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Slant ruled surfaces generated by the striction curves of the hyper-dual curves

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Abstract. In this paper, exploiting some considerable properties of hyper-dual curves and the concept of unit hyper-dual sphere, it is shown that each striction curve of the hyper-dual curve denotes two slant ruled surfaces in $\mathbb{R}^3$. Moreover, these slant ruled surfaces have a common striction curve. Then, it is proved that the normal vectors of these slant ruled surfaces are orthogonal along the common base curve. Consequently, an example is given to verify the obtained results.

1. Introduction

The algebra of dual numbers, whose properties given in [11], was firstly introduced by Clifford as a generalization of real numbers in [3]. Dual vectors, represented by $\mathbb{D}^3$, were applied by E. Study mapping which says that there exists one-to-one correspondence between the directed lines in $\mathbb{R}^3$ and the points of unit dual sphere in $S^2_{\mathbb{D}}$ in [23].
In mathematics, especially in geometry, ruled surfaces, which are generated by the motion of a straight line in [6], have significiant application areas such as engineering, computer-aided design, etc. Using E. Study mapping, in [16], a correspondence among the dual curves mentioned in [18] and [22] on $S^2_{\mathbb{D}}$, the tangent bundle of unit 2-sphere, TS^2, and non-cylindirical ruled surfaces was given. Then, in [12], each curves on TS^2 were corresponded to the ruled surfaces in $\mathbb{R}^3$. Furthermore, the relationship between the developability conditions of these ruled surfaces and their striction curves was analyzed. Motivating this research, in [15], each natural lift curve, which is obtained by the unit tangent vectors of the main curve, was corresponded to the ruled surface in $\mathbb{R}^3$.
The concept of slant helix by saying that the normal lines of the curve make a constant angle with a fixed direction was introduced in [13]. Inspring some properties of slant helix, a ruled surface has an orthonormal base, which is called Frenet frame of the ruled surface, along its striction line. By considering the Frenet vectors of a ruled surface, the definitions of some special ruled surfaces, where the Frenet vectors make a constant angle with some fixed directions in the space, were defined as a slant ruled surface in [20]. In [17], some significiant theorems about slant ruled surfaces were proved in $\mathbb{R}^3$ in detail. In [19], slant ruled surfaces were defined in Minkowski 3-space by using E. Study mapping for directed spacelike and timelike lines in Minkowski 3-space $\mathbb{R}^3_1$ in [24]. In [14], the definition of the slant ruled surface was examined by considering Frenet frames given in [21] and using the correspondence among the subset of the tangent

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bundle of the unit 2-sphere, $T\bar{M}$, unit dual sphere, DS^2 and slant ruled surface in $\mathbb{R}^3$.
The hyper-dual numbers were introduced for solving some derivative problems in [19, 20] and [7–10]. In [4], it was expressed that two dual numbers consists of the hyper-dual number. In [5], hyper-dual numbers were applied to rigid body motion in kinematics. In [1], the hyper-dual numbers were applied to kinematics and geometric representations of hyper-dual numbers were denoted.
Motivating in [2], this paper is designed to fill the gap for the slant ruled surfaces generated by the striction curves of the hyper-dual curves. Hence, in this paper, it is explained that each striction curve of the hyper-dual curve on the subset of unit hyper-dual sphere $S^2_{\tilde{\mathbb{D}}_1}$ represents two slant ruled surfaces in $\mathbb{R}^3$. Then, it is shown that these slant ruled surfaces intersect along a common base curve and their rulings are orthogonal. Moreover, it is seen that each striction curve of dual curve on unit dual sphere $S^2_{\mathbb{D}}$ denotes a slant ruled surface in $\mathbb{R}^3$ whereas each striction curve of hyper-dual curve on $S^2_{\tilde{\mathbb{D}}_1}$ indicates two slant ruled surfaces in $\mathbb{R}^3$ such that these two slant ruled surfaces intersect along a common base curve. Also, an example is given to verify the obtained results.
This paper is organized as follows: In Section 2, some basic properties of dual numbers and hyper-dual numbers are given. In Section 3, the Frenet frame of the slant ruled surface and some characterizations for slant ruled surfaces are mentioned. In Section 4, the relation between hyper-dual numbers and ruled surfaces is denoted in [2]. In Section 5, the slant ruled surfaces constructed by the striction curves of hyper-dual curves are acquired. Also, some fundamental theorems are proved in detail. In Section 6, an example is given to support the main results explicitly. In Section 7, the important results are pointed out summarily.

2. Preliminaries

In this section, some definitions and properties of dual and hyper-dual numbers are, respectively, given.

2.1. Dual Numbers

$$\mathbb{D} = \{X = x + \varepsilon x^* : x, x^* \in \mathbb{R}, \varepsilon \neq 0\} \tag{1}$$

denotes the set of all dual numbers, where ε is dual unit satisfying $\varepsilon^2 = 0$.
The set of dual vectors is given by

$$\mathbb{D}^3 = \{\vec{X} = \vec{x} + \varepsilon \vec{x}^* : \vec{x}, \vec{x}^* \in \mathbb{R}^3\} \tag{2}$$

and $\vec{X}$ is also called dual vector in $\mathbb{D}^3$. For any dual vectors $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ and $\vec{Y} = \vec{y} + \varepsilon\vec{y}^*$, the scalar and vector products are defined as follows:

$$\langle \vec{X}, \vec{Y}\rangle_D = \langle \vec{x}, \vec{y}\rangle + \varepsilon(\langle \vec{x}, \vec{y}^*\rangle + \langle \vec{x}^*, \vec{y}\rangle), \tag{3}$$
$$\vec{X} \times_D \vec{Y} = \vec{x} \times \vec{y} + \varepsilon(\vec{x} \times \vec{y}^* + \vec{x}^* \times \vec{y}). \tag{4}$$

Here, "$\langle , \rangle$" and "$\times$" are the usual scalar and vector products in $\mathbb{R}^3$.
The modulus of the dual vector $\vec{X}$ is defined by

$$|\vec{X}|_D = |\vec{x}| + \varepsilon \frac{\langle \vec{x}, \vec{x}^*\rangle}{|\vec{x}|}, \ \ |\vec{x}| \neq 0. \tag{5}$$

If $|\vec{X}|_D = 1$ (that means $|\vec{x}| = 1$ and $\langle \vec{x}, \vec{x}^*\rangle = 0$), then $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ is called a unit dual vector. The unit dual sphere, which consists of all unit dual vectors, is

$$S^2_{\mathbb{D}} = \{\vec{X} = \vec{x} + \varepsilon\vec{x}^* : |\vec{X}|_D = 1, \ \ \vec{X} \in \mathbb{D}^3\}. \tag{6}$$

Theorem 2.1 (E. Study mapping). *There exists one-to-one correspondence between the points of the unit dual sphere $S^2_{\mathbb{D}}$ and the directed lines in $\mathbb{R}^3$, see [23].*

Also, the scalar product of $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ and $\vec{Y} = \vec{y} + \varepsilon\vec{y}^*$ is

$$\langle \vec{X}, \vec{Y} \rangle_D = \cos\varphi = \cos\theta + \varepsilon\theta^* \sin\theta, \tag{7}$$

where $\varphi = \theta + \varepsilon\theta^*$ denotes a dual angle, see [23]. If d_1 and d_2 are the directed lines in $\mathbb{R}^3$ to the unit dual vectors $\vec{X}$ and $\vec{Y}$, respectively, then θ is the angle between $\vec{x}$ and $\vec{y}$. Furthermore, $|\theta^*|$ denotes the shortest distance between d_1 and d_2.
The vector product of $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ and $\vec{Y} = \vec{y} + \varepsilon\vec{y}^*$ is

$$\vec{X} \times_D \vec{Y} = \vec{N} \sin\varphi. \tag{8}$$

Here, $\sin\varphi = \sin\theta + \varepsilon\theta^* \cos\theta$ and $\vec{N} = \frac{\vec{X}\times_D\vec{Y}}{|\vec{X}\times_D\vec{Y}|_D}$ is the common orthogonal direction vector to the dual vectors $\vec{X}$ and $\vec{Y}$, directed from $\vec{x}$ to $\vec{y}$. For more information about dual numbers, see [11].

