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Dependence of eigenvalues of discontinuous Sturm-Liouville operators with distributional potentials

Jianfang Qin[a], Zhaowen Zheng[b,c,*], Kun Li[c], Zhiyu Li[d]

[a]*Shanxi University of Electronic Science and Technology, Linfen, Shanxi 041000, P. R. China*
[b] *School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong 510665, P. R. China*
[c]*School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, P. R. China*
[d]*School of Mathematics and Information Science, Shandong Technology and Business University, Yantai, 264003, P. R. China*

Abstract. This paper deals with a regular Sturm-Liouville problem with distributional potential and transmission condition. We obtain the differentiability of the eigenvalues on parameters. Moreover, we give the derivative formulas of the eigenvalues with respect to the parameters.

1. Introduction

Sturm-Liouville theory has been widely applied in various fields such as engineering, physics, finance. With various practical problems arose in the fields of physics and medicine, many problems need to be converted to differential operators with interior discontinuity. For instance, the heat conduction and mass transfer problems [21], string vibration with nodes [20], and diffraction problems of light [24]. In order to describe the connection between two sides of discontinuous points, some conditions need to be added, such conditions are often referred to point interactions, transmission or interface conditions. In recent years, such problems with interior discontinuity have attracted the attention of many researchers and have made great progress [1, 3, 4, 8, 15, 16, 25].

As we all know, the classical Sturm-Liouville theory is the main mathematical tool to describe the state of microscopic particles in quantum mechanics. However, the description of the interaction between microscopic particles needs to be studied by using the Sturm-Liouville problem with distributional potentials. Recently, Sturm-Liouville problem with distributional potential function has attracted the attention and discussion of a large number of mathematicians [6, 7, 18, 19, 23, 26]. In particular, Eckhardt et al. [6] presented a systematical development of Weyl-Titchmarsh theory for singular differential operators associated with the following differential expression

$$\tau f = \frac{1}{r}\Big(-(p[f' + sf])' + ps[f' + sf] + qf\Big) \quad \text{on } J = (a, b), \tag{1}$$

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* Corresponding author: Zhaowen Zheng
Email addresses: qjf0528@126.com (Jianfang Qin), zhwzheng@126.com (Zhaowen Zheng), qslikun@163.com (Kun Li), zyli@qfnu.edu.cn (Zhiyu Li)

where $p,\ s,\ q,\ r$ are real-valued, $\frac{1}{p},\ s,\ q,\ r \in L(J,\mathbb{R}),\ p>0,\ r>0$ a.e. on J, and the differential operator determined by the differential expression (1) in space $L_r^2(J)$ is called Sturm-Liouville operator with distributional potential. In 2014, Yan and Shi [26] discussed the continuous dependence of the n-th eigenvalue of the self-adjoint Sturm-Liouville operator with distributional potentials, and the oscillatory properties of the eigenfunctions were also considered. Recently, Uğurlu [23] investigated the properties of eigenvalues of the Sturm-Liouville problem with distributional potential and obtained the dependence of the eigenvalues with respect to the elements. As we know, the dependence of eigenvalues is the theoretical basis of numerical calculation of eigenvalues in differential operator theory. For example, the codes SLEUTH [9] and SLEIGN2 [2].

The dependence of eigenvalues on parameters has received extensive attention by researchers. In 1999, Kong and Zettl [12, 13] considered the continuous differentiable dependence of eigenvalues of second-order Sturm-Liouville problem on parameters. Later, these results were extended to Sturm-Liouville problem with eigenparameter dependent boundary condition or transmission condition, and higher order differential operators [10, 14, 17, 22, 28, 29]. Up to now, the dependence of eigenvalues of discontinuous Sturm-Liouville problem with distributional potentials has not been studied. In this paper, we not only discuss the continuity and differentiability of eigenvalues but also give differential expressions of these eigenvalues with respect to parameters.

The rest of the paper is organized as follows: Section 2 introduces a discontinuous Sturm-Liouville problem with distributional potential and give some basic properties of the problem. Section 3 proves the continuity of the eigenvalues. Finally, the differentiability of eigenvalues and the corresponding derivative formulas are presented in Section 4.

2. Preliminaries

Consider the differential equation

$$\frac{1}{r(x)}\Big(-(p(x)[f'(x)+s(x)f(x)])' + p(x)s(x)[f'(x)+s(x)f(x)] + q(x)f(x)\Big) = \nu f(x) \ \text{ on } S \tag{2}$$

with boundary condition

$$AF(a)+BF(b)=0 \tag{3}$$

and transmission condition

$$CF(c-)+DF(c+)=0, \tag{4}$$

where

$$S=[a,c)\cup(c,b],\ -\infty<a<b<+\infty,\ \frac{1}{p},s,q,r\in L^1(S,\mathbb{R}),\ p>0,r>0 \text{ a.e. on } S, \tag{5}$$

$\nu\in\mathbb{C}$ is the spectral parameter, $F(x)=(f(x),f^{[1]}(x))^T$, and $f^{[1]}=p[f'+sf]$ is the first order quasi-derivative of f. A,B are 2×2 complex matrices, C,D are 2×2 real matrices, $\det C=\rho>0, \det D=\theta>0$ and satisfy

$$\operatorname{rank}(A|B)=2, \tag{6}$$

$$\theta AE_0A^*=\rho BE_0B^*,\ \theta CE_0C^*=\rho DE_0D^*, E_0=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{7}$$

where A^* is the conjugate transpose of A.

It is well known that the self-adjoint boundary conditions (3), (6), (7) can be divided into three disjoint and mutually exclusive subclasses [27]. In this paper, we study the following three canonical representations:

1. Separated boundary conditions

$$f(a)\cos\gamma - f^{[1]}(a)\sin\gamma = 0,\ \gamma\in[0,\pi), \tag{8}$$

$$f(b)\cos\varphi - f^{[1]}(b)\sin\varphi = 0,\ \varphi\in(0,\pi]. \tag{9}$$

2. Real coupled boundary conditions

$$F(b) = KF(a). \tag{10}$$

3. Complex coupled boundary conditions

$$F(b) = e^{i\tau}KF(a), \tag{11}$$

where $\tau\in(-\pi,0)$ or $\tau\in(0,\pi)$, K is 2×2 real matrix with $\det K = \frac{\rho}{\theta}$, $\theta KE_0K^* = \rho E_0$.

