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Global well-posedness and general energy decay of solutions for fully dynamic and electrostatic piezoelectric beams with a viscoelastic damping

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Abstract. In this article, we study a one-dimensional system of fully dynamic and electrostatic piezoelectric beams with a magnetic effect in the presence of a viscoelastic damping term acting on the mechanical equation. Under suitable assumptions on the kernel g, we prove the global existence and uniqueness of the solution by using the Faedo-Galerkin approximations. And by constructing a suitable Lyapunov functional, we establish a general energy decay result. Furthermore, our result does not depend on any relationship between system parameters.

1. Introduction

The term (piezoelectricity) comes from the Greek root (piezen), which means to press or squeeze. Therefore, piezoelectricity is the result of a coupling between the mechanical and electrical properties of a material. Thus, the term (piezoelectricity) designates the property exhibited by certain bodies of being electrically polarized, i.e., of generating an electric field or potential, under the action of a mechanical constraint. This is called the (direct piezoelectric effect), because the inverse piezoelectric effect is also observed. An electric voltage applied to a material having piezoelectric properties leads to a modification of the dimensions of this material. Piezoelectric materials such as quartz, barium titanate, and Rochelle salt have the necessary capacity to transform mechanical energy into electro–magnetic energy under mechanical stress. In 1880, this phenomenon was discovered by the brothers Pierre and Jacques Curie. Furthermore, the latter is known as the direct piezoelectric effect. The same materials can convert electro–magnetic energy to mechanical energy, a phenomenon known as the (converse piezoelectric effect), which was discovered in 1881 by Gabriel Lippmann [22]. Furthermore, as mechanical energy is converted into electric energy, a small portion of it is converted into magnetic energy [14]. This last energy has a relatively small effect on the general dynamics, and there exist models that neglect magnetic effects such as piezoelectric beams.

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Furthermore, this magnetic contribution may limit the system's performance; for example, the magnetic effect can cause oscillations in the output, which leads to system instability in closed loops [15, 24]. In addition, quartz was first used to develop imaging techniques. Perhaps the best–known example of the application of the piezoelectric effect is found in the watch industry. Indeed, piezoelectricity is used to manufacture watches–the famous quartz watches–and clocks. Thanks to the voltage provided by a battery, the quartz crystal begins to vibrate, which allows time to be measured. In the modeling of piezoelectric systems, three main effects and their interrelationships should be taken into account: mechanical, electrical, and magnetic. Mechanical effects are generally modeled through Kirchhoff, Euler–Bernoulli, or Mindlin–Timoshenko small displacement assumptions; see, for example, [23]. There are mainly three approaches for including electrical and magnetic effects: electrostatic, quasi–static, and fully dynamic [21]. Electrostatic and quasi–static approaches are widely employed; see, for example, [4, 8, 9]. These models totally exclude magnetic effects and their coupling with electrical and mechanical effects. Although the mechanical equations in an electrostatic approach are dynamic, the electrical effects are stationary. The quasi–static approach still excludes magnetic effects, but electric charges have time dependence. The electromechanical coupling is not dynamic. Morris et al. [14] using a variational approach to introduce the following coupled model of piezoelectric beams with magnetic effects

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \text{ in } (0,L)\times(0,\infty), \end{cases} \tag{1}$$

where the positive parameters ρ, α, γ, μ, β and L represent, respectively, the mass density, elastic stiffness, piezoelectric coefficient, magnetic permeability, water resistance coefficient of the beam, and length of the beam. Furthermore, the relationship is considered

$$\alpha = \alpha_1 + \gamma^2\beta \text{ with } \alpha_1 > 0. \tag{2}$$

The system (1) is subjected to the following initial and boundary conditions

$$\begin{cases} v(0,t) = p(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \\ \beta p_x(L,t) - \gamma\beta v_x(L,t) = -V(t)/h, \\ (v, v_t, p, p_t)(x,0) = (v_0, v_1, p_0, p_1)(x), \end{cases} \tag{3}$$

where $V(t)$ is the voltage applied at the electrode, and h is the thickness of the beam. Ramos et al. [16] studied the one–dimensional piezoelectric beams system with magnetic effects given by

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, \text{ in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \text{ in } (0,L)\times(0,T), \end{cases}$$

with the following initial and boundary conditions

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) + \xi_1 \frac{v_t(L,t)}{h} = 0, & 0 < t < T, \\ p(0,t) = \beta p_x(L,t) - \gamma\beta v_x(L,t) + \xi_2 \frac{p_t(L,t)}{h} = 0, & 0 < t < T, \\ (v, v_t, p, p_t)(x,0) = (v_0, v_1, p_0, p_1)(x), & 0 < x < L. \end{cases}$$

The authors established, using terms of feedback at the boundary, that the system is exponentially stable regardless of any condition on the coefficients of the system, and exponential stability is equivalent to exact observability at the boundary. Ramos et al. [17] considered the following piezoelectric beams with magnetic effects

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta v_t = 0, & \text{in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & \text{in } (0,L)\times(0,T), \end{cases} \tag{4}$$

and the system (4) is subjected to the following initial and boundary conditions:

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & 0 \le t \le T, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, & 0 \le t \le T, \\ (v, v_t, p, p_t)(x,0) = (v_0, v_1, p_0, p_1)(x), & 0 \le x \le L. \end{cases} \tag{5}$$

They show that irrespective of the model's physical parameters, the dissipation produced by damping δv_t is strong enough to stabilize the system solution (4)–(5) exponentially. They also presented the results of numerical simulations made using the explicit finite difference method. Freitas et al. in [7] studied the following nonlinear piezoelectric beams system with magnetic effects and a delay term

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + f_1(v,p) + v_t = h_1, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + f_2(v,p) + \mu_1 p_t + \mu_2 p_t(x,t-\tau) = h_2, \end{cases}$$

where $(x,t) \in (0,L)\times(0,T)$, the functions $f_1(v,p)$ and $f_2(v,p)$ are nonlinear source terms, h_1 and h_2 represent external forces, whereas v_t and p_t denote damping in displacement and magnetic current, respectively. This system is subjected to the following initial and boundary conditions

$$\begin{cases} (v,v_t,p,p_t)(x,0) = (v_0,v_1,p_0,p_1)(x), & x \in (0,L), \\ v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = 0, & t \in (0,\infty). \end{cases}$$

The authors proved that the dynamical system associated with the solution of the system possesses global and exponential attractors. Freitas et al. in [6] considered the following semi–linear, partially–damped, and fully–dynamic piezoelectric beam model

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta v_t + f(v) = 0, & \text{in } (0,L)\times\mathbb{R}^+, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & \text{in } (0,L)\times\mathbb{R}^+, \end{cases} \tag{6}$$

and the system (6) is accompanied by the following boundary and initial conditions

$$\begin{cases} (v,v_t,p,p_t)(x,0) = (v_0,v_1,p_0,p_1)(x), & x \in (0,L), \\ v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = 0, & t \in (0,\infty), \end{cases} \tag{7}$$

where δv_t $(\delta > 0)$ is the frictional dissipation and $f(v)$ is the nonlinear structural forcing. The authors presented the major results for the long–time dynamics of (6)–(7). The initial result concerns the existence of smooth global attractors with finite fractal dimension. The second result is about the upper semi–continuity of attractors with respect to the magnetic permeability parameter $\mu \to 0$. Soufyane et al. [20] studied the system (1) subjected to the nonlinear damping and nonlinear delay terms that work on the mechanical equation. This work is a generalization of the recent result obtained by Ramos et al. [19]. The authors established an energy decay rate using a perturbed energy method and some properties of convex functions, as well as appropriate assumptions on the weight of the delay. Ramos et al., [18] considered the one–dimensional piezoelectric beam model with second sound, that is, the model includes the thermal effect given by Cattaneo's law of heat conduction. The authors established the system's well–posedness using semigroup theory, and by exploiting the energy method with multiplier techniques, they showed that the system is exponentially stable. In addition, this result is obtained without depending on any relationship between the coefficients. Santos et al., [5] studied the system (1) by inserting the term past history in the equation $(1)_1$. The authors, by using the semigroup theory of linear operators, obtained the existence and uniqueness of a solution and, by constructing an appropriate Lyapunov function, established that the energy associated with the system is exponentially stable. On the other hand, in [1] Afilal et al. studied the following piezoelectric beams with magnetic effects and localized damping

