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W-core inverses in a ring with involution

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Abstract. In this paper, we shall give some new properties and characterizations of w-core inverses in a unital $*$-ring, providing a new construction method of group inverse and Moore-Penrose inverse. We also present several new constructions of w-core invertible elements.

1. Introduction

The idea of the core inverse has been intensively studied by a number of academics. It was initially proposed by Baksalary and Trenkler in the context of complex matrices [1] and then expanded by Rakić et al. [21] to the case of elements in rings with involution. Subsequently, the core inverse was extended to several new classes of generalized inverses such as the core-EP inverse of square complex matrices [20], the DMP inverse of square complex matrices [14], the pseudo core inverse of $*$-ring elements [4] and the e-core inverse of $*$-ring elements [16]. Moreover, their characteristics and properties have been investigated which one can refer to [3, 23–25] and the references therein.

Recently, Zhu et al. [26] introduced a new type of generalized inverses, called the w-core inverse, extending Moore-Penrose inverses, core inverses and core-EP inverses. Jin et al. [9] gave some new characterizations on w-core inverses in a unital $*$-ring R. Yang and Zhu [22] established necessary and sufficient conditions for the existence of the w-core inverse of a regular element by units in a unital $*$-ring R and derived the existence criterion of the w-core inverse of the product of three elements, which were employed into 2×2 matrices over a ring as applications. Moreover, Zhu et al. [27] defined one-sided versions of 'w-core inverse', right w-core invertible, and gave several characterizations for this type of generalized inverses. They also presented the relationships among the right w-core inverses, right inverses along an element, right (b, c)-inverses and right annihilator (b, c)-inverses.

Motivated by these results, this paper mainly provides some novel methods to characterize w-core inverse in $*$-ring and apply its properties to construct the relationship among the group inverse, Moore-Penrose inverse and EP elements. The rest of this paper is organized as follows. In Section 2, we give some properties of w-core inverses and establish the characterizations of EP elements. Let $a \in R^{\#} \bigcap R^{+}$. It is shown that $a \in R^{EP}$ if and only if $a(a^{+})^* \in R^{EP}$ if and only if $a(a^{+})^*(a^{\#})^* a \in R^{EP}$. We also provide one new way to construct the group inverse of finite product of an element and its generalized inverses. In Section 3,

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several new methods are given to represent different w-core invertible elements by the combinations of $a, a^+, a^\#, a^*$ and $a^n (n \in N^+)$.

For the convenience of reader, let us now recall several basic notions of generalized inverses in a ring.

Let R be an associative ring and $a \in R$. If there exists $b \in R$ such that $a = aba$, then a is called a regular element, and b is called an inner inverse of a. Clearly, bab is also an inner inverse of a.

If there exists $a^\# \in R$ such that

$$aa^\# a = a,\ a^\# aa^\# = a^\#,\ aa^\# = a^\# a,$$

then a is called a group invertible element and $a^\#$ is called the group inverse of a [7, 11, 12], and it is uniquely determined by these equalities. We write $R^\#$ to denote the set of all group invertible elements of R.

If a map $* : R \to R$ satisfies

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^* a^* \text{ for all } a,\ b \in\ R,$$

then R is said to be an involution ring or a $*$−ring.

Let R be a $*$-ring and $a \in R$. If there exists $a^+ \in R$ such that

$$a = aa^+ a,\ a^+ = a^+ aa^+,\ (aa^+)^* = aa^+,\ (a^+ a)^* = a^+ a,$$

then a is called a Moore-Penrose invertible element, and a^+ is called the Moore-Penrose inverse of a [5, 6]. Let R^+ denote the set of all Moore-Penrose invertible elements of R.

If $a \in R^\# \cap R^+$ and $a^\# = a^+$, then a is called an EP element. On the studies of EP, the readers can refer to [2, 6, 8, 10, 13, 15, 17–19].

2. Properties of w-core inverses and constructions of group inverses

In this section, we mainly give some properties of w-core inverses and establish the characterizations of EP elements. To this end, we first propose the following definition.

Definition 2.1. *[26] Let R be a $*$-ring and $a, w \in R$. If there exists $x \in R$ such that*

$$x = awx^2,\ a = xawa,\ (awx)^* = awx,$$

then a is called w-core invertible and x is called the w-core inverse of a. Denote by $a_w^{\text{\textcircled{\#}}} = x$.

In particular, if a is a 1-core invertible element, then a is called core invertible [21] and x is called the core inverse of a and denote it by $a^{\text{\textcircled{\#}}}$. For example, if $a \in R^\# \cap R^+$, then it is easy to show the following results:

(1) $a_{a^\#}^{\text{\textcircled{\#}}} = aa^+$;

(2) $a^{\text{\textcircled{\#}}} = a^\# aa^+$;

(3) $(a^\#)^{\text{\textcircled{\#}}} = a^2 a^+$;

(4) $(a^+)^{\text{\textcircled{\#}}} = (aa^\#)^* a$;

(5) $(a^+)_{(a^\#)^*}^{\text{\textcircled{\#}}} = a^* a$;

(6) $(a^*)^{\text{\textcircled{\#}}} = (a^\#)^* a^+ a$;

(7) $(a^*)_{(a^\#)^*}^{\text{\textcircled{\#}}} = a^+ a$;

(8) $(a^*)_{(a^\#)^* a^+}^{\text{\textcircled{\#}}} = (aa^\#)^* a$.