2.2. *Hyper dual numbers*

$$\tilde{\mathbb{D}} = \{\mathbb{X} = x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3 : x_0, x_1, x_2, x_3 \in \mathbb{R}\} \tag{9}$$

is called the set of hyper-dual numbers, where ε_1 and ε_2 are dual units satisfying

$$\varepsilon_1^2 = \varepsilon_2^2 = (\varepsilon_1\varepsilon_2)^2 = 0$$

and

$$\varepsilon_1 \neq \varepsilon_2, \varepsilon_1 \neq 0, \varepsilon_2 \neq 0, \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 \neq 0.$$

For $\mathbb{X} = x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3$ and $\mathbb{Y} = y_0 + \varepsilon_1 y_1 + \varepsilon_2 y_2 + \varepsilon_1\varepsilon_2 y_3$, the addition and multiplication rules are given as follows:

$$\begin{aligned} \mathbb{X} + \mathbb{Y} &= (x_0 + y_0) + \varepsilon_1(x_1 + y_1) + \varepsilon_2(x_2 + y_2) + \varepsilon_1\varepsilon_2(x_3 + y_3), \\ \mathbb{X}\mathbb{Y} &= (x_0 y_0) + \varepsilon_1(x_0 y_1 + x_1 y_0) + \varepsilon_2(x_0 y_2 + x_2 y_0) + \varepsilon_1\varepsilon_2(x_0 y_3 + x_1 y_2 + x_2 y_1 + x_3 y_0). \end{aligned}$$

Moreover, the multiplicative-inverse of $\mathbb{X} = x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3$ is

$$\mathbb{X}^{-1} = \frac{1}{\mathbb{X}} = \frac{1}{x_0} - \varepsilon_1\frac{x_1}{x_0^2} - \varepsilon_2\frac{x_2}{x_0^2} + \varepsilon_1\varepsilon_2(-\frac{x_3}{x_0^2} + \frac{2x_1x_2}{x_0^3}), \;\; x_0 \neq 0. \tag{10}$$

A hyper-dual number $\mathbb{X} = x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3$ can be written according to two dual numbers as

$$\mathbb{X} = X + \varepsilon^* X^*. \tag{11}$$

Here, $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^*$ and $X = x_0 + \varepsilon x_1$, $X^* = x_2 + \varepsilon x_3 \in \mathbb{D}$. For two hyper-dual numbers $\mathbb{X} = x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3$ and $\mathbb{Y} = y_0 + \varepsilon_1 y_1 + \varepsilon_2 y_2 + \varepsilon_1\varepsilon_2 y_3$, the addition and multiplication rules are, respectively, calculated by

$$\mathbb{X} + \mathbb{Y} = (X + Y) + \varepsilon^*(X^* + Y^*), \tag{12}$$
$$\mathbb{X}\mathbb{Y} = XY + \varepsilon^*(XY^* + X^*Y). \tag{13}$$

An alternative representation of the multiplicative-inverse of $\mathbb{X} = x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_1\varepsilon_2 x_3 = X + \varepsilon^* X^*$ expressed by Eq. (10) can be given by

$$\mathbb{X}^{-1} = \frac{1}{X} - \varepsilon^* \frac{X^*}{X^2}, \;\; x_0 \neq 0. \tag{14}$$

If the real vectors $\vec{x}_0$ and $\vec{p}\times\vec{x}_0$ in a dual vector $\hat{X}=\vec{x}_0+\varepsilon(\vec{p}\times\vec{x}_0)$ to the dual vectors $\hat{X}$ and $\hat{P}\times_D\hat{X}$, then we get

$$\tilde{X}=\hat{X}+\varepsilon^*(\hat{P}\times_D\hat{X}). \tag{15}$$

For $\tilde{X}=\hat{X}+\varepsilon^*(\hat{P}\times_D\hat{X})$ and $\tilde{Y}=\hat{Y}+\varepsilon^*(\hat{P}\times_D\hat{Y})$, scalar and vector products are, respectively, denoted by

$$\langle\tilde{X},\tilde{Y}\rangle_{HD} = |\tilde{X}|_D|\tilde{Y}|_D\cos\tilde{\varphi}, \tag{16}$$

$$\tilde{X}\times_{HD}\tilde{Y} = |\tilde{X}|_D|\tilde{Y}|_D\vec{n}\sin\tilde{\varphi}. \tag{17}$$

Here, $\tilde{\varphi}$ denotes a hyper-dual angle and $\vec{n}$ is common perpendicular direction vector to $\tilde{X}$ and $\tilde{Y}$, directed from $\hat{X}$ to $\hat{Y}$. For more information about hyper-dual numbers, see [4, 5].

3. Differential geometry of slant ruled surfaces

In this section, some basic definitions and properties about slant ruled surfaces mentioned in [21] are given.
Let I be an open interval in $\mathbb{R}$. Let $\beta=\vec{\beta}(u)$ also be a regular curve in in $\mathbb{R}^3$ and $\vec{q}=\vec{q}(u)$ be a unit direction vector of an oriented line in $\mathbb{R}^3$. The parametric representation of a ruled surface $\bar{\phi}$ is given as follows:

$$\vec{r}(u,s)=\vec{\beta}(u)+s\vec{q}(u), \tag{18}$$

where $\beta(u)$ is base curve and $\vec{q}(u)$ is rulling, respectively. The unit normal vector $\vec{m}$ is

$$m=\frac{\vec{r}_u\times\vec{r}_s}{|\vec{r}_u\times\vec{r}_s|}=\frac{(\vec{\beta}'+s\vec{q}')\times\vec{q}}{\sqrt{|\langle\vec{\beta}',\vec{q}\rangle^2-\langle\vec{q},\vec{q}\rangle\langle\vec{\beta}'+s\vec{q},\vec{\beta}'+s\vec{q}\rangle|}}. \tag{19}$$

Along a rulling $u=u_1$, we write

$$\vec{a}=\lim_{s\to\infty}\vec{m}(u_1,s)=\frac{\vec{q}'\times\vec{q}}{|\vec{q}'|}. \tag{20}$$