Let $H = L^2_r([a,c))\oplus L^2_r((c,b])$ be a weight Hilbert space with inner product

$$\langle f,g\rangle = \rho\int_a^c f\bar{g}r\mathrm{d}x + \theta\int_c^b f\bar{g}r\mathrm{d}x$$

for any $f,g\in H$. Define an operator $\mathcal{P}$ in H with domain

$$\begin{aligned} D(\mathcal{P}) =&\{f\in H: f, f^{[1]}\in \mathrm{AC}(S), AF(a)+BF(b)=0,\\ &CF(c-)+DF(c+)=0, \mathcal{P}f\in H\}.\\ \mathcal{P}f =& r^{-1}[-(f^{[1]})' + sf^{[1]} + qf], f\in D(\mathcal{P}). \end{aligned}$$

Note that the eigenvalues of the operator $\mathcal{P}$ are consistent with those of the problem (2)-(4).

For each $f,g\in H$, we define the modified Wronski determinant $W(f,g)(x) = \begin{vmatrix} f & g\\ f^{[1]} & g^{[1]}\end{vmatrix}$. For any two solutions f and g of equation (2), by a direct calculation we can verify that $W(f,g)(x)$ is constant on $[a,c)$ and $(c,b]$, respectively.

Lemma 2.1. *The operator $\mathcal{P}$ is symmetric in H.*

Proof. For all $f,g\in D(\mathcal{P})$, using (2) and integrating by parts, we get

$$\begin{aligned} \langle\mathcal{P}f,g\rangle =& \rho\int_a^c -(p[f'+sf])'\bar{g} + ps[f'+sf]\bar{g} + qf\bar{g}\mathrm{d}x\\ &+\theta\int_c^b -(p[f'+sf])'\bar{g} + ps[f'+sf]\bar{g} + qf\bar{g}\mathrm{d}x\\ =&[-\rho p[f'+sf]\bar{g}]_a^{c-} + \rho\int_a^c p[f'+sf]\bar{g}' + ps[f'+sf]\bar{g} + qf\bar{g}\mathrm{d}x\\ &+[-\theta p[f'+sf]\bar{g}]_{c+}^b + \theta\int_c^b p[f'+sf]\bar{g}' + ps[f'+sf]\bar{g} + qf\bar{g}\mathrm{d}x\\ =&\langle f,\mathcal{P}g\rangle + \rho[f\bar{g}^{[1]} - f^{[1]}\bar{g}]_a^{c-} + \theta[f\bar{g}^{[1]} - f^{[1]}\bar{g}]_{c+}^b. \end{aligned} \tag{12}$$

It follows from (7) that

$$\rho A^{*-1}E_0A^{-1} = \theta B^{*-1}E_0B^{-1},\ \rho C^{*-1}E_0C^{-1} = \theta D^{*-1}E_0D^{-1}. \tag{13}$$

Then by (13) and boundary condition (3), we have

$$\begin{aligned}\rho[f\bar{g}^{[1]} - f^{[1]}\bar{g}](a) =& \rho G^*(a)E_0F(a)\\ =&\rho((A^{-1}BG)^*E_0A^{-1}BF)(b)\\ =&\rho(G^*B^*A^{*-1}E_0A^{-1}BF)(b)\\ =&\theta(G^*B^*B^{*-1}E_0B^{-1}BF)(b)\\ =&\theta G^*(b)E_0F(b)\\ =&\theta[f\bar{g}^{[1]} - f^{[1]}\bar{g}](b).\end{aligned} \tag{14}$$

Similarly, by (13) and transmission condition (4), we have

$$\rho[f\bar{g}^{[1]} - f^{[1]}\bar{g}](c-) = \theta[f\bar{g}^{[1]} - f^{[1]}\bar{g}](c+). \tag{15}$$

Substituting (14) and (15) into (12), we obtain

$$\langle \mathcal{P}f, g\rangle = \langle f, \mathcal{P}g\rangle.$$

Therefore, the operator $\mathcal{P}$ is symmetric. □

Lemma 2.2. *The operator $\mathcal{P}$ is a self-adjoint operator in H.*

Proof. The proof is similar to that in [8, 25], here we omit the details. □

To prove that the eigenvalues of the problem (2), (4), (8)-(9) are simple, we first define two fundamental solutions of equation (2) as follows

$$\phi(x,\nu) = \begin{cases} \phi_1(x,\nu), & x\in[a,c),\\ \phi_2(x,\nu), & x\in(c,b],\end{cases} \quad \chi(x,\nu) = \begin{cases} \chi_1(x,\nu), & x\in[a,c),\\ \chi_2(x,\nu), & x\in(c,b],\end{cases}$$

where $\phi_1(x,\nu)$, $\phi_2(x,\nu)$ are solutions of equation (2) on $[a,c)$ and $(c,b]$ respectively, and satisfying the following initial conditions

$$\begin{pmatrix} \phi_1(a,\nu)\\ \phi_1^{[1]}(a,\nu)\end{pmatrix} = \begin{pmatrix}\sin\gamma\\ \cos\gamma\end{pmatrix}, \tag{16}$$

and

$$\begin{pmatrix} \phi_2(c+,\nu)\\ \phi_2^{[1]}(c+,\nu)\end{pmatrix} = -D^{-1}C\begin{pmatrix} \phi_1(c-,\nu)\\ \phi_1^{[1]}(c-,\nu)\end{pmatrix}. \tag{17}$$

Similarly, we can also define $\chi_2(x,\nu)$, $\chi_1(x,\nu)$ are solutions of equation (2) on $(c,b]$ and $[a,c)$ respectively, and satisfying the following initial conditions

$$\begin{pmatrix} \chi_2(b,\nu)\\ \chi_2^{[1]}(b,\nu)\end{pmatrix} = \begin{pmatrix}\sin\varphi\\ \cos\varphi\end{pmatrix}, \tag{18}$$

and

$$\begin{pmatrix} \chi_1(c-,\nu)\\ \chi_1^{[1]}(c-,\nu)\end{pmatrix} = -C^{-1}D\begin{pmatrix} \chi_2(c+,\nu)\\ \chi_2^{[1]}(c+,\nu)\end{pmatrix}. \tag{19}$$

Let the modified Wronskians be $\omega_i(\nu) := W(\phi_i(x,\nu),\chi_i(x,\nu)) = \phi_i\chi_i^{[1]} - \phi_i^{[1]}\chi_i$, $i = 1,2$, where $\omega_i(\nu)$ are entire functions of parameter ν and independent on variable x. By direct calculation, we have $\omega_2(\nu) = \frac{\rho}{\theta}\omega_1(\nu)$.

Let $\omega(\nu) = \omega_1(\nu)$, then $\omega(\nu) = \frac{\theta}{\rho}\omega_2(\nu)$, and $\omega(\nu)$ is an entire function of parameter ν.