Unless otherwise specified in the remaining sections of this article, we designate that R is a $*$-ring, $a, w, x \in R$ and that corresponding inverses exist.

First, it follows from [9, Lemma 2.1 and Corollary 2.2] that the following lemma is easily verified.

Lemma 2.2. *The following two statements are valid:*
(1) *If* $(aw)^{\textcircled{\#}} = x$, *then for any integer* $n \geq 0$, *we have*
$x^n(aw)^{n+1} = aw = awx^n(aw)^n = (aw)^n x^n aw;$
$(aw)^n x^{n+1} = x = x^n(aw)^n x;$
$(awx)^* = awx = (aw)^{n+1}x^{n+1};$
$xaw = x^{n+1}(aw)^{n+1}.$
(2) *If* $a_w^{\textcircled{\#}} = x$, *then* $(aw)^{\textcircled{\#}} = x$ *and* $awx^2 = x = xawx,\ xawa = a = awxa,\ (awx)^* = awx.$

Theorem 2.3. *If* $(aw)^{\textcircled{\#}} = x$, *then* $(aw)^n x \in R^{EP}$ *and* $((aw)^n x)^\# = ((aw)^n x)^+ = (aw)^2 x^{n+1}$ *for each integer* $n \geq 0$. *Especially,* $x \in R^{EP}$ *and* $x^\# = x^+ = (aw)^2 x$.

Proof. By Lemma 2.2, we have
$((aw)^n x)((aw)^2 x^{n+1}) = (aw)^n(x(aw)^2)x^{n+1} = (aw)^{n+1}x^{n+1} = awx;$
$((aw)^2 x^{n+1})((aw)^n x) = (aw)^2 x(x^n(aw)^n x) = (aw)^2 x^2 = awx;$
$((aw)^n x)((aw)^2 x^{n+1})((aw)^n x) = (awx)((aw)^n x) = ((aw)x(aw)^n)x = (aw)^n x;$
$((aw)^2 x^{n+1})((aw)^n x)((aw)^2 x^{n+1}) = awx((aw)^2 x^{n+1}) = (aw)^2 x^{n+1}.$
Noting that $(awx)^* = awx$. Then

$$(aw)^n x \in R^{EP} \text{ with } ((aw)^n x)^\# = ((aw)^n x)^+ = (aw)^2 x^{n+1}.$$

□

Theorem 2.4. *If* $(aw)^{\textcircled{\#}} = x$, *then* $x^n aw \in R^\#$ *and* $(x^n aw)^\# = x^2(aw)^{n+1}$ *for each integer* $n \geq 0$. *Especially,* $aw \in R^\#$ *with* $(aw)^\# = x^2 aw$.

Proof. By Lemma 2.2, one gets
$(x^n aw)(x^2(aw)^{n+1}) = x^n(awx^2)(aw)^{n+1} = x^{n+1}(aw)^{n+1} = xaw;$
$(x^2(aw)^{n+1})(x^n aw) = (x^2(aw)^3)(aw)^{n-2}x^n aw = (aw)^{n-1}x^n aw = xaw;$
$(x^n aw)(x^2(aw)^{n+1})(x^n aw) = (xaw)(x^n aw) = x^n aw;$
$(x^2(aw)^{n+1})(x^n aw)(x^2(aw)^{n+1}) = xaw(x^2(aw)^{n+1}) = x^2(aw)^{n+1}.$
Hence, $x^n aw \in R^\#$ and $(x^n aw)^\# = x^2(aw)^{n+1}$. □

Theorem 2.5. *Suppose that* $a \in R^\# \bigcap R^+$. *Then (1)* $a^{\textcircled{\#}}_{aa^\# a^* a^\# a} = (a^+)^* a^+;$
(2) $a^{\textcircled{\#}}_{(a^+)^*} = aa^\# a^* a^\# aa^+.$

Proof. By Definition 2.1, we easily check

$$aaa^\# a^* a^\# a(a^+)^* a^+ = aa^*(a^+)^* a^+ = aa^+ = (aa^+)^* = (aaa^\# a^* a^\# a(a^+)^* a^+)^*,$$

$$aaa^\# a^* a^\# a((a^+)^* a^+)^2 = (aa^+)(a^+)^* a^+ = (a^+)^* a^+,$$

and

$$((a^+)^* a^+)a(aa^\# a^* a^\# a)a = (a^+)^* a^* a = a.$$

These imply $a^{\textcircled{\#}}_{aa^\# a^* a^\# a} = (a^+)^* a^+$. The proof is completed.
Similarly, we can show (2). □

Combining Theorem 2.4 and Theorem 2.5, we can easily get the following corollary, which gives a new way to construct the group inverse of finite product of an element and its generalized inverses.

Corollary 2.6. *If* $a \in R^\# \bigcap R^+$, *then (1)* $(aa^* a^\# a)^\# = (a^+)^* a^\#;$
(2) $(a(a^+)^*)^\# = aa^\# a^* a^\#.$

Proof. (1) By Theorem 2.5(1), we obtain $a^{\textcircled{\#}}_{aa^{\#}a^{*}a^{\#}a} = (a^{+})^{*}a^{+}$. Then, it follows from Lemma 2.2(2) and Theorem 2.4 that $(aaa^{\#}a^{*}a^{\#}a)^{\#} = (aa^{*}a^{\#}a)^{\#} = ((a^{+})^{*}a^{+})^{2}aa^{*}a^{\#}a = (a^{+})^{*}a^{+}(a^{+})^{*}a^{+}aa^{*}a^{\#}a = (a^{+})^{*}a^{\#}$.