The point, where m is orthogonal to $\vec{a}$, is called the striction point (or central point) and denoted by C. The set of striction points of all rullings is called a striction curve of the surface. The parametric representation of the striction curve $\vec{\bar{\beta}}(u)=\bar{\beta}(u)$ on the ruled surface is denoted by

$$\bar{\beta}(u)=\vec{\beta}(u)-\frac{\langle\vec{q}'(u),\vec{\beta}'(u)\rangle}{\langle\vec{q}'(u),\vec{q}'(u)\rangle}\vec{q}(u). \tag{21}$$

The vector $\vec{h}=\vec{a}\times\vec{q}$ is defined as the central normal vector which is the surface normal along the striction curve. Therefore, the set $\{C;\vec{q},\vec{h},\vec{a}\}$ is called Frenet frame of the ruled surface $\bar{\phi}$, where C is the central point and $\vec{q},\vec{h},\vec{a}$ are unit vectors of ruling, central normal and central tangent, respectively.
For the Frenet formulas of $\bar{\phi}$ of the striction curve, we have

$$\begin{pmatrix}\vec{q}'\\ \vec{h}'\\ \vec{a}'\end{pmatrix}=\begin{pmatrix}0 & k_1 & 0\\ -k_1 & 0 & k_2\\ 0 & -k_2 & 0\end{pmatrix}\begin{pmatrix}\vec{q}\\ \vec{h}\\ \vec{a}\end{pmatrix},$$

where k_1 and k_2 are curvature and torsion of $\bar{\phi}$, respectively. Now, we will introduce $\vec{q}-,\vec{h}-,\vec{a}-$ slant ruled surfaces as follows:

Definition 3.1. *Let $\bar{\phi}$ be the ruled surface in $\mathbb{R}^3$ given by the parametrization as*

$$\bar{\phi}(u,s) = \bar{\beta}(u) + s\vec{q}(u), \tag{22}$$

where $\bar{\beta}(u)$ is the striction curve of $\bar{\phi}(u,s)$. Let $\{\vec{q},\vec{h},\vec{a},k_1,k_2\}$ be Frenet operators of $\bar{\phi}$. The following equation exists

$$\langle \vec{q},\vec{p}\rangle = \cos\theta = constant; \quad \theta \neq \frac{\pi}{2}$$

if the rulling makes constant angle θ with a fixed non-zero direction $\vec{p}$ in the space. $\bar{\phi}$ is called a $\vec{q}-$ slant ruled surface.

Definition 3.2. *Let $\bar{\phi}$ be the ruled surface in $\mathbb{R}^3$ given by the parametrization as*

$$\bar{\phi}(u,s) = \bar{\beta}(u) + s\vec{q}(u), \tag{23}$$

where $\bar{\beta}(u)$ is the striction curve of $\bar{\phi}$. Let $\{\vec{q},\vec{h},\vec{a},k_1,k_2\}$ be Frenet operators of $\bar{\phi}$. The following equation exists

$$\langle \vec{h},\vec{p}\rangle = \cos\varphi = constant; \quad \varphi \neq \frac{\pi}{2}$$

if the central normal vector makes constant angle φ with a fixed non-zero direction $\vec{p}$ in the space. $\bar{\phi}$ is called a $\vec{h}-$ slant ruled surface.

Definition 3.3. *Let $\bar{\phi}$ be the ruled surface in $\mathbb{R}^3$ given by the parametrization as*

$$\bar{\phi}(u,s) = \bar{\beta}(u) + s\vec{q}(u), \tag{24}$$

where $\bar{\beta}(u)$ is the striction curve of $\bar{\phi}$. Let $\{\vec{q},\vec{h},\vec{a},k_1,k_2\}$ be Frenet operators of $\bar{\phi}$. The following equation exists

$$\langle \vec{a},\vec{p}\rangle = \cos\mu = constant; \quad \mu \neq \frac{\pi}{2}$$

if the central tangent vector makes constant angle μ with a fixed non-zero direction $\vec{p}$ in the space. $\bar{\phi}$ is called a $\vec{a}-$ slant ruled surface.

4. Hyper-dual numbers and ruled surfaces

In this section, considering some basic properties of hyper-dual numbers, each elements on $S^2_{\tilde{\mathbb{D}}_1}$, which is the subset of the unit-hyper dual sphere $S^2_{\tilde{\mathbb{D}}}$, denote two intersecting and orthogonal directed lines in $\mathbb{R}^3$. Additionally, the correspondence between the unit hyper-dual curve on $S^2_{\tilde{\mathbb{D}}_1}$ and two ruled surfaces in $\mathbb{R}^3$ is given. All results are given in [2].

For the hyper-dual number $\mathbb{X} = X + \varepsilon^* X^*$, the square root is

$$\sqrt{\mathbb{X}} = \sqrt{X} + \varepsilon^* \frac{X^*}{2\sqrt{X}}, \ \ x_0 > 0. \tag{25}$$

The set of all hyper-dual vectors is

$$\tilde{\mathbb{D}}^3 = \{\tilde{\mathbb{X}} = \hat{X} + \varepsilon^*\hat{X}^* : \hat{X}, \hat{X}^* \in \mathbb{D}^3\}. \tag{26}$$

Also, each element of this set is called a hyper-dual vector. For any hyper-dual vectors $\tilde{\mathbb{X}} = \hat{X} + \varepsilon^*\hat{X}^* = x_0 + \varepsilon x_1 + \varepsilon^* x_2 + \varepsilon\varepsilon^* x_3$ and $\tilde{\mathbb{Y}} = \hat{Y} + \varepsilon^*\hat{Y}^* = y_0 + \varepsilon y_1 + \varepsilon^* y_2 + \varepsilon\varepsilon^* y_3$, the scalar and vector products are, respectively,

$$\begin{aligned}
\langle \tilde{\mathbb{X}}, \tilde{\mathbb{Y}}\rangle_{HD} &= \langle \hat{X}, \hat{Y}\rangle_D + \varepsilon^*(\langle \hat{X}, \hat{Y}^*\rangle_D + \langle \hat{X}^*, \hat{Y}\rangle_D) \\
&= \langle x_0, y_0\rangle + \varepsilon(\langle x_0, y_1\rangle + \langle x_1, y_0\rangle) + \varepsilon^*(\langle x_0, y_2\rangle + \langle x_2, y_0\rangle) \\
&+ \varepsilon\varepsilon^*(\langle x_0, y_3\rangle + \langle x_1, y_2\rangle + \langle x_2, y_1\rangle + \langle x_3, y_0\rangle)
\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbb{X}} \times_{HD} \tilde{\mathbb{Y}} &= \hat{X} \times_D \hat{Y} + \varepsilon^*(\hat{X} \times_D \hat{Y}^* + \hat{X}^* \times_D \hat{Y}) \\ &= x_0 \times y_0 \varepsilon(x_0 \times y_1 + \langle x_1 \times y_0) + \varepsilon^*(x_0 \times y_2 + x_2 \times y_0) \\ &+ \varepsilon\varepsilon^*(x_0 \times y_3 + \langle x_1 \times y_2 + x_2 \times y_1 + x_3 \times y_0).\end{aligned}$$

