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Note on the local spectral theory for Drazin invertible operators

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Abstract. In this paper we continue the analysis undertaken in [6] where we have investigated the transmission of some local spectral properties from R to its Drazin inverse S, when this does exist. In this paper we consider a similar problem for unbounded operators.

1. Introduction and results

In the sequel we shall give the relevant definitions concerning the local spectral theory forr an $(T, D(T))$ closed linear operator in $\mathcal{H}$ and we extend some of the results established in the bounded case to an unbounded linear operator. First we begin with some preliminary notations and remarks.

Let $(T, D(T))$ be a (possibly unbounded) closed linear operator in $\mathcal{H}$. Clearly we define $D(T^2) := \{x \in D(T) : \quad Tx \in D(T)\}$ and, in general, for $n \geq 2$ we put $D(T^n) := \{x \in D(T^{n-1}) : \quad T^{n-1}x \in D(T)\}$ and $T^n(x) = T(T^{n-1}x)$. It is worth mentioning that nothing guarantees, in general, that $D(T^k)$ does not reduce to the null subspace $\{0\}$, for some $k \in \mathbb{N}$. For this reason powers of an unbounded operator could be of little use in many occasions. Throughout this paper if $\mathcal{D}$ is linear subspace of $\mathcal{H}$ a function defined on an open set Ω of $\mathbb{C}$, $f : \Omega \to D$ is analytic if $f : \Omega \to \mathcal{H}$ is analytic and $f^n(x) \in D$ for every $x \in \Omega$, and $n \in \mathbb{N}$.

Let $(T, D(T))$ be a closed linear operator in $\mathcal{H}$. As usual, the spectrum of $(T, D(T))$ is defined as the set

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a bijection of } D(T) \text{ onto } \mathcal{H}\}.$$

The set $\rho(T) = \mathbb{C} \setminus \sigma(T)$ is called the resolvent set of $(T, D(T))$, while the map $R(\lambda, T) : \rho(T) \ni \lambda \mapsto (\lambda I - T)^{-1}$ is called the resolvent of $(T, D(T))$.

It is well known that, if T is a bounded everywhere defined operator, $\sigma(T)$ is a compact subset of the complex plane. The viceversa is not true: there exist closed unbounded operators whose spectrum is a bounded subset of $\mathbb{C}$. Thus, the spectral radius of an unbounded operator can be finite.

Definition 1.1. Let $(T, D(T))$ be a closed operator in $\mathcal{H}$.

- A point $\lambda \in \mathbb{C}$ is said to be in the *local resolvent set* of $x \in \mathcal{H}$, denoted by $\rho_T(x)$, if there exist an open neighborhood $\mathcal{U}$ of λ in $\mathbb{C}$ and an analytic function $f : \mathcal{U} \to D(T)$ which satisfies

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in \mathcal{U}. \quad (1)$$

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- The *local spectrum* $\sigma_T(x)$ of T at $x \in \mathcal{H}$ is the set defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ and obviously $\sigma_T(x) \subseteq \sigma(T)$, and $\sigma_T(x)$ is a closed subset of $\mathbb{C}$.

Definition 1.2. Let $(T, D(T)), D := D(T)$, be a closed linear operator in $\mathcal{H}$ such that $T^n(D) \subseteq D$. The *hyperrange* of T is the subspace

$$T^\infty(D) := \bigcap_{n \in \mathbf{N}} T^n(D) =: \mathcal{R}^\infty(T)$$

.

Now, let us introduce two classical quantities associated with an operator. To every linear operator T on a vector space D there correspond the two chains:

$$\{0\} = \ker T^0 \subseteq \ker T \subseteq \ker T^2 \cdots .$$

and

$$D = T^0(D) \supseteq T(D) \supseteq T^2(D) \cdots .$$

The *ascent* of T is the smallest positive integer $p = p(T)$, whenever it exists, such that $\ker T^p = \ker T^{p+1}$. If such p does not exist we let $p = +\infty$. Analogously, the *descent* of T is defined to be the smallest integer $q = q(T)$, whenever it exists , such that $T^{q+1}(D) = T^q(D)$. If such q does not exist we let $q = +\infty$.

Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{D})$ the set of all closable linear operators from $\mathcal{D}$ to $\mathcal{D}$ and $\mathcal{L}^\dagger(\mathcal{D})$ be the space consisting of all its elements which leave, together with their adjoints, the domain $\mathcal{D}$ invariant. Then $\mathcal{L}(\mathcal{D})$ is a algebra with respect to the usual operations and $\mathcal{L}^\dagger(\mathcal{D})$ is a subalgebra of $\mathcal{L}(\mathcal{D})$.