(2) By Theorem 2.5(2), we know that $a^{\textcircled{\#}}_{(a^{+})^{*}} = aa^{\#}a^{*}a^{\#}aa^{+}$. Then, applying Lemma 2.2(2) and Theorem 2.4, we have $(a(a^{+})^{*})^{\#} = (aa^{\#}a^{*}a^{\#})^{2}a(a^{+})^{*} = aa^{\#}a^{*}a^{\#}aa^{\#}a^{*}a^{\#}a(a^{+})^{*} = aa^{\#}a^{*}a^{\#}$. □

Clearly, if $a \in R^{\#} \bigcap R^{+}$, then $a^{+} = (a^{+}a)a^{\#}(aa^{+})$. Hence, Corollary 2.6 inspires us to give the following corollary.

Corollary 2.7. *If $a \in R^{\#} \bigcap R^{+}$, then (1) $(aa^{*}a^{\#}a)^{+} = a^{+}a(a^{+})^{*}a^{+}$;*
(2) $(a(a^{+})^{*})^{+} = a^{*}a^{\#}aa^{+}$.

Proof. (1) Due to $(aa^{*}a^{\#}a)^{+} = a^{+}a(aa^{*}a^{\#}a)^{\#}aa^{+}$ and Corollary 2.6(1), we know that

$$(aa^{*}a^{\#}a)^{+} = a^{+}a(a^{+})^{*}a^{\#}aa^{+} = a^{+}a(a^{+})^{*}a^{+}.$$

(2) Similarly, employing Corollary 2.6(2), we deduce that

$$(a(a^{+})^{*})^{+} = a^{+}a(a(a^{+})^{*})^{\#}aa^{+} = a^{+}a(aa^{\#}a^{*}a^{\#})aa^{+} = a^{*}a^{\#}aa^{+}.$$

□

Noting that $(a^{+}a(a^{+})^{*}a^{+})^{\#} = (aa^{\#})^{*}(a^{+}a(a^{+})^{*}a^{+})^{+}(aa^{\#})^{*}$ and $(a^{*}a^{\#}aa^{+})^{\#} = (aa^{\#})^{*}(a^{*}a^{\#}aa^{+})^{+}(aa^{\#})^{*}$. Then Corollary 2.7 induces the following corollary.

Corollary 2.8. *If $a \in R^{\#} \bigcap R^{+}$, then (1) $(a^{+}a(a^{+})^{*}a^{+})^{\#} = (aa^{\#})^{*}aa^{*}a^{\#}a(aa^{\#})^{*}$;*
(2) $(a^{*}a^{\#}aa^{+})^{\#} = (aa^{\#})^{*}a^{2}a^{+}(a^{\#})^{*}$.

Proof. (1) It follows from $(a^{+}a(a^{+})^{*}a^{+})^{\#} = (aa^{\#})^{*}(a^{+}a(a^{+})^{*}a^{+})^{+}(aa^{\#})^{*}$ and Corollary 2.7(1) that

$$(a^{+}a(a^{+})^{*}a^{+})^{\#} = (aa^{\#})^{*}(a^{+}a(a^{+})^{*}a^{+})^{+}(aa^{\#})^{*} = (aa^{\#})^{*}((aa^{*}a^{\#}a)^{+})^{+}(aa^{\#})^{*} = (aa^{\#})^{*}aa^{*}a^{\#}a(aa^{\#})^{*}$$

(2) According to $(a^{*}a^{\#}aa^{+})^{\#} = (aa^{\#})^{*}(a^{*}a^{\#}aa^{+})^{+}(aa^{\#})^{*}$ and Corollary 2.7(2), we can deduce that

$$\begin{aligned}(a^{*}a^{\#}aa^{+})^{\#} &= (aa^{\#})^{*}(a^{*}a^{\#}aa^{+})^{+}(aa^{\#})^{*} = (aa^{\#})^{*}((a(a^{+})^{*})^{+})^{+}(aa^{\#})^{*} = (aa^{\#})^{*}a(a^{+})^{*}(aa^{\#})^{*}\\ &= (aa^{\#})^{*}a(aa^{\#}a^{+})^{*} = (aa^{\#})^{*}a(a^{\#}aa^{+})^{*} = (aa^{\#})^{*}a(aa^{+})^{*}(a^{\#})^{*} = (aa^{\#})^{*}a^{2}a^{+}(a^{\#})^{*}.\end{aligned}$$

□

Theorem 2.9. *Let $a \in R^{\#} \bigcap R^{+}$. Then $a \in R^{EP}$ if and only if $a(a^{+})^{*} \in R^{EP}$.*

Proof. $\Rightarrow$ If $a \in R^{EP}$, then $aa^{\#} = a^{+}a$. By Corollary 2.6, we can get

$$(a(a^{+})^{*})^{\#} = aa^{\#}a^{*}a^{\#} = a^{+}aa^{*}a^{\#} = a^{*}a^{+}.$$

Applying Corollary 2.7, we have $(a(a^{+})^{*})^{+} = a^{*}a^{\#}aa^{+} = a^{*}a^{+}$. These imply $(a(a^{+})^{*})^{\#} = (a(a^{+})^{*})^{+}$. Hence, $a(a^{+})^{*} \in R^{EP}$.

