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Generalized Drazin-g-meromorphic spectrum for operator matrices

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Abstract. For $C \in \mathcal{L}(K,H)$, $B \in \mathcal{L}(K)$ and $A \in \mathcal{L}(H)$, let M_C be the operator matrix defined on $H \oplus K$ by $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, whereas K and H are complex Hilbert spaces. In this paper, we demonstrate that $\sigma(M_C) = \sigma(B) \cup \sigma(A)$ is equivalent to

$$\sigma_{gD(g\mathcal{M})}(M_C) = \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)$$

whereas $\sigma_{gD(g\mathcal{M})}(.)$ is the generalized Drazin-g-meromorphic spectrum [9]. Also, we used the local spectral theory to give a sufficient condition to have the last equality.

1. Introduction

Let K and H denote infinite dimensional complex Hilbert spaces and $\mathcal{L}(H,K)$ denotes the set of all linear bounded operators from H into K. We write $\mathcal{L}(H)$ instead of $\mathcal{L}(H,H)$, when $H = K$. Let $A \in \mathcal{L}(H)$, we denote by $\sigma(A)$, $\sigma_{ap}(A)$, $\sigma_{su}(A)$, A^*, the spectrum, the approximate point spectrum, the surjective spectrum and the adjoint operator of A.

Remember that an operator $A \in \mathcal{L}(X)$ is said to possess the single valued extension property (SVEP for short) at λ if there exists V an open neighborhood of λ such that for any open subset $W \subseteq V$ the only analytic solution of the equation $(A-\mu)f(\mu) = 0$ for all $\mu \in W$ is the function $f \equiv 0$. Let $S(A)$ be the set of all $\lambda \in \mathbb{C}$ such that A does not admit the SVEP at λ. Evidently, if $T - \lambda$ possesses the SVEP at 0, then T possesses the SVEP at λ (See [1]). A is said to possess the SVEP if A possesses the SVEP at all $\lambda \in \mathbb{C}$, in this particular situation $S(A) = \emptyset$. Note that $\sigma(A) = S(A) \cup \sigma_{su}(A)$.

In the Drazin sense, $A \in \mathcal{L}(H)$ is invertible if we can find $B \in \mathcal{L}(H)$ such that

$$AB = BA,\ B^2A = B \text{ and } BA^2 - B \text{ is nilpotent.}$$

A generalization of this concept is given by J.J. Koliha [6], in fact $A \in \mathcal{L}(H)$ is called Koliha-Drazin invertible (or generalized Drazin invertible) if there exists $B \in \mathcal{L}(H)$ such that

$$AB = BA,\ B^2A = B \text{ and } BA^2 - B \text{ is quasinilpotent,}$$

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which is equivalent to $0 \notin acc\sigma(A)$.

The Drazin spectrum and the generalized Drazin spectrum are defined, respectively, by

$$\sigma_D(A) = \{\lambda \in \mathbb{C}, A - \lambda I \text{ is not Drazin invertible }\},$$
$$\sigma_{gD}(A) = \{\lambda \in \mathbb{C}, A - \lambda I \text{ is not Koliha-Drazin invertible}\},$$

with $\rho_D(A) = \mathbb{C} \setminus \sigma_D(A)$ and $\rho_{gD}(A) = \mathbb{C} \setminus \sigma_{gD}(A)$.

Keep in mind that an operator $A \in \mathcal{L}(H)$ is supposedly g-meromorphic if every non-zero point of its spectrum is an isolated point ($A \in (g\mathcal{M})$ for short) which is equivalent to $\sigma_{gD}(A) \subseteq \{0\}$ [9].