The norm of $\tilde{\mathbb{X}} = \hat{X} + \varepsilon^* \hat{X}^*$ is

$$\begin{aligned}N_{\tilde{\mathbb{X}}} &= \langle \tilde{\mathbb{X}}, \tilde{\mathbb{X}} \rangle_{HD} = |\hat{X}|_D^2 + 2\varepsilon^* \langle \hat{X}, \hat{X}^* \rangle_D \\ &= |x_0|^2 + 2(\varepsilon \langle x_0, x_1 \rangle + \varepsilon^* \langle x_0, x_2 \rangle + \varepsilon\varepsilon^*(\langle x_0, x_3 \rangle + \langle x_1, x_2 \rangle)).\end{aligned}$$

Moreover, the modulus of $\tilde{\mathbb{X}} = \hat{X} + \varepsilon^* \hat{X}^*$ is

$$\begin{aligned}|\tilde{\mathbb{X}}|_{HD} &= \sqrt{\langle \tilde{\mathbb{X}}, \tilde{\mathbb{X}} \rangle_{HD}} = |\hat{X}|_D + \varepsilon^* \frac{\langle \hat{X}, \hat{X}^* \rangle_D}{|\hat{X}|_D} \\ &= |x_0| + \varepsilon \frac{\langle x_0, x_1 \rangle}{|x_0|} + \varepsilon^* \frac{\langle x_0, x_2 \rangle}{|x_0|} \\ &+ \varepsilon\varepsilon^* \left(\frac{\langle x_0, x_3 \rangle}{|x_0|} + \frac{\langle x_1, x_2 \rangle}{|x_0|} - \frac{\langle x_0, x_1 \rangle \langle x_0, x_2 \rangle}{|x_0|^3}\right),\end{aligned}$$

where $|x_0| \neq 0$. If $|\tilde{\mathbb{X}}|_{HD} = 1$, then $\tilde{\mathbb{X}} = \hat{X} + \varepsilon^* \hat{X}^*$ is called a unit hyper-dual vector.

Definition 4.1. *The unit dual sphere $S^2_{\tilde{\mathbb{D}}}$ is defined as*

$$S^2_{\tilde{\mathbb{D}}} = \{\tilde{\mathbb{X}} = \hat{X} + \varepsilon^* \hat{X}^* : |\tilde{\mathbb{X}}|_{HD} = 1; \hat{X}, \hat{X}^* \in \mathbb{D}^3\}, \tag{27}$$

which consists of all unit hyper-dual vectors.

Theorem 4.2. *Let us consider a subset of unit hyper-dual sphere $S^2_{\tilde{\mathbb{D}}}$ as*

$$S^2_{\tilde{\mathbb{D}}_1} = \{\tilde{\mathbb{X}} = \hat{X} + \varepsilon^* \hat{X}^* : |\hat{X}|_D = 1, \tilde{\mathbb{X}} \in S^2_{\tilde{\mathbb{D}}}\} \subseteq S^2_{\tilde{\mathbb{D}}}. \tag{28}$$

Then, there exists an isomorphism between the points of $S^2_{\tilde{\mathbb{D}}_1}$ and any two intersecting perpendicular directed lines in $\mathbb{R}^3$.

A dual curve in $\mathbb{D}^3$ is given by $A(u) = \vec{a}(u) + \varepsilon \vec{a}^*(u)$, where $\vec{a}(u) = (a_1(u), a_2(u), a_3(u))$ and $\vec{a}^*(u) = (a_1^*(u), a_2^*(u), a_3^*(u))$ are vectors in $\mathbb{R}^3$. If $|A(u)|_D = 1$, then the dual curve $A(u)$ is on the unit dual sphere $S^2_{\mathbb{D}}$.
Assume that $A(u) = \vec{a}(u) + \varepsilon \vec{a}^*(u)$ is a dual curve on $S^2_{\mathbb{D}}$. The ruled surface corresponding to $A(u)$ in $\mathbb{R}^3$ is given as follows:

$$\phi(u, s) = \vec{a}(u) \times \vec{a}^*(u) + s\vec{a}(u), \quad u \in \mathbb{R}, \quad s \in \mathbb{R}. \tag{29}$$

Here, $\alpha(u) = \vec{a}(u) \times \vec{a}^*(u)$ is base curve and $\vec{a}(u)$ is the director curve of $\phi(u, s)$, see [].

Definition 4.3. *A hyper-dual curve in $\mathbb{D}^3$ is defined by*

$$\begin{aligned}\tilde{A} : I \subseteq \mathbb{R} &\rightarrow \mathbb{D}^3 \\ u &\longmapsto \tilde{A}(u) = \hat{A}(u) + \varepsilon^* \hat{A}^*(u), \quad I \subseteq \mathbb{R}\end{aligned}$$

and if $|\tilde{A}(u)|_{HD} = 1$, then $\tilde{A}(u)$ is a hyper-dual curve on $S^2_{\tilde{\mathbb{D}}}$. Furthermore, $\tilde{A}(u)$ is a hyper-dual curve on $S^2_{\tilde{\mathbb{D}}}$ and $|\hat{A}^*(u)|_D = 1$, then $\tilde{A}(u)$ is a hyper-dual curve on $S^2_{\tilde{\mathbb{D}}_1}$.

Theorem 4.4. *Assume that $\tilde{A}(u) = \hat{A}(u) + \varepsilon^* \hat{A}^*(u)$ is a hyper-dual curve on $S^2_{\tilde{\mathbb{D}}_1}$. Then, each hyper-dual curve $\tilde{A}(u)$ represents two ruled surfaces in $\mathbb{R}^3$ such that these surfaces have a common base curve and the position vectors of their director curves are perpendicular.*

Theorem 4.5. *Assume that $\phi_1(u,s_1) = z(u)+s_1\vec{a}_0(u)$ and $\phi_2(u,s_2) = z(u)+s_1\vec{a}_2(u)$ are the ruled surfaces corresponding to the hyper-dual curve $\tilde{A}(u) = \hat{A}(u) + \varepsilon^*\hat{A}^*(u)$ on $S^2_{\tilde{\mathbb{D}}_1}$, where $\hat{A}(u) = \vec{a}_0(u) + \varepsilon\vec{a}_1(u)$ and $\hat{A}^*(u) = \vec{a}_2(u) + \varepsilon\vec{a}_3(u)$. Then, the normal vectors of $\phi_1(u,s_1)$ and $\phi_2(u,s_2)$ are orthogonal along the common base curve $z(u)$ iff the velocity vector $\frac{d}{du}z(u) = z'(u)$ is orthogonal to $\vec{a}_0(u)$ or $\vec{a}_2(u)$.*

Proposition 4.6. *Let $\phi_1(u,s_1) = z(u) + s_1\vec{a}_0(u)$ and $\phi_2(u,s_2) = z(u) + s_1\vec{a}_2(u)$ be the ruled surfaces corresponding to the hyper-dual curve $\tilde{A}(u) \in S^2_{\tilde{\mathbb{D}}_1}$ such that their normal vectors are perpendicular along their common base curve $z(u)$. If $z(u)$ is the principal curve of $\phi_1(u,s_1)$ (resp., $\phi_2(u,s_2)$), then $z(u)$ is also the principal curve of $\phi_2(u,s_2)$ (resp., $\phi_1(u,s_1)$).*