$\Leftarrow$ If $a(a^{+})^{*} \in R^{EP}$, then $(a(a^{+})^{*})^{\#} = (a(a^{+})^{*})^{+}$. By Corollary 2.6 and Corollary 2.7, we have

$$aa^{\#}a^{*}a^{\#} = a^{*}a^{\#}aa^{+}.$$

Multiplying the equality on the left by $(a^{+})^{*}$, we obtain $a^{\#} = a^{\#}aa^{+}$. Thus $a \in R^{EP}$. □

Replacing a in the above theorem by $a(a^{+})^{*} \in R^{EP}$, the following corollary holds.

Corollary 2.10. *Let $a \in R^{\#} \bigcap R^{+}$. Then $a \in R^{EP}$ if and only if $a(a^{+})^{*}aa^{+}(a^{\#})^{*}a \in R^{EP}$.*

Proof. According to Theorem 2.9, we obtain $a \in R^{EP}$ if and only if $a(a^+)^* \in R^{EP}$. Thus $a \in R^{EP}$ if and only if $a(a^+)^*((a(a^+)^*)^+)^* \in R^{EP}$. It follows from Corollary 2.7 that $(a(a^+)^*)^+ = a^*a^\# aa^+$. Hence, $a(a^+)^*((a(a^+)^*)^+)^* = a(a^+)^*(a^*a^\# aa^+)^* = a(a^+)^*aa^+(a^\#)^*a \in R^{EP}$.

Therefore $a \in R^{EP}$ if and only if $a(a^+)^*aa^+(a^\#)^*a \in R^{EP}$. □

Lemma 2.11. *Let $a \in R^\# \bigcap R^+$. Then $(a(a^+)^*(a^\#)^*a)^+ = a^+a^*a^*a^\# aa^+$.*

Proof. Notice that

$$(a(a^+)^*(a^\#)^*a)(a^+a^*a^*a^\# aa^+) = a(a^+)^*(a^\#)^*a^*a^*a^\# aa^+ = aa^+$$
$$= (aa^+)^* = ((a(a^+)^*(a^\#)^*a)a^+a^*a^*a^\# aa^+)^*.$$

Similarly,

$$(a^+a^*a^*a^\# aa^+)(a(a^+)^*(a^\#)^*a) = a^+a = (a^+a)^* = ((a^+a^*a^*a^\# aa^+)(a(a^+)^*(a^\#)^*a))^*.$$

Then, we have

$$(a^+a^*a^*a^\# aa^+)(a(a^+)^*(a^\#)^*a)(a^+a^*a^*a^\# aa^+) = a^+a(a^+a^*a^*a^\# aa^+) = a^+a^*a^*a^\# aa^+$$

and

$$(a(a^+)^*(a^\#)^*a)(a^+a^*a^*a^\# aa^+)(a(a^+)^*(a^\#)^*a) = a(a^+)^*(a^\#)^*a.$$

Thus $(a(a^+)^*(a^\#)^*a)^+ = a^+a^*a^*a^\# aa^+$. □

It is well known that $a^\# = (a^\# a)a^+(aa^\#)$. Hence, Lemma 2.11 induces us to give the following corollary.

Corollary 2.12. *Let $a \in R^\# \bigcap R^+$. Then $(a(a^+)^*(a^\#)^*a)^\# = a^\# aa^+a^*a^*a^\#$.*

Proof. Since $a^\# = (a^\# a)a^+(aa^\#)$, we have

$$(a(a^+)^*(a^\#)^*a)^\# = (a^\# a)(a(a^+)^*(a^\#)^*a)^+(aa^\#).$$

According to Lemma 2.11, we directly deduce that

$$(a(a^+)^*(a^\#)^*a)^\# = (a^\# a)a^+a^*a^*a^\# aa^+(aa^\#) = a^\# aa^+a^*a^*a^\#.$$

□

Theorem 2.13. *Let $a \in R^\# \bigcap R^+$. Then $a \in R^{EP}$ if and only if $a(a^+)^*(a^\#)^*a \in R^{EP}$.*

Proof. ⇒ Assume that $a \in R^{EP}$. Then $a^+ = a^\#$. By Lemma 2.11 and Corollary 2.12, we have

$$(a(a^+)^*(a^\#)^*a)^+ = a^+a^*a^*a^\# aa^+ = a^+a^*a^*a^+,$$

and

$$(a(a^+)^*(a^\#)^*a)^\# = a^\# aa^+a^*a^*a^\# = a^+a^*a^*a^+,$$

which implies $(a(a^+)^*(a^\#)^*a)^\# = (a(a^+)^*(a^\#)^*a)^+$. Hence, $a(a^+)^*(a^\#)^*a \in R^{EP}$.

⇐ Assume that $a(a^+)^*(a^\#)^*a \in R^{EP}$. Then $(a(a^+)^*(a^\#)^*a)^\# = (a(a^+)^*(a^\#)^*a)^+$. According to Lemma 2.11 and Corollary 2.12, we have

$$a^+a^*a^*a^\# aa^+ = a^\# aa^+a^*a^*a^\#.$$

Multiplying the equality on the right by $a^2a^+((a^\#)^*)^2$, we have $a^+ = a^\# aa^+$ which implies $a \in R^{EP}$. □

Evidently, $a \in R^{EP}$ if and only if $a^\# \in R^{EP}$ if and only if $a^+ \in R^{EP}$. Hence, Theorem 2.13 includes the following corollary.