Recently, S. Č Živković-Zlatanović [9] presented and studied a new extended inverse concept to expand the Koliha-Drazin idea to "generalized Drazin-g-meromorphic invertible". In fact, $A \in \mathcal{L}(H)$ is said to be generalized Drazin-g-meromorphic invertible if there exists $B \in \mathcal{L}(H)$ such that

$$AB = BA, \ BAB = B \text{ and } A^2B - A \text{ is } g\text{-meromorphic}.$$

The generalized Drazin-g-meromorphic spectrum is defined by

$$\sigma_{gD(g\mathcal{M})}(T) = \{\lambda \in \mathbb{C}, A - \lambda I \text{ is not generalized Drazin-}g\text{-meromorphic invertible }\},$$

and we write $\rho_{gD(g\mathcal{M})}(A) = \mathbb{C} \setminus \sigma_{gD(g\mathcal{M})}(A)$.

An interesting characterization of this class is given by the following lemma.

Lemma 1.1. *[9] Let $A \in \mathcal{L}(H)$. The following statements are equivalent.*

1. *A is generalized Drazin-g-meromorphic invertible.*
2. *$0 \notin acc\sigma_{gD}(A)$.*

Let E be a compact subset of $\mathbb{C}$, we denote by $accE$, $isoE$, ∂E, $\eta(E)$ and E^c be the accumulation points of E, the isolated points of E, the boundary of E, the polynomially convex hull and the complement of E, respectively.

In the last two decades an extensive study of 2×2 upper triangular operator matrices has been carried out. The research was primarily motivated by the following fact: If $T \in \mathcal{L}(H)$ and F is closed, complemented and T-invariant subspace of H, then T may be expressed as

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : F \oplus F^{\perp} \longrightarrow F \oplus F^{\perp}.$$

Throughout the remainder of this paper, $(A, B) \in \mathcal{L}(H) \times \mathcal{L}(K)$ and $C \in \mathcal{L}(K, H)$. The upper triangular operator matrix $M_C \in \mathcal{L}(H \oplus K)$ represents a bounded linear operator on the Hilbert space $H \oplus K$ given by:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

When it comes to infinite dimensional spaces, H. K. Du and J. Pan [4] showed that the inclusion $\sigma(M_C) \subset \sigma(B) \cup \sigma(A)$ may be strict. A few years later other authors [5] were able to prove the following theorem.

Theorem 1.2. *[5] Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$. For all $C \in \mathcal{L}(K, H)$, we have*

$$\sigma(M_C) \cup W = \sigma(B) \cup \sigma(A), \tag{1}$$

where W is the union of certain holes in $\sigma(M_C)$ such that $W \subseteq \sigma(B) \cap \sigma(A)$.

Subsequently, several mathematicians have generalized this result for other spectra. As examples we have the following two results:

Theorem 1.3. *[12] Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$. For all $C \in \mathcal{L}(K, H)$, we have*

$$\sigma_D(M_C) \cup W_D = \sigma_D(B) \cup \sigma_D(A), \tag{2}$$

where W_D is the union of certain holes in $\sigma_D(M_C)$ and $W_D \subseteq \sigma_D(A) \cap \sigma_D(B)$.

Theorem 1.4. *[11] Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$. For all $C \in \mathcal{L}(K, H)$, we have*

$$\sigma_{gD}(M_C) \cup W_{gD} = \sigma_{gD}(B) \cup \sigma_{gD}(A), \tag{3}$$

where W_{gD} is the union of certain holes in $\sigma_{gD}(M_C)$ and $W_{gD} \subseteq \sigma_{gD}(B) \cap \sigma_{gD}(A)$.

Generally, there are many research papers that have studied this type of operator matrices, including [3], [8], [2], and [7].

In this paper, we prove the following hole-filling property:

$$\sigma_{gD(g\mathcal{M})}(M_C) \cup W_{gD(g\mathcal{M})} = \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B),$$

where $W_{gD(g\mathcal{M})}$ is the union of certain holes in $\sigma_{gD(g\mathcal{M})}(M_C)$ which happen to be subsets of $\sigma_{gD(g\mathcal{M})}(A) \cap \sigma_{gD(g\mathcal{M})}(B)$. Also, we will give a sufficient condition, related to the SVEP, to have the equality

$$\sigma_{gD(g\mathcal{M})}(M_C) = \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B).$$

2. Main results

The following lemma is important and it is widely used in the proofs of our main results.