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \alpha(x) v_t = 0, \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \text{ in } (0,L)\times(0,\infty). \end{cases} \tag{8}$$

This system is accompanied by the following initial and boundary conditions:

$$\begin{cases} (v,v_t,p,p_t)(x,0) = (v_0,v_1,p_0,p_1)(x), & x \in (0,L), \\ v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & t \in (0,\infty), \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, & t \in (0,\infty). \end{cases} \tag{9}$$

The authors, by using a damping mechanism acting only on one component and on a small part of the beam, established that the system (8)–(9) is exponentially stable. Other problems related to piezoelectric systems can be found in the following references [2, 10–13, 21]. Motivated and inspired by the above works, in this paper we consider the following system:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \int_0^t g(t-s)v_{xx}(s)\,ds = 0, \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \text{ in } (0,L)\times(0,\infty). \end{cases} \tag{10}$$

This system is accompanied by the following initial and boundary conditions:

$$\begin{array}{ll} (v, v_t, p, p_t)(x,0) = (v_0, v_1, p_0, p_1)(x), & x \in (0,L), \\ v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = 0, & t \in (0,\infty), \end{array} \tag{11}$$

where $v = v(x,t)$ is the longitudinal displacement of the center line and $p = p(x,t)$ is the total load of the electric displacement along the transverse direction at each point x. v_0, v_1, p_0 and p_1 are the initial data that are assumed to belong to a suitable functional space. The coefficients ρ, α, γ, μ and β are constitutive constants, which are positive. Throughout this article, we will suppose that (2) is satisfied and the relaxation function g meets the following assumptions:

(H1) $g : [0,\infty) \to [0,\infty)$ is a non–increasing differentiable function such that

$$g(0) > 0,\ \alpha_1 - \int_0^\infty g(s)\,ds = \alpha_1 - g_0 > 0. \tag{12}$$

(H2) There exists a non–increasing differentiable function $\vartheta : [0,\infty) \to (0,\infty)$ satisfying

$$g'(t) \le -\vartheta(t)\, g(t), \forall t \in \mathbb{R}_+. \tag{13}$$

Moreover, along this paper, we use the following notation:

$$(g \circ v_x)(t) = \int_0^L \int_0^t g(t-s)\,(v_x(t) - v_x(s))^2\, ds dx.$$

The outline of this paper is as follows: In Section 2, we state and prove the well–posedness of the problem (10)–(11). In Section 3, we state and prove our stability result. Finally, in Section 4, the electrostatic/quasi–static equations are investigated for general energy decay.

2. The Global Well-Posedness of the Problem

In this section, by using the classical Faedo–Galerkin approximations, we will prove the existence and uniqueness of solutions for (10)–(11). To achieve this, we use the Sobolev space $\tilde{H}^1(0,L)$ and the standard Lebesgue space $L^2(0,L)$, with their usual scalar products and norms. Let us define the space $\mathcal{H}$ as follows:

$$\mathcal{H} = \tilde{H}^1(0,L) \times L^2(0,L) \times \tilde{H}^1(0,L) \times L^2(0,L),$$

where

$$\tilde{H}^1(0,L) = \left\{ u \in H^1(0,L) : u(0) = 0 \right\},$$

and

$$\tilde{H}^2(0,L) = \left\{ u \in H^2(0,L) : u_x(L) = 0 \right\}.$$

Obtaining the well–posedness of (10)–(11) is provided by the following theorem.

Theorem 2.1. *Let $(v_0, v_1), (p_0, p_1) \in \tilde{H}^1(0,L) \times L^2(0,L)$ be given. Assume that g satisfies hypothesis (H1). Then, problem (10)–(11) has a unique global strong solution*

$$v, p \in C\left(\mathbb{R}_+; \tilde{H}^2(0,L) \cap \tilde{H}^1(0,L)\right) \cap C^1\left(\mathbb{R}_+; \tilde{H}^1(0,L)\right) \cap C^2\left(\mathbb{R}_+; L^2(0,L)\right). \tag{14}$$

Proof. The proof is given by the Faedo–Galerkin method.

Step 1. Approximate problem. Let $\left\{\omega_j\right\}_{j=1}^{\infty}$ be an orthogonal basis in $\tilde{H}^2(0,L) \cap \tilde{H}^1(0,L)$ which is orthonormal in $L^2(0,L)$, and also $\left\{\omega_j\right\}_{j=1}^{\infty}$ constituted by the eigenfunctions of the operator $-\partial_{xx}(.)$, to the eigenvalue $\left\{\lambda_j\right\}$, that is

$$-\partial_{xx}\omega_j = \lambda_j \omega_j \,,\ 1 \leq j \leq n.$$

Now, for every integer $n \in \mathbb{N}$, we define the finite–dimensional subspace by

$$V_n := span\left\{\omega_1, \omega_2, ..., \omega_n\right\},\ n \geq 1.$$

If the initial data $(v_0, v_1, p_0, p_1) \in \mathcal{H}$, we are looking for functions $h_j^n, L_j^n \in C^2([0,T])$, such that the following approximations are satisfied

$$v^n(x,t) := \sum_{j=1}^{n} h_j^n(t)\,\omega_j(x),\ p^n(x,t) := \sum_{j=1}^{n} L_j^n(t)\,\omega_j(x), \tag{15}$$

check the following approximate problem in V_n

$$\begin{cases} \rho\left(v_{tt}^n, u\right) + \alpha\left(v_x^n, u_x\right) - \gamma\beta\left(p_x^n, u_x\right) - \left(\int_0^t g(t-s)\, v_x^n(s)\, ds, u_x\right) = 0,\ \forall u \in V_n, \\ \mu\left(p_{tt}^n, v\right) + \beta\left(p_x^n, v_x\right) - \gamma\beta\left(v_x^n, v_x\right) = 0,\ \forall v \in V_n, \\ v^n(.,0) = v_0^n,\ v_t^n(.,0) = v_1^n,\ p^n(.,0) = p_0^n,\ p_t^n(.,0) = p_1^n, \end{cases} \tag{16}$$

and

$$\begin{cases} v_0^n := \sum_{j=1}^n \left(v_0, \omega_j\right)_{L^2(0,L)} \omega_j \underset{n\to\infty}{\longrightarrow} v_0 \text{ strongly in } \tilde{H}^1(0,L), \\ v_1^n := \sum_{j=1}^n \left(v_1, \omega_j\right)_{L^2(0,L)} \omega_j \underset{n\to\infty}{\longrightarrow} v_1 \text{ strongly in } L^2(0,L), \\ p_0^n := \sum_{j=1}^n \left(p_0, \omega_j\right)_{L^2(0,L)} \omega_j \underset{n\to\infty}{\longrightarrow} p_0 \text{ strongly in } \tilde{H}^1(0,L), \\ p_1^n := \sum_{j=1}^n \left(p_1, \omega_j\right)_{L^2(0,L)} \omega_j \underset{n\to\infty}{\longrightarrow} p_1 \text{ strongly in } L^2(0,L). \end{cases} \tag{17}$$

This brings us to a system of linear ordinary differential equations (ODEs) with these two unknown functions, h_j^n and L_j^n. The application of the basic ODE theory yields the existence of a unique C^2–solution (h_j^n, L_j^n) on the maximal interval $[0, t_n)$ for all $n \geq 1$. Then, thanks to the next a priori estimates that follow, it implies that, in fact, $t_n = T$ for any $T > 0$.