Corollary 2.14. *Let $a \in R^{\#} \bigcap R^{+}$. Then the following statements are equivalent:*
(1) $a \in R^{EP}$;
(2) $a^{+}a^{*}a^{*}a^{\#}aa^{+} \in R^{EP}$;
(3) $a^{\#}aa^{+}a^{*}a^{*}a^{\#} \in R^{EP}$.

Proof. (1) ⇒ (2) Since $a \in R^{EP}$, $a(a^{+})^{*}(a^{\#})^{*}a \in R^{EP}$ by Theorem 2.13, it follows that $(a(a^{+})^{*}(a^{\#})^{*}a)^{+} \in R^{EP}$. By Lemma 2.11, $a^{+}a^{*}a^{*}a^{\#}aa^{+} \in R^{EP}$.

(2) $\Longrightarrow$ (3) Assume that $a^{+}a^{*}a^{*}a^{\#}aa^{+} \in R^{EP}$. Then $(a^{+}a^{*}a^{*}a^{\#}aa^{+})^{+} \in R^{EP}$. By Lemma 2.11, one gets $a(a^{+})^{*}(a^{\#})^{*}a \in R^{EP}$. Hence, $(a(a^{+})^{*}(a^{\#})^{*}a)^{\#} \in R^{EP}$. By Corollary 2.12, one yields $a^{\#}aa^{+}a^{*}a^{*}a^{\#} \in R^{EP}$.

(3) ⇒ (1) Assume that $a^{\#}aa^{+}a^{*}a^{*}a^{\#} \in R^{EP}$. Then $(a^{\#}aa^{+}a^{*}a^{*}a^{\#})^{\#} = (a^{\#}aa^{+}a^{*}a^{*}a^{\#})^{+}$, it follows from Corollary 2.12 and $(a^{\#}aa^{+}a^{*}a^{*}a^{\#})^{+} = a^{+}a^{2}(a^{+})^{*}(a^{\#})^{*}a^{2}a^{+}$ that

$$a(a^{+})^{*}(a^{\#})^{*}a = a^{+}a^{2}(a^{+})^{*}(a^{\#})^{*}a^{2}a^{+}.$$

Multiplying the equality on the right by aa^{+}, we have

$$a(a^{+})^{*}(a^{\#})^{*}a = a(a^{+})^{*}(a^{\#})^{*}a^{2}a^{+}.$$

Multiplying the last equality on the left by $a^{+}a^{*}a^{*}a^{\#}$, we obtain $a^{+}a = a^{+}a^{2}a^{+}$. Thus $a \in R^{EP}$. □

Corollary 2.15. *Let $a \in R^{\#} \bigcap R^{+}$. Then (1) $(a^{+}a^{*})^{\#} = (a^{\#})^{*}a(aa^{\#})^{*}$;*
(2) $(a^{+}a^{*})^{+} = aa^{+}(a^{\#})^{*}a$;
(3) $(a^{+}a^{*}a)^{+} = a^{+}(a^{\#})^{*}a$.

Proof. (1) By Corollary 2.6, we have $(a(a^{+})^{*})^{\#} = aa^{\#}a^{*}a^{\#}$. Thus

$$((a(a^{+})^{*})^{\#})^{*} = (aa^{\#}a^{*}a^{\#})^{*} = (a^{\#})^{*}a(aa^{\#})^{*},$$

and

$$((a(a^{+})^{*})^{\#})^{*} = ((a(a^{+})^{*})^{*})^{\#} = (a^{+}a^{*})^{\#}.$$

Therefore $(a^{+}a^{*})^{\#} = (a^{\#})^{*}a(aa^{\#})^{*}$.

(2) Noting that $(a^{+}a^{*})^{+} = aa^{+}(a^{+}a^{*})^{\#}a^{+}a$. Then, by (1), we have

$$(a^{+}a^{*})^{+} = aa^{+}(a^{\#})^{*}a(aa^{\#})^{*}a^{+}a = aa^{+}(a^{\#})^{*}a.$$

(3) By (2), $(a^{+}a^{*})(aa^{+}(a^{\#})^{*}a) = a^{+}a = (a^{+}a^{*}a)(a^{+}(a^{\#})^{*}a)$. Hence, $(a^{+}a^{*}a)^{+} = a^{+}(a^{\#})^{*}a$. □

Noting that

$$(a^{\#})^{*}a(aa^{\#})^{*} = (a^{\#})^{*}a(a^{\#})^{*}a^{*} = (a^{\#})^{*}a(a^{\#})^{*}aa^{+}a^{*} = ((a^{\#})^{*}a)^{2}a^{+}a^{*},$$

and

$$(a^{+}a^{*}a)^{\#} = (a^{+}a^{*}a)^{+} = a^{+}(a^{\#})^{*}a = (a^{+}(a^{\#})^{*}a)a^{+}a = (a^{+}(a^{\#})^{*}a)(a^{+}(a^{\#})^{*}a^{*})a = (a^{+}(a^{\#})^{*}a)(a^{+}(a^{\#})^{*}a)(a^{+}a^{*}a).$$

Then Theorem 2.4 and Corollary 2.15 inspires us to give the following corollary, which proof is routine.

Corollary 2.16. *Let $a \in R^{\#} \bigcap R^{+}$. Then (1) $(a^{+})^{\text{\textcircled{\#}}}_{a^{*}} = (a^{\#})^{*}a$;*
(2) $(a^{+})^{\text{\textcircled{\#}}}_{a^{*}a} = a^{+}(a^{\#})^{*}a$.

It is easy to show the following lemma.