Lemma 2.1. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, and $C \in \mathcal{L}(K, H)$. The following statements hold:*

1. *Operators M_C, A and B are all invertible if any two of them are.*
2. *Operators M_C, A and B are all generalized Drazin invertible if any two of them are.*
3. *Operators M_C, A and B are all generalized Drazin-g-meromorphic invertible if any two of them are.*

Proof. For (1), see [5, Theorem 2], and for (2), see [11, Lemma 2.4].

(3): It suffices to show that M_C and A are generalized Drazin-g-meromorphic invertible implies that B is generalized Drazin-g-meromorphic invertible. If M_C and A are generalized Drazin-g-meromorphic invertible, that is $0 \notin acc\sigma_{gD}(M_C)$ and $0 \notin acc\sigma_{gD}(A)$, then there exists $\epsilon > 0$ such that $M_C - \lambda I$ and $A - \lambda I$ are generalized Drazin invertible for all $\lambda, 0 < |\lambda| < \epsilon$. By (*ii*), we have that $B - \lambda I$ is generalized Drazin invertible for all $\lambda, 0 < |\lambda| < \epsilon$. Thus, $0 \notin acc\sigma_{gD}(B)$. So, B is generalized Drazin-g-meromorphic invertible. □

Lemma 2.2. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$. For all $C \in \mathcal{L}(K, H)$, we have*

$$\sigma_{gD(g\mathcal{M})}(M_C) \subseteq \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B).$$

Proof. Without loss of generality let $0 \notin \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B)$, then there exists $\varepsilon > 0$ such that $B - \lambda I$ and $A - \lambda I$ are generalized Drazin invertible for any λ, $0 < |\lambda| < \varepsilon$. According to Lemma 2.1, $M_C - \lambda I$ is generalized Drazin invertible for any λ, $0 < |\lambda| < \varepsilon$. Hence $0 \notin \sigma_{gD(g\mathcal{M})}(M_C)$. □

The inclusion $\sigma_{gD(g\mathcal{M})}(M_C) \subset \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B)$ may be strict as shown in the following example.

Example 2.3. *(Cf. [4, Example 3]) Let $A, B, C \in \mathcal{L}(l^2(\mathbb{N}))$ be defined by:*

$$Ae_n = e_{n+1} \text{ for all } n \in \mathbb{N},$$

$$B = A^*, \text{ and}$$

$$Cx = < x, e_0 > e_0 \text{ for all } x \in l^2(\mathbb{N}),$$

whereas $\{e_n\}_{n\in\mathbb{N}}$ is the orthonormal basis of $l^2(\mathbb{N})$. We have $\sigma_{gD(g\mathcal{M})}(A) = \{\lambda \in \mathbb{C}; |\lambda| \leq 1\}$. M_C is unitary, so $\sigma_{gD(g\mathcal{M})}(M_C) \subseteq \{\mu \in \mathbb{C}; |\mu| = 1\}$. Then $0 \notin \sigma_{gD(g\mathcal{M})}(M_C)$, nonetheless $0 \in \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B)$.

We can now present our first main result.

Theorem 2.4. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$. For all $C \in \mathcal{L}(K, H)$, we have*

$$\sigma_{gD(g\mathcal{M})}(M_C) \cup W_{gD(g\mathcal{M})} = \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B),$$

where $W_{gD(g\mathcal{M})}$ is the union of certain holes in $\sigma_{gD(g\mathcal{M})}(M_C)$ and $W_{gD(g\mathcal{M})} \subseteq \sigma_{gD(g\mathcal{M})}(A) \cap \sigma_{gD(g\mathcal{M})}(B)$.