First a priori estimate. Let $u = v_t^n$ in $(16)_1$, $v = p_t^n$ in $(16)_2$, and adding the obtained results, we get

$$\frac{d}{dt}\frac{1}{2}\int_0^L \left[\rho\left|v_t^n\right|^2 + \mu\left|p_t^n\right|^2 + \alpha_1\left|v_x^n\right|^2 + \beta\left|\gamma v_x^n - p_x^n\right|^2\right] dx - \int_0^L v_{tx}^n \int_0^t g(t-s)\, v_x^n(s)\, ds dx \ = \ 0. \tag{18}$$

Then

$$\begin{aligned} -\int_0^L v_{tx}^n \int_0^t g(t-s)\, v_x^n(s)\, ds dx &= \int_0^L v_{tx}^n \int_0^t g(t-s)\left(v_x^n(t) - v_x^n(s)\right) ds dx - \int_0^t g(s) ds \int_0^L v_{tx}^n v_x^n dx \\ &= \frac{1}{2}\frac{d}{dt}\left(g \circ v_x^n\right) - \frac{1}{2}\frac{d}{dt}\int_0^t g(s)\, ds \int_0^L \left|v_x^n\right|^2 dx \\ &\quad - \frac{1}{2}\left(g' \circ v_x^n\right) + \frac{1}{2}g(t)\int_0^L \left|v_x^n\right|^2 dx, \end{aligned} \tag{19}$$

substituting (19) in (18), we obtain

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left[\int_0^L\left(\rho\left|v_t^n\right|^2+\mu\left|p_t^n\right|^2+\left[\alpha_1-\int_0^t g(s)\,ds\right]\left|v_x^n\right|^2+\beta\left|\gamma v_x^n-p_x^n\right|^2\right)dx+(g\circ v_x^n)\right]\\
&=\ \frac{1}{2}(g'\circ v_x^n)-\frac{1}{2}g(t)\int_0^L\left|v_x^n\right|^2dx\ \le\ 0.
\end{aligned}$$

For any $n\geq 1$ and $t\geq 0$, integration over $(0,t)$ yields

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^L\left(\rho\left|v_t^n\right|^2+\mu\left|p_t^n\right|^2+\left[\alpha_1-\int_0^t g(s)\,ds\right]\left|v_x^n\right|^2+\beta\left|\gamma v_x^n-p_x^n\right|^2\right)dx+(g\circ v_x^n)\right]\\
&\le\ \frac{1}{2}\int_0^L\left[\rho\left|v_1^n\right|^2+\mu\left|p_1^n\right|^2+\left[\alpha_1-\int_0^t g(s)\,ds\right]\left|\left(v_0^n\right)_x\right|^2+\beta\left|\gamma\left(v_0^n\right)_x-\left(p_0^n\right)_x\right|^2\right]dx.
\end{aligned}$$

Now, according (17) the following sequences $\left(v_0^n\right)_{n\in\mathbb{N}},\left(v_1^n\right)_{n\in\mathbb{N}},\left(p_0^n\right)_{n\in\mathbb{N}},\left(p_1^n\right)_{n\in\mathbb{N}}$ converge, then we can find a positive constant C independent of n such that

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^L\left(\rho\left|v_t^n\right|^2+\mu\left|p_t^n\right|^2+\left[\alpha_1-\int_0^t g(s)\,ds\right]\left|v_x^n\right|^2+\beta\left|\gamma v_x^n-p_x^n\right|^2\right)dx+(g\circ v_x^n)\right]\\
&\le\ \frac{1}{2}\int_0^L\left[\rho\left|v_1^n\right|^2+\mu\left|p_1^n\right|^2+\left[\alpha_1-\int_0^t g(s)\,ds\right]\left|\left(v_0^n\right)_x\right|^2dx+\beta\left|\gamma\left(v_0^n\right)_x-\left(p_0^n\right)_x\right|^2\right]dx \qquad (20)\\
&\le\ C.
\end{aligned}$$

Then $t_n=T$, for all $T>0$.
Second a priori estimate. From (15), as $h_j^n,L_j^n\in C^2([0,T])$, and as

$$\left\{\omega_j\right\}_{j=1}^{\infty}\subset\tilde{H}^2(0,L)\cap\tilde{H}^1(0,L)\subset H^1(0,L)\hookrightarrow C(0,L),$$

with ($\hookrightarrow$) representing the continuous embedding. Then we have

$$v^n,p^n\in C^2\left(0,T;\tilde{H}^2(0,L)\cap\tilde{H}^1(0,L)\right), \tag{21}$$

and according to (21), we can get

$$\int_0^L\left(\left|v_{xx}^n(x,t)\right|^2+\left|p_{xx}^n(x,t)\right|^2\right)dx<\infty,\ \forall t\in[0,T]. \tag{22}$$

Step 3: The limit process.
By exploiting (20)–(21), we arrive at

$$\begin{cases}
(v^n)_{n\in\mathbb{N}^*}\ \text{is bounded in}\ L^\infty\left(0,T;\tilde{H}^2(0,L)\cap\tilde{H}^1(0,L)\right),\\
\left(v_t^n\right)_{n\in\mathbb{N}^*}\ \text{is bounded in}\ L^\infty\left(0,T;L^2(0,L)\right),\\
(p^n)_{n\in\mathbb{N}^*}\ \text{is bounded in}\ L^\infty\left(0,T;\tilde{H}^2(0,L)\cap\tilde{H}^1(0,L)\right),\\
\left(p_t^n\right)_{n\in\mathbb{N}^*}\ \text{is bounded in}\ L^\infty\left(0,T;L^2(0,L)\right).
\end{cases} \tag{23}$$

By applying the Aubin–Lions–Simon theorem (theorem **II.5.16,** [3]), because

$$\begin{cases}
\text{The embedding of } \tilde{H}^1(0,L)\ \text{in}\ L^2(0,L)\ \text{is continuous},\\
\text{The embedding of } \tilde{H}^2(0,L)\cap\tilde{H}^1(0,L)\ \text{in}\ \tilde{H}^1(0,L)\ \text{is compact}.
\end{cases}$$

Then, we can find that the embedding of $\mathbb{E}_{\infty,\infty}$ and $\tilde{\mathbb{E}}_{\infty,\infty}$ in $C(0,T;\tilde{H}^1(0,L))$ is compact, with

$$\mathbb{E}_{\infty,\infty}=\left\{v^n/\ v^n\in L^\infty\left(0,T;\tilde{H}^2(0,L)\cap\tilde{H}^1(0,L)\right),\ v_t^n=\frac{dv^n}{dt}\in L^\infty\left(0,T;L^2(0,L)\right)\right\},$$

and

$$\tilde{\mathbb{E}}_{\infty,\infty}=\left\{p^n/\ p^n\in L^\infty\left(0,T;\tilde{H}^2(0,L)\cap\tilde{H}^1(0,L)\right),\ p_t^n=\frac{dp^n}{dt}\in L^\infty\left(0,T;L^2(0,L)\right)\right\}.$$

We conclude from (23) that $(v^n)_{n\in\mathbb{N}^*}$ and $(p^n)_{n\in\mathbb{N}^*}$ are bounded in $\mathbb{E}_{\infty,\infty}$, $\tilde{\mathbb{E}}_{\infty,\infty}$ respectively. Then there exist $\left(v^k\right)_{k\in\mathbb{N}^*}$ and $\left(p^k\right)_{k\in\mathbb{N}^*}$ two subsequences of $(v^n)_{n\in\mathbb{N}^*}$ and $(p^n)_{n\in\mathbb{N}^*}$, respectively, such that

$$v^k\overset{k\to\infty}{\longrightarrow}v \text{ and } p^k\overset{k\to\infty}{\longrightarrow}p \text{ strongly in } C(0,T;\tilde{H}^1(0,L)). \tag{24}$$

Now, we define the operator $\mathcal{A}=\partial_{xx}(.)$ as follows

$$\mathcal{A}:H^2(0,L)\subset H^1(0,L)\to L^2(0,L)\quad \text{with } \mathcal{A}v^k=v_{xx}^k \ \text{ and }\ \mathcal{A}p^k=p_{xx}^k,$$

since

$$\mathcal{A}v^k=v_{xx}^k=\gamma_1 v^k \text{ and } \mathcal{A}p^k=p_{xx}^k=\gamma_2 p^k,$$

such that γ_1 and γ_2 are both eigenvalues of the operator $\partial_{xx}(.)$, we can now conclude from (24) that

$$\begin{cases} v_{xx}^k=\gamma_1 v^k\overset{k\to\infty}{\longrightarrow}\gamma_1 v=\xi_1 \text{ strongly in } C(0,T;L^2(0,L)),\\ p_{xx}^k=\gamma_2 p^k\overset{k\to\infty}{\longrightarrow}\gamma_2 p=\xi_2 \text{ strongly in } C(0,T;L^2(0,L)).\end{cases} \tag{25}$$

According to (24) and (25) and as the operator $\partial_{xx}(.)$ is closed, we directly get

$$v\in H^2(0,L)\ \text{ with } \xi_1=v_{xx} \text{ and } p\in H^2(0,L)\ \text{ with } \xi_2=p_{xx}.$$

