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Generalizations of recent singular value inequalities for sums of products of matrices

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Abstract. Given four $n \times n$ complex matrices A, B, X, Y, finding possible bounds for the singular values of the new matrix $AX + YB$ has been of interest.

In this paper, we discuss this interest and prove some new bounds that sharpen recently found bounds in the literature.

Applications of the obtained results include bounds for unitarily invariant bounds and bounds for the real part of certain matrix forms.

1. Introduction

For the rest of this paper, upper case letters will denote elements of the algebra $\mathbb{M}_n$ of all $n \times n$ complex matrices, with identity I. The conjugate transpose of $A \in \mathbb{M}_n$ is denoted by A^*. If $A^* = A$, we say that A is Hermitian, and if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$, we say that A is positive semidefinite. In the latter case, we write $A \geq O$, where O is the zero element in $\mathbb{M}_n$. The absolute value of $A \in \mathbb{M}_n$ is denoted by $|A|$ and is defined as the unique positive semidefinite root of A^*A. The singular values of A are the eigenvalues of $|A|$. These singular values are usually enumerated in non-increasing order, counting multiplicities. Thus, we write $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$.

A matrix norm $|||\cdot|||$ on $\mathbb{M}_n$ is said to be unitarily invariant if it satisfies $|||UAV||| = |||A|||$, for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. It is known that unitarily invariant norms are increasing functions of the singular values. That is, if $s_j(A) \leq s_j(B)$ for all $j = 1, 2, \ldots, n$, for certain $A, B \in \mathbb{M}_n$, then $|||A||| \leq |||B|||$ for all unitarily invariant norms.

The usual operator (or the spectral) norm and the Schatten $p-$norms are among the most useful examples of unitarily invariant norms on $\mathbb{M}_n$. These are defined, respectively, for $A \in \mathbb{M}_n$ by

$$\|A\| = s_1(A) \text{ and } \|A\|_p = \left(\sum_{j=1}^{n} s_j(A)^p\right)^{\frac{1}{p}}, \ p \geq 1.$$

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Two other equivalent definitions for $\|A\|$ are $\|A\| = \sup_{\|x\|=1} \|Ax\|$ and $\|A\| = \sup_{\|x\|=\|y\|=1} \left|\langle Ax, y\rangle\right|$. Related to the latter form, the numerical radius of a matrix A is defined by $\omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|$. The numerical radius is a norm, but it is not a matrix norm (that is, it is not sub-multiplicative), nor is it unitarily invariant. However, it is equivalent to the usual operator norm, where we have the relation [15, Theorem 1.3-1]

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|.$$

To see related results to the above inequality, including its refinements and generalizations, we refer the interested reader to [1, 18, 22, 25, 27].

We have the following lemma among the most basic bounds for the singular values.

Lemma 1.1. *[11]. Let $A, B \in \mathbb{M}_n$. Then*

$$2s_j(AB^*) \leq s_j(A^*A + B^*B) \tag{1}$$

for $j = 1, 2, ..., n$.

In [20], it is shown that if $A, B \in \mathbb{M}_n$ are such that $A, B \geq O$, then

$$|||A + B||| \leq |||A \oplus B||| + \left|\left|\left|A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2}\right|\right|\right|, \tag{2}$$

where the notation $\oplus$ refers to the direct sum of A and B, which is defined by $\begin{bmatrix} A & O \\ O & B \end{bmatrix} \in \mathbb{M}_{2n}$. In particular, considering the spectral norm and the Schatten p-norms, respectively, (2) implies

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \left\|A^{1/2}B^{1/2}\right\| \tag{3}$$

and

$$\|A + B\|_p \leq \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p} + 2^{1/p}\left\|A^{1/2}B^{1/2}\right\|_p \tag{4}$$

for $p \geq 1$. Inequality (3) has been a celebrated refinement of a well-known result [13, Lemma 3.3].