Let $\alpha(T) := \dim \ker T$ and $\beta(T) := \operatorname{codim} T(X)$. The class of all *upper semi-Fredholm operators* is defined by

$$\Phi_+(X) := \{T \in \mathcal{L}(\mathcal{D}) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\},$$

while the class all *lower semi-Fredholm operators* is defined by

$$\Phi_-(X) := \{T \in \mathcal{L}(\mathcal{D}) : \beta(T) < \infty\}.$$

If $T \in \Phi_+(X) \cup \Phi_-(X)$ the *index* of T is defined by $\operatorname{ind} T = \alpha(T) - \beta(T)$. It is well known that if $\beta(T) < \infty$ then $T(X)$ is closed. An operator, in general, is said to be *bounded below* if is injective and has closed range. The *approximate point spectrum* is defined by

$$\sigma_{\rm ap}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

while the *surjectivity spectrum* is defined as

$$\sigma_{\rm s}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\}.$$

If T^* denotes the *dual* of T it is well known that $\sigma_{\rm ap}(T) = \sigma_{\rm s}(T^*)$ and $\sigma_{\rm s}(T) = \sigma_{\rm ap}(T^*)$. Let $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ the class of all *Fredholm operators* . An operator $T \in \mathcal{L}(\mathcal{D})$ is said to be a *Weyl operator* if $T \in \Phi(X)$ and $\operatorname{ind} T = 0$, $T \in \mathcal{L}(\mathcal{D})$ is said to be *upper semi-Weyl* if $T \in \Phi_+(X)$ and $\operatorname{ind} T \leq 0$, $T \in \mathcal{L}(\mathcal{D})$ is said to be *lower semi-Weyl* if $T \in \Phi_-(X)$ and $\operatorname{ind} T \geq 0$. Denote by $\sigma_{\rm w}(T)$, $\sigma_{\rm uw}(T)$ and $\sigma_{\rm lw}(T)$ the Weyl spectrum, the upper semi-Weyl spectrum and the lower semi-Weyl spectrum , respectively. Evidently,

$$\sigma_{\rm uw}(T) \subseteq \sigma_{\rm ap}(T) \quad \text{and} \quad \sigma_{\rm lw}(T) \subseteq \sigma_{\rm s}(T)$$

holds for every $T \in \mathcal{L}(\mathcal{D})$. There is a duality:

$$\sigma_{\rm uw}(T) = \sigma_{\rm lw}(T^*) \quad \text{and} \quad \sigma_{\rm lw}(T) = \sigma_{\rm uw}(T^*),$$

The *ascent* of $T \in \mathcal{L}(\mathcal{D})$ is the smallest positive integer $p = p(T)$, whenever it exists, such that $\ker T^p = \ker T^{p+1}$. If such p does not exist we let $p = +\infty$. Analogously, the *descent* of T is defined to be the smallest integer $q = q(T)$, whenever it exists , such that $T^{q+1}(H) = T^q(H)$. If such q does not exist we set $q = +\infty$. Note that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Moreover λ is a pole of the resolvent if and only if $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. If $\alpha(\lambda I - T) < \infty$ and λ is a pole then λ is said to have *finite rank*. An operator $T \in \mathcal{L}(\mathcal{D})$ is said to be *Browder* if $T \in \Phi(X)$ and $p(T) = q(T) < \infty$. $T \in \mathcal{L}(\mathcal{D})$ is said to be *upper semi-Browder* if $T \in \Phi_+(X)$ and $p(\lambda I - T) < \infty$, while $T \in \mathcal{L}(\mathcal{D})$ is said to be *lowerr semi-Browder* if $T \in \Phi_-(X)$ and $q(\lambda I - T) < \infty$.

The *Browder spectrum*, the *upper semi-Browder spectrum*, the *lower semi-Browder spectrum* are denoted by $\sigma_{\mathrm{b}}(T)$, $\sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{lb}}(T)$, respectively. Note that if λ is a spectral point for which $\lambda I - T$ is Browder then λ is an isolated point of $\sigma(T)$.