Then, we obtain

$$v^k\overset{k\to\infty}{\longrightarrow}v \text{ and } p^k\overset{k\to\infty}{\longrightarrow}p \text{ strongly in } C\left(0,T;\tilde{H}^2(0,L)\cap\tilde{H}^1(0,L)\right). \tag{26}$$

Now from (21) and (26), we conclude that

$$v,p\in C\left(0,T;\tilde{H}^2(0,L)\cap\tilde{H}^1(0,L)\right). \tag{27}$$

By using (21) and (24), and exploiting the theorem of dominated convergence, we arrive at

$$\begin{cases} \left\|v_t^k-v_t\right\|_W=\left\|\frac{d}{dt}v^k-v_t\right\|_W\overset{k\to\infty}{\longrightarrow}0,\\ \left\|p_t^k-p_t\right\|_W=\left\|\frac{d}{dt}p^k-p_t\right\|_W\overset{k\to\infty}{\longrightarrow}0,\end{cases}$$

where $W=C(0,T;\ \tilde{H}^1(0,L))$, then we conclude that

$$v_t^k\overset{k\to\infty}{\longrightarrow}v_t \text{ and } p_t^k\overset{k\to\infty}{\longrightarrow}p_t \text{ strongly in } X=C(0,T;\ \tilde{H}^1(0,L)),\ \forall T>0. \tag{28}$$

Now, from (21) and (28), we conclude that

$$v,p\in C^1\left(0,T;\tilde{H}^1(0,L)\right). \tag{29}$$

Finally, by exploiting (21) and (24) and using the theorem of dominated convergence, we get

$$\begin{cases} \left\|v_{tt}^k - v_{tt}\right\|_Y = \left\|\frac{d^2}{dt^2}v^k - v_{tt}\right\|_Y \overset{k\to\infty}{\longrightarrow} 0, \\ \left\|p_{tt}^k - p_{tt}\right\|_Y = \left\|\frac{d^2}{dt^2}p^k - p_{tt}\right\|_Y \overset{k\to\infty}{\longrightarrow} 0, \end{cases}$$

where $Y = C(0,T;\ L^2(0,L))$, then we deduce that

$$v_{tt}^k \overset{k\to\infty}{\longrightarrow} v_{tt} \text{ and } p_{tt}^k \overset{k\to\infty}{\longrightarrow} p_{tt} \text{ strongly in } C(0,T;L^2(0,L)). \tag{30}$$

Now from (21) and (30), we conclude that

$$v, p \in C^2\left(0,T;L^2(0,L)\right). \tag{31}$$

Then by passing the limit in (16)–(17) and exploiting (27), (29) and (31), we deduce that, the problem (10)–(11) has a strong solution that satisfies (14). For the uniqueness of the solution, we suppose that (v,p) and (v_1,p_1) are two pairs of strong solutions to problem (10)–(11), then the pair $(\mathbb{V},\mathbb{P}) = (v - v_1, p - p_1)$ satisfies

$$\begin{cases} \rho\mathbb{V}_{tt} - \alpha\mathbb{V}_{xx} + \gamma\beta\mathbb{P}_{xx} + \int_0^t g(t-s)\mathbb{V}_{xx}(s)\,ds = 0, & \text{in } (0,L)\times(0,\infty), \\ \mu\mathbb{P}_{tt} - \beta\mathbb{P}_{xx} + \gamma\beta\mathbb{V}_{xx} = 0, & \text{in } (0,L)\times(0,\infty), \\ (\mathbb{V},\mathbb{V}_t,\mathbb{P},\mathbb{P}_t)(x,0) = 0, & x\in(0,L), \\ \mathbb{V}(0,t) = \mathbb{V}_x(L,t) = \mathbb{P}(0,t) = \mathbb{P}_x(L,t) = 0, & t\in(0,\infty). \end{cases} \tag{32}$$

Multiplying $(32)_1$ by $\mathbb{V}_t$ and $(32)_2$ by $\mathbb{P}_t$, then integration by parts over $(0,L)$ and the boundary conditions, and thanks to a method similar to that used in the first a priori estimate, we obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\left[\rho\|\mathbb{V}_t\|_2^2 + \mu\|\mathbb{P}_t\|_2^2 + \left[\alpha_1 - \int_0^t g(s)\,ds\right]\|\mathbb{V}_x\|_2^2 + \beta\left\|\gamma\mathbb{V}_x - \mathbb{P}_x\right\|_2^2 + (g\circ\mathbb{V}_x)\right] \\ &= \frac{1}{2}(g'\circ\mathbb{V}_x) - \frac{1}{2}g(t)\|\mathbb{V}_x\|_2^2 \le 0. \end{aligned}$$

Now, by integrating over $(0,t)$, we get

$$\rho\|\mathbb{V}_t\|_2^2 + \mu\|\mathbb{P}_t\|_2^2 + \left[\alpha_1 - \int_0^t g(s)\,ds\right]\|\mathbb{V}_x\|_2^2 + \beta\left\|\gamma\mathbb{V}_x - \mathbb{P}_x\right\|_2^2 + (g\circ\mathbb{V}_x) \le 0,$$

this implies that

$$(\mathbb{V},\mathbb{P}) = (v - v_1, p - p_1) = (0,0).$$

Finally, we get

$$(v,p) = (v_1,p_1).$$

As a result, problem (10)–(11) has a unique strong solution. The proof of the theorem is finished. □

3. General decay

In this section, we state and prove a general decay result for system (10)–(11) using the energy method. We define the following energy functional:

$$E(t) := \frac{1}{2}\int_0^L \left[\rho v_t^2 + \left(\alpha_1 - \int_0^t g(s)\,ds\right)v_x^2 + \mu p_t^2 + \beta(\gamma v_x - p_x)^2\right]dx + \frac{1}{2}(g\circ v_x). \tag{33}$$

The main result of this section is the following theorem.

Theorem 3.1. *Assume that (H1) and (H2) hold. Then, the energy functional defined by (33) satisfies*

$$E(t) \leq \lambda_0 e^{-\lambda_1 \int_0^t \vartheta(s)ds}, \ \forall t \geq 0, \tag{34}$$

where λ_0 and λ_1 are positive constants. To achieve our goal, we need the following lemmas.

Lemma 3.2. *The energy functional defined by (33) satisfies*

$$E'(t) = \frac{1}{2}(g' \circ v_x) - \frac{1}{2}g(t)\int_0^L v_x^2 dx \leq \frac{1}{2}(g' \circ v_x) \leq 0. \tag{35}$$

Proof. Multiplying $(10)_1$ by v_t and $(10)_2$ by p_t, then integration by parts over $(0, L)$ and the boundary conditions, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left[\rho v_t^2 + \mu p_t^2 + \alpha_1 v_x^2 + \beta(\gamma v_x - p_x)^2\right] dx - \int_0^L v_{tx}\int_0^t g(t-s)v_x(x,s)\,dsdx = 0. \tag{36}$$

Meanwhile, estimate the last term of (36) as follows

$$\begin{aligned} -\int_0^L v_{xt}\int_0^t g(t-s)v_x(x,s)\,dsdx &= \int_0^L v_{tx}\int_0^t g(t-s)\left(v_x(t) - v_x(s)\right)dsdx - \int_0^t g(s)ds\int_0^L v_{tx}v_x dx \\ &= \frac{1}{2}\frac{d}{dt}(g \circ v_x) - \frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\int_0^L v_x^2 dx - \frac{1}{2}(g' \circ v_x) + \frac{1}{2}g(t)\int_0^L v_x^2 dx. \end{aligned} \tag{37}$$

Simple substitution of (37) into (36) gives (35). □

Lemma 3.3. *Let (v, p) be the solution of system (10)–(11). Then the functional*

$$F_1(t) := \rho\int_0^L v_t v dx + \gamma\mu\int_0^L p_t v dx,$$

satisfies, for all $\varepsilon_1 > 0$, the estimate

$$F_1'(t) \leq -\frac{\alpha_0}{2}\int_0^L v_x^2 dx + \left(\rho + \frac{\gamma^2\mu^2}{4\varepsilon_1}\right)\int_0^L v_t^2 dx + \varepsilon_1\int_0^L p_t^2 dx + C_1(g \circ v_x), \tag{38}$$

where

$$\alpha_0 = \alpha_1 - \int_0^t g(s)ds > 0, \ C_1 = \frac{1}{2\alpha_0}\left(\int_0^t g(s)ds\right).$$