Latter, even (3) has been refined in [20], where it was shown that if $A, B \geq O$, then

$$\|A + B\| \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\left\|A^{1/2}B^{1/2}\right\|^2}\right). \tag{5}$$

In [2], a new generalization of (2), (3) and (4) has been given, where the authors showed that if $A, B, X \in \mathbb{M}_n$ are such that $A, B \geq O$, then

$$|||AX + XB||| \leq \frac{1}{2}|||K||| + \frac{1}{2}|||L||| + |||M|||, \tag{6}$$

where

$$K = A \oplus XBX^*$$

$$L = B \oplus X^*AX$$

and

$$M = A^{1/2}XB^{1/2} \oplus A^{1/2}XB^{1/2}.$$

Upon restricting the norm to the spectral norm or the $p-$norm, the inequality (6) reads as follows

$$\|AX + XB\| \leq \frac{1}{2}\max\{\|A\|, \|XBX^*\|\} + \frac{1}{2}\max\{\|B\|, \|X^*AX\|\} + \left\|A^{1/2}XB^{1/2}\right\|, \tag{7}$$

and for $p \geq 1$,

$$\|AX + XB\|_p \leq \frac{1}{2}\left\{\left(\|A\|_p^p + \|XBX^*\|_p^p\right)^{1/p} + \left(\|B\|_p^p + \|X^*AX\|_p^p\right)^{1/p}\right\} + 2^{1/p}\left\|A^{1/2}XB^{1/2}\right\|_p. \tag{8}$$

In the same reference, it is shown that, for the same matrices,

$$\|AX + XB\| \leq \frac{1}{4}\left(\|W_1\| + \|W_2\| + W_4\right), \tag{9}$$

where

$$W_1 = A + A^{1/2}|X^*|^2 A^{1/2}, W_2 = B + B^{1/2}|X|^2 B^{1/2}, W_3 = A^{1/2}XB^{1/2}$$

and

$$W_4 = \sqrt{(\|W_1\| - \|W_2\|)^2 + 16\|W_3\|^2}.$$

In that reference, the authors showed that (9) is a generalization of (5).

In this paper, we intend to show new generalizations of (2), (3), and (4) from another point of view. Moreover, we give generalizations of (6), (7), (8) and (9).

We give some recent progress in the same direction as the current work. In [4], it is shown that if $A, B, X \in \mathbb{M}_n$ are such that $A, B \geq O$, then for $j = 1, 2, ..., n$,

$$s_j(AX + XB) \leq s_j(C \oplus D) \tag{10}$$

where

$$C = \frac{1}{2}A + \frac{1}{2}A^{1/2}|X^*|^2 A^{1/2} + \left|B^{1/2}X^*A^{1/2}\right|$$

and

$$D = \frac{1}{2}B + \frac{1}{2}B^{1/2}|X|^2 B^{1/2} + \left|A^{1/2}XB^{1/2}\right|.$$

In the same reference, a variant of (10) was shown as follows, for $j = 1, 2, ..., n$,

$$s_j(AX - XB) \leq s_j(W_1 \oplus W_2), \tag{11}$$

where

$$W_1 = \frac{1}{2}A + \frac{1}{2}A^{1/2}|X^*|^2 A^{1/2} \text{ and } W_2 = \frac{1}{2}B + \frac{1}{2}B^{1/2}|X|^2 B^{1/2}.$$

Further generalizations of (10) and (11) were given in [5] in the form

$$s_j(AX + YB) \leq s_j(P_1 \oplus P_2) \tag{12}$$

for $j = 1, 2, ..., n$, where

$$P_1 = \frac{1}{2}A + \frac{1}{2}A^{1/2}|X^*|^2 A^{1/2} + \left|\frac{1}{2}B^{1/2}X^*A^{1/2} + \frac{1}{2}B^{1/2}Y^*A^{1/2}\right|$$

and

$$P_2 = \frac{1}{2}B + \frac{1}{2}B^{1/2}|Y|^2 B^{1/2} + \left|\frac{1}{2}A^{1/2}XB^{1/2} + \frac{1}{2}A^{1/2}YB^{1/2}\right|.$$

Additionally, it is shown that for $j = 1, 2, ..., n$,

$$s_j(AX - YB) \leq s_j(Z \oplus Q) \tag{13}$$

where

$$Z = Z_1 + |Z_2|,\ Z_1 = \frac{1}{2}A + \frac{1}{2}A^{1/2}|X^*|^2 A^{1/2},\ Z_2 = \frac{1}{2}B^{1/2}Y^*A^{1/2} - \frac{1}{2}B^{1/2}X^*A^{1/2},$$

$$Q = Q_1 + |Q_2|, Q_1 = \frac{1}{2}B + \frac{1}{2}B^{1/2}|Y|^2 B^{1/2} \text{ and } Q_2 = \frac{1}{2}A^{1/2}YB^{1/2} - \frac{1}{2}A^{1/2}XB^{1/2}.$$