Lemma 2.3. *The number ν is an eigenvalue of the problem (2), (4), (8)-(9) if and only if $\omega(\nu) = 0$.*

Proof. Let ν be an eigenvalue of the problem (2), (4), (8)-(9), and $f(x, \nu)$ be the corresponding eigenfunction. Assume that $\omega(\nu) \neq 0$, then

$$W(\phi_i(x, \nu), \chi_i(x, \nu)) \neq 0, \ i = 1, 2.$$

Thus $\phi_i(x, \nu)$ and $\chi_i(x, \nu)$ are linearly independent. The solution $f(x, \nu)$ of (2) can be writeen as

$$f(x, \nu) = \begin{cases} l_1\phi_1(x, \nu) + l_2\chi_1(x, \nu), & x \in [a, c), \\ l_3\phi_2(x, \nu) + l_4\chi_2(x, \nu), & x \in (c, b], \end{cases} \tag{20}$$

where l_i, $i = 1, 2, 3, 4$ are not all zero. Since $f(x, \nu)$ satisfies conditions (8)-(9), we have $l_2 = 0$, $l_3 = 0$. Substituting (20) into the transmission condition (4), then l_1 and l_4 satisfy the following equations

$$C\begin{pmatrix} l_1\phi_1(c-, \nu) \\ l_1\phi_1^{[1]}(c-, \nu) \end{pmatrix} + D\begin{pmatrix} l_4\chi_2(c+, \nu) \\ l_4\chi_2^{[1]}(c+, \nu) \end{pmatrix} = 0.$$

By (17), we have

$$\begin{pmatrix} l_1\phi_2(c+, \nu) \\ l_1\phi_2^{[1]}(c+, \nu) \end{pmatrix} - \begin{pmatrix} l_4\chi_2(c+, \nu) \\ l_4\chi_2^{[1]}(c+, \nu) \end{pmatrix} = 0.$$

Since $\omega_2(\nu) = \phi_2\chi_2^{[1]} - \phi_2^{[1]}\chi_2 \neq 0$, the equations have only trivial solutions $l_1 = 0, l_4 = 0$. Such a contradiction proves $\omega(\nu) = 0$.

Conversely, if $\omega(\nu) = 0$, then $W(\phi_1(x, \nu), \chi_1(x, \nu)) = 0$ for all $x \in [a, c)$. Therefore $\phi_1(x, \nu)$ and $\chi_1(x, \nu)$ are linearly dependent, that is,

$$\chi_1(x, \nu) = k_1\phi_1(x, \nu), \ \ x \in [a, c)$$

for some $k_1 \neq 0$. Then

$$\begin{aligned} \chi(a, \nu)\cos\gamma - \chi^{[1]}(a, \nu)\sin\gamma =& \chi_1(a, \nu)\cos\gamma - \chi_1^{[1]}(a, \nu)\sin\gamma \\ =& k_1(\phi_1(a, \nu)\cos\gamma - \phi_1^{[1]}(a, \nu)\sin\gamma) \\ =& 0. \end{aligned}$$

Therefore $\chi(x, \nu)$ satisfies the boundary condition (8).

By (18) and (19), we can get that $\chi(x, \nu)$ satisfies conditions (4) and (9). Hence $\chi(x, \nu)$ is a corresponding eigenfunction for the eigenvalue ν of the problem (2), (4), (8)-(9). □

Lemma 2.4. *The eigenvalues of the problem (2), (4), (8)-(9) are simple.*

Proof. Let $\nu = u + iv$. For simplicity, let $\phi = \phi(x, \nu)$, $\phi_{1\nu} = \frac{\partial\phi_1}{\partial\nu}$, $\phi'_{1\nu} = \frac{\partial\phi'_1}{\partial\nu}$. Differentiating the equation $\mathcal{P}\chi = \nu\chi$ with respect to ν, yields

$$\mathcal{P}\chi_\nu = \nu\chi_\nu + \chi.$$

Then

$$\langle\mathcal{P}\chi_\nu, \phi\rangle - \langle\chi_\nu, \mathcal{P}\phi\rangle = \langle\nu\chi_\nu + \chi, \phi\rangle - \langle\chi_\nu, \nu\phi\rangle = \langle\chi, \phi\rangle + 2iv\langle\chi_\nu, \phi\rangle. \tag{21}$$

Using integration by parts and (16)-(19), we have

$$\begin{aligned} \langle\mathcal{P}\chi_\nu, \phi\rangle - \langle\chi_\nu, \mathcal{P}\phi\rangle =& \rho[\chi_{1\nu}\bar{\phi_1}^{[1]} - \chi_{1\nu}^{[1]}\bar{\phi_1}]_a^{c-} + \theta[\chi_{2\nu}\bar{\phi_2}^{[1]} - \chi_{2\nu}^{[1]}\bar{\phi_2}]_{c+}^{b} \\ =& \rho(\chi_\nu^{[1]}(a, \nu)\sin\gamma - \chi_\nu(a, \nu)\cos\gamma). \end{aligned} \tag{22}$$

Since $\omega(\nu)$ is independent of x and by (16), we obtain

$$\begin{aligned}\omega'(\nu)|_{x=a} = \frac{d\omega_1(\nu)}{d\nu}\Big|_{x=a} &= \frac{d}{d\nu}(\phi_1(a,\nu)\chi_1^{[1]}(a,\nu) - \phi_1^{[1]}(a,\nu)\chi_1(a,\nu)) \\ &= \chi_{1\nu}^{[1]}(a,\nu)\sin\gamma - \chi_{1\nu}(a,\nu)\cos\gamma.\end{aligned} \tag{23}$$

It follows from (21)-(23) that

$$\omega'(\nu) = \frac{1}{\rho}(\langle\chi,\phi\rangle + 2i\nu\langle\chi_\nu,\phi\rangle). \tag{24}$$

Let ν_0 be an eigenvalue of the problem (2), (4), (8)-(9). Since the operator $\mathcal{P}$ is self-adjoint, ν_0 is real. By Lemma 2.3, $\omega(\nu_0) = 0$, then there exist $c_j \neq 0$, such that

$$\chi_j(x,\nu_0) = c_j\phi_j(x,\nu_0), \quad j = 1,2.$$

By the transmission condition (4), we have

$$\begin{aligned}\begin{pmatrix}\chi_2(c+,\nu_0)\\ \chi_2^{[1]}(c+,\nu_0)\end{pmatrix} &= -D^{-1}C\begin{pmatrix}\chi_1(c-,\nu_0)\\ \chi_1^{[1]}(c-,\nu_0)\end{pmatrix} \\ &= -c_1D^{-1}C\begin{pmatrix}\phi_1(c-,\nu_0)\\ \phi_1^{[1]}(c-,\nu_0)\end{pmatrix} \\ &= c_1\begin{pmatrix}\phi_2(c+,\nu_0)\\ \phi_2^{[1]}(c+,\nu_0)\end{pmatrix},\end{aligned}$$

then $c_1 = c_2 \neq 0$ and $\chi(x,\nu_0) = c_1\phi(x,\nu_0)$. The equation (24) can be expressed as

$$\omega'(\nu_0) = \frac{1}{\rho}\langle\chi,\phi\rangle = \frac{c_1}{\rho}\langle\phi,\phi\rangle \neq 0.$$

Thus ν_0 is simple. □

Remark 2.5. *The spectrum of $\mathcal{P}$ is composed of isolated eigenvalues, and the problem (2), (4), (8)-(9) has only simple and real eigenvalues. Moreover, we can obtain that the problem (2), (4) and (10) has only real eigenvalues, each of which may be simple or double. The problem (2), (4) and (11) has only simple and real eigenvalues.*

3. Continuity of eigenvalues and eigenfunctions

In this section, we present the continuity of eigenvalues on the parameters in the discontinuous Sturm-Liouville problem with distributional potential (2)-(4).

