) \right| + \left| \left(1 - \frac{\mu}{a_n}\right) - \frac{a_{n-1}}{a_n} \right| \\ &\leq \frac{|a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu}{|a_n|} + \frac{|a_n - \mu - a_{n-1}|}{|a_n|} \\ &= \frac{|a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu + (a_{n-1} - a_n + \mu)}{|a_n|} \\ &= \frac{|a_0| - a_0 + 2a_\lambda - a_n + 2\sigma + 2\mu}{|a_n|}. \end{aligned}$$

This shows that the point q belongs to the region defined by (1).

By taking $\sigma = 0 = \mu$ in Theorem 3.3, we get the following result:

Corollary 3.5. *If* $T(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \leq a_{n-1} \leq \cdots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \cdots \geq a_1 \geq a_0,$$

where λ *is an integer such that* $0 \leq \lambda \leq n-1$, *then all the zeros of* $T(q)$ *lie in*

$$\left| q + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{|a_0| - a_0 + 2a_\lambda - a_{n-1}}{|a_n|}.$$

Remark 3.6. We observe that Corollary 3.5 reduces to Theorem 2.7 by assuming $a_v > 0$ for $0 \leq v \leq n$.

Taking $a_v > 0$ for $0 \leq v \leq n$, in Theorem 3.3, we get the following result:

Corollary 3.7. *Let* $T(q) = \sum_{v=0}^{n} q^v a_v$ *be a polynomial of degree n in the quaternionic variable q and with positive real coefficients. If for some non negative real numbers* μ *and* σ, *we have*

$$a_n - \mu \leq a_{n-1} \leq \cdots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \cdots \geq a_1 \geq a_0 - \sigma,$$

where λ *is an integer such that* $0 \leq \lambda \leq n-1$, *then all the zeros of* $T(q)$ *lie in*

$$\left| q + \frac{a_{n-1}}{a_n} - \left(1 - \frac{\mu}{a_n}\right) \right| \leq \frac{2a_\lambda - a_{n-1} + 2\sigma + \mu}{a_n}.$$

Taking $\mu = (1-\rho)a_n$ and $\sigma = (1-\tau)a_0$, where $0 < \rho \leq 1$ and $0 < \tau \leq 1$ in Corollary 3.7, we get, the following result:

Corollary 3.8. *Let* $T(q) = \sum_{v=0}^{n} q^v a_v$ *be a polynomial of degree n in the quaternionic variable q and with real coefficients. If for some* $0 < \rho \leq 1$ *and* $0 < \tau \leq 1$, *we have*

$$0 < \rho a_n \leq a_{n-1} \leq \cdots \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq \cdots \geq a_1 \geq \tau a_0 \geq 0,$$

where λ *is an integer such that* $0 \leq \lambda \leq n-1$, *then all the zeros of* $T(q)$ *lie in*

$$\left| q + \frac{a_{n-1}}{a_n} - \rho \right| \leq \frac{2a_\lambda - a_{n-1} + 2(1-\tau)a_0 + (1-\rho)a_n}{a_n}.$$

For $\lambda = n-1$ in Corollary 3.8, we get the following result:

Corollary 3.9. *Let* $T(q) = \sum_{v=0}^{n} q^v a_v$ *be a polynomial of degree n in the quaternionic variable q and with real coefficients. If for some* $0 < \rho \leq 1$ *and* $0 < \tau \leq 1$, *we have*

$$0 < \rho a_n \leq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq \tau a_0 \geq 0,$$

then all the zeros of $T(q)$ *lie in*

$$\left| q + \frac{a_{n-1}}{a_n} - \rho \right| \leq \frac{a_{n-1} + 2(1-\tau)a_0 + (1-\rho)a_n}{a_n}.$$

For $\rho = 1$ in Corollary 3.9, we get the following result:

Corollary 3.10. *If* $T(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree n where q is a quaternionic variable with real coefficients satisfying*

$$0 < a_n \leq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq \tau a_0 \geq 0,$$

where $0 < \tau \leq 1$. *Then all the zeros of* $T(q)$ *lie in*

$$\left| q + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{a_{n-1}}{a_n} + \frac{2a_0(1-\tau)}{a_n}.$$