5. Slant ruled surfaces constructed by the striction curves of hyper-dual curves

In this section, the striction curve of a hyper dual curve is defined. Then, it is shown that each striction curve of the dual curve represents two slant ruled surfaces in $\mathbb{R}^3$. Moreover, some results for intersection of these slant ruled surfaces and Bertrand and Mannheimm offsets for these slant ruled surfaces are indicated. A dual striction curve in $\mathbb{D}^3$ is defined by

$$\begin{array}{rcl} \tilde{\Gamma} : I \subseteq \mathbb{R} & \rightarrow & \mathbb{D}^3 \\ u & \longmapsto & \tilde{\Gamma}(u) = (q_1(u) + \varepsilon\vartheta_1^*(u), q_2(u) + \varepsilon\vartheta_2^*(u), q_2(u) + \varepsilon\vartheta_2^*(u)) \\ & = & \vec{q}(u) + \varepsilon\vartheta^*(u), \;\; I \subseteq \mathbb{R}, \end{array}$$

where $\vec{q}(u) = (q_1(u), q_2(u), q_3(u))$ and $\vartheta^*(u) = (\vartheta_1^*(u), \vartheta_2^*(u), \vartheta_3^*(u))$ are vectors in $\mathbb{R}^3$ and if $|\tilde{\Gamma}(u)|_D = 1$, then $\tilde{\Gamma}(u)$ is on $S^2_{\mathbb{D}}$, see [22].

Definition 5.1. *A ruled surface $\bar{\phi}(u,s) = \vec{q}(u) \times \vec{\vartheta}(u) + s\vec{q}(u)$ in $\mathbb{R}^3$ is called a $\vec{q}$ (resp., $\vec{h}-, \vec{a}-$) slant ruled surface if the following conditions are satisfied:*
(*i*) *The base curve $\beta(u) = \vec{q}(u) \times \vec{\vartheta}(u)$ of $\bar{\phi}(u,s)$ must be*

$$\begin{array}{rcl} \vec{\beta}(u) & = & \vec{q}(u) \times \vec{\vartheta}^*(u) \\ & = & (\vec{q}(u) \times \vec{\vartheta}(u)) - \dfrac{\langle (\vec{q}(u) \times \vec{\vartheta}(u))', \vec{q}'(u)\rangle}{\langle \vec{q}'(u), \vec{q}'(u)\rangle}\vec{q}(u), \quad \vec{\vartheta}^* \subseteq \vec{\vartheta}. \end{array}$$

(*ii*) *The equations $\langle \vec{q}(u), \vec{\vartheta}^*(u)\rangle = 0$ and $|\vec{q}(u)| = 1$ are satisfied.*
(*iii*) *$\vec{q}-$ (resp., $\vec{h}-, \vec{a}-$) must make a constant angle with a fixed non-zero direction.*

Let $\hat{\Gamma}(u) = \vec{q}(u) + \varepsilon\vec{\vartheta}^*(u)$ be a striction curve of the dual curve $\bar{\Gamma}(u) = \vec{q}(u) + \varepsilon\vec{\vartheta}(u)$ on $S^2_{\mathbb{D}}$ with parameter u. In $\mathbb{R}^3$, the slant ruled surface acquired by $\hat{\Gamma}(u)$ is denoted by

$$\hat{\phi}(u,s) = \vec{q}(u) \times \vec{\vartheta}^*(u) + s\vec{q}(u), \tag{30}$$

where the base curve is

$$\vec{\beta}(u) = \vec{q}(u) \times \vec{\vartheta}^*(u) \tag{31}$$

and $\vec{q}(u)$ is the director curve of $\hat{\phi}$, [14].

Definition 5.2 (Striction curve of a hyper-dual curve). *A striction curve of a hyper dual curve in $\tilde{\mathbb{D}}^3$ is defined as*

$$\begin{array}{rcl} \hat{\Gamma} : I \subseteq \mathbb{R} & \rightarrow & \tilde{\mathbb{D}}^3 \\ u & \rightarrow & \hat{\Gamma}(u) = \bar{\Gamma}_1(u) + \varepsilon\bar{\Gamma}_2(u), \;\; I \subseteq \mathbb{R}. \end{array}$$

Here, $\bar{\Gamma}_1(u)$ and $\bar{\Gamma}_2(u)$ are differentiable striction curves in $\mathbb{D}^3$, then $\hat{\Gamma}(u)$ is said to be differentiable. Also, if $|\hat{\Gamma}(u)|_{HD} = 1$, then $\hat{\Gamma}(u)$ is a striction curve of a hyper-dual curve on unit hyper-dual sphere $S^2_{\tilde{\mathbb{D}}}$. And if $\hat{\Gamma}(u)$ is a striction curve of a hyper-dual curve on $S^2_{\tilde{\mathbb{D}}}$ and $|\bar{\Gamma}_2(u)|_D = 1$, then $\hat{\Gamma}(u)$ is a striction curve of a hyper-dual curve on $S^2_{\tilde{\mathbb{D}}_1}$.

Theorem 5.3. *Let $\hat{\Gamma}(u) = \bar{\Gamma}_1(u) + \varepsilon\bar{\Gamma}_2(u)$ be a the striction curve of the hyper-dual curve on $S^2_{\tilde{\mathbb{D}}_1}$. Then, each striction curve of the dual curve $\hat{\Gamma}(u)$ denotes two slant ruled surfaces in $\mathbb{R}^3$ such that these slant ruled surfaces have a common striction curve and the position vectors of their director curves are orthogonal to each other.*

Proof. From the striction curve of the hyper dual curve $\hat{\Gamma}(u) = \bar{\Gamma}_1(u) + \varepsilon\bar{\Gamma}_2(u) \in S^2_{\tilde{\mathbb{D}}_1}$, $\bar{\Gamma}_1(u)$ and $\bar{\Gamma}_1(u)$ are the dual striction curves on $S^2_{\mathbb{D}}$. Hence, these curves are expressed by, respectively,

$$\bar{\Gamma}_1(u) = \vec{q}_1(u) + \varepsilon\vec{\vartheta}^*_1(u) \tag{32}$$

and

$$\bar{\Gamma}_2(u) = \vec{q}_2(u) + \varepsilon\vec{\vartheta}^*_2(u), \tag{33}$$

where $\vec{q}_1(u), \vec{\vartheta}^*_1(u), \vec{\vartheta}^*_2(u), \vec{\vartheta}_3(u) \in \mathbb{R}^3$. Also, the scalar product of $\bar{\Gamma}_1(u)$ and $\bar{\Gamma}_2(u)$ is given by

$$\langle\bar{\Gamma}_1(u), \bar{\Gamma}_2(u)\rangle_D = \langle\vec{q}_1(u), \vec{q}_2(u)\rangle + \varepsilon(\langle\vec{q}_1(u), \vec{\vartheta}^*_2(u)\rangle + \langle\vec{q}_2(u), \vec{\vartheta}^*_1(u)\rangle). \tag{34}$$

Since $\hat{\Gamma}(u)$ is the striction curve of the hyper-dual curve on $S^2_{\tilde{\mathbb{D}}_1}$, it is also the striction curve of the hyper-dual curve on $S^2_{\mathbb{D}}$. Therefore, $\langle\bar{\Gamma}_1(u), \bar{\Gamma}_2(u)\rangle_D = 0$. That is,

$$\langle\vec{q}_1(u), \vec{q}_2(u)\rangle = 0, \ \ \langle\vec{q}_1(u), \vec{\vartheta}^*_2(u)\rangle + \langle\vec{q}_2(u), \vec{\vartheta}^*_1(u)\rangle = 0. \tag{35}$$