Proof. We have

$$(\sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)) = \sigma_{gD(g\mathcal{M})}(M_C) \cup \{\sigma_{gD(g\mathcal{M})}(B) \cap \sigma_{gD(g\mathcal{M})}(A)\}. \tag{4}$$

Indeed,

$$\sigma_{gD(g\mathcal{M})}(M_C) \cup \{\sigma_{gD(g\mathcal{M})}(A) \cap \sigma_{gD(g\mathcal{M})}(B)\} \subseteq (\sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B))$$

holds for every $C \in \mathcal{L}(K, H)$. Now, we have

$$\begin{aligned}
&\lambda \notin \sigma_{gD(g\mathcal{M})}(M_C) \cup \{\sigma_{gD(g\mathcal{M})}(B) \cap \sigma_{gD(g\mathcal{M})}(A)\} \\
\Longleftrightarrow\ &\lambda \in \{\rho_{gD(g\mathcal{M})}(M_C) \cap \rho_{gD(g\mathcal{M})}(A)\} \text{ or } \lambda \in \{\rho_{gD(g\mathcal{M})}(M_C) \cap \rho_{gD(g\mathcal{M})}(B)\}, \\
\Longleftrightarrow\ &\lambda \in \rho_{gD(g\mathcal{M})}(A) \text{ and } \lambda \in \rho_{gD(g\mathcal{M})}(B), \quad \text{(Lemma 2.1)} \\
\Longleftrightarrow\ &\lambda \notin \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B),
\end{aligned}$$

which give the opposite inclusion.

According to [11, Theorem 2.1], we have

$$\eta(\sigma_{gD}(A) \cup \sigma_{gD}(B)) = \eta(\sigma_{gD}(M_C)).$$

From [8, Lemma 2.5], we have

$$\eta(\sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B)) = \eta(\sigma_{gD(g\mathcal{M})}(M_C)). \tag{5}$$

Therefore (5) says that the passage from $\sigma_{gD(g\mathcal{M})}(M_C)$ to $\sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)$ is the filling in certain of the holes in $\sigma_{gD(g\mathcal{M})}(M_C)$. Moreover, equality (4) ensures that

$$(\sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)) \backslash \sigma_{gD(g\mathcal{M})}(M_C) \subset \sigma_{gD(g\mathcal{M})}(A) \cap \sigma_{gD(g\mathcal{M})}(B).$$

It follows that the filling in certain of the holes in $\sigma_{gD(g\mathcal{M})}(M_C)$ should occur in $\sigma_{gD(g\mathcal{M})}(B) \cap \sigma_{gD(g\mathcal{M})}(A)$. □

The following two results are immediately obtained from Theorem 2.4.

Corollary 2.5. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, and $C \in \mathcal{L}(K, H)$. If the interior of $\sigma_{gD(g\mathcal{M})}(B) \cap \sigma_{gD(g\mathcal{M})}(A)$ is empty, then we have*

$$\sigma_{gD(g\mathcal{M})}(M_C) = \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A), \textit{ for every } C \in \mathcal{L}(K, H). \tag{6}$$

Theorem 2.6. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, and $C \in \mathcal{L}(K, H)$. The two statements that follow are equivalent,*

1. $\sigma(M_C) = \sigma(B) \cup \sigma(A)$,
2. $\sigma_{gD(g\mathcal{M})}(M_C) = \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)$.

Proof. First, we show that $W \subseteq W_{gD(g\mathcal{M})}$. Indeed, let $\lambda \in W$ then $\lambda \in \sigma(A) \cup \sigma(B)$ and $\lambda \notin \sigma(M_C)$. So, $\lambda \notin \sigma_{gD(g\mathcal{M})}(M_C)$. Suppose that $\lambda \notin \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B) = acc(\sigma_{gD}(A) \cup \sigma_{gD}(B))$, that is

$$\lambda \in iso(\sigma_{gD}(B) \cup \sigma_{gD}(A)) \cup (\rho_{gD}(B) \cap \rho_{gD}(A)).$$

-If $\lambda \in \rho_{gD}(B) \cap \rho_{gD}(A)$. Since $\lambda \in \sigma(B) \cup \sigma(A)$, it is not difficult to see that $\lambda \in iso(\sigma(B) \cup \sigma(A)) \subseteq iso\sigma(A) \cup iso\sigma(B)$. Hence $\lambda \in \partial\sigma(B) \cup \partial\sigma(A) \subseteq \sigma_{ap}(B) \cup \sigma_{su}(A) \subseteq \sigma(M_C)$ which is absurd.

