Filomat 38:27 (2024), 9493–9500
https://doi.org/10.2298/FIL2427493L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the phase transition property of one random graph model

Anshui Li[a], Jiajia Wang[a], Huajun Zhang[a,*]

[a] *School of Mathematics, Physics and Information, Shaoxing University, Shaoxing, 312000, P.R.China*

Abstract. In this paper, we introduce one model named random connection model $RG(n, \alpha, \beta)$ defined as follows: the vertex set is $\mathbb{Z}_2^n$, and two vertices $u, v \in \mathbb{Z}_2^n$ are adjacent with probability $\alpha^{n-H(u,v)}\beta^{H(u,v)}$, in which $H(u, v)$ is the Hamming distance between u and v, and $\alpha, \beta \in (0, 1)$ are some fixed constants. This model can be regarded as the discrete counterpart of the random connection model which is given by Penrose (1991) via the Euclidean distance. We obtain some phase transition properties of $RG(n, \alpha, \beta)$ with the help of some asymptotic results by the First and Second moment arguments; and some possible generalizations of the results mentioned in this paper are discussed in the last section.

1. Introduction

The stochastic Kronecker graph model was proposed by Leskovec et al. [7] basing on Kronecker matrix multipications as a model that captures many properties of real-world networks. Fix integer $n > 0$, and $0 < \alpha, \beta, \gamma < 1$, we define a 2×2 matrix

$$\mathbf{P} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}.$$

The stochastic Kronecker graph $K(n, P)$ is a graph whose vertex set is given by the set $\mathbb{Z}_2^n$ of all binary strings of length n. For any vertex u we denote by u_k its k-th digit. Then the probability that a pair of vertices $\{u, v\}$ are connected by an edge is

$$\mathbb{P}_{u,v} = \prod_{k=1}^{n} \mathbf{P}_{u_k, v_k}$$

independently of the presence or absence of any other edge. Compared with the classic Erdös-Rényi random graphs, the stochastic Kronecker graphs are *inhomogeneous* graph model: the presence or absence of an edge is relevant to its two endpoints. As a consequence, the properties of the stochastic Kronecker graph $K(n, P)$ is different from those of the classic Erdös-Rényi model.

As most of the results mentioned in this paper are asymptotic, and to make this paper more readable, we list the standard asymptotic notation used in the sequel as follows:

2020 *Mathematics Subject Classification*. Primary 05C80; Secondary 05D40.
Keywords. Random connection model, Isolated vertices, Phase transition property, Hamming distance, the Giant Component.
Received: 13 September 2023; Revised: 14 January 2024; Accepted: 13 June 2024
Communicated by Dora Selesi
* Corresponding author: Huajun Zhang
Email addresses: anshuili@usx.edu.cn (Anshui Li), zoewang9527@163.com (Jiajia Wang), huajunzhang@usx.edu.cn (Huajun Zhang)

- $f(n) = O(g(n)) : f \leq Kg$ for sufficiently large n and some absolute positive constant K.
- $f(n) = \Omega(g(n))$: if $g(n) = O(f(n))$.
- $f(n) = \Theta(g(n))$: if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
- w.h.p.: A sequence of events A_n, $n = 1, 2, \cdots$, is said to occur *with high probability* if $\lim_{n\to\infty} \mathbb{P}(A_n) = 1$.

A lot of researches were done by a couple of authors since the stochastic Kronecker graph models was first introduced. Mahdian and Xu [8] studied the connectivity, giant components, constant diameter and also searchability of this models, mainly on the setting $0 < \alpha \leq \beta \leq \gamma < 1$; later Horn and Radcliffe [4] managed to prove that the threshold of giant component without the restriction $0 < \alpha \leq \beta \leq \gamma < 1$; Radcliffe and Young [10] also investigated the connectivity and giant component of stochastic Kronecker graphs via $k \times k$ generating matrix; Very recently, Kang et at. [6] studied the degree distribution of this model and showed that it does not feature a power law degree distribution for any parameters $0 \leq \alpha, \beta, \gamma \leq 1$ w.h.p..

We mention two results on connectivity and percolation properties of stochastic Kronecker graphs related to our main theorems in Section 2.