4. Proofs of the main results

Proof of Theorem 3.1. Consider the product

$$\begin{aligned} T(q) * (1-q) &= a_0 + q(a_1 - a_0) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1}a_n \\ &= a_0 - q\sigma + q(a_1 - a_0 + \sigma) + q^2(a_2 - a_1) + \ldots q^\lambda(a_\lambda - a_{\lambda-1}) \\ &\quad + q^{\lambda+1}(a_{\lambda+1} - a_\lambda) + \cdots + q^n\mu - q^n(a_{n-1} - a_n + \mu) - q^{n+1}a_n \\ &= \phi(q) - q^{n+1}a_n, \end{aligned}$$

where

$$\begin{aligned} \phi(q) &= a_0 - q\sigma + q(a_1 - a_0 + \sigma) + q^2(a_2 - a_1) + \cdots + q^\lambda(a_\lambda - a_{\lambda-1}) \\ &\quad + q^{\lambda+1}(a_{\lambda+1} - a_k) + \cdots + q^n\mu - q^n(a_{n-1} - a_n + \mu). \end{aligned}$$

For $|q| = 1$, we get

$$\begin{aligned} |\phi(q)| &= |a_0 - q\sigma + q(a_1 - a_0 + \sigma) + q^2(a_2 - a_1) + \cdots + q^\lambda(a_\lambda - a_{\lambda-1}) \\ &\quad + q^{\lambda+1}(a_{\lambda+1} - a_\lambda) + \cdots + q^n\mu - q^n(a_{n-1} - a_n + \mu)| \\ &\leq |a_0| + |\sigma| + |a_1 - a_0 + \sigma| + |a_2 - a_1| + \cdots + |a_\lambda - a_{\lambda-1}| \\ &\quad + |a_{\lambda+1} - a_\lambda| + \cdots + |\mu| + |a_{n-1} - a_n + \mu| \\ &= |a_0| + \sigma + (a_1 - a_0 + \sigma) + (a_2 - a_1) + \cdots + (a_\lambda - a_{\lambda-1}) \\ &\quad + (a_\lambda - a_{\lambda+1}) + \cdots + \mu + (a_{n-1} - a_n + \mu) \\ &= |a_0| + 2\sigma - a_0 + 2a_\lambda - a_n + 2\mu. \end{aligned}$$

Notice that

$$\begin{aligned} \max_{|q|=1} \left| q^n * \phi\left(\frac{1}{q}\right) \right| &= \max_{|q|=1} \left| q^n \phi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} \left| \phi\left(\frac{1}{q}\right) \right| \\ &= \max_{|q|=1} |\phi(q)|, \end{aligned}$$

it follows that $q^n * \phi(1/q)$ has the same bound on $|q| = 1$ as ϕ i.e.,

$$\left| q^n * \phi\left(\frac{1}{q}\right) \right| = \left| q^n \phi\left(\frac{1}{q}\right) \right| \leq 2a_\lambda - a_n + |a_0| - a_0 + 2\sigma + 2\mu \quad \text{for } |q| = 1.$$

Since $q^n * \phi(1/q)$ is a polynomial and so is regular in $|q| \leq 1$, it follows by the Maximum Modulus Theorem (Theorem 2.3), that

$$\left| q^n \phi\left(\frac{1}{q}\right) \right| \leq 2a_\lambda - a_n + |a_0| - a_0 + 2\sigma + 2\mu \quad \text{for } |q| \leq 1.$$

Replacing q by $1/q$, we have for $|q| \geq 1$,

$$|\phi(q)| \leq |q|^n[2a_\lambda - a_n + |a_0| - a_0 + 2\sigma + 2\mu]. \tag{3}$$

Thus, for $|q| \geq 1$, we have

$$\begin{aligned}|T(q)*(1-q)| &= |\phi(q) - q^{n+1}a_n| \\ &\geq |q|^{n+1}|a_n| - |\phi(q)| \\ &\geq |q|^{n+1}|a_n| - |q|^n[2a_\lambda - a_n + |a_0| - a_0 + 2\sigma + 2\mu], \quad \text{(by (3))} \\ &= |q|^n[|q||a_n| - \{2a_\lambda - a_n + |a_0| - a_0 + 2\sigma + 2\mu\}].\end{aligned}$$

Hence, if

$$|q| > \frac{2a_\lambda - a_n + |a_0| - a_0 + 2\sigma + 2\mu}{|a_n|},$$

then $|T(q)*(1-q)| > 0$, that is $T(q)*(1-q) \neq 0$. Since by Theorem 2.2, the only zeros of $T(q)*(1-q)$ are $q = 1$ and the zeros of $T(q)$, therefore, $T(q) \neq 0$ for

$$|q| > \frac{2a_\lambda - a_n + |a_0| - a_0 + 2\sigma + 2\mu}{|a_n|}.$$

Thus all zeros of $T(q)$ lie in

$$|q| \leq \frac{2a_\lambda - a_n + |a_0| - a_0 + 2\sigma + 2\mu}{|a_n|},$$

which completes the proof of Theorem 3.1.