-If $\lambda \in iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))$, then

$$\begin{aligned} iso(\sigma_{gD}(A) \cup \sigma_{gD}(B)) &\subseteq iso\sigma_{gD}(A) \cup iso\sigma_{gD}(B) \\ &\subseteq \partial\sigma(A) \cup \partial\sigma(B) \\ &\subseteq \sigma_{ap}(A) \cup \sigma_{su}(B) \\ &\subseteq \sigma(M_C). \end{aligned}$$

Then $\lambda \in \sigma(M_C)$, contradiction. Thus,

$$\lambda \in \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B) \setminus \sigma_{gD(g\mathcal{M})}(M_C),$$

by Theorem 2.4, $\lambda \in W_{gD(g\mathcal{M})}$, so $W \subseteq W_{gD(g\mathcal{M})}$, which shows the inclusion.

Now, if $\sigma_{gD(g\mathcal{M})}(M_C) = \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)$, then $W_{gD(g\mathcal{M})} = \emptyset$, which implies that $W = \emptyset$. Consequently, $\sigma(M_C) = \sigma(B) \cup \sigma(A)$.

Conversely, if $\sigma(B) \cup \sigma(A) = \sigma(M_C)$, by [11, Theorem 2.2] we have $\sigma_{gD}(B) \cup \sigma_{gD}(A) = \sigma_{gD}(M_C)$. Now, let $\lambda \notin \sigma_{gD(g\mathcal{M})}(M_C)$, without losing generality, take $0 \notin \sigma_{gD(g\mathcal{M})}(M_C)$ then there exists $\varepsilon > 0$ such that $M_C - \lambda I$ is generalized Drazin-g-meromorphic invertible, for all λ, $0 < |\lambda| < \varepsilon$, hence $0 \notin \sigma_{gD}(M_C) = \sigma_{gD}(B) \cup \sigma_{gD}(A)$. Thus both $B - \lambda I$ and $A - \lambda I$ are generalized Drazin invertible for any λ, $0 < |\lambda| < \varepsilon$. Therefore $0 \notin \sigma_{gD(g\mathcal{M})}(A)) \cup \sigma_{gD(g\mathcal{M})}(B))$. Since $\sigma_{gD(g\mathcal{M})}(M_C) \subseteq \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)$ always holds, then $\sigma_{gD(g\mathcal{M})}(M_C) = \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)$.
□

According to Theorem 2.6, [10, Proposition 3.6] and [11, Theorem 2.2], we can conclude the following conclusion.

Corollary 2.7. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, and $C \in \mathcal{L}(K, H)$. The following claims are equivalent:*

1. $\sigma(M_C) = \sigma(B) \cup \sigma(A)$.
2. $\sigma_D(M_C) = \sigma_D(B) \cup \sigma_D(A)$.
3. $\sigma_{gD}(M_C) = \sigma_{gD}(B) \cup \sigma_{gD}(A)$.
4. $\sigma_{gD(g\mathcal{M})}(M_C) = \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)$.

According to the proof of Theorem 2.6, we have $W \subset W_{gD(g\mathcal{M})}$. The following example show that this inclusion may be strict in general.

Example 2.8. *Define $P, Q, R \in \mathcal{L}(l^2(\mathbb{N}))$ by*

$$P(x_1, x_2, x_3, ...) = (0, x_1, x_2, ...),$$
$$Q(x_1, x_2, x_3, ...) = (x_2, x_3, x_4 ...),$$

and

$$R(x_1, x_2, x_3, ...) = (x_1, 0, 0, ...).$$

Let $A = P \in \mathcal{L}(l^2(\mathbb{N}))$, $C = (R, 0) \in \mathcal{L}(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}), l^2(\mathbb{N}))$ and $B = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))$. Let

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{L}(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}) \oplus l^2(\mathbb{N})).$$