Exploiting Eq. (30), the slant ruled surface generated by $\bar{\Gamma}_1(u) = \vec{q}_1(u) + \varepsilon\vec{\vartheta}^*_1(u)$ and $\bar{\Gamma}_2(u) = \vec{q}_2(u) + \varepsilon\vec{\vartheta}^*_2(u)$ is, respectively,

$$\hat{\phi}_1(u, s_1) = \vec{q}_1(u) \times \vec{\vartheta}^*_1(u) + s_1\vec{q}_1(u), \ \ s_1 \in \mathbb{R}, \tag{36}$$

$$\hat{\phi}(u, s_2) = \vec{q}_2(u) \times \vec{\vartheta}^*_2(u) + s_2\vec{q}_2(u), \ \ s_2 \in \mathbb{R}. \tag{37}$$

The base curves of $\hat{\phi}_1(u, s_1)$ and $\hat{\phi}_2(u, s_2)$ are written, respectively,

$$\begin{aligned}\bar{\beta}_1(u) &= \vec{q}_1(u) \times \vec{\vartheta}^*_1(u),\\ \bar{\beta}_1(u) &= (\vec{q}_1(u) \times \vec{\vartheta}_1(u)) - \frac{\langle(\vec{q}_1(u) \times \vec{\vartheta}_1(u))', \vec{q}_1'(u)\rangle}{\langle\vec{q}_1'(u), \vec{q}_1'(u)\rangle}\vec{q}_1(u), \ \ \vec{\vartheta}^*_1 \subseteq \vec{\vartheta}_1\end{aligned}$$

and

$$\begin{aligned}\bar{\beta}_2(u) &= \vec{q}_2(u) \times \vec{\vartheta}^*_2(u),\\ \bar{\beta}_2(u) &= (\vec{q}_2(u) \times \vec{\vartheta}_2(u)) - \frac{\langle(\vec{q}_2(u) \times \vec{\vartheta}_2(u))', \vec{q}_2'(u)\rangle}{\langle\vec{q}_2'(u), \vec{q}_2'(u)\rangle}\vec{q}_2(u), \ \ \vec{\vartheta}^*_2 \subseteq \vec{\vartheta}_2.\end{aligned}$$

Also, $\vec{q}_1 -$ (*resp.*, $\vec{h}_1-, \vec{a}_1-$) (resp., $\vec{q}_2 -$ (*resp.*, $\vec{h}_2-, \vec{a}_2-$)) must make a constant angle with a fixed non-zero direction.

For $u = u_0$, assume that $m_{u_0}(s_1)$ and $n_{u_0}(s_2)$ are the lines of $\hat{\phi}_1(u_0, s_1)$ and $\hat{\phi}_2(u_0, s_2)$, respectively. Also, $m_u(s_1)$ and $n_u(s_2)$ are the rullings of these slant ruled surfaces for all $u \in I$. Additionally, $m_{u_0}(s_1)$ denotes a line related to the unit dual vector $\bar{\Gamma}_1(u_0) = \vec{q}_1(u_0) + \varepsilon\vec{\vartheta}^*_1(u_0)$ and $n_{u_0}(s_2)$ also denotes a line related to the unit dual vector $\bar{\Gamma}_2(u_0) = \vec{q}_2(u_0) + \varepsilon\vec{\vartheta}^*_2(u_0)$.

Since $\hat{\Gamma}(u_0) = \bar{\Gamma}_1(u_0) + \varepsilon\bar{\Gamma}_2(u_0) \in D_1S^2$, $\hat{\Gamma}(u_0)$ indicates two intersecting orthogonal lines (which are $m_{u_0}(s_1)$ and $n_{u_0}(s_2)$) in $\mathbb{R}^3$. Let $k(u)$ be the intersection point of the lines $m_u(s_1)$ and $n_u(s_2)$ for all $u \in I$. From E. Study mapping, the moments of $\vec{q}_1(u)$ and $\vec{q}_2(u)$ with respect to the origin is calculated by

$$\vartheta_1^* = k(u) \times \vec{q}_1(u), \tag{38}$$
$$\vartheta_2^* = k(u) \times \vec{q}_2(u), \tag{39}$$

respectively. Then, we obtain

$$\begin{aligned}\hat{\phi}_1(u,s_1) &= \vec{q}_1(u) \times \vec{\vartheta}_1^*(u) + s_1\vec{q}_1(u)\\ &= \vec{q}_1(u) \times (k(u) \times \vec{q}_1(u)) + s_1\vec{q}_1(u)\\ &= \langle\vec{q}_1(u),\vec{q}_1(u)\rangle k(u) - \langle\vec{q}_1(u),k(u)\rangle\vec{q}_1(u) + s_1\vec{q}_1(u)\\ &= k(u) - \langle\vec{q}_1(u),k(u)\rangle\vec{q}_1(u) + s_1\vec{q}_1(u)\\ &= k(u) + (s_1 - \langle\vec{q}_1(u),k(u)\rangle)\vec{q}_1(u),\end{aligned} \tag{40}$$

where $\langle\vec{q}_1(u),\vec{q}_1(u)\rangle = 1$. Substituting $v_1 = s_1 - \langle\vec{q}_1(u),k(u)\rangle$ in Eq. (40), we have

$$\hat{\phi}_1(u,v_1) = \vec{q}_1(u) \times \vec{\vartheta}_1^*(u) + v_1\vec{q}_1(u), \quad v_1 \in \mathbb{R}. \tag{41}$$

We also obtain

$$\hat{\phi}_2(u,v_2) = \vec{q}_1(u) \times \vec{\vartheta}_1^*(u) + v_2\vec{q}_1(u), \quad v_2 \in \mathbb{R}. \tag{42}$$

From Eqs. (41) and (42), it is deduced that the slant ruled surfaces $\hat{\phi}_1(u,v_1)$ and $\hat{\phi}_2(u,v_2)$ have a common striction curve which is $k(u)$. And from Eq. (35), it is easily seen that the position vectors of $\vec{q}_1(u)$ and $\vec{q}_2(u)$ of the slant ruled surfaces $\hat{\phi}_1(u,v_1)$ and $\hat{\phi}_2(u,v_2)$ are orthogonal. □

Theorem 5.4. *Assume that $\hat{\phi}_1(u,v_1) = \vec{q}_1(u) \times \vec{\vartheta}_1^*(u) + v_1\vec{q}_1(u)$ and $\hat{\phi}_2(u,v_2) = \vec{q}_1(u) \times \vec{\vartheta}_1^*(u) + v_2\vec{q}_1(u)$ are the slant ruled surfaces related to the striction curve of the hyper dual curve $\hat{\Gamma}(u) = \bar{\Gamma}_1(u) + \varepsilon\bar{\Gamma}_2(u)$ on $S^2_{\tilde{\mathbb{D}}_1}$, where $\bar{\Gamma}_1(u) = \vec{q}_1(u) + \varepsilon\vec{\vartheta}_1^*(u)$ and $\bar{\Gamma}_2(u) = \vec{q}_2(u) + \varepsilon\vec{\vartheta}_2^*(u)$. Then, the normal vectors of these surfaces are orthogonal along the common base curve $k(u)$ iff the velocity vector $\frac{d}{du}k(u) = k'(u)$ is orthogonal to $\vec{q}_1(u)$ or $\vec{q}_2(u)$.*