Lemma 2.17. *Let w_1 and $w_2 \in R$. If $a^{\text{\textcircled{\#}}}_{w_1w_2} = x$, then $(aw_1)^{\text{\textcircled{\#}}}_{w_2} = x$.*

Corollary 2.16 and Lemma 2.17 lead to the following corollary.

Corollary 2.18. *Let $a \in R^{\#} \bigcap R^{+}$. Then (1) $(a^{+}a^{+}a)_{a^*}^{\#\!\!\!\bigcirc} = (a^{\#})^* a$;*
(2) $(a^{+}a^{*})_{a}^{\#\!\!\!\bigcirc} = a^{+}(a^{\#})^* a$.

Corollary 2.19. *Let $a \in R^{\#} \bigcap R^{+}$. Then $(a^*)_{a^{\#}}^{\#\!\!\!\bigcirc} = a^{+}a^2(a^{+})^*$.*

Proof. According to Definition 2.1, we can verify directly

$$a^*a^{\#}a^{+}a^2(a^{+})^* = a^*a^{\#}a(a^{+})^* = a^{+}a = (a^{+}a)^* = (a^*a^{\#}a^{+}a^2(a^{+})^*)^*,$$

$$a^*a^{\#}(a^{+}a^2(a^{+})^*)^2 = a^{+}a(a^{+}a^2(a^{+})^*) = a^{+}a^2(a^{+})^*$$

and

$$a^{+}a^2(a^{+})^*a^*a^{\#}a^* = a^{+}aa^* = a^*.$$

Hence, we have $(a^*)_{a^{\#}}^{\#\!\!\!\bigcirc} = a^{+}a^2(a^{+})^*$. □

By Corollary 2.19, we have $a^*a^{\#}a^{+}a^2(a^{+})^* = a^*a^{\#}a(a^{+})^* = a^*a^{\#}a(aa^{+}(a^{+})^*)$. This gives us the following enlightenms.

Corollary 2.20. *Let $a \in R^{\#} \bigcap R^{+}$. The following statements are valid.*
(1) $(a^*)_{aa^{\#}}^{\#\!\!\!\bigcirc} = a^{+}a(a^{+})^*$;
(2) $(a^*aa^{\#})^{\#} = (a^*aa^{\#})^{+} = (a^{+}a(a^{+})^*)^2a^*aa^{\#} = a^{+}a(a^{+})^*$;
(3) $(a^{+})_{a(a^{+})^*}^{\#\!\!\!\bigcirc} = a^*aa^{\#}$;
(4) $(a^{+}a)_{(a^{+})^*}^{\#\!\!\!\bigcirc} = a^*aa^{\#}$;
(5) $(a^*a)_{a^{\#}}^{\#\!\!\!\bigcirc} = a^{+}a(a^{+})^*$.

Proof. (1) According to Definition 2.1, we can verify directly

$$a^*aa^{\#}a^{+}a(a^{+})^* = a^*aa^{\#}(a^{+})^* = a^{+}a = (a^{+}a)^* = (a^*aa^{\#}a^{+}a(a^{+})^*)^*,$$

$$a^*aa^{\#}(a^{+}a(a^{+})^*)^2 = a^{+}a(a^{+}a(a^{+})^*) = a^{+}a(a^{+})^*$$

and

$$a^{+}a(a^{+})^*a^*aa^{\#}a^* = a^{+}aa^* = a^*.$$

Hence, we have $(a^*)_{aa^{\#}}^{\#\!\!\!\bigcirc} = a^{+}a(a^{+})^*$.

(2) If $(a^*)_{aa^{\#}}^{\#\!\!\!\bigcirc} = a^{+}a(a^{+})^*$, then it follows from Lemma 2.2(2) and Theorem 2.4 that

$$(a^*aa^{\#})^{\#} = (a^{+}a(a^{+})^*)^2a^*aa^{\#} = a^{+}a(a^{+})^*a^{+}a = a^{+}a(a^{+})^*.$$

Since $(a^*aa^{\#})^{\#} = a^{+}a(a^{+})^*$, we get

$$a^*aa^{\#}(a^{+}a(a^{+})^*) = a^*(a^{+})^* = a^{+}a = (a^{+}a)^* = (a^*aa^{\#}(a^{+}a(a^{+})^*))^*,$$

and then $(a^{+}a(a^{+})^*)a^*aa^{\#} = a^{+}a = (a^{+}a)^* = ((a^{+}a(a^{+})^*)a^*aa^{\#})^*$. That is $(a^*aa^{\#})^{+} = a^{+}a(a^{+})^*$.

(3) Similarly, employing Definition 2.1, we obtain

$$a^{+}a(a^{+})^*a^*aa^{\#} = a^{+}aaa^{+}aa^{\#} = a^{+}a = (a^{+}a)^* = (a^{+}a(a^{+})^*a^*aa^{\#})^*,$$

$$a^{+}a(a^{+})^*(a^*aa^{\#})^2 = a^{+}a(a^*aa^{\#}) = a^*aa^{\#}$$

and

$$a^*aa^{\#}a^{+}a(a^{+})^*a^{+} = a^*(a^{+})^*a^{+} = a^{+}.$$

Hence, $(a^+)^{\textcircled{\#}}_{a(a^+)^*} = a^*aa^\#$.