We have $\sigma(M_C) = \{\lambda \in \mathbb{C}, |\lambda| = 1\} \cup \{0\}, \sigma(B) = \sigma(A) = \{\lambda \in \mathbb{C}, |\lambda| \leq 1\}$, *then* $\sigma_{gD(g\mathcal{M})}(M_C) = \{\lambda \in \mathbb{C}, |\lambda| = 1\}, \sigma_{gD(g\mathcal{M})}(B) = \sigma_{gD(g\mathcal{M})}(A) = \{\lambda \in \mathbb{C}, |\lambda| \leq 1\}$. *So,*

$$W = \{\lambda \in \mathbb{C}, 0 < |\lambda| < 1\} \text{ and } W_{gD(g\mathcal{M})} = \{\lambda \in \mathbb{C}, |\lambda| < 1\}.$$

Consequently, $W_{gD(g\mathcal{M})} \neq W$.

The subsequent theorem, however, provides an adequate condition for the equality.

Theorem 2.9. *Let* $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, *and* $C \in \mathcal{L}(K, H)$. *If* $iso\partial W = \emptyset$, *then*

$$W = W_{gD} = W_{gD(g\mathcal{M})}.$$

Proof. Assume that $iso\partial W = \emptyset$, from [11, Theorem 2.3] we have $W = W_{gD}$, hence $iso\partial W_{gD} = \emptyset$, so

$$iso\sigma_{gD}(M_C) = iso(\sigma_{gD}(A) \cup \sigma_{gD}(B)) \subseteq iso\sigma_{gD}(A) \cup iso\sigma_{gD}(B).$$

Let $\lambda \in iso\sigma_{gD}(M_C)$, then $\lambda \in iso\sigma_{gD}(A)$ or $\lambda \in iso\sigma_{gD}(B)$. If $\lambda \in iso\sigma_{gD}(A)$, then $A - \lambda$ is generalized Drazin-g-meromorphic invertible but not generalized Drazin invertible. According to Lemma 2.1, $B - \lambda$ is generalized Drazin-g-meromorphic invertible, then $\lambda \in iso\sigma_{gD}(B) \cup \rho_{gD}(B)$. Similarly, we have $\lambda \in iso\sigma_{gD}(B) \implies \lambda \in iso\sigma_{gD}(A) \cup \rho_{gD}(A)$. Which entails that:

$$iso\sigma_{gD}(M_C) \subseteq (iso\sigma_{gD}(A) \cap iso\sigma_{gD}(B)) \cup (iso\sigma_{gD}(A) \cap \rho_{gD}(B)) \cup (iso\sigma_{gD}(B) \cap \rho_{gD}(A)).$$

Furthermore, Lemma 2.1 ensure that

$$\begin{aligned}
&(iso\sigma_{gD}(A) \cap iso\sigma_{gD}(B)) \cup (iso\sigma_{gD}(A) \cap \rho_{gD}(B)) \cup (iso\sigma_{gD}(B) \cap \rho_{gD}(A)) \\
&\subseteq \partial\sigma_{gD}(A) \cup \partial\sigma_{gD}(B) \\
&\subseteq acc\sigma_{ap}(B) \cup acc\sigma_{su}(A) \\
&\subseteq \sigma_{gD}(M_C)
\end{aligned}$$

and

$$(iso\sigma_{gD}(A) \cap iso\sigma_{gD}(B)) \cup (iso\sigma_{gD}(A) \cap \rho_{gD}(B)) \cup (iso\sigma_{gD}(B) \cap \rho_{gD}(A)) \subseteq iso\sigma_{gD}(M_C).$$

According to the above, we have

$$(iso\sigma_{gD}(A) \cap iso\sigma_{gD}(B)) \cup (iso\sigma_{gD}(A) \cap \rho_{gD}(B)) \cup (iso\sigma_{gD}(B) \cap \rho_{gD}(A)) = iso\sigma_{gD}(M_C).$$

Hence

$$iso\sigma_{gD}(M_C) \cap (\sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B)) = \emptyset.$$