Proof of Theorem 3.3. As in Theorem 3.1, we have

$$\begin{aligned}T(q)*(1-q) &= a_0 + q(a_1 - a_0) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1}a_n \\ &= a_0 - q\sigma + q(a_1 - a_0 + \sigma) + q^2(a_2 - a_1) + \cdots + q^\lambda(a_\lambda - a_{\lambda-1}) \\ &\quad + q^{\lambda+1}(a_{\lambda+1} - a_\lambda) + \cdots + q^n\mu - q^n(qa_n + a_{n-1} - a_n + \mu) \\ &= \phi(q) - q^n(qa_n + a_{n-1} - a_n + \mu),\end{aligned}$$

where here

$$\begin{aligned}\phi(q) &= a_0 - q\sigma + q(a_1 - a_0 + \sigma) + q^2(a_2 - a_1) + \cdots + q^\lambda(a_\lambda - a_{\lambda-1}) \\ &\quad + q^{\lambda+1}(a_{\lambda+1} - a_\lambda) + \cdots + q^{n-1}(a_{n-1} - a_{n-2}) + q^n\mu.\end{aligned}$$

For $|q| = 1$, we have

$$\begin{aligned}|\phi(q)| &= |a_0 - q\sigma + q(a_1 - a_0 + \sigma) + q^2(a_2 - a_1) + \cdots + q^\lambda(a_\lambda - a_{\lambda-1}) \\ &\quad + q^{\lambda+1}(a_{\lambda+1} - a_\lambda) + \cdots + q^{n-1}(a_{n-1} - a_{n-2}) + q^n\mu| \\ &\leq |a_0| + |\sigma| + |a_1 - a_0 + \sigma| + |a_2 - a_1| + \cdots + |a_\lambda - a_{\lambda-1}| + |a_{\lambda+1} - a_\lambda| + \dots \\ &\quad + |a_{n-1} - a_{n-2}| + |\mu| \\ &= |a_0| + \sigma + (a_1 - a_0 + \sigma) + (a_2 - a_1) + \cdots + (a_\lambda - a_{\lambda-1}) + (a_\lambda - a_{\lambda+1}) + \dots \\ &\quad + (a_{n-2} - a_{n-1}) + \mu \\ &= |a_0| + 2\sigma + 2a_\lambda - a_{n-1} + \mu - a_0.\end{aligned}$$

As in the proof of Theorem 3.1, we have for $|q| \geq 1$,

$$|\phi(q)| \leq |q|^n[|a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu].$$

Thus, for $|q| \geq 1$, we have

$$\begin{aligned}|T(q) * (1-q)| &= |\phi(q) - q^n((qa_n + a_{n-1} - a_n + \mu)| \\ &\geq |q^n|\,|(qa_n + a_{n-1} - a_n + \mu)| - |\phi(q)| \\ &\geq |q^n|[|(qa_n + a_{n-1} - a_n + \mu)| - \{|a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu\}].\end{aligned}$$

Hence, if

$$|qa_n + a_{n-1} - a_n + \mu| > |a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu,$$

then $|T(q) * (1-q)| > 0$. Equivalently, if

$$\left|q + \frac{a_{n-1}}{a_n} - \left(1 - \frac{\mu}{a_n}\right)\right| > \frac{|a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu}{|a_n|},$$

then $|T(q) * (1-q)| > 0$, that is $T(q) * (1-q) \neq 0$. By Theorem 2.2, the only zeros of $T(q) * (1-q)$ are $q = 1$ and the zeros of $T(q)$, therefore, $T(q) \neq 0$ for

$$\left|q + \frac{a_{n-1}}{a_n} - \left(1 - \frac{\mu}{a_n}\right)\right| > \frac{|a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu}{|a_n|}.$$

In other words, all zeros of $T(q)$ lie in

$$\left|q + \frac{a_{n-1}}{a_n} - \left(1 - \frac{\mu}{a_n}\right)\right| \leq \frac{|a_0| - a_0 + 2a_\lambda - a_{n-1} + 2\sigma + \mu}{|a_n|},$$

which completes the proof of Theorem 3.3.

Conclusion: The classic Eneström-kakeya theorem and its various generalizations give explicit upper bounds for the moduli of the zeros of complex polynomials having a monotone sequence of of non-negative real coefficients. Here, we constructed a framework to establish various generalizations to the classical result, namely, theorems that derive bounds for the moduli of the zeros of polynomials of a quaternionic variable with coefficients located on only one side of the variable. We used the recently extended maximum modulus theorem and the zero sets of a regular product to establish our results.

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