Theorem 1.1 ([8]). *For $0 < \alpha \leq \beta \leq \gamma < 1$. The necessary and sufficient condition for stochastic Kronecker graphs to be connected w.h.p. is $\beta + \gamma > 1$ or $\gamma = \beta = 1, \alpha = 0$.*

Theorem 1.2 ([4]). *The necessary and sufficient condition for stochastic Kronecker graphs to have a giant component of size $\Theta(2^n)$ w.h.p. is $(\alpha + \gamma)(\beta + \gamma) > 1$, or $(\alpha + \gamma)(\beta + \gamma) = 1$ and $\alpha + \gamma > \beta + \gamma$.*

In fact, the main task in [4] is to cancel the restriction on condition $0 \leq \alpha \leq \beta \leq \gamma \leq 1$ for the same conclusion in [8].

In this paper, one new model named random connection graph, which can be regarded as some variant of the stochastic Kronecker graphs mentioned in [7], is proposed and several interesting results are given. In particular, we prove the transition property of the isolated vertices of this random graph, and also obtain one result on percolation property for the components of this model.

The rest of this paper is organized as follows: our model and the main results are shown in Section 2; and the main results are proved in Section 3. The paper is concluded with some discussion in Section 4.

2. One novel model and main result

In this section, we give our model first.

Definition 2.1 (Random connection model). *Suppose n is fixed. Fix two parameters $\alpha, \beta \in [0, 1]$. A random connection model $RG(n, \alpha, \beta)$ is a random graph with vertex set $\mathbb{Z}_2^n$, where two vertices $u, v \in \mathbb{Z}_2^n$ are connected by an edge with probability $\alpha^{n-H(u,v)}\beta^{H(u,v)}$ independently with all other pairs, in which $H(u, v)$ is the hamming distance between u and v.*

From the definition above, the probability that two vertices $u, v \in \mathbb{Z}_2^n$ are neighbors is only dependent on the discrete distance of these two vertices. From this point of view, we can consider this model as the discrete counterpart of random connection model proposed by Penrose [9]: the probability that two vertices $x, y \in \mathbb{R}^d$ are neighbors is $g(|y - x|)$ for some proper function g and $| \cdot |$ is the Euclidean distance in $\mathbb{R}^d$.

However, there are no obvious same coupling in our random connection model in general, but it does have some interesting properties. One of the interesting observations goes as follows:

- If $H(u, v) \geq H(u, w)$ and $\alpha \geq \beta$, then
$$\mathbb{P}[u \sim w] \geq \mathbb{P}[u \sim v].$$
It has **intuitive explanation**: the smaller the distance between two vertices are, the more likely they are neighbors.

- If $H(u,v) \geq H(u,w)$ and $\beta \geq \alpha$, then

$$\mathbb{P}[u \sim w] \leq \mathbb{P}[u \sim v].$$

It has **intuitive explanation**: the larger the distance between two vertices are, the more likely they are neighbors.

In other words, both the distance and the parameters matter for the structures of this graph.

It turns out that this model also enjoys some phase transition property. To be more precise, we obtained the phase transition of the isolated vertices in random connection model $RG(n,\alpha,\beta)$.

Theorem 2.2. *Let $0 < \beta \leq \alpha < 1$ and $\mathcal{P}$ be the property that a graph $RG(n,\alpha,\beta)$ contains at least one isolated vertex. Then*

$$\lim_{n\to\infty} \mathbb{P}\left(RG(n,\alpha,\beta) \in \mathcal{P}\right) = \begin{cases} 0 & \text{if } \alpha+\beta > 1, \\ 1 & \text{if } \alpha+\beta < 1. \end{cases}$$

Moreover, comparing our model with the classic Erdös-Rényi random graph model $G(n,p)$, we can easily get the following result regarding to the "large component":

Theorem 2.3. *The random connection graph $RG(n,\alpha,\beta)$ will only have components of size n w.h.p.when* $\max\{\alpha,\beta\} < 1/2$*; The random connection graph $RG(n,\alpha,\beta)$ will have one giant component of size same order as 2^n w.h.p. when* $\min\{\alpha,\beta\} > 1/2$.