According to Lemma 2.1,

$$\begin{aligned}
&(iso\sigma_{gD}(A) \cup iso\sigma_{gD}(B)) \setminus iso\sigma_{gD}(M_C) \\
&= (iso\sigma_{gD}(A) \setminus iso\sigma_{gD}(M_C)) \cup (iso\sigma_{gD}(B) \setminus iso\sigma_{gD}(M_C)) \\
&\subseteq \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sigma_{gD}(A) \cup \sigma_{gD}(B) \\
&= \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B) \cup iso\sigma_{gD}(A) \cup iso\sigma_{gD}(B) \\
&= \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B) \cup iso\sigma_{gD}(M_C) \cup [(iso\sigma_{gD}(A) \cup iso\sigma_{gD}(B)) \setminus iso\sigma_{gD}(M_C)] \\
&= \sigma_{gD(g\mathcal{M})}(A) \cup \sigma_{gD(g\mathcal{M})}(B) \cup iso\sigma_{gD}(M_C) \\
&= \sigma_{gD(g\mathcal{M})}(M_C) \cup W_{gD(g\mathcal{M})} \cup iso\sigma_{gD}(M_C) \\
&= \sigma_{gD(g\mathcal{M})}(M_C) \cup iso\sigma_{gD}(M_C) \cup W_{gD(g\mathcal{M})}
\end{aligned}$$

and

$$\begin{aligned}&[\sigma_{gD(g\mathcal{M})}(M_C) \cup iso\sigma_{gD}(M_C)] \cap W_{gD(g\mathcal{M})}\\&= [\sigma_{gD(g\mathcal{M})}(M_C) \cap W_{gD(g\mathcal{M})}] \cup [iso\sigma_{gD}(M_C) \cap W_{gD(g\mathcal{M})}]\\&\subseteq [\sigma_{gD(g\mathcal{M})}(M_C) \cap W_{gD(g\mathcal{M})}] \cup [iso\sigma_{gD}(M_C) \cap (\sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A))]\\&= \emptyset.\end{aligned}$$

Moreover,

$$\begin{aligned}\sigma_{gD}(B) \cup \sigma_{gD}(A) &= \sigma_{gD}(M_C) \cup W_{gD}\\&= \sigma_{gD(g\mathcal{M})}(M_C) \cup iso\sigma_{gD}(M_C) \cup W_{gD}\end{aligned}$$

and

$$\sigma_{gD}(M_C) \cap W_{gD} = \emptyset.$$

We get that $W_{gD} = W_{gD(g\mathcal{M})}$, therefore $W = W_{gD(g\mathcal{M})}$.
□

Recall that for $T \in \mathcal{L}(H)$ we have $S(T) \cup S(T^*) \subset \sigma_{gD(g\mathcal{M})}(T)$ [9]. Using the SVEP, we found the following result:

Lemma 2.10. *Let $A \in \mathcal{L}(K)$ and $B \in \mathcal{L}(K)$. We have*

1. $S(B) \cap S(A^*) \subseteq \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A)$, *and*
2. $[S(A^*) \cap S(B)] \cap [iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))]^c = S(B) \cap S(A^*)$.

Proof. For (1), let $\lambda \in \rho_{gD(g\mathcal{M})}(A)$. Then $A - \lambda I$ is generalized Drazin-g-meromorphic. According to [9, Theorem 3.10], A^* has SVEP at λ. So, $\lambda \in S(A^*)^c$. As a result, we have $\rho_{gD(g\mathcal{M})}(A) \subseteq S(A^*)^c$. By a similar argument, we can conclude that $\rho_{gD(g\mathcal{M})}(B) \subseteq S(B)^c$. Hence

$$\rho_{gD(g\mathcal{M})}(B) \cap \rho_{gD(g\mathcal{M})}(A) \subseteq S(B)^c \cap S(A^*)^c.$$

Consequently,

$$S(B) \cap S(A^*) \subseteq \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A).$$