Proof. Taking the derivative of F_1, using (10) and integrating by parts over $(0, L)$ and using the boundary conditions in (11), we get

$$F_1'(t) = -\alpha_1\int_0^L v_x^2 dx + \int_0^L v_x\int_0^t g(t-s)v_x(x,s)\,dsdx + \rho\int_0^L v_t^2 dx + \gamma\mu\int_0^L p_t v_t dx. \tag{39}$$

Using Young's and Cauchy–Schwarz inequalities, it gives

$$\begin{aligned} &\int_0^L v_x\int_0^t g(t-s)v_x(x,s)\,dsdx = \int_0^t g(s)ds\int_0^L v_x^2 dx - \int_0^L v_x\int_0^t g(t-s)\left(v_x(t) - v_x(s)\right)dsdx \\ &\leq \left(\delta_1 + \int_0^t g(s)ds\right)\int_0^L v_x^2 dx + \frac{1}{4\delta_1}\left(\int_0^t g(s)ds\right)g \circ v_x, \end{aligned} \tag{40}$$

$$\gamma\mu\int_0^L p_t v_t dx \le \varepsilon_1\int_0^L p_t^2 dx + \frac{\gamma^2\mu^2}{4\varepsilon_1}\int_0^L v_t^2 dx. \tag{41}$$

By substituting (40) and (41) into (39), we find

$$F_1'(t) \le -\left(\alpha_1 - \delta_1 - \int_0^t g(s)ds\right)\int_0^L v_x^2 dx + \left(\rho + \frac{\gamma^2\mu^2}{4\varepsilon_1}\right)\int_0^L v_t^2 dx + \varepsilon_1\int_0^L p_t^2 dx + \frac{1}{4\delta_1}\left(\int_0^t g(s)ds\right) g\circ v_x.$$

Let $\alpha_0 = \alpha_1 - \int_0^t g(s)ds > 0$, and letting $\delta_1 = \frac{\alpha_0}{2}$, gives (38). □

Lemma 3.4. *Let (v, p) be the solution of the system (10)–(11). Then the functional*

$$F_2(t) := -\rho\int_0^L v_t \int_0^t g(t-s)(v(t) - v(s))\, dsdx,$$

satisfies, for all $\varepsilon_2 > 0$, the estimate

$$F_2'(t) \le -\frac{\rho c_0}{2}\int_0^L v_t^2 dx + \varepsilon_2\int_0^L v_x^2 dx + \varepsilon_2\int_0^L (\gamma v_x - p_x)^2 dx + C_2(\varepsilon_2)(g\circ v_x) - \frac{\rho g(0)}{2c_0}(g'\circ v_x), \tag{42}$$

where

$$c_0 = \int_0^{t_0} g(s)ds,\ C_2(\varepsilon_2) = \left(\frac{\alpha_1^2}{2\varepsilon_2} + \frac{1}{2\varepsilon_2}\left(\int_0^t g(s)ds\right)^2 + \frac{\gamma^2\beta^2}{4\varepsilon_2} + 1\right)\int_0^t g(s)ds.$$

Proof. By differentiating F_2, then using $(10)_1$, integrating by parts over $(0, L)$ and using the boundary conditions in (11), we find

$$\begin{aligned} F_2'(t) &= -\rho\int_0^t g(s)ds\int_0^L v_t^2 dx + \alpha_1\int_0^L v_x\int_0^t g(t-s)(v_x(t) - v_x(s))\, dsdx \\ &- \int_0^L\int_0^t g(t-s)v_x(s)\, ds\int_0^t g(t-s)(v_x(t) - v_x(s))\, dsdx \\ &+ \gamma\beta\int_0^L(\gamma v_x - p_x)\int_0^t g(t-s)(v_x(t) - v_x(s))\, dsdx - \rho\int_0^L v_t\int_0^t g'(t-s)(v(t) - v(s))\, dsdx. \end{aligned} \tag{43}$$

Using Young's, Cauchy–Schwarz and Poincaré's inequalities. So, for any $\varepsilon_2 > 0$, we obtain

$$\begin{aligned} &\alpha_1\int_0^L v_x\int_0^t g(t-s)(v_x(t) - v_x(s))\, dsdx \le \frac{\varepsilon_2}{2}\int_0^L v_x^2 dx + \frac{\alpha_1^2}{2\varepsilon_2}\int_0^L\left(\int_0^t g(t-s)(v_x(t) - v_x(s))\, ds\right)^2 dx \\ &\le \frac{\varepsilon_2}{2}\int_0^L v_x^2 dx + \left(\frac{\alpha_1^2}{2\varepsilon_2}\int_0^t g(s)ds\right) g\circ v_x, \end{aligned} \tag{44}$$

$$\begin{aligned} &-\int_0^L\int_0^t g(t-s)v_x(s)\, ds\int_0^t g(t-s)(v_x(t) - v_x(s))\, dsdx \\ &= -\int_0^t g(s)ds\int_0^L v_x\int_0^t g(t-s)(v_x(t) - v_x(s))\, dsdx + \int_0^L\left(\int_0^t g(t-s)(v_x(t) - v_x(s))\, ds\right)^2 dx \\ &\le \frac{\varepsilon_2}{2}\int_0^L v_x^2 dx + \left(\frac{1}{2\varepsilon_2}\left(\int_0^t g(s)ds\right)^3 + \int_0^t g(s)ds\right) g\circ v_x, \end{aligned} \tag{45}$$

$$-\rho\int_0^L v_t\int_0^t g'(t-s)\left(v\left(t\right)-v\left(s\right)\right)dsdx\leq\rho\delta_2\int_0^L v_t^2dx-\frac{\rho g(0)}{4\delta_2}g'\circ v_x. \tag{46}$$

Using similar calculations as in (44), we obtain

$$\gamma\beta\int_0^L\left(\gamma v_x-p_x\right)\int_0^t g(t-s)\left(v_x\left(t\right)-v_x\left(s\right)\right)dsdx\leq\varepsilon_2\int_0^L\left(\gamma v_x-p_x\right)^2dx+\left(\frac{\gamma^2\beta^2}{4\varepsilon_2}\int_0^t g(s)ds\right)g\circ v_x. \tag{47}$$

Inserting (44)–(47) into (43), we end up with

$$\begin{aligned}F_2'(t)&\leq-\left(\rho\int_0^t g(s)ds-\rho\delta_2\right)\int_0^L v_t^2dx+\varepsilon_2\int_0^L v_x^2dx+\varepsilon_2\int_0^L\left(\gamma v_x-p_x\right)^2dx\\&+\left(\left(\frac{\alpha_1^2}{2\varepsilon_2}+\frac{1}{2\varepsilon_2}\left(\int_0^t g(s)ds\right)^2+1+\frac{\gamma^2\beta^2}{4\varepsilon_2}\right)\int_0^t g(s)ds\right)g\circ v_x-\frac{\rho g(0)}{4\delta_2}g'\circ v_x.\end{aligned} \tag{48}$$

Using assumption (H1), for any $t\geq t_0>0$, we have

$$c_0=\int_0^{t_0}g(s)ds\leq\int_0^t g(s)ds.$$

Consequently, by taking $\delta_2=\dfrac{c_0}{2}$ we obtain (42). □

Lemma 3.5. *Let (v,p) be the solution of the system (10)–(11). Then the functional*

$$F_3\left(t\right):=\rho\int_0^L v_t\left(\gamma v-p\right)dx+\gamma\mu\int_0^L p_t\left(\gamma v-p\right)dx,$$

satisfies, for all $\varepsilon_3>0$, the estimate

$$F_3'\left(t\right)\leq-\frac{\gamma\mu}{2}\int_0^L p_t^2dx+C_3\int_0^L v_t^2dx+C_4\left(\varepsilon_3\right)\int_0^L v_x^2dx\ +\varepsilon_3\int_0^L\left(\gamma v_x-p_x\right)^2dx+C_5\left(\varepsilon_3\right)\left(g\circ v_x\right), \tag{49}$$

where

$$C_3=\gamma\rho+\gamma^3\mu+\frac{\rho^2}{\gamma\mu},\ C_4\left(\varepsilon_3\right)=\frac{\alpha_1^2}{2\varepsilon_3}+\frac{1}{\varepsilon_3}\left(\int_0^t g(s)ds\right)^2,\ C_5\left(\varepsilon_3\right)=\frac{1}{\varepsilon_3}\left(\int_0^t g(s)ds\right).$$