Let $\delta_1(x,\nu)$, $\sigma_1(x,\nu)$ be the linearly independent solutions of equation (2) on $[a,c)$ satisfying the following initial conditions

$$\begin{pmatrix}\delta_1(a,\nu) & \sigma_1(a,\nu)\\ \delta_1^{[1]}(a,\nu) & \sigma_1^{[1]}(a,\nu)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{25}$$

In view of dependency properties of the solutions on the parameter, we get that $\delta_1(x,\nu)$, $\sigma_1(x,\nu)$ are entire functions of parameter ν for a fixed x. Then, the Wronskian of functions $\delta_1(x,\nu)$, $\sigma_1(x,\nu)$ is an entire function of parameter ν and is independent of x.
Denote $\omega_1 = W(\delta_1(x,\nu),\sigma_1(x,\nu))$, then we get

$$\begin{aligned}\omega_1 &= \begin{vmatrix}\delta_1(x,\nu) & \sigma_1(x,\nu)\\ \delta_1^{[1]}(x,\nu) & \sigma_1^{[1]}(x,\nu)\end{vmatrix} \\ &= \begin{vmatrix}\delta_1(a,\nu) & \sigma_1(a,\nu)\\ \delta_1^{[1]}(a,\nu) & \sigma_1^{[1]}(a,\nu)\end{vmatrix} \\ &= 1.\end{aligned}$$

Let $\delta_2(x,\nu)$, $\sigma_2(x,\nu)$ be the solutions of equation (2) on $(c,b]$ satisfying the following initial condition

$$C\begin{pmatrix} \delta_1(c-,\nu) & \sigma_1(c-,\nu) \\ \delta_1^{[1]}(c-,\nu) & \sigma_1^{[1]}(c-,\nu) \end{pmatrix} + D\begin{pmatrix} \delta_2(c+,\nu) & \sigma_2(c+,\nu) \\ \delta_2^{[1]}(c+,\nu) & \sigma_2^{[1]}(c+,\nu) \end{pmatrix} = 0.$$

Let $\omega_2(x,\nu)$ be the Wronskian of functions $\delta_2(x,\nu)$, $\sigma_2(x,\nu)$, then $\omega_2(x,\nu)$ is an entire function of ν and is independent of x. By direct calculation, we obtain

$$\omega_2 = \frac{\rho}{\theta} \neq 0.$$

Therefore, the functions $\delta_2(x,\nu)$, $\sigma_2(x,\nu)$ are the linearly independent solutions of equation (2) on $(c,b]$.

Lemma 3.1. *The number ν is an eigenvalue of* (2)-(4) *if and only if*

$$\Delta(\nu) = \det(A + B\Psi(b,\nu)) = 0,$$

where $\Psi(b,\nu) = \begin{pmatrix} \delta_2(b,\nu) & \sigma_2(b,\nu) \\ \delta_2^{[1]}(b,\nu) & \sigma_2^{[1]}(b,\nu) \end{pmatrix}$.

Proof. The proof is similar to that of Lemma 2.3, here we omit it. □

To discuss the dependence of eigenvalues, we introduce a Banach space

$$\mathcal{Z} = M_{2\times2}(\mathbb{C}) \times M_{2\times2}(\mathbb{C}) \times M_{2\times2}(\mathbb{R}) \times M_{2\times2}(\mathbb{R}) \times L(S) \times L(S) \times L(S) \times L(S)$$

with its norm

$$\|\epsilon\| = \|A\| + \|B\| + \|C\| + \|D\| + \int_a^c \Big(\frac{1}{|p|} + |s| + |q| + |r|\Big) + \int_c^b \Big(\frac{1}{|p|} + |s| + |q| + |r|\Big)$$

for any $\epsilon = (A, B, C, D, \frac{1}{p}, s, q, r) \in \mathcal{Z}$. Let

$$\Xi = \{\epsilon \in \mathcal{Z} : (5), (6), (7) \text{ hold}\}.$$

Theorem 3.2. *Let $\epsilon_0 = (A_0, B_0, C_0, D_0, \frac{1}{p_0}, s_0, q_0, r_0) \in \Xi$ and $\nu(\epsilon_0)$ be an eigenvalue of* (2)-(4) *with ϵ_0. Then ν is continuous at ϵ_0. In other words, for any $\varepsilon > 0$, there exists an $\eta > 0$ such that*

$$|\nu(\epsilon) - \nu(\epsilon_0)| < \varepsilon,$$

if $\epsilon = (A, B, C, D, \frac{1}{p}, s, q, r) \in \Xi$ satisfies

$$\begin{aligned}\|\epsilon - \epsilon_0\| =& \|A - A_0\| + \|B - B_0\| + \|C - C_0\| + \|D - D_0\| \\ &+ \int_a^c \Big(|\frac{1}{p} - \frac{1}{p_0}| + |s - s_0| + |q - q_0| + |r - r_0|\Big) \\ &+ \int_c^b \Big(|\frac{1}{p} - \frac{1}{p_0}| + |s - s_0| + |q - q_0| + |r - r_0|\Big) < \eta.\end{aligned}$$

Proof. By Lemma 3.1, $\nu(\epsilon_0)$ is an eigenvalue of (2)-(4) if and only if $\Delta(\epsilon_0, \nu(\epsilon_0)) = 0$. Given any $\epsilon \in \Xi$, $\Delta(\epsilon, \nu)$ is an entire function of ν and is continuous in ϵ (see [11, Theorems 2.7, 2.8]). Due to $\nu(\epsilon_0)$ is an isolated eigenvalue, we can conclude that $\Delta(\epsilon_0, \nu)$ is not a constant in ν. Therefore, there exists $\vartheta > 0$ such that $\Delta(\epsilon_0, \nu) \neq 0$ for $\nu \in S_\vartheta := \{\nu \in \mathbb{C} : |\nu - \nu(\epsilon_0)| = \vartheta\}$. In view of the continuity of the roots of an equation as a function of parameters [5], the conclusion holds. □