(4) According to Definition 2.1, it is easy to check that

$$a^+a(a^+)^*a^*aa^\# = a^+aaa^+aa^\# = a^+a = (a^+a)^* = (a^+a(a^+)^*a^*aa^\#)^*,$$

$$a^+a(a^+)^*(a^*aa^\#)^2 = a^+a(a^*aa^\#) = a^*aa^\#$$

and

$$a^*aa^\#a^+a(a^+)^*a^+a = a^*aa^\#(a^+)^*a^+a = a^+a.$$

Hence, $(a^+a)^{\textcircled{\#}}_{(a^+)^*} = a^*aa^\#$.

(5) Similar to the arguments in (1), (3) and (4), we get

$$a^*aa^\#a^+a(a^+)^* = a^*aa^\#(a^+)^* = a^*(a^+)^* = a^+a = (a^+a)^* = (a^*aa^\#a^+a(a^+)^*)^*,$$

$$a^*aa^\#(a^+a(a^+)^*)^2 = a^+a(a^+a(a^+)^*) = a^+a(a^+)^*$$

and

$$a^+a(a^+)^*a^*aa^\#a^*a = a^+aa^*a = a^*a.$$

Hence, $(a^*a)^{\textcircled{\#}}_{a^\#} = a^+a(a^+)^*$. □

3. Constructions of w-core invertible elements

In this section, we mainly provide some new methods to construct different w-core invertible elements.

Theorem 3.1. *Let $a^{\textcircled{\#}}_w = x$. Then $(awa)^{\textcircled{\#}}_{wx} = x$.*

Proof. Since $a^{\textcircled{\#}}_w = x$, $x = awx^2 = xawx$, $a = xawa = awxa$, $(awx)^* = awx$. Then we have

$$awawx \cdot x^2 = aw(awx^2)x = awx^2 = x,$$

$$x(awa)wx(awa) = (xawa)wa = awa,$$

and

$$(awa)(wx)x = aw(awx^2) = awx = (awx)^* = ((awa)(wx)x)^*.$$

Hence, $(awa)^{\textcircled{\#}}_{wx} = x$. □

Corollary 3.2. *Let $a \in R^\# \bigcap R^+$. Then $(aa^*a)^{\textcircled{\#}}_{a^+} = (a^+)^*a^+$.*

Proof. By Definition 2.1, we know that

$$\begin{aligned}
&aa^*(a^+)^*a^+ = aa^+ = (aa^+)^* = (aa^*(a^+)^*a^+)^*,\\
&aa^*(a^+)^*a^+(a^+)^*a^+ = aa^+(a^+)^*a^+ = (a^+)^*a^+,\\
&(a^+)^*a^+aa^*(a^+)^*a^+ = (a^+)^*a^*a = a,
\end{aligned}$$

which implies that $a^{\textcircled{\#}}_{a^*} = (a^+)^*a^+$. Then we have $(aa^*a)^{\textcircled{\#}}_{a^*(a^+)^*a^+} = (a^+)^*a^+$ by Theorem 3.1, and then $(aa^*a)^{\textcircled{\#}}_{a^+} = (a^+)^*a^+$. □

Corollary 3.3. *Let $a \in R^\# \bigcap R^+$. Then*

(1) $(a(a^+)^*a)^{\textcircled{\#}}_{a^\#aa^+} = aa^\#a^*a^\#aa^+$;

(2) $(a(a^+)^*aa^+)^\# = (a(a^+)^*a(a^\#aa^+))^\# = aa^\#a^*a^\#aa^+$.

Proof. (1) Since $a^{\textcircled{\#}}_{(a^+)^*} = aa^\# a^* a^\# aa^+$ by Theorem 2.5, it follows that $(a(a^+)^*a)^{\textcircled{\#}}_{(a^+)^*aa^\# a^* a^\# aa^+} = aa^\# a^* a^\# aa^+$ by Theorem 3.1, and then $(a(a^+)^*a)^{\textcircled{\#}}_{a^\# aa^+} = aa^\# a^* a^\# aa^+$.

(2) Since $(a(a^+)^*a)^{\textcircled{\#}}_{a^\# aa^+} = aa^\# a^* a^\# aa^+$, it follows from Lemma 2.2(2) and Theorem 2.4 that

$$(a(a^+)^*aa^\# aa^+)^\# = (aa^\# a^* a^\# aa^+)^2 a(a^+)^* a(a^\# aa^+).$$

Hence, $(a(a^+)^*aa^+)^\# = (aa^\# a^* a^\# aa^+)(aa^\# a^* a^\# aa^+)a(a^+)^* a(a^\# aa^+) = aa^\# a^* a^\# aa^+$. □

Theorem 3.4. *Let $a^{\textcircled{\#}}_w = x$. Then $a^{\textcircled{\#}}_{wx} = awx$.*

Proof. From $a^{\textcircled{\#}}_w = x$, we have $a = xawa = awxa$, $x = awx^2 = xawx$ and $(awx)^* = awx$. It follows from Definition 2.1 that

$$awx(awx) = (awxa)wx = awx = (awx)^* = (awx(awx))^*,$$

$$awx(awx)^2 = awx(awx)(awx) = awx(awx) = awx$$

and

$$(awx)a(wx)a = awxa = a.$$

Hence, $a^{\textcircled{\#}}_{wx} = awx$. □

Corollary 3.5. *Let $a \in R^\# \bigcap R^+$. Then $a^{\textcircled{\#}}_{a^\# aa^+} = aa^+$.*