3. Proof of the main results

3.1. Proof of Theorem 2.2

We will prove Theorem 2.2 in this section by several steps. The main idea goes as follows: we first estimate the probability that one vertex is isolated, and then we can prove the theorem by First and Second Moment arguments.

3.1.1. Estimation of the probability of the isolation

Proposition 3.1. *The expected degree of a vertex u in $RG(n,\alpha,\beta)$ is $d(u) = (\alpha+\beta)^n$.*

Proof. Fix a vertex $u \in \mathbb{Z}_2^n$, we then define 2^n random variables as follows:

$$X_v = \begin{cases} 1 & \text{if } v \sim u, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D(u)$ be the degree of u, then

$$D(u) = \sum_{v\in\mathbb{Z}_2^n} X_v$$

With

$$\mathbb{E}(X_v) = \mathbb{P}(u \sim v) = \alpha^{n-H(u,v)}\beta^{H(u,v)}$$

and the linearity of expectation, we can get the expected degree $d(u)$ of u as follows:

$$\begin{aligned} d(u) = \mathbb{E}(D(u)) &= \sum_{v\in\mathbb{Z}_2^n} \alpha^{n-H(u,v)}\beta^{H(u,v)} \\ &= \alpha^n \sum_{v\in\mathbb{Z}_2^n} \left(\frac{\beta}{\alpha}\right)^{H(u,v)} \\ &= \alpha^n \sum_{k=0}^{n} \binom{n}{k}\left(\frac{\beta}{\alpha}\right)^{k} \\ &= \alpha^n \left(1+\frac{\beta}{\alpha}\right)^{n} \\ &= (\alpha+\beta)^n. \end{aligned} \tag{1}$$

□

Before showing the proof, we first give one result on the probability of being isolated for one vertex. Fix a vertex $v \in \mathbb{Z}_2^n$, we then define one random variable as follows

$$I_v = \begin{cases} 1 & \text{if } v \text{ is an isolated vertex in } RG(n,\alpha,\beta), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.2. *For fixed $u \in \mathbb{Z}_2^n$, we have*

$$\mathbb{P}(I_u = 1) = \prod_{i=0}^{n}\left(1-\alpha^n(\frac{\beta}{\alpha})^i\right)^{\binom{n}{i}}$$

Proof. With the definition of $RG(n,\alpha,\beta)$, u is one isolated vertex iff all other vertices(including itself) have no edge with it. The probability that a vertex v with distance $H(u,v) = k$ from u does not connect with v is $\left(1-\alpha^n(\frac{\beta}{\alpha})^k\right)$. All these events are independent with each other, so we consider this issue regarding all the 2^n vertices, we have that the probability that the vertex u is isolated should be

$$\prod_{i=0}^{n}\left(1-\alpha^n(\frac{\beta}{\alpha})^i\right)^{\binom{n}{i}}.$$

□

Then we give some asymptotic result when n is sufficiently large for the probability mentioned above:

Lemma 3.3. *For large n, the probability that one vertex u be isolated is*

$$\mathbb{P}(I_u = 1) \sim e^{-(\alpha+\beta)^n}$$

Proof. For fixed $u \in \mathbb{Z}_2^n$, we have the following result:

$$\mathbb{P}(I_u = 1) = \prod_{i=0}^{n}\left(1-\alpha^n(\frac{\beta}{\alpha})^i\right)^{\binom{n}{i}} = \exp\left(\sum_{k=0}^{n}\binom{n}{k}\log\left(1-\alpha^n(\frac{\beta}{\alpha})^k\right)\right)$$

Then we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\log\left(1-\alpha^n(\frac{\beta}{\alpha})^k\right) &\sim \sum_{k=0}^{n}\binom{n}{k}(-\alpha^n(\frac{\beta}{\alpha})^k)\\ &= -\alpha^n\sum_{k=0}^{n}\binom{n}{k}(\frac{\beta}{\alpha})^k\\ &= -\alpha^n\sum_{k=0}^{n}\binom{n}{k}(\frac{\beta}{\alpha})^k 1^{n-k}\\ &= -\alpha^n(1+\frac{\beta}{\alpha})^n\\ &= -(\alpha+\beta)^n\end{aligned} \tag{2}$$

In other words, the probability for a vertex to be an isolated one is approximately

$$\exp\left(\sum_{k=0}^{n}\binom{n}{k}\log\left(1-\alpha^n(\frac{\beta}{\alpha})^k\right)\right)\sim e^{-(\alpha+\beta)^n}$$

□

3.1.2. Proof of Theorem 2.2

Then we show some classical techniques named First Moment Method and Second Moment Method which will be used in the following proof, see [1] for more information.