Proof. The normal vectors of $\hat{\phi}_1(u,v_1)$ and $\hat{\phi}_2(u,v_2)$ are, respectively, denoted by

$$\vec{N}_1(u,v_1) = \vec{q}_1(u) \times (k'(u + v_1\vec{q}_1'(u))), \tag{43}$$
$$\vec{N}_2(u,v_2) = \vec{q}_2(u) \times (k'(u + v_2\vec{q}_2'(u))). \tag{44}$$

Due to the intersection along the common base curve $k(u)$ in the case of $v_1 = v_2 = 0$, the normal vectors of $\hat{\phi}_1(u,v_1)$ and $\hat{\phi}_2(u,v_2)$ are given by

$$\vec{N}_1(u,0) = \vec{q}_1(u) \times k'(u), \tag{45}$$
$$\vec{N}_2(u,0) = \vec{q}_2(u) \times k'(u) \tag{46}$$

for all $u \in I$. Thus, the scalar product of these vectors are represented as follows:

$$\langle\vec{N}_1(u,0),\vec{N}_2(u,0)\rangle = -\langle\vec{q}_1(u),k'(u)\rangle\langle k'(u),\vec{q}_2(u)\rangle. \tag{47}$$

It is deduced that $\vec{N}_1$ and $\vec{N}_2$ are orthogonal along $k(u)$ iff $\langle\vec{q}_1(u),k'(u) = 0\rangle$ or $\langle k'(u),\vec{q}_2(u) = 0\rangle$. □

Proposition 5.5. *Let $\hat{\phi}_1(u,v_1) = \vec{q}_1(u) \times \vec{\vartheta}_1^*(u) + v_1\vec{q}_1(u)$ and $\hat{\phi}_2(u,v_2) = \vec{q}_1(u) \times \vec{\vartheta}_1^*(u) + v_2\vec{q}_1(u)$ be the slant ruled surfaces related to the striction curve of the hyper dual curve $\hat{\Gamma}(u) = \bar{\Gamma}_1(u) + \varepsilon\bar{\Gamma}_2(u)$ on $S^2_{\tilde{\mathbb{D}}_1}$ such that their normal vectors are orthogonal along their common base curve $k(u)$. If $k(u)$ is the principal curve of $\hat{\phi}_1(u,v_1)$ (resp., $\hat{\phi}_2(u,v_2)$), $k(u)$ is also the principal curve of $\hat{\phi}_1(u,v_1)$ (resp., $\hat{\phi}_2(u,v_2)$).*

Proof. Assume that $k(u)$ is a common curve of $\hat{\phi}_1(u,v_1)$ and $\hat{\phi}_2(u,v_2)$. Let $\{\vec{t}_1(u),\vec{y}_1(u),\vec{n}_1(u)\}$ and $\{\vec{t}_2(u),\vec{y}_2(u),\vec{n}_2(u)\}$ be the Darboux frames along the curve $k(u)$, respectively. Namely,

$$\begin{aligned}
\vec{t}_1(u,0) &= t_2(u) = \frac{d}{du}k(u) = k^{'}(u) = \vec{t}(u),\\
\vec{n}_1(u,0) &= \vec{q}_1(u)\times k^{'}(u) = \vec{q}_1(u)\times\vec{t}(u),\\
\vec{n}_2(u,0) &= \vec{q}_2(u)\times k^{'}(u) = \vec{q}_2(u)\times\vec{t}(u),\\
\vec{y}_1(u,0) &= \vec{n}_1(u)\times\vec{t}_1(u) = \vec{n}_1(u,0)\times\vec{t}(u),\\
\vec{y}_2(u,0) &= \vec{n}_2(u)\times\vec{t}_2(u) = \vec{n}_2(u,0)\times\vec{t}(u).
\end{aligned}$$

Additionally, we get

$$\frac{d}{du}\vec{n}_1(u,0) = -k_{n_1}\vec{t}(u) - t_{g_1}\vec{y}_1(u,0), \tag{48}$$

$$\frac{d}{du}\vec{n}_2(u,0) = -k_{n_2}\vec{t}(u) - t_{g_2}\vec{y}_1(u,0). \tag{49}$$

Here, k_{n_1}, k_{n_2} are the normal curvatures and t_{g_1}, t_{g_2} are geodesic torsions. $k(u)$ is the principal curve if $t_{g_1} = 0$ or $t_{g_2} = 0$. Since $\vec{n}_1$ and $\vec{n}_2$ are orthogonal, we write

$$\langle\vec{n}_1(u,0),\vec{n}_2(u,0)\rangle = 0. \tag{50}$$

Differentiating Eq. (50), we obtain

$$\frac{d}{du}\langle\vec{n}_1(u,0),\vec{n}_2(u,0)\rangle = \langle\frac{d}{du}\vec{n}_1(u,0),\vec{n}_2(u,0)\rangle + \langle\vec{n}_1(u,0),\frac{d}{du}\vec{n}_2(u,0)\rangle. \tag{51}$$

Exploiting Eqs. (48-50), we calculate

$$\langle -k_{n_1}\vec{t}(u) - t_{g_1}\vec{y}_1(u,0),\vec{n}_2(u,0)\rangle + \langle\vec{n}_1(u,0), -k_{n_2}\vec{t}(u) - t_{g_2}\vec{y}_1(u,0)\rangle = 0. \tag{52}$$

Also, we have

$$-t_{g_1}\langle\vec{y}_1(u,0),\vec{n}_2(u,0)\rangle - t_{g_2}\langle\vec{n}_1(u,0),\vec{y}_2(u,0)\rangle = 0. \tag{53}$$

It means that

$$-t_{g_1}\langle\vec{y}_1(u,0),\vec{n}_2(u,0)\rangle = t_{g_2}\langle\vec{n}_1(u,0),\vec{y}_2(u,0)\rangle. \tag{54}$$

Consequently, if $t_{g_1} = 0$ (resp., $t_{g_2} = 0$), then $t_{g_2} = 0$ (resp., $t_{g_1} = 0$). □

Remark 5.6. *Let $\hat{\phi}_1(u,v_1) = \vec{q}_1(u)\times\vec{\vartheta}^*_1(u) + v_1\vec{q}_1(u)$ and $\hat{\phi}_2(u,v_2) = \vec{q}_1(u)\times\vec{\vartheta}^*_1(u) + v_2\vec{q}_1(u)$ be two intersecting slant ruled surfaces with Frenet frames $\{\vec{q}_1,\vec{h}_1,\vec{a}_1\}$ and $\{\vec{q}_2,\vec{h}_2,\vec{a}_2\}$, respectively. If $\hat{\phi}_1(u,v_1)$ and $\hat{\phi}_2(u,v_2)$ have common central normals (that means $\vec{h}_1 = \vec{h}_2$) at the corresponding points of their striction lines, $\hat{\phi}_1(u,v_1)$ and $\hat{\phi}_2(u,v_2)$ are called Bertrand offsets.*