Proof. By Theorem 2.5, we know $a^{\textcircled{\#}}_{(a^+)^*} = aa^\# a^* a^\# aa^+$. Thus, by Theorem 3.4, we have

$$a^{\textcircled{\#}}_{(a^+)^*aa^\# a^* a^\# aa^+} = a(a^+)^*aa^\# a^* a^\# aa^+.$$

Since $(a^+)^*aa^\# a^* a^\# aa^+ = (a^+)^* a^* a^\# aa^+ = a^\# aa^+$ and

$$a(a^+)^*aa^\# a^* a^\# aa^+ = a(a^+)^* a^* a^\# aa^+ = aa^+,$$

which implies $a^{\textcircled{\#}}_{a^\# aa^+} = aa^+$. □

Theorem 3.6. *Let $a^{\textcircled{\#}}_w = x$. Then $(aw)^{\textcircled{\#}}_x = awx$.*

Proof. Since $a^{\textcircled{\#}}_w = x$, we have $awx^2 = x = xawx$, $xawa = a = awxa$ and $(awx)^* = awx$.

By Definition 2.1, we have

$$awx(awx) = awx = (awx)^* = (awx(awx))^*,$$

$$awx(awx)^2 = awx(awx)(awx) = awxawx = awx,$$

and

$$(awx)(aw)xaw = awxaw = aw.$$

Hence, $(aw)^{\textcircled{\#}}_x = awx$. □

Corollary 3.7. *Let $a \in R^+$ and $a^{\textcircled{\#}}_w = x$. Then $(aw)^{\textcircled{\#}}_x = aa^+$.*

Proof. Since $a^{\textcircled{\#}}_w = x$, we have $awx^2 = x = xawx$, $xawa = a = awxa$ and $(awx)^* = awx$. Thus, $(awxaa^+)^* = (aa^+)^*(awx)^* = aa^+awx = awx = (awx)^* = ((awxaa^+)^*)^* = awxaa^+$, $(awx(aa^+)^2) = awxaa^+ = aa^+$ and $aa^+awxaw = awxaw = aw$. That is $(aw)^{\textcircled{\#}}_x = aa^+$. □

Theorem 3.8. *Let $a \in R^{\#} \bigcap R^{+}$. Then*
(1) $a^{\textcircled{\#}}_{a^*a} = a^{\#}(a^{+})^{*}a^{+}$;
(2) $(aa^*a)^{\textcircled{\#}}_{a^{\#}(a^{+})^{*}a^{+}} = aa^{+}$;
(3) $(aa^*a)^{\#} = a^{\#}(a^{+})^{*}a^{+}a^{\#}a$;
(4) $(aa^*a)^{+} = a^{+}aa^{\#}(a^{+})^{*}a^{+}a^{\#}aaa^{+} = a^{+}(a^{+})^{*}a^{+}$;
(5) $(a^{\#}(a^{+})^{*}a^{+})^{\#} = (a^{\#}(a^{+})^{*}a^{+})^{+} = aa^*a^2a^{+}$.

Proof. (1) It follows from Definition 2.1 that

$$aa^*aa^{\#}(a^{+})^{*}a^{+} = aa^*(a^{+})^{*}a^{+} = aa^{+} = (aa^{+})^{*} = (aa^*aa^{\#}(a^{+})^{*}a^{+})^{*},$$

$$aa^*a(a^{\#}(a^{+})^{*}a^{+})^2 = aa^{+}(a^{\#}(a^{+})^{*}a^{+}) = a^{\#}(a^{+})^{*}a^{+}$$

and

$$a^{\#}(a^{+})^{*}a^{+}aa^*aa = a^{\#}(a^{+})^{*}a^*a^2 = a.$$

Thus, $a^{\textcircled{\#}}_{a^*a} = a^{\#}(a^{+})^{*}a^{+}$.

(2) Combining $a^{\textcircled{\#}}_{a^*a} = a^{\#}(a^{+})^{*}a^{+}$ with Theorem 3.6, we directly deduce that

$$(aa^*a)^{\textcircled{\#}}_{a^{\#}(a^{+})^{*}a^{+}} = aa^*aa^{\#}(a^{+})^{*}a^{+} = aa^*(a^{+})^{*}a^{+} = aa^{+}.$$

(3) Since $a^{\textcircled{\#}}_{a^*a} = a^{\#}(a^{+})^{*}a^{+}$, it follows from Lemma 2.2(2) that $(aa^*a)^{\#} = a^{\#}(a^{+})^{*}a^{+}$. By Theorem 2.4, we obtain $(aa^*a)^{\#} = (a^{\#}(a^{+})^{*}a^{+})^2aa^*a = a^{\#}(a^{+})^{*}a^{+}a^{\#}a$.

(4) It is well known that $a^{+} = a^{+}aa^{\#}aa^{+}$. Then we have $(aa^*a)^{+} = a^{+}a(aa^*a)^{\#}aa^{+}$. By (3), we obtain $(aa^*a)^{+} = a^{+}a(a^{\#}(a^{+})^{*}a^{+}a^{\#}a)aa^{+} = a^{+}(a^{+})^{*}a^{+}$.

(5) By (1) and Lemma 2.2(2), we know $(aa^*a)^{\#} = a^{\#}(a^{+})^{*}a^{+}$. Then according to Theorem 2.3, we get

$$(a^{\#}(a^{+})^{*}a^{+})^{\#} = (a^{\#}(a^{+})^{*}a^{+})^{+} = aa^*a(aa^*a)a^{\#}(a^{+})^{*}a^{+} = aa^*a^2a^{+}.$$

□

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