Proof. Taking the derivative of F_3, using (10) and integration by parts over $(0,L)$, we obtain

$$\begin{aligned}F_3'\left(t\right)&=-\alpha_1\int_0^L v_x\left(\gamma v_x-p_x\right)dx+\int_0^L\left(\gamma v_x-p_x\right)\int_0^t g(t-s)v_x\left(x,s\right)dsdx\\&+\gamma\rho\int_0^L v_t^2dx-\gamma\mu\int_0^L p_t^2dx+\gamma^2\mu\int_0^L v_tp_tdx-\rho\int_0^L v_tp_tdx.\end{aligned} \tag{50}$$

Using Young's inequality, we get for $\varepsilon_3>0$

$$-\alpha_1\int_0^L v_x\left(\gamma v_x-p_x\right)dx\leq\frac{\alpha_1^2}{2\varepsilon_3}\int_0^L v_x^2dx+\frac{\varepsilon_3}{2}\int_0^L\left(\gamma v_x-p_x\right)^2dx, \tag{51}$$

$$\begin{aligned}&\int_0^L\left(\gamma v_x-p_x\right)\int_0^t g(t-s)v_x\left(x,s\right)dsdx\leq\frac{\varepsilon_3}{2}\int_0^L\left(\gamma v_x-p_x\right)^2dx+\frac{1}{2\varepsilon_3}\int_0^L\left(\int_0^t g(t-s)v_x\left(x,s\right)ds\right)^2dx\\&\leq\frac{\varepsilon_3}{2}\int_0^L\left(\gamma v_x-p_x\right)^2dx+\frac{1}{\varepsilon_3}\left(\int_0^t g(s)ds\right)^2\int_0^L v_x^2dx+\frac{1}{\varepsilon_3}\left(\int_0^t g(s)ds\right)g\circ v_x,\end{aligned} \tag{52}$$

$$\gamma^2\mu\int_0^L v_tp_tdx \leq \frac{\gamma\mu}{4}\int_0^L p_t^2dx + \gamma^3\mu\int_0^L v_t^2dx, \tag{53}$$

$$-\rho\int_0^L v_tp_tdx \leq \frac{\gamma\mu}{4}\int_0^L p_t^2dx + \frac{\rho^2}{\gamma\mu}\int_0^L v_t^2dx. \tag{54}$$

Inserting (51)–(54) into (50), we get (49). □

Lemma 3.6. *Let (v,p) be the solution of system (10)–(11). Then the functional*

$$F_4(t) := \mu\int_0^L p_tpdx,$$

satisfies,

$$F_4'(t) \leq -\frac{\beta}{2}\int_0^L (\gamma v_x - p_x)^2\,dx + \frac{\gamma^2\beta}{2}\int_0^L v_x^2dx + \mu\int_0^L p_t^2dx. \tag{55}$$

Proof. By differentiating F_4, using $(10)_2$ and integrating by parts over $(0,L)$ and using the boundary conditions in (11), we have

$$F_4'(t) = -\beta\int_0^L (\gamma v_x - p_x)^2\,dx + \gamma\beta\int_0^L v_x(\gamma v_x - p_x)\,dx + \mu\int_0^L p_t^2dx. \tag{56}$$

Using Young's inequality, we obtain

$$\gamma\beta\int_0^L v_x(\gamma v_x - p_x)\,dx \leq \frac{\gamma^2\beta}{2}\int_0^L v_x^2dx + \frac{\beta}{2}\int_0^L (\gamma v_x - p_x)^2\,dx. \tag{57}$$

Inserting (57) into (56), we get (55). □

Now, we are ready to prove a general decay result.

Proof. (**Of Theorem** 3.1) Let

$$L(t) := NE(t) + \sum_{i=1}^{4} N_iF_i(t),\ \forall t \geq 0, \tag{58}$$

where N, N_1, N_2, N_3 and N_4 are positive real numbers to be chosen appropriately later. By simple routine computations, applying Young's, Poincaré's, and Cauchy–Schwarz inequalities, it follows that $L \sim E$ in the sense that there exist two positive constants, c_1 and c_2, such that

$$c_1E(t) \leq L(t) \leq c_2E(t),\ \ \forall t \geq 0. \tag{59}$$

Now, taking the derivative of (58) and recalling (35), (38), (42), (49) and (55), we obtain

$$\begin{aligned}
L'(t) \leq & -\left(\frac{\rho c_0}{2}N_2 - \left(\rho + \frac{\gamma^2\mu^2}{4\varepsilon_1}\right)N_1 - C_3N_3\right)\int_0^L v_t^2dx \\
& -\left(\frac{\alpha_0}{2}N_1 - \varepsilon_2N_2 - C_4(\varepsilon_3)N_3 - \frac{\gamma^2\beta}{2}N_4\right)\int_0^L v_x^2dx \\
& -\left(\frac{\gamma\mu}{2}N_3 - \varepsilon_1N_1 - \mu N_4\right)\int_0^L p_t^2dx \\
& -\left(\frac{\beta}{2}N_4 - \varepsilon_2N_2 - \varepsilon_3N_3\right)\int_0^L (\gamma v_x - p_x)^2\,dx \\
& + (N_1C_1 + N_2C_2(\varepsilon_2) + N_3C_5(\varepsilon_3))(g \circ v_x) \\
& + \left(\frac{N}{2} - \frac{\rho g(0)}{2c_0}N_2\right)(g' \circ v_x).
\end{aligned}$$

By considering

$$\varepsilon_i = \frac{1}{N_i}, i = 1, 2, 3,$$

we arrive at

$$L'(t) \leq -\eta_1 \int_0^L v_t^2 dx - \eta_2 \int_0^L v_x^2 dx - \eta_3 \int_0^L p_t^2 dx - \eta_4 \int_0^L (\gamma v_x - p_x)^2 dx + \eta_5 (g \circ v_x) + \eta_6 (g' \circ v_x), \quad (60)$$

where

$$\begin{cases} \eta_1 = \frac{\rho c_0}{2} N_2 - \left(\rho + \frac{\gamma^2 \mu^2 N_1}{4}\right) N_1 - \left(\gamma\rho + \gamma^3 \mu + \frac{\rho^2}{\gamma\mu}\right) N_3, \\ \eta_2 = \frac{\alpha_0}{2} N_1 - \left(\frac{\alpha_1^2}{2} + \left(\int_0^t g(s)ds\right)^2\right) N_3^2 - \frac{\gamma^2\beta}{2} N_4 - 1, \\ \eta_3 = \frac{\gamma\mu}{2} N_3 - \mu N_4 - 1, \\ \eta_4 = \frac{\beta}{2} N_4 - 2, \\ \eta_5 = \left(\frac{\alpha_1^2 N_2}{2} + \frac{N_2}{2}\left(\int_0^t g(s)ds\right)^2 + \frac{\gamma^2\beta^2 N_2}{4} + 1\right) N_2 \int_0^t g(s)ds + \left(\frac{N_1}{2\alpha_0} + N_3^2\right) \int_0^t g(s)ds, \\ \eta_6 = \frac{N}{2} - \frac{\rho g(0)}{2c_0} N_2. \end{cases}$$

At this stage, we choose our different constants. First, choosing N_4 large enough such that

$$\eta_4 = \frac{\beta}{2} N_4 - 2 > 0.$$

Then, we pick N_3 large enough such that

$$\eta_3 = \frac{\gamma\mu}{2} N_3 - \mu N_4 - 1 > 0.$$

Furthermore, we choose N_1 large enough so that

$$\eta_2 = \frac{\alpha_0}{2} N_1 - \left(\frac{\alpha_1^2}{2} + \left(\int_0^t g(s)ds\right)^2\right) N_3^2 - \frac{\gamma^2\beta}{2} N_4 - 1 > 0.$$

After that, we choose N_2 large enough so that

$$\eta_1 = \frac{\rho c_0}{2} N_2 - \left(\rho + \frac{\gamma^2 \mu^2 N_1}{4}\right) N_1 - \left(\gamma\rho + \gamma^3 \mu + \frac{\rho^2}{\gamma\mu}\right) N_3 > 0.$$