Definition 1.3. The operator $(T, D(T))$ is said to have *the single valued extension property* at $\lambda_o \in \mathbb{C}$ (abbreviated SVEP at λ_o), if for every open disc $\mathbf{D}_{\lambda_o}$ centered at λ_o the only analytic function $f : \mathbf{D}_{\lambda_o} \to D(T)$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \tag{2}$$

is the function $f \equiv 0$.
An unbounded linear operator $(T, D(T))$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

Following [1] if $(T, D(T))$ be closed linear operator in $\mathcal{H}$ for every subset Ω of $\mathbb{C}$, the *analytic spectral subspace* of T associated with Ω is the set

$$X_T(\Omega) := \{x \in \mathcal{H} : \sigma_T(x) \subseteq \Omega\}.$$

Remark 1.4. If T is globally defined ($D(T) = \mathcal{H}$) and bounded then the SVEP may be easily characterized by means of the subspace $X_T(\emptyset)$ through the equivalence of the following statements[18]:

(i) T has the SVEP.

(ii) If $\sigma_T(x) = \emptyset$ then $x = 0$, i.e. $X_T(\emptyset) = \{0\}$.

(iii) $X_T(\emptyset)$ is closed.

Given a (possibly unbounded) linear operator $(T, D(T))$ and a closed set $F \subseteq \mathbb{C}$, let $\mathfrak{X}_T(F)$ consist of all $x \in \mathcal{H}$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to D(T)$ that satisfies

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in \mathbb{C} \setminus F. \tag{3}$$

Clearly, the identity $X_T(F) = \mathfrak{X}_T(F)$ holds for all closed sets $F \subseteq \mathbb{C}$ whenever T has SVEP.

■

The following proposition generalizes partially the result of remark (1.4).

Theorem 1.5. *Every closed linear operator $(T, D(T))$ such that $X_T(\emptyset) = \{0\}$ has the SVEP.*

Definition 1.6. *The* quasi-nilpotent part *of an operator $T \in \mathcal{L}(\mathcal{D})$ is the set*

$$H_0(T) := \{x \in D : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\},$$

while the analytic core *of T is the set $K(T) := X_T(\mathbb{C} \setminus \{0\})$.*

Let $\mathcal{N}^\infty(T) := \bigcup_{k=1}^{\infty} \ker T^k$. For every $n \in \mathbb{N}$, we have the increasing chain of range-type subspaces

$$X_T(\emptyset) \subseteq K(T) \subseteq \mathcal{R}^\infty(T) \subseteq R(T^n) \subseteq R(T).$$

This result will be one of our principal tools.

Theorem 1.7. *For every operator $T \in \mathcal{L}(\mathcal{D})$ and $\lambda \in \mathbb{C}$, the following assertions are equivalent:*

(i) *T has SVEP at λ;*

(ii) $\ker(\lambda I - T) \cap X_T(\emptyset) = \{0\}$*;*

(iii) $\mathcal{N}^\infty(\lambda I - T) \cap X_T(\emptyset) = \{0\}$.

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory [15] in the case of the Banach algebra of bounded linear operators $L(X)$.

In this work a similar definition is given but in a completely different context. $R \in \mathcal{L}(\mathcal{D})$ is said to be *Drazin invertible* (with a finite index) if there exist two closed invariant subspaces Y and Z such that $\mathcal{D} = Y \oplus Z$ and, with respect to this decomposition,

$$R = R_1 \oplus R_2, \quad \text{with } R_1 := R|Y \text{ nilpotent and } R_2 := R|Z \text{ invertible.} \tag{4}$$

Note that the Drazin inverse S of an operator, if it exists, may be represented, with respect to the decomposition $\mathcal{D} = Y \oplus Z$, as the directed sum

$$S := 0 \oplus S_2 \quad \text{with } S_2 := {R_2}^{-1}. \tag{5}$$

Indeed, if n is such that $R_1^n = 0$ then it is easy to check that S satisfies the equalities (6).

$$RS = SR, \quad SRS = S, \quad R^n SR = R^n. \tag{6}$$

Trivially

$$\sigma(R) = \sigma(R_1) \cup \sigma(R_2) = \{0\} \cup \sigma(R_2).$$

If $0 \notin \sigma(R)$ then $\sigma(R) = \sigma(R_2)$ and analogously $\sigma(S) = \sigma(S_2)$, while if $0 \in \sigma(R)$ then $\sigma(R) \setminus \{0\} = \sigma(R_2)$, since $0 \notin \sigma(R_2)$. Analogous arguments shows that, $\sigma(S) \setminus \{0\} = \sigma(S_2)$. Since S_2 is the inverse of R_2 we then conclude that the nonzero part of the spectrum of S is given by the reciprocals of the nonzero points of the spectrum of R, i.e.