Remark 5.7. *Let $\hat{\phi}_1(u,v_1) = \vec{q}_1(u)\times\vec{\vartheta}^*_1(u) + v_1\vec{q}_1(u)$ and $\hat{\phi}_2(u,v_2) = \vec{q}_1(u)\times\vec{\vartheta}^*_1(u) + v_2\vec{q}_1(u)$ be two intersecting slant ruled surfaces with Frenet frames $\{\vec{q}_1,\vec{h}_1,\vec{a}_1\}$ and $\{\vec{q}_2,\vec{h}_2,\vec{a}_2\}$, respectively. If $\vec{a}_1 = \vec{h}_2$, then $\hat{\phi}_2(u,v_2)$ is called a Mannheim offset of $\hat{\phi}_1(u,v_1)$. Moreover, $\hat{\phi}_1(u,v_1)$ and $\hat{\phi}_2(u,v_2)$ are called Mannheim offsets.*

6. Examples

In this section, an example is given to support the main results clearly.

Example 6.1. *Let us consider the striction curve of the hyper-dual curve* $\hat{\Gamma}(u) = \bar{\Gamma}_1(u) + \varepsilon\bar{\Gamma}_2(u)$, *where* $\bar{\Gamma}_1(u) = \vec{q}_1(u) + \varepsilon\vec{\vartheta}_1^*(u)$ *and* $\bar{\Gamma}_2(u) = \vec{q}_2(u) + \varepsilon\vec{\vartheta}_2^*(u)$. *Here,*

$$\begin{aligned} \vec{q}_1(u) &= (1,0,0) \\ \vec{\vartheta}_1^*(u) &= (0,\sin u,-\cos u), \\ \vec{q}_2(u) &= (0,\sin u,\cos u), \\ \vec{\vartheta}_2^*(u) &= (\cos 2u,0,0). \end{aligned}$$

Since $|\bar{\Gamma}_1(u)|_D = |\bar{\Gamma}_2(u)|_D = 1$ *and* $\langle\bar{\Gamma}_1(u),\bar{\Gamma}_2(u)\rangle_D = 0$, $\hat{\Gamma}(u)$ *is a striction curve of the hyper-dual curve on* $S^2_{\tilde{\mathbb{D}}_1}$, *and* $\bar{\Gamma}_1(u)$ *and* $\bar{\Gamma}_2(u)$ *are dual striction curves on* $S^2_{\mathbb{D}}$. *Hence, the ruled surfaces corresponding to the curves* $\bar{\Gamma}_1(u) = \vec{q}_1(u) + \varepsilon\vec{\vartheta}_1^*(u)$ *and* $\bar{\Gamma}_2(u) = \vec{q}_2(u) + \varepsilon\vec{\vartheta}_2^*(u)$ *are, respectively,*

$$\hat{\phi}_1(u,u_1) = (0,\cos u,\sin u) + u_1(1,0,0), \tag{55}$$
$$\hat{\phi}_2(u,u_2) = (0,\cos u\cos 2u,-\cos u\cos 2u) + u_2(0,\sin u,\cos u), \tag{56}$$

where $u \in I = (0,\pi)$ *and* $u_1,u_2 \in \mathbb{R}$. *For* $u = u_0$, *assume that* $m_{u_0}(u_1)$ *and* $n_{u_0}(u_2)$ *are the lines of* $\hat{\phi}_1(u_0,u_1)$ *and* $\hat{\phi}_2(u_0,u_2)$, *respectively. Moreover, for all* $u \in I$, *the intersection point of the lines* $m_u(u_1)$ *and* $n_u(u_2)$ *will be taken as*

$$k(u) = (0,\cos u,\sin u). \tag{57}$$

Therefore, the ruled surfaces are given by

$$\hat{\phi}_1(u,v_1) = (0,\cos u,\sin u) + v_1(1,0,0), \tag{58}$$
$$\hat{\phi}_2(u,v_2) = (0,\cos u,\sin u) + v_2(0,\sin u,\cos u), \tag{59}$$

Hence, $\hat{\phi}_1(u,v_1)$ *and* $\hat{\phi}_2(u,v_2)$ *have a common striction curve* $k(u) = (0,\cos u,\sin u)$. *We see that* $\langle\vec{q}_1(u),\vec{q}_2(u)\rangle = 0$. *Thus, the position vectors of the director curves* $\vec{q}_1(u)$ *and* $\vec{q}_2(u)$ *of the slant ruled surfaces* $\hat{\phi}_1(u,v_1)$ *and* $\hat{\phi}_2(u,v_2)$ *are perpendicular.*

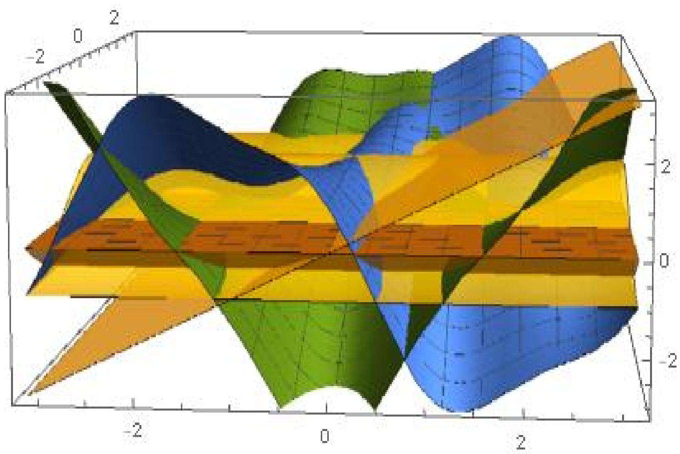

Figure 1: Geometric representation of two slant ruled surfaces in $\mathbb{R}^3$ corresponding to the striction curve of the hyper-dual curve $\hat{\Gamma}(u)$

Also, the velocity vector $k'(u) = (0,-\sin u,\cos u)$ *is orthogonal to* $\vec{q}_2(u) = (0,\sin u,\cos u)$. *Additionally, according to Theorem (5.4), the normal vectors of these slant ruled surfaces are orthogonal.*

7. Conclusion

In literature, there is not any studies about the intersection of two slant ruled surfaces with the perspective of hyper-dual numbers. The aim of this study is to fill the gap the intersection of two slant ruled surfaces by using the hyper-dual numbers. Therefore, the following results are obtained:

1. Each striction curve of the hyper-dual curve on the subset of unit hyper-dual sphere $S^2_{\tilde{\mathbb{D}}_1}$ denotes two slant ruled surfaces in $\mathbb{R}^3$.
2. These slant ruled surfaces intersect along a common base curve and their rulings are orthogonal.
3. Each striction curve of dual curve on unit dual sphere $S^2_{\mathbb{D}}$ indicates a slant ruled surface in $\mathbb{R}^3$ while each striction curve of hyper-dual curve on $S^2_{\tilde{\mathbb{D}}_1}$ indicates two slant ruled surfaces in $\mathbb{R}^3$.
4. Bertrand and Mannheimm offsets of two intersecting slant ruled surfaces are pointed out.
5. An example is given to verify the obtained results.

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