Lemma 3.3. *For $h \in S \cup \{c-, c+\}$, the unique solution of the initial value problem*

$$\begin{cases} -(f^{[1]})' + sf^{[1]} + qf = \nu rf, \\ \quad f(h) = e, \quad f^{[1]}(h) = t \end{cases} \tag{26}$$

$f = f(\cdot, h, e, t, \frac{1}{p}, s, q, r, C, D)$ is continuous with respect to each of these variables and satisfies the transmission conditions (4). More precisely, given any $\varepsilon > 0$, there exists an $\eta > 0$ such that if

$$|h - h_0| + |e - e_0| + |t - t_0| + \int_a^c \Big(|\frac{1}{p} - \frac{1}{p_0}| + |s - s_0| + |q - q_0| + |r - r_0|\Big)$$
$$+ \int_c^b \Big(|\frac{1}{p} - \frac{1}{p_0}| + |s - s_0| + |q - q_0| + |r - r_0|\Big) + \|C - C_0\| + \|D - D_0\| < \eta,$$

then

$$|f(x, h, e, t, \frac{1}{p}, s, q, r, C, D) - f(x, h_0, e_0, t_0, \frac{1}{p_0}, s_0, q_0, r_0, C_0, D_0)| < \varepsilon,$$
$$|f^{[1]}(x, h, e, t, \frac{1}{p}, s, q, r, C, D) - f^{[1]}(x, h_0, e_0, t_0, \frac{1}{p_0}, s_0, q_0, r_0, C_0, D_0)| < \varepsilon,$$

uniformly for $\forall x \in S$.

Proof. The proof is similar as that in [26]. □

Lemma 3.4. *Let $\epsilon_0 = (A_0, B_0, C_0, D_0, \frac{1}{p_0}, s_0, q_0, r_0)$ and $\nu = \nu(\epsilon)$ be an eigenvalue of* (2)-(4)*. Suppose that $\nu(\epsilon_0)$ is simple, then there is a neighborhood $\mathcal{M}$ of $\epsilon_0 \in \Xi$ such that $\nu(\epsilon)$ is simple for any $\epsilon \in \mathcal{M}$.*

Proof. Suppose that $\nu(\epsilon_0)$ is simple, then $\Delta'(\nu(\epsilon_0)) \neq 0$. By Theorem 3.2, the conclusion follows since $\Delta(\nu)$ is an entire function of ν. □

We call $m \in H$ is a normalized eigenfunction of (2)-(4), if the eigenfunction m satisfies

$$\langle m, m \rangle = \rho \int_a^c m\bar{m}r\mathrm{d}x + \theta \int_c^b m\bar{m}r\mathrm{d}x = 1.$$

Now we give the continuity of the eigenfunctions.

Theorem 3.5. *Suppose that $\nu(\epsilon)$ $(\epsilon \in \Xi)$ is an eigenvalue of* (2)-(4) *with multiplicity h $(h = 1, 2)$, if $h = 2$, we suppose further that there is a neighborhood $\mathcal{M}$ of $\epsilon_0 \in \Xi$ such that $\nu(\epsilon)$ is double for any $\epsilon \in \mathcal{M}$. Let $m(\cdot, \epsilon_0) \in H$ be the normalized eigenfunctions for $\nu(\epsilon_0)$. Then there exist normalized eigenfunctions $m(\cdot, \epsilon) \in H$ for $\nu(\epsilon)$ such that*

$$m(\cdot, \epsilon) \to m(\cdot, \epsilon_0),\ m^{[1]}(\cdot, \epsilon) \to m^{[1]}(\cdot, \epsilon_0), \tag{27}$$

as $\epsilon \to \epsilon_0$ hold uniformly on S.

Proof. (i) If $\nu(\epsilon_0)$ is simple, then by Lemma 3.4, there exists a neighborhood $\mathcal{M}$ of ϵ_0 such that $\nu(\epsilon)$ is simple for $\forall \epsilon \in \mathcal{M}$. Let $m(\cdot, \epsilon)$ be an eigenfunction for $\nu(\epsilon)$ with

$$\|M(x_0, \epsilon)\| = |m(x_0, \epsilon)| + |m^{[1]}(x_0, \epsilon)| = 1,\ m(x, \epsilon) > 0,$$

for some $x_0 \in S$ and $x \to x_0$, where $M(\cdot, \epsilon) = (m(\cdot, \epsilon), m^{[1]}(\cdot, \epsilon))^T$. Next, we prove

$$M(x_0, \epsilon) \to M(x_0, \epsilon_0),\ \ \epsilon \to \epsilon_0,\ \ \epsilon \in \Xi. \tag{28}$$

If (28) does not hold, then we can find a sequence $\epsilon_k \to \epsilon_0$ such that

$$M(x_0, \epsilon_k) \to F,\ \ \epsilon_k \to \epsilon_0,\ \ \epsilon \in \Xi,$$

where F and $M(x_0, \epsilon_0)$ are linearly independent vectors. Let $G(x)$ be the vector solution of (2) with $\epsilon = \epsilon_0$, $\nu = \nu(\epsilon_0)$ and the initial condition $G(x_0) = F$. According to Lemma 3.3, $M(x, \epsilon_k) \to G(x)$ uniformly hold on S. More precisely,

$$M(a, \epsilon_k) \to G(a),\ M(b, \epsilon_k) \to G(b),\ M(c-, \epsilon_k) \to G(c-),\ M(c+, \epsilon_k) \to G(c+).$$

Since $M(\cdot, \epsilon_k)$ satisfies the conditions

$$A_k M(a, \epsilon_k) + B_k M(b, \epsilon_k) = 0,\ \ C_k M(c-, \epsilon_k) + D_k M(c+, \epsilon_k) = 0,$$

Taking $k \to \infty$, we have

$$A_0 G(a) + B_0 G(b) = 0,\ \ C_0 G(c-) + D_0 G(c+) = 0.$$

Then $G(x)$ is a vector eigenfunction for $\epsilon = \epsilon_0$, $\nu = \nu(\epsilon_0)$, which is a contradiction since $\nu(\epsilon_0)$ is simple. In view of Lemma 3.3, $m(x, \epsilon) \to m(x, \epsilon_0)$, $m^{[1]}(x, \epsilon) \to m^{[1]}(x, \epsilon_0)$ as $\epsilon \to \epsilon_0$ and $x \in S$.

(ii) If $\nu(\epsilon)$ is double for all ϵ in some neighborhood $\mathcal{M}$ of ϵ_0. When we find a linear combination of two linearly independent eigenfunctions satisfying any initial conditions, then we can find the eigenfunctions of $\nu(\epsilon)$ and these eigenfunctions satisfy the same initial conditions at x_0 for some $x_0 \in S$. Similarly, we can get (27) as (i). The proof is finished. □

4. Differential expression of eigenvalues

In this section, we not only prove that the eigenvalues of the operator $\mathcal{P}$ are differentiable with respect to the parameters but also give the corresponding derivative expressions.