$$\sigma(S) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma(R) \setminus \{0\}\}. \tag{7}$$

The spectral mapping theorem holds also for the approximate point spectrum, so, by using similar arguments, we obtain a similar equality for the upper and Lower semi-Browder spectrum, Browder spectrum, *B-Weyl spectrum*, approximate point spectra, i.e.

$$\sigma_\sharp(S) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_\sharp(R) \setminus \{0\}\}, \tag{8}$$

where

$$\sigma_\sharp(S) \quad \text{is} \quad \sigma_{\mathrm{ub}}(S), \sigma_{\mathrm{lb}}(S), \sigma_{\mathrm{bw}}(S), \sigma_{\mathrm{a}}(S), \sigma_{\mathrm{wa}}(S) \quad \text{or also} \quad \sigma_{\mathrm{w}}(S)$$

In this paper we have continued a study the relationship between the local spectral properties of an operator R and the local spectral properties of its Drazin inverse S, following the previous paper ([6]) relating to bounded operators.

2. Local spectral theory of Drazin inverse

The following results, in this section, are a simple consequence of the passage of SVEP and polaroid property. Many local spectral properties are preserved by the Riesz functional calculus. An easy consequence is that the local spectral properties considered are transmitted from T to T^{-1} in the case that T is invertible. In [6] it is demonstrated how local spectral properties are transmitted to the inverse Drazin. In this article we consider a similar problem for unbounded operators.

Theorem 2.1. *Suppose that $R \in \mathcal{L}(\mathcal{D})$ is Drazin invertible with Drazin inverse S. If R is* upper semi-Browder operator *then S is* upper semi-Browder operator.

Proof. According the decomposition $\mathcal{D} = Y \oplus Z$, $R_1 := R|Y$ nilpotent and $R_2 := R|Z$ invertible, and the Drazin inverse $S := 0 \oplus S_2$, with $S_2 := R_2^{-1}$, then $KerR = KerR_1$. By hypothesis $\infty > \alpha(R) = \alpha(R_1)$, then for every $l \in \mathbb{N}$, $\alpha(R^l) < \infty$. Since R_1 is nilpotent there exist $l \in \mathbb{N}$ such that $Ker{R_1}^l = Y$ therefore the dimension of Y is finite. Thus $Y = KerS$ is closed and has dimension finite and at the same time the range of S is the closed subspace Z. Moreover, since the Drazin inverse S is also Drazin invertible, then $p(S) = q(S) < \infty$.

Theorem 2.2. *Suppose that $R \in \mathcal{L}(\mathcal{D})$ is Drazin invertible with Drazin inverse S. If R is* lower-Browder operator *then S is* lower Browder operator.

Proof. According the decomposition $\mathcal{D} = Y \oplus Z$, $R_1 := R|Y$ nilpotent and $R_2 := R|Z$ invertible, and the Drazin inverse $S := 0 \oplus S_2$, with $S_2 := R_2^{-1}$, then $\beta(R) < \infty$. For every $l \in \mathbb{N}$, $\beta(R^l) < \infty$ (see Lemma 2.2 [?]). Since R_1 is nilpotent there exist $l \in \mathbb{N}$ such that ${R_1}^l = 0$. Clearly $R^l = 0 \oplus R_2^l$. Therefore $\beta(R^l) = \beta({R_2}^l) = codimZ < \infty$. Thus $\beta(S) = codimZ$ and moreover since the Drazin inverse S is also Drazin invertible then $p(S) = q(S) < \infty$. So the proof is complete $S \in \mathcal{B}_-(X)$.

An easy consequence, of (2.1) and (2.8), is the following

Corollary 2.3. *Suppose that R is Drazin invertible with Drazin inverse S. If R is Browder operator then S is Browder operator.*

Theorem 2.4. *Suppose that $R \in \mathcal{L}(\mathcal{D})$ is Drazin invertible with Drazin inverse S. If $R \in \Phi_\pm(X)$ then S is* Weyl operator.

Proof. Suppose that $R \in \Phi_\pm(X)$ and R Drazin invertible with Drazin inverse S then by Theorem 3.4 of [2] $R \in \Phi(X)$. Then there exist two closed invariant subspaces Y and Z of X such that $X = Y \oplus Z$, $R_1 := R|Y$ is nilpotent, $R_2 := R|Z$ is invertible and the Drazin inverse of R is given by $S := 0 \oplus S_2$, with $S_2 := R_2^{-1}$. Similarly to the previous theorems it is easy to prove that $\alpha(S) = dimY < \infty$ and $\beta(S) = codimZ < \infty$. Therefore $S \in \Phi(X)$ and ind $S = 0$.