For (2), let $\lambda \in S(B) \cap S(A^*)$. Hence B and A^* have SVEP at λ. According to [9, Theorem 3.10], $\lambda \in acc\sigma_{gD}(B) \cap acc\sigma_{gD}(A) \subseteq acc\sigma_{gD}(B) \cup acc\sigma_{gD}(A)$. Since $acc\sigma_{gD}(B) \cup acc\sigma_{gD}(A) = acc(\sigma_{gD}(B) \cup \sigma_{gD}(A))$ (See [11, Lemma 2.2]), we have $\lambda \in acc(\sigma_{gD}(B) \cup \sigma_{gD}(A))$. Thus $\lambda \in [iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))]^c$. As a result,

$$S(A^*) \cap S(B) \subseteq [iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))]^c.$$

Consequently,

$$S(A^*) \cap S(B) \subseteq [S(A^*) \cap S(B)] \cap [iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))]^c.$$

The other inclusion is obvious. □

Theorem 2.11. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$ and, $C \in \mathcal{L}(K, H)$. Then*

$$\sigma_{gD(g\mathcal{M})}(M_C) \cup [S(B) \cap S(A^*)] = \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A).$$

Proof. It follows from [10, Theorem 3.2] that

$$\sigma_{gD}(M_C) \cup [S(B) \cap S(A^*)] = \sigma_{gD}(A) \cup \sigma_{gD}(B) \text{ for all } C \in \mathcal{B}(K, H).$$

Hence,

$$\sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A) = \{\sigma_{gD}(M_C) \cup [S(B) \cap S(A^*)]\} \cap \{iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))\}^c.$$

By Lemma 2.10, we have

$$S(B) \cap S(A^*) \subseteq \sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A),$$

$$\text{and } [S(A^*) \cap S(B)] \cap \{iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))\}^c = S(B) \cap S(A^*).$$

From Theorem 1.4, we have $\sigma_{gD}(M_C) \cap W_{gD} = \emptyset$, then $\lambda \in iso\sigma_{gD}(M_C)$ implies that there exists a neighborhood V of λ such that $V \cap \sigma_{gD}(M_C) = \{\lambda\}$. Put $U = V \cap W^c_{gD}$, then $[\sigma_{gD}(M_C) \cup W_{gD}] \cap U = \{\lambda\}$. Therefore, $\lambda \in iso(\sigma_{gD}(M_C) \cup W_{gD})$. Whence

$$iso\sigma_{gD}(M_C) \subseteq iso(\sigma_{gD}(B) \cup \sigma_{gD}(A)).$$

Hence

$$\begin{aligned}
&\sigma_{gD}(M_C) \cap \{iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))\}^c \\
&= (iso\sigma_{gD}(M_C) \cup \sigma_{gD(g\mathcal{M})}(M_C)) \cap \{iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))\}^c \\
&= \sigma_{gD(g\mathcal{M})}(M_C) \cap \{iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))\}^c \\
&\subseteq \sigma_{gD(g\mathcal{M})}(M_C).
\end{aligned}$$

So,

$$\begin{aligned}
&\sigma_{gD(g\mathcal{M})}(B) \cup \sigma_{gD(g\mathcal{M})}(A) \\
&= \{\sigma_{gD}(M_C) \cup [S(B) \cap S(A^*)]\} \cap \{iso(\sigma_{gD}(B) \cup \sigma_{gD}(A))\}^c \\
&\subseteq \sigma_{gD(g\mathcal{M})}(M_C) \cup [S(B) \cap S(A^*)].
\end{aligned}$$

□

We obtain the following corollary from Theorem 2.11.

Corollary 2.12. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$. If $S(B) \cap S(A^*) = \emptyset$, then for every $C \in \mathcal{L}(K, H)$ we have*

$$\sigma_{gD(g\mathcal{M})}(M_C) = \sigma_{gD}(B) \cup \sigma_{gD(g\mathcal{M})}(A).(**)$$

Specifically, if B or A^* have the SVEP, then equality $(**)$ hold.

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