Lemma 3.4 (First Moment Method). *Let X be a non-negative integer valued random variable. Then*

$$\mathbb{P}(X>0)\le \mathbb{E}X.$$

Lemma 3.5 (Second Moment Method). *Let X be a non-negative integer valued random variable. Then*

$$\mathbb{P}(X>0)\ge \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}=1-\frac{\operatorname{Var}X}{\mathbb{E}X^2}.$$

It is time to to give the proof of Theorem 2.2.

Proof of Theorem 2.2: We first show the first part of Theorem 2.2 by the First Moment argument: the random graph will have no isolated vertex w.h.p. when $\alpha+\beta>1$.

Let X_0 be the number of isolated vertices in the random connection model $RG(n,\alpha,\beta)$. Then X_0 can be represented as the sum of indicator random variables

$$X_0=\sum_{v\in\mathbb{Z}_2^n} I_v$$

where

$$I_v=\begin{cases}1 & \text{if } v \text{ is an isolated vertex in } RG(n,\alpha,\beta),\\ 0 & \text{otherwise.}\end{cases}$$

So

$$\begin{aligned}\mathbb{E}(X_0) &= \sum_{v\in\mathbb{Z}_2^n} \mathbb{E}I_v \\ &= \sum_{v\in\mathbb{Z}_2^n} \mathbb{P}(I_v = 1) \\ &\sim \sum_{v\in\mathbb{Z}_2^n} e^{-(\alpha+\beta)^n} \\ &= \frac{2^n}{e^{(\alpha+\beta)^n}} \to 0.\end{aligned} \tag{3}$$

The First Moment Method implies that $X_0 = 0$ w.h.p. Which means, there are no isolated vertices at all as $n \to \infty$ when $\alpha + \beta > 1$.

Then we bound the second moment of X_0 as follows:

$$\begin{aligned}\mathbb{E}X_0^2 &= \mathbb{E}(\sum_{v\in\mathbb{Z}_2^n} I_v)^2 \\ &= \sum_{v,u\in\mathbb{Z}_2^n} \mathbb{E}(I_u I_v) \\ &= \sum_{v,u\in\mathbb{Z}_2^n} \mathbb{P}(I_u = 1, I_v = 1) \\ &= \sum_{u\neq v} \mathbb{P}(I_u = 1, I_v = 1) + \sum_{u=v} \mathbb{P}(I_u = 1, I_v = 1) \\ &= \sum_{u\neq v} \mathbb{P}(I_u = 1, I_v = 1) + \mathbb{E}X_0 \\ &= \sum_{u\neq v} \mathbb{P}(I_u = 1|I_v = 1)\mathbb{P}(I_v = 1) + \mathbb{E}X_0 \\ &= \sum_{u\neq v}\sum_{k=1}^{n} \mathbb{P}(I_u = 1, H(u,v) = k|I_v = 1)\mathbb{P}(I_v = 1) + \mathbb{E}X_0 \qquad \text{(Condition on the distance } H(u,v)\text{)} \\ &= \sum_{u\neq v}\sum_{k=1}^{n} \mathbb{P}(I_u = 1|H(u,v) = k, I_v = 1)\mathbb{P}(H(u,v) = k|I_v = 1)\mathbb{P}(I_v = 1) + \mathbb{E}X_0\end{aligned} \tag{4}$$

We then consider the events $(H(u,v) = k|I_v = 1)$ and $(I_u = 1|H(u,v) = k, I_v = 1)$ separately.