An easy consequence is the following:

Corollary 2.5. *Suppose that $R \in \mathcal{L}(\mathcal{D})$ is Drazin invertible with Drazin inverse S.*

- *If R is* Weyl operator *then S is* Weyl operator.
- *If R is* Fredholm operator *then S is* Fredholm operator.

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_o \in \mathbf{C}$ (abbreviated SVEP at λ_o), if for every open disc $\mathbf{D}_{\lambda_o}$ centered at λ_o the only analytic function $f : \mathbf{D}_{\lambda_o} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \tag{9}$$

is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbf{C}$.

In [6] we have shown that the SVEP is transmitted to the Drazin inverse. In this paper we prove that is also transmitted locally. Clearly R and S have the SVEP in 0 indeed $p(R) < \infty$ and $p(S) < \infty$ (see [2]).

Theorem 2.6. *Suppose that $R \in L(X)$ is Drazin invertible with Drazin inverse S. If R has SVEP at $\lambda_0 \neq 0$ then S has SVEP at $\frac{1}{\lambda_0}$.*

Proof. (i) By Theorem 1.3 of [7] for every $0 \neq y + z = x \in Ker(\lambda_0 I - R)$, we have $\sigma_R(x) = \{\lambda_o\}$.

If $x = y + z \in Ker(\frac{1}{\lambda_0}I - S)$ where $S := 0 \oplus S_2$, with $S_2 := R_2^{-1}$ then $z \in Ker(\frac{1}{\lambda_0}I - S_2)$. Moreover $\frac{1}{\lambda_0}z - S_2 z = 0$, hence, $S_2 z = \frac{1}{\lambda_0}z$ and evidently $R_2 S_2 z = \frac{1}{\lambda_0}R_2 z$ thus $R_2 z = \lambda_0 z$. Clearly $z \in Ker(\lambda_0 I - R_2)$, then since R has SVEP at λ_0 then R_2 has SVEP at λ_0 by [2, Theorem 2.9], and by Theorem 1.3 of [7] $\sigma_{R_2}(z) = \{\lambda_o\}$. Consequently since S_2 is the inverse of R_2, from the spectral mapping theorem of the local spectrum (see [18] Theorem 3.3) applied to the function $f(\lambda) := \frac{1}{\lambda}$, we have

$$\sigma_S(x) = \sigma_0(y) \cup \sigma_{S_2}(z) = \sigma_{S_2}(z) = \{\frac{1}{\lambda} : \lambda \in \sigma_{R_2}(z)\} = \frac{1}{\lambda_0} \quad \text{for all } x \in \mathcal{D}.$$

Theorem 2.7. *Suppose that $R \in L(\mathcal{D})$ is Drazin invertible with Drazin inverse S. If R satisfies the Browder's theorem then S satisfies the the Browder's theorem.*

Proof. If R satisfies the Browder's theorem then $\sigma_w(R) = \sigma_b(R)$. By (2.5) and (2.1),(2.8), $0 \in \sigma_w(R)$ if and only if $0 \in \sigma_w(S)$ and also $0 \in \sigma_b(R)$ if and only if $0 \in \sigma_b(S)$. Then, by hypothesis, if $0 \in \sigma_w(S)$ then $0 \in \sigma_w(R) = \sigma_b(R)$, then $0 \in \sigma_b(S)$. Analogously if $0 \in \sigma_b(S)$ then $0 \in \sigma_b(S)$.

If $0 \neq \lambda \in \sigma_w(S)$ then $\frac{1}{\lambda} \in \sigma_w(R) = \sigma_b(R)$ by (7). Therefore $\lambda \in \sigma_b(S)$.

Theorem 2.8. *Suppose that $R \in L(\mathcal{D})$ is Drazin invertible with Drazin inverse S. If R satisfies the* Weyl's theorem *then S satisfies the the* Weyl's theorem.

Proof. If R satisfies the the Weyl's theorem then

$$\sigma(R) \setminus \sigma_w(R) = \pi_{00}(R). Q.E.D. \tag{10}$$

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Data accessibility statement

This work does not have any experimental data.

Competing interests statement

I have no competing interests.

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