Finally, we choose N very large enough so that

$$\eta_6 = \frac{N}{2} - \frac{\rho g(0)}{2c_0} N_2 > 0.$$

Consequently, there exist some positive constants, k_1 and k_2, such that

$$L'(t) \leq -k_1 E(t) + k_2 (g \circ v_x), \quad \forall t \geq 0. \quad (61)$$

By multiplying (61) by $\vartheta(t)$, we get

$$\vartheta(t) L'(t) \leq -k_1 \vartheta(t) E(t) + k_2 \vartheta(t) (g \circ v_x), \quad \forall t \geq 0. \quad (62)$$

Now, by using assumption (H2), we have the following estimate

$$\vartheta(t)(g \circ v_x) = \vartheta(t) \int_0^L \int_0^t g(t-s)(v_x(t) - v_x(s))^2 \, dsdx \leq \int_0^L \int_0^t \vartheta(t-s) g(t-s)(v_x(t) - v_x(s))^2 \, dsdx$$
$$\leq -\int_0^L \int_0^t g'(t-s)(v_x(t) - v_x(s))^2 \, dsdx = -(g' \circ v_x) \leq -2E'(t).$$

Thus, (62) becomes

$$\vartheta(t) L'(t) \leq -k_1 \vartheta(t) E(t) - 2k_2 E'(t), \quad \forall t \geq 0,$$

which can be rewritten as

$$(\vartheta(t) L(t) + 2k_2 E(t))' - \vartheta'(t) L(t) \leq -k_1 \vartheta(t) E(t), \quad \forall t \geq 0,$$

next, from the fact that $\vartheta'(t) \leq 0$, we find

$$(\vartheta(t) L(t) + 2k_2 E(t))' \leq -k_1 \vartheta(t) E(t), \quad \forall t \geq 0.$$

Through (59), we easily arrive at

$$\mathcal{L}(t) = (\vartheta(t) L(t) + 2k_2 E(t)) \backsim E(t). \tag{63}$$

Consequently, we have

$$\mathcal{L}'(t) \leq -\lambda_1 \vartheta(t) \mathcal{L}(t), \quad \forall t \geq 0, \tag{64}$$

for some positive constant λ_1. By integrating (64) over $(0, t)$, we get

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\lambda_1 \int_0^t \vartheta(s) ds}, \quad \forall t \geq 0. \tag{65}$$

Consequently, (34) is established by combining (65) and (63). The proof is complete. □

4. General decay result for the electrostatic/quasi–static equations

Because Maxwell's equations neglect the magnetic effects, the electrostatic equations are given as follows

$$\begin{cases} \rho v_{tt} - \alpha_1 v_{xx} + \int_0^t g(t-s) v_{xx}(s) \, ds = 0, & \text{in } (0, L) \times (0, \infty), \\ v(0, t) = v_x(L, t) = 0, & t \in (0, \infty), \\ (v, v_t)(x, 0) = (v_0, v_1)(x), & x \in (0, L). \end{cases} \tag{66}$$

The energy of (66) has been defined by

$$\mathcal{E}(t) := \frac{1}{2} \int_0^L \left[\rho v_t^2 + \left(\alpha_1 - \int_0^t g(s) \, ds \right) v_x^2 \right] dx + \frac{1}{2} (g \circ v_x), \tag{67}$$

and it satisfies

$$\mathcal{E}'(t) = \frac{1}{2} (g' \circ v_x) - \frac{1}{2} g(t) \int_0^L v_x^2 dx \leq \frac{1}{2} (g' \circ v_x) \leq 0, \ \forall t \geq 0. \tag{68}$$

Theorem 4.1. *Assume that (H1) and (H2) hold. Then, the energy functional defined by (67) satisfies*

$$\mathcal{E}(t) \leq \Upsilon_0 e^{-\Upsilon_1 \int_0^t \vartheta(s) ds}, \ \forall t \geq 0, \tag{69}$$

where Υ_0 and Υ_1 are positive constants. To achieve our goal, we need the following lemmas.

Lemma 4.2. *Let v be the solution of system (66). Then the functional*

$$\mathcal{F}_1(t) := \rho \int_0^L v_t v dx,$$

satisfies,

$$\mathcal{F}_1'(t) \leq -\frac{\alpha_0}{2} \int_0^L v_x^2 dx + \rho \int_0^L v_t^2 dx + C_1 (g \circ v_x), \tag{70}$$

where

$$\alpha_0 = \alpha_1 - \int_0^t g(s)ds > 0,\ C_1 = \frac{1}{2\alpha_0}\left(\int_0^t g(s)ds\right).$$

Proof. Taking the derivative of $\mathcal{F}_1$, using (66) and integrating by parts over $(0, L)$ and using the boundary conditions in $(66)_2$, we get

$$\mathcal{F}_1'(t) = -\alpha_1 \int_0^L v_x^2 dx + \int_0^L v_x \int_0^t g(t-s) v_x(x,s)\, ds dx + \rho \int_0^L v_t^2 dx. \tag{71}$$

Using Young's and Cauchy–Schwarz inequalities, it gives

$$\begin{aligned}\int_0^L v_x \int_0^t g(t-s) v_x(x,s)\, dsdx &= \int_0^t g(s)ds \int_0^L v_x^2 dx - \int_0^L v_x \int_0^t g(t-s)\left(v_x(t) - v_x(s)\right) dsdx \\ &\leq \left(\delta_1 + \int_0^t g(s)ds\right) \int_0^L v_x^2 dx + \frac{1}{4\delta_1}\left(\int_0^t g(s)ds\right) g \circ v_x. \end{aligned} \tag{72}$$

By substituting (72) into (71), we get

$$\mathcal{F}_1'(t) \leq -\left(\alpha_1 - \delta_1 - \int_0^t g(s)ds\right) \int_0^L v_x^2 dx + \rho \int_0^L v_t^2 dx + \frac{1}{4\delta_1}\left(\int_0^t g(s)ds\right) g \circ v_x.$$

Let $\alpha_0 = \alpha_1 - \int_0^t g(s)ds > 0$, and letting $\delta_1 = \dfrac{\alpha_0}{2}$, gives (70). □

Lemma 4.3. *Let v be the solution of the system (66). Then the functional*

$$\mathcal{F}_2(t) := -\rho \int_0^L v_t \int_0^t g(t-s)\left(v(t) - v(s)\right) dsdx,$$

satisfies, for all $\varepsilon_2 > 0$, the estimate

$$\mathcal{F}_2'(t) \leq -\frac{\rho c_0}{2} \int_0^L v_t^2 dx + \varepsilon_2 \int_0^L v_x^2 dx + C_2(\varepsilon_2)(g \circ v_x) - \frac{\rho g(0)}{2c_0}(g' \circ v_x), \tag{73}$$

where

$$c_0 = \int_0^{t_0} g(s)ds,\ C_2(\varepsilon_2) = \left(\frac{\alpha_1^2}{2\varepsilon_2} + \frac{1}{2\varepsilon_2}\left(\int_0^t g(s)ds\right)^2 + 1\right)\int_0^t g(s)ds.$$

Proof. By differentiating $\mathcal{F}_2$, then using (66), integrating by parts over $(0, L)$ and using the boundary conditions, we get

$$\begin{aligned}\mathcal{F}_2'(t) = &-\rho \int_0^t g(s)ds \int_0^L v_t^2 dx + \alpha_1 \int_0^L v_x \int_0^t g(t-s)\left(v_x(t) - v_x(s)\right) dsdx \\ &- \int_0^L \int_0^t g(t-s) v_x(s)\, ds \int_0^t g(t-s)\left(v_x(t) - v_x(s)\right) dsdx - \rho \int_0^L v_t \int_0^t g'(t-s)\left(v(t) - v(s)\right) dsdx. \end{aligned} \tag{74}$$