(i) $(H(u,v) = k|I_v = 1)$: Given v, $H(u,v) = k$ means u is one of the $\binom{n}{k}$ vertices which has a distance of k with v (it does not depend on whether v is isolated or not), i.e.,

$$\mathbb{P}(H(u,v) = k|I_v = 1) = \frac{\binom{n}{k}}{2^n}.$$

(ii) $(I_u = 1|H(u,v) = k, I_v = 1)$: Given that v is isolated and u has a distance k with v,u is an isolated vertex means that all other $n-1$ verties (except v, since v connects with no vertex) donot have edges with u at all. By Lemma 3.2, we have

$$\mathbb{P}(I_u = 1|H(u,v) = k, I_v = 1) = \frac{\prod_{i=0}^{n}\left(1 - \alpha^n(\frac{\beta}{\alpha})^i\right)^{\binom{n}{i}}}{1 - \alpha^n(\frac{\beta}{\alpha})^k}.$$

Combining all the corresponding formulas into equation 4, we get

$$
\begin{aligned}
\mathbb{E}X_0^2 &= \sum_{u\neq v}\sum_{k=1}^{n} \frac{\prod_{i=0}^{n}\left(1-\alpha^n(\frac{\beta}{\alpha})^i\right)^{\binom{n}{i}}}{1-\alpha^n(\frac{\beta}{\alpha})^k}\frac{\binom{n}{k}}{2^n}\mathbb{P}(I_v=1)+\mathbb{E}X_0 \\
&\sim \sum_{u\neq v}\sum_{k=1}^{n} \frac{e^{-(\alpha+\beta)^n}}{e^{-\alpha^n(\frac{\beta}{\alpha})^k}}\frac{\binom{n}{k}}{2^n}e^{-(\alpha+\beta)^n}+\mathbb{E}X_0 \qquad \text{(By Lemma 3.3 and } 1-x\sim e^{-x} \text{ when } x\to 0) \\
&= \sum_{u\neq v}\frac{1}{e^{(\alpha+\beta)^{2n}}}\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}e^{\alpha^n(\frac{\beta}{\alpha})^k}+\mathbb{E}X_0 \\
&\leq \sum_{u\neq v}\frac{1}{e^{(\alpha+\beta)^{2n}}}\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}e^{\alpha^n}+\mathbb{E}X_0 \qquad \text{(Since } \beta\leq\alpha) \\
&\sim \sum_{u\neq v}\frac{1}{e^{(\alpha+\beta)^{2n}}}\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}+\mathbb{E}X_0 \qquad \text{(Since } \alpha<1 \text{ and let } n \text{ be large enough)} \\
&\leq \frac{4^n}{e^{(\alpha+\beta)^{2n}}}+\mathbb{E}X_0 \\
&= (1+o(1))(\mathbb{E}X_0)^2+\mathbb{E}X_0.
\end{aligned} \tag{5}
$$

Then, by the Second Moment Method, we have

$$
\begin{aligned}
\mathbb{P}(X_0>0) &\geq \frac{(\mathbb{E}X_0)^2}{\mathbb{E}X_0^2} \\
&\geq \frac{(\mathbb{E}X_0)^2}{1+o(1)((\mathbb{E}X_0)^2)+\mathbb{E}X_0} \\
&= \frac{1}{(1+o(1))+(\mathbb{E}X_0)^{-1}} \\
&= 1-o(1),
\end{aligned} \tag{6}
$$

since $\mathbb{E}X_0\to\infty$ as $n\to\infty$ when $(\alpha+\beta)<1$, which is easy to check from equation 3. Hence $\mathbb{P}(X_0>0)\to 1$ when $(\alpha+\beta)<1$ as $n\to\infty$. In other words, there will be some isolated vertices in $RG(n,\alpha,\beta)$ w.h.p when $\alpha+\beta<1$.

■

3.2. Proof of Theorem 2.3

Our proof will be based on some corresponding results on the percolation property of the classic Erdös-Rényi random graph model $G(n,p)$.

Proof of Theorem 2.3: For the classic Erdös-Rényi random graph model $G(n,p)$, the critical probability is $p_c=1/n$ for the following percolation property:

- For $p<p_c$, all connected components will have size $O(\ln n)$.
- For $p>p_c$, there is precisely one infinite component called giant component of size $\Omega(n)$.

The complicated proof can be checked in several great books, see [2, 3, 5].