Definition 4.1. *[12] Let Υ, $\mathbb{Z}$ be Banach spaces. A map $T\colon \Upsilon \to \mathbb{Z}$ is differentiable at a point $y \in \Upsilon$, if there exists a bounded linear operator $\mathrm{d}T_y : \Upsilon \to \mathbb{Z}$ such that for $\varsigma \in \Upsilon$*

$$|T(y + \varsigma) - T(y) - \mathrm{d}T_y(\varsigma)| = o(\varsigma), as\ \ \varsigma \to 0.$$

Theorem 4.2. *Let the hypotheses of Theorem 3.5 hold. $\nu(\epsilon)$ is an eigenvalue of the operator $\mathcal{P}$ with $\epsilon \in \Xi$ and $m = m(\cdot, \epsilon) \in H$ is the corresponding normalized eigenfunction for $\nu(\epsilon)$, then ν is differentiable with respect to γ, φ, τ, K, C, D, $\frac{1}{p}$, s, q, r.*

(I) Let all parameters of ϵ be fixed except γ and $\nu(\gamma) := \nu(\epsilon)$. Then

$$\nu'(\gamma) = -\rho \csc^2 \gamma |m(a)|^2. \tag{29}$$

(II) Let all parameters of ϵ be fixed except φ and $\nu(\varphi) := \nu(\epsilon)$. Then

$$\nu'(\varphi) = \theta \csc^2 \varphi |m(b)|^2. \tag{30}$$

(III) Let all parameters of ϵ be fixed except τ and $\nu(\tau) := \nu(\epsilon)$. Then

$$\nu'(\tau) = -2\theta \Im(m(b)\bar{m}^{[1]}(b)). \tag{31}$$

(IV) Let all parameters of ϵ be fixed except K and $\nu(K) := \nu(\epsilon)$. For all I satisfying $\det(K + I) = \frac{\rho}{\theta}$ and $\theta(K + I)E_0(K + I)^* = \rho E_0$. Then

$$\mathrm{d}\nu_K(I) = \theta M^*(b) E_0 I K^{-1} M(b). \tag{32}$$

(V) Let all parameters of ϵ be fixed except C and $\nu(C) := \nu(\epsilon)$. For all I satisfying $\det(C+I) = \rho$ in the neighborhood of C and $\theta(C + I)E_0(C + I)^* = \rho D E_0 D^*$. Then

$$\mathrm{d}\nu_C(I) = \rho M^*(c-) I^* C^{*-1} E_0 M(c-). \tag{33}$$

(VI) Let all parameters of ϵ be fixed except D and $\nu(D) := \nu(\epsilon)$. For all I satisfying $\det(D+I) = \theta$ *in the neighborhood of D and $\theta C E_0 C^* = \rho(D+I)E_0(D+I)^*$. Then*

$$\mathrm{d}\nu_D(I) = -\theta M^*(c+)I^* D^{*-1} E_0 M(c+). \tag{34}$$

(VII) Let all parameters of ϵ be fixed except p and $\nu(\frac{1}{p}) := \nu(\epsilon)$. Then

$$\mathrm{d}\nu_{\frac{1}{p}}(\varsigma) = -\rho\int_a^c \varsigma|m^{[1]}|^2\mathrm{d}x - \theta\int_c^b \varsigma|m^{[1]}|^2\mathrm{d}x,\ \ \varsigma\in L(S,\mathbb{R}). \tag{35}$$

(VIII) Let all parameters of ϵ be fixed except s and $\nu(s) := \nu(\epsilon)$. Then

$$\mathrm{d}\nu_s(\varsigma) = 2\rho\int_a^c \varsigma\Re(m\bar{m}^{[1]})\mathrm{d}x + 2\theta\int_c^b \varsigma\Re(m\bar{m}^{[1]})\mathrm{d}x,\ \ \varsigma\in L(S,\mathbb{R}). \tag{36}$$

(IX) Let all parameters of ϵ be fixed except q and $\nu(q) := \nu(\epsilon)$. Then

$$\mathrm{d}\nu_q(\varsigma) = \rho\int_a^c \varsigma|m|^2\mathrm{d}x + \theta\int_c^b \varsigma|m|^2\mathrm{d}x,\ \ \varsigma\in L(S,\mathbb{R}). \tag{37}$$

(X) Let all parameters of ϵ be fixed except r and $\nu(r) := \nu(\epsilon)$. Then

$$\mathrm{d}\nu_r(\varsigma) = -\nu(r)\cdot(\rho\int_a^c \varsigma|m|^2\mathrm{d}x + \theta\int_c^b \varsigma|m|^2\mathrm{d}x),\ \ \varsigma\in L(S,\mathbb{R}). \tag{38}$$

Proof. Let all the data of ϵ be fixed except one and $\nu(\epsilon_0)$ be the eigenvalue satisfying Theorem 3.2 when $\|\epsilon-\epsilon_0\| < \eta$ for sufficiently small $\eta > 0$. For the above ten cases, we replace $\nu(\epsilon_0)$ by $\nu(\gamma+\varsigma)$, $\nu(\varphi+\varsigma)$, $\nu(\tau+\varsigma)$, $\nu(K+I)$, $\nu(C+I)$, $\nu(D+I)$, $\nu(\frac{1}{p}+\varsigma)$, $\nu(s+\varsigma)$, $\nu(q+\varsigma)$, $\nu(r+\varsigma)$, respectively. Let n be the corresponding normalized eigenfunction.

We prove (29) firstly. For $\varsigma\in\mathbb{R}$ sufficiently small, by (2), we get

$$-(p[m'+sm])' + ps[m'+sm] + qm = \nu(\gamma)rm, \tag{39}$$

$$-(p[\bar{n}'+s\bar{n}])' + ps[\bar{n}'+s\bar{n}] + q\bar{n} = \nu(\gamma+\varsigma)r\bar{n}. \tag{40}$$

It follows from (39) and (40) that

$$\begin{aligned}[\nu(\gamma+\varsigma)-\nu(\gamma)]m\bar{n}r = &-(p[\bar{n}'+s\bar{n}])'m + ps[\bar{n}'+s\bar{n}]m\\ &+(p[m'+sm])'\bar{n} - ps[m'+sm]\bar{n}.\end{aligned}$$

Then

$$\begin{aligned}[\nu(\gamma+\varsigma)-\nu(\gamma)]\langle m,n\rangle =&[\nu(\gamma+\varsigma)-\nu(\gamma)]\Big[\rho\int_a^c m\bar{n}r\mathrm{d}x + \theta\int_c^b m\bar{n}r\mathrm{d}x\Big]\\ =&\rho[-m\bar{n}^{[1]}+m^{[1]}\bar{n}]_a^{c-} + \theta[-m\bar{n}^{[1]}+m^{[1]}\bar{n}]_{c+}^{b}.\end{aligned} \tag{41}$$

The boundary condition (9) yields

$$m(b)\cos\varphi - m^{[1]}(b)\sin\varphi = 0,\ \ \bar{n}(b)\cos\varphi - \bar{n}^{[1]}(b)\sin\varphi = 0.$$

Thus, $-m(b)\bar{n}^{[1]}(b) + m^{[1]}(b)\bar{n}(b) = 0$.