Using Young's, Cauchy–Schwarz and Poincaré's inequalities. So, for any $\varepsilon_2 > 0$, we obtain

$$\begin{aligned} &\alpha_1 \int_0^L v_x \int_0^t g(t-s)\left(v_x(t)-v_x(s)\right) dsdx \\ &\leq \frac{\varepsilon_2}{2}\int_0^L v_x^2 dx + \frac{\alpha_1^2}{2\varepsilon_2}\int_0^L \left(\int_0^t g(t-s)\left(v_x(t)-v_x(s)\right) ds\right)^2 dx \\ &\leq \frac{\varepsilon_2}{2}\int_0^L v_x^2 dx + \left(\frac{\alpha_1^2}{2\varepsilon_2}\int_0^t g(s)ds\right) g\circ v_x, \end{aligned} \tag{75}$$

$$\begin{aligned} &-\int_0^L \int_0^t g(t-s)v_x(s)\,ds \int_0^t g(t-s)\left(v_x(t)-v_x(s)\right) dsdx \\ &= -\int_0^t g(s)ds \int_0^L v_x \int_0^t g(t-s)\left(v_x(t)-v_x(s)\right) dsdx + \int_0^L \left(\int_0^t g(t-s)\left(v_x(t)-v_x(s)\right) ds\right)^2 dx \\ &\leq \frac{\varepsilon_2}{2}\int_0^L v_x^2 dx + \left(\frac{1}{2\varepsilon_2}\left(\int_0^t g(s)ds\right)^3 + \int_0^t g(s)ds\right) g\circ v_x, \end{aligned} \tag{76}$$

$$-\rho \int_0^L v_t \int_0^t g'(t-s)\left(v(t)-v(s)\right) dsdx \leq \rho\delta_2 \int_0^L v_t^2 dx - \frac{\rho g(0)}{4\delta_2} g'\circ v_x. \tag{77}$$

Inserting (75)–(77) into (74), we end up with

$$\begin{aligned} \mathcal{F}_2'(t) \leq &-\left(\rho\int_0^t g(s)ds - \rho\delta_2\right)\int_0^L v_t^2 dx + \varepsilon_2 \int_0^L v_x^2 dx + \left(\left(\frac{\alpha_1^2}{2\varepsilon_2} + \frac{1}{2\varepsilon_2}\left(\int_0^t g(s)ds\right)^2 + 1\right)\int_0^t g(s)ds\right) g\circ v_x \\ &-\frac{\rho g(0)}{4\delta_2} g'\circ v_x. \end{aligned}$$

Using assumption (H1), for any $t \geq t_0 > 0$, we have

$$c_0 = \int_0^{t_0} g(s)ds \leq \int_0^t g(s)ds.$$

Consequently, by taking $\delta_2 = \dfrac{c_0}{2}$ we obtain (73).

Now, for $\widehat{N}$ sufficiently large, we build the functional of Lyapunov $\widehat{L}$ as follows:

$$\widehat{L}(t) := \widehat{N}\mathcal{E}(t) + \sum_{i=1}^{2} \widehat{N}_i \mathcal{F}_i(t), \ \forall t \geq 0, \tag{78}$$

where $\widehat{N}$, $\widehat{N}_1$, and $\widehat{N}_2$, are positive real numbers to be chosen appropriately later. By using the same calculations used in the proof of theorem 3.1. It is clear that $\widehat{L} \sim \mathcal{E}$.

Now, taking the derivative of (78) and recalling (68), (70), and (73), we obtain

$$\begin{aligned} \widehat{L}'(t) \leq &-\left(\frac{\rho c_0}{2}\widehat{N}_2 - \rho\widehat{N}_1\right)\int_0^L v_t^2 dx - \left(\frac{\alpha_0}{2}\widehat{N}_1 - \varepsilon_2\widehat{N}_2\right)\int_0^L v_x^2 dx + \left(\widehat{N}_1 C_1 + \widehat{N}_2 C_2(\varepsilon_2)\right)(g\circ v_x) \\ &+\left(\frac{\widehat{N}}{2} - \frac{\rho g(0)}{2c_0}\widehat{N}_2\right)(g'\circ v_x). \end{aligned}$$

By taking $\varepsilon_2 = \frac{1}{\widehat{N}_2}$, we arrive at

$$\widehat{L}'(t) \leq -\zeta_1 \int_0^L v_t^2 dx - \zeta_2 \int_0^L v_x^2 dx + \zeta_3 (g \circ v_x) + \zeta_4 (g' \circ v_x),$$

where

$$\begin{cases} \zeta_1 = \dfrac{\rho c_0}{2}\widehat{N}_2 - \rho \widehat{N}_1, \\ \zeta_2 = \dfrac{\alpha_0}{2}\widehat{N}_1 - 1, \\ \zeta_3 = \left(\frac{\alpha_1^2 \widehat{N}_2}{2} + \frac{\widehat{N}_2}{2}\left(\int_0^t g(s)ds\right)^2 + 1\right)\widehat{N}_2 \int_0^t g(s)ds + \frac{\widehat{N}_1}{2\alpha_0}\int_0^t g(s)ds, \\ \zeta_4 = \dfrac{\widehat{N}}{2} - \frac{\rho g(0)}{2c_0}\widehat{N}_2. \end{cases}$$

At this stage, we choose our different constants. First, choosing $\widehat{N}_1$ large enough such that

$$\zeta_2 = \frac{\alpha_0}{2}\widehat{N}_1 - 1 > 0.$$

Then, we pick $\widehat{N}_2$ large enough such that

$$\zeta_1 = \frac{\rho c_0}{2}\widehat{N}_2 - \rho \widehat{N}_1 > 0.$$

Finally, we choose $\widehat{N}$ very large enough so that

$$\zeta_4 = \frac{\widehat{N}}{2} - \frac{\rho g(0)}{2c_0}\widehat{N}_2 > 0.$$

Consequently, there exist some positive constants, $\widehat{k}_1$ and $\widehat{k}_2$, such that

$$\widehat{L}'(t) \leq -\widehat{k}_1 \mathcal{E}(t) + \widehat{k}_2 (g \circ v_x), \quad \forall t \geq 0. \tag{79}$$

By multiplying (79) by $\vartheta(t)$, we get

$$\vartheta(t)\widehat{L}'(t) \leq -\widehat{k}_1 \vartheta(t) \mathcal{E}(t) + \widehat{k}_2 \vartheta(t)(g \circ v_x), \quad \forall t \geq 0. \tag{80}$$

Now, by using assumption (H2), we have the following estimate

$$\begin{aligned} \vartheta(t)(g \circ v_x) = \vartheta(t) \int_0^L \int_0^t g(t-s)(v_x(t) - v_x(s))^2 dsdx &\leq \int_0^L \int_0^t \vartheta(t-s) g(t-s)(v_x(t) - v_x(s))^2 dsdx \\ &\leq -\int_0^L \int_0^t g'(t-s)(v_x(t) - v_x(s))^2 dsdx = -(g' \circ v_x) \leq -2\mathcal{E}'(t). \end{aligned}$$

Thus, (80) becomes

$$\vartheta(t)\widehat{L}'(t) \leq -\widehat{k}_1 \vartheta(t) \mathcal{E}(t) - 2\widehat{k}_2 \mathcal{E}'(t), \quad \forall t \geq 0,$$

which can be rewritten as

$$\left(\vartheta(t)\widehat{L}(t) + 2\widehat{k}_2 \mathcal{E}(t)\right)' - \vartheta'(t)\widehat{L}(t) \leq -\widehat{k}_1 \vartheta(t) \mathcal{E}(t), \quad \forall t \geq 0,$$

next, from the fact that $\vartheta'(t) \leq 0$, we find

$$\left(\vartheta(t)\widehat{L}(t) + 2\widehat{k}_2 \mathcal{E}(t)\right)' \leq -\widehat{k}_1 \vartheta(t) \mathcal{E}(t), \quad \forall t \geq 0.$$

Through $\widehat{L} \sim \mathcal{E}$, we easily arrive at

$$\widehat{\mathcal{L}}(t) = \left(\vartheta(t)\widehat{L}(t) + 2\widehat{k_2}\mathcal{E}(t)\right) \backsim \mathcal{E}(t). \tag{81}$$

Consequently, we have

$$\widehat{\mathcal{L}}'(t) \leq -\Upsilon_1 \vartheta(t)\widehat{\mathcal{L}}(t), \quad \forall t \geq 0, \tag{82}$$

for some positive constant Υ_1. By integrating (82) over $(0,t)$, we get

$$\widehat{\mathcal{L}}(t) \leq \widehat{\mathcal{L}}(0)e^{-\Upsilon_1 \int_0^t \vartheta(s)ds}, \quad \forall t \geq 0. \tag{83}$$

Consequently, (69) is established by combining (83) and (81). The proof is complete. □

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