For any two fixed $u,v\in\mathbb{Z}_2^n$, we have

$$\min\{\alpha^n,\beta^n\}\leq\mathbb{P}[u\sim v]=\alpha^{n-H(u,v)}\beta^{H(u,v)}\leq\max\{\alpha^n,\beta^n\}.$$

Of course, for a fixed triplet (u, v, w) of vertices of $\mathbb{Z}_2^n$, $H(u, w), H(v, w), H(u, v)$ are not independent, which is different from the classic Erdös-Rényi random graph model $G(n, p)$. However, we can have the following observation:

$$\min\{\alpha^n, \beta^n\} \leq \mathbb{P}[u \sim v | u \sim w, v \sim w] \leq \max\{\alpha^n, \beta^n\}.$$

In other words, the event u and v have common neighbor w only means the probability of u and v being adjacent will be larger or smaller depending on the value of α and β, but this probability can be bounded.

One can get easily that

- $\mathbb{P}[u \sim v] < \frac{1}{2^n}$ and $\mathbb{P}[u \sim v | u \sim w, v \sim w] < \frac{1}{2^n}$ when $\max\{\alpha, \beta\} < 1/2$;
- $\mathbb{P}[u \sim v] > \frac{1}{2^n}$ and $\mathbb{P}[u \sim v | u \sim w, v \sim w] > \frac{1}{2^n}$ when $\min\{\alpha, \beta\} > 1/2$.

Comparing with the percolation property of classical Erdös-Rényi random graph $G(\mathbb{Z}_2^n, p)$ when $p < \frac{1}{2^n}$ and $p > \frac{1}{2^n}$ respectively, we complete our proof. ∎

4. Discussion

This paper proposes one novel random connection model and gives proofs of some phase transition properties of this model. Though it can be considered as one special case of the classic stochastic Kronecker graph models proposed by Leskovec et al. [7] basing on Kronecker matrix multipications as a model that captures many properties of real-world networks, some of the results appeared already for classic stochastic Kronecker graphs can not be transferred into this special case trivially. There are many interesting properties related to $RG(n, \alpha, \beta)$ which can be explored in the future. We just mention one interesting conjecture as follows which can be considered as the strong version of Theorem 2.3:

Conjecture 4.1. *The random connection graph $RG(n, \alpha, \beta)$ will only have components of size n w.h.p.when $\alpha+\beta < 1$; The random connection graph $RG(n, \alpha, \beta)$ will have one giant component of size same order as 2^n w.h.p. when $\alpha+\beta > 1$. In other words, the critical value of the giant component of $RG(n, \alpha, \beta)$ is $\alpha + \beta = 1$.*

Acknowledgments

A.Li's research was supported by Natural Science Foundation of China (grant number 11901145).

References

[1] N. Alon and J. H. Spencer. *The Probabilistic Method, Fourth ed.* Wiley, New York, 2015.
[2] B Bollobas. *Random Graphs, Second ed.* Cambridge University Press, 2003.
[3] A. Frieze and M Karoński. *Introduction to Random Graphs*. Cambridge University Press, 2015.
[4] Paul Horn and Mary Radcliffe. Giant components in kronecker graphs. *Random. Struct. Algor*, 40(3):365–387, 2012.
[5] S. Janson, T Luczak, and A. Rucinski. *Random Graphs*. Wiley, New York, 1997.
[6] M. Kang, M Karoński, C. Koch, and T. Makai. Properties of stochastic kronecker graphs. *J.Comb.*, 6(4):395–432, 2015.
[7] Jurij Leskovec, Deepayan Chakrabarti, Jon Kleinberg, and Christos Faloutsos. Realistic, mathematically tractable graph generation and evolution, using kronecker multiplication. In *Knowledge Discovery in Databases: PKDD 2005*, pages 133–145. Springer, 2005.
[8] Mohammad Mahdian and Ying Xu. Stochastic kronecker graphs. *Random. Struct. Algor*, 38(4):453–466, 2011.
[9] Mathew D Penrose. On a continuum percolation model. *Adv. App. Prob*, pages 536–556, , 1991.
[10] M. Radcliffe and S. J. Young. Connectivity and giant component of stochastic kronecker graphs. *J.Comb.*, 6(4):385–397, 2013.