By the transmission condition (4), we obtain

$$\begin{aligned}&\rho[-m\bar{n}^{[1]}+m^{[1]}\bar{n}](c-) - \theta[-m\bar{n}^{[1]}+m^{[1]}\bar{n}](c+)\\ =&-\rho(N^*E_0M)(c-) + \theta(N^*E_0M)(c+)\\ =&-\rho((C^{-1}DN)^*E_0C^{-1}DM)(c+) + \theta(N^*E_0M)(c+)\\ =&-\rho(N^*D^*C^{*-1}E_0C^{-1}DM)(c+) + \theta(N^*E_0M)(c+)\\ =&-\theta(N^*E_0M)(c+) + \theta(N^*E_0M)(c+) = 0,\end{aligned}$$

where $M(c+) = (m(c+), m^{[1]}(c+))^T$. Analogously, the boundary condition (8) implies that

$$m(a)\cos\gamma - m^{[1]}(a)\sin\gamma = 0,\ \ \bar{n}(a)\cos(\gamma+\varsigma) - \bar{n}^{[1]}(a)\sin(\gamma+\varsigma) = 0.$$

Therefore

$$m^{[1]}(a) = \cot\gamma m(a),\ \ \bar{n}^{[1]}(a) = \cot(\gamma+\varsigma)\bar{n}(a).$$

It follows that (41) can be transformed into

$$\begin{aligned}[\nu(\gamma+\varsigma) - \nu(\gamma)]\langle m, n\rangle =& -\rho[-m\bar{n}^{[1]} + m^{[1]}\bar{n}](a)\\ =&\rho[\cot(\gamma+\varsigma) - \cot\gamma]m(a)\bar{n}(a).\end{aligned} \tag{42}$$

Dividing both sides of (42) by ς and taking the limit as $\varsigma \to 0$, then in view of Theorem 3.5, we have

$$\nu'(\gamma) = -\rho\csc^2\gamma|m(a)|^2.$$

Then (29) follows. Similarly, we can get (30).

Secondly, we prove (31). For $\varsigma \in \mathbb{R}$, it follows from (2) and (4) that

$$\begin{aligned}[\nu(\tau+\varsigma) - \nu(\tau)]\langle m, n\rangle =&\rho[-m\bar{n}^{[1]} + m^{[1]}\bar{n}]_a^{c-} + \theta[-m\bar{n}^{[1]} + m^{[1]}\bar{n}]_{c+}^b\\ =&\rho[m\bar{n}^{[1]} - m^{[1]}\bar{n}](a) - \theta[m\bar{n}^{[1]} - m^{[1]}\bar{n}](b)\\ =&\rho(N^*E_0M)(a) - \theta(N^*E_0M)(b).\end{aligned} \tag{43}$$

By complex coupled boundary conditions (11), we have

$$\begin{aligned}N^*(b)E_0 =&e^{-i(\tau+\varsigma)}N^*(a)K^*E_0\\ =&e^{-i(\tau+\varsigma)}N^*(a)E_0(-E_0)K^*E_0\\ =&e^{-i(\tau+\varsigma)}N^*(a)E_0\frac{\rho}{\theta}K^{-1}.\end{aligned} \tag{44}$$

Thus

$$N^*(a)E_0 = \frac{\theta}{\rho}e^{i(\tau+\varsigma)}N^*(b)E_0K. \tag{45}$$

Combining (11), (43) and (45), we get

$$[\nu(\tau+\varsigma) - \nu(\tau)]\langle m, n\rangle = \theta e^{i(\tau+\varsigma)}N^*(b)E_0KM(a) - \theta N^*(b)E_0e^{i\tau}KM(a). \tag{46}$$

Then by virtue of Theorem 3.5, we obtain

$$\nu'(\tau) = -2\theta\Im(m(b)\bar{m}^{[1]}(b)).$$

Thirdly, we prove (32). For $I \in M_{2\times2}(\mathbb{R})$. Similar to the proof of (45), one gets

$$N^*(a)E_0 = \frac{\theta}{\rho}e^{i\tau}N^*(b)E_0(K+I). \tag{47}$$

Then

$$\begin{aligned}[\nu(K+I) - \nu(K)]\langle m, n\rangle =&\rho(N^*E_0M)(a) - \theta(N^*E_0M)(b)\\ =&\rho\cdot\frac{\theta}{\rho}e^{i\tau}N^*(b)E_0(K+I)M(a) - \theta N^*(b)E_0e^{i\tau}KM(a)\\ =&\theta e^{i\tau}N^*(b)E_0IM(a)\\ =&\theta N^*(b)E_0IK^{-1}M(b).\end{aligned} \tag{48}$$

Let $I \to 0$, by Theorem 3.5 we have

$$d\nu_K(I) = \theta M^*(b)E_0IK^{-1}M(b).$$

Next, we prove (33). For $I \in M_{2\times 2}(\mathbb{R})$, by (2) and (14), we obtain

$$[\nu(C+I) - \nu(C)]\langle m, n\rangle = \rho[-m\bar{n}^{[1]} + m^{[1]}\bar{n}](c-) + \theta[m\bar{n}^{[1]} - m^{[1]}\bar{n}](c+).$$

By (4), we have

$$CM(c-) + DM(c+) = 0,$$

$$(C+I)N(c-) + DN(c+) = 0.$$

Thus

$$\begin{aligned}
&[\nu(C+I) - \nu(C)]\langle m, n\rangle \\
=& -\rho(N^*E_0M)(c-) + \theta(N^*E_0M)(c+) \\
=& -\rho(N^*E_0M)(c-) + \theta((D^{-1}(C+I)N)^*E_0D^{-1}CM)(c-) \\
=& -\rho(N^*E_0M)(c-) + \theta(N^*(C+I)^*D^{*-1}E_0D^{-1}CM)(c-) \\
=& -\rho(N^*E_0M)(c-) + \rho(N^*(C+I)^*C^{*-1}E_0C^{-1}CM)(c-) \\
=& \rho(N^*I^*C^{*-1}E_0M)(c-).
\end{aligned}$$

Let $I \to 0$, by Theorem 3.5 we get

$$d\nu_C(I) = \rho M^*(c-)I^*C^{*-1}E_0M(c-).$$

Then (33) follows. Similarly, we can get (34).

To prove (35), let $\frac{1}{p} + \varsigma = \frac{1}{p_0}$, $\varsigma \in L(S, \mathbb{R})$. Then $p - p_0 = pp_0\varsigma$. Using (2), we obtain

$$\begin{aligned}
\left[\nu\left(\frac{1}{p} + \varsigma\right) - \nu\left(\frac{1}{p}\right)\right]\langle m, n\rangle =& \left[\nu\left(\frac{1}{p} + \varsigma\right) - \nu\left(\frac{1}{p}\right)\right]\left[\rho\int_a^c m\bar{n}r\mathrm{d}x + \theta\int_c^b m\bar{n}r\mathrm{d}x\right] \\
=& \rho[m^{[1]}\bar{n} - m\bar{n}^{[1]}]_a^{c-} + \theta[m^{[1]}\bar{n} - m\bar{n}^{[1]}]_{c+}^b \\
&+ \rho\int_a^c (p_0 - p)(m' + sm)(\bar{n}' + s\bar{n})\mathrm{d}x \\
&+ \theta\int_c^b (p_0 - p)(m' + sm)(\bar{n}' + s\bar{n})\mathrm{d}x \\
=& \rho[m^{[1]}\bar{n} - m\bar{n}^{[1]}]_a^{c-} + \theta[m^{[1]}\bar{n} - m\bar{n}^{[1]}]_{c+}^b \\
&- \rho\int_a^c \varsigma[p(m' + sm)][p_0(\bar{n}' + s\bar{n})]\mathrm{d}x \\
&- \theta\int_c^b \varsigma[p(m' + sm)][p_0(\bar{n}' + s\bar{n})]\mathrm{d}x,
\end{aligned}$$

where $\bar{n}^{[1]} = p_0(\bar{n}' + s\bar{n})$.

Using the conditions (3) and (4), we get

$$\left[\nu\left(\frac{1}{p} + \varsigma\right) - \nu\left(\frac{1}{p}\right)\right]\langle m, n\rangle = -\rho\int_a^c \varsigma m^{[1]}\bar{n}^{[1]}\mathrm{d}x - \theta\int_c^b \varsigma m^{[1]}\bar{n}^{[1]}\mathrm{d}x.$$

Let $\varsigma \to 0$, then in view of Theorem 3.5, we obtain

$$d\nu_{\frac{1}{p}}(\varsigma) = -\rho\int_a^c \varsigma|m^{[1]}|^2\mathrm{d}x - \theta\int_c^b \varsigma|m^{[1]}|^2\mathrm{d}x.$$

Now, we prove (36). For $\varsigma \in L(S, \mathbb{R})$. Using (2), one gets

$$\begin{aligned}
[\nu(s+\varsigma)-\nu(s)]\langle m,n\rangle =&[\nu(s+\varsigma)-\nu(s)]\Big[\rho\int_a^c m\bar{n}r\mathrm{d}x+\theta\int_c^b m\bar{n}r\mathrm{d}x\Big]\\
=&\rho[m^{[1]}\bar{n}-m\bar{n}^{[1]}]_a^{c-}+\theta[m^{[1]}\bar{n}-m\bar{n}^{[1]}]_{c+}^b\\
&+\rho\int_a^c \bar{n}^{[1]}[m'+(s+\varsigma)m]-m^{[1]}[\bar{n}'+s\bar{n}]\mathrm{d}x\\
&+\theta\int_c^b \bar{n}^{[1]}[m'+(s+\varsigma)m]-m^{[1]}[\bar{n}'+s\bar{n}]\mathrm{d}x\\
=&\rho[m^{[1]}\bar{n}-m\bar{n}^{[1]}]_a^{c-}+\theta[m^{[1]}\bar{n}-m\bar{n}^{[1]}]_{c+}^b\\
&+\rho\int_a^c \varsigma m\bar{n}^{[1]}+\varsigma m^{[1]}\bar{n}\mathrm{d}x+\theta\int_c^b \varsigma m\bar{n}^{[1]}+\varsigma m^{[1]}\bar{n}\mathrm{d}x.
\end{aligned}$$

By (3)-(4), we find that

$$\begin{aligned}
[\nu(s+\varsigma)-\nu(s)]\langle m,n\rangle =&[\nu(s+\varsigma)-\nu(s)]\Big[\rho\int_a^c m\bar{n}r\mathrm{d}x+\theta\int_c^b m\bar{n}r\mathrm{d}x\Big]\\
=&\rho\int_a^c \varsigma m\bar{n}^{[1]}+\varsigma m^{[1]}\bar{n}\mathrm{d}x+\theta\int_c^b \varsigma m\bar{n}^{[1]}+\varsigma m^{[1]}\bar{n}\mathrm{d}x.
\end{aligned}\tag{49}$$

Let $\varsigma \to 0$, we have

$$\mathrm{d}\nu_s(\varsigma)=2\rho\int_a^c \varsigma\Re(m\bar{m}^{[1]})\mathrm{d}x+2\theta\int_c^b \varsigma\Re(m\bar{m}^{[1]})\mathrm{d}x.$$

Finally, we prove (37). For $\varsigma \in L(S, \mathbb{R})$. Using (2), one gets

$$\begin{aligned}
[\nu(q+\varsigma)-\nu(q)]\langle m,n\rangle =&[\nu(q+\varsigma)-\nu(q)]\Big[\rho\int_a^c m\bar{n}r\mathrm{d}x+\theta\int_c^b m\bar{n}r\mathrm{d}x\Big]\\
=&\rho[m^{[1]}\bar{n}-m\bar{n}^{[1]}]_a^{c-}+\theta[m^{[1]}\bar{n}-m\bar{n}^{[1]}]_{c+}^b\\
&+\rho\int_a^c (q+\varsigma)m\bar{n}\mathrm{d}x+\theta\int_c^b (q+\varsigma)m\bar{n}\mathrm{d}x\\
&-\rho\int_a^c qm\bar{n}\mathrm{d}x-\theta\int_c^b qm\bar{n}\mathrm{d}x.
\end{aligned}$$

Similarly, it follows from (3)-(4) that

$$[\nu(q+\varsigma)-\nu(q)]\langle m,n\rangle=\rho\int_a^c \varsigma m\bar{n}\mathrm{d}x+\theta\int_c^b \varsigma m\bar{n}\mathrm{d}x.$$

Thus

$$\mathrm{d}\nu_q(\varsigma)=\rho\int_a^c \varsigma|m|^2\mathrm{d}x+\theta\int_c^b \varsigma|m|^2\mathrm{d}x.$$

Then (37) holds. Using the similar method, we can get (38), here we omit the details. □

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