Clearly, (12) implies, for any unitarily invariant norm $|||\cdot|||$,

$$2|||AX + YB||| \leq |||P_1 \oplus P_2|||. \tag{14}$$

In the same reference [5], the following generalization of (9) was given

$$\|AX + YB\| \leq \frac{1}{4}(\|R_1\| + \|R_2\| + R_4), \tag{15}$$

where

$$R_1 = A + A^{1/2}|X^*|^2 A^{1/2},\ R_2 = B + B^{1/2}|Y|^2 B^{1/2},\ R_3 = A^{1/2}XB^{1/2} + A^{1/2}YB^{1/2}$$

and

$$R_4 = \sqrt{(\|R_1\| - \|R_2\|)^2 + 4\|R_3\|^2}.$$

We refer interested readers to [6–9, 16, 17, 23, 24, 26, 29] for more information on inequalities related to singular values and unitarily invariant norms. As mentioned earlier, this paper aims to present new generalizations of some earlier known results. Additionally, generalizations (10) (11), (12), (13), (14) and (15) will be given. Some applications that involve unitarily invariant norms, usual operator norm, the numerical radius, and block matrices will be given, too.

2. The first singular value bounds with its variants and applications

We begin this section by presenting the following generalization of (10) and (12).

Theorem 2.1. *Let $A, B, X, Y \in \mathbb{M}_n$ be such that $A, B \geq O$, and let $r, s \in [0, 1]$. Then, for $j = 1, 2, ..., n$*

$$2s_j(AX + YB) \leq s_j(E \oplus F), \tag{16}$$

where

$$E = E_1 + |E_2|,\ F = F_1 + |F_2|,$$

$$E_1 = |t|^2 A^{2-2r} + \frac{1}{|t|^2}A^r|X^*|^2 A^r,\ E_2 = \frac{t}{\bar{t}}B^sY^*A^{1-r} + \frac{t}{\bar{t}}B^{1-s}X^*A^r,$$

$$F_1 = |t|^2 B^{2-2s} + \frac{1}{|t|^2}B^s|Y|^2 B^s,\ F_2 = E_2^*,$$

and t is any nonzero complex number with conjugate $\bar{t}$.

Proof. For the given parameters, let $K = \begin{bmatrix} tA^{1-r} & \frac{1}{t}YB^s \\ O & O \end{bmatrix}$ and $L^* = \begin{bmatrix} \frac{1}{t}A^rX & O \\ tB^{1-s} & O \end{bmatrix}$. Then

$$KL^* = AX + YB,\ K^*K = \begin{bmatrix} |t|^2 A^{2-2r} & \frac{\bar{t}}{t}A^{1-r}YB^s \\ \frac{t}{\bar{t}}B^sY^*A^{1-r} & \frac{1}{|t|^2}B^s |Y|^2 B^s \end{bmatrix},$$

and

$$L^*L = \begin{bmatrix} \frac{1}{|t|^2}A^r |X^*|^2 A^r & \frac{\bar{t}}{t}A^rXB^{1-s} \\ \frac{t}{\bar{t}}B^{1-s}X^*A^r & |t|^2 B^{2-2s} \end{bmatrix}.$$

Applying (1) yields

$$\begin{aligned} & 2s_j(AX+YB) \\ \leq\ & s_j\left(\begin{bmatrix} |t|^2 A^{2-2r} & \frac{\bar{t}}{t}A^{1-r}YB^s \\ \frac{t}{\bar{t}}B^sY^*A^{1-r} & \frac{1}{|t|^2}B^s |Y|^2 B^s \end{bmatrix} + \begin{bmatrix} \frac{1}{|t|^2}A^r |X^*|^2 A^r & \frac{\bar{t}}{t}A^rXB^{1-s} \\ \frac{t}{\bar{t}}B^{1-s}X^*A^r & |t|^2 B^{2-2s} \end{bmatrix}\right) \\ =\ & s_j\left(\begin{bmatrix} |t|^2 A^{2-2r} + \frac{1}{|t|^2}A^r |X^*|^2 A^r & \frac{\bar{t}}{t}A^{1-r}YB^s + \frac{\bar{t}}{t}A^rXB^{1-s} \\ \frac{t}{\bar{t}}B^sY^*A^{1-r} + \frac{t}{\bar{t}}B^{1-s}X^*A^r & \frac{1}{|t|^2}B^s |Y|^2 B^s + |t|^2 B^{2-2s} \end{bmatrix}\right) \\ =\ & s_j\begin{bmatrix} E_1 & F_2 \\ E_2 & F_1 \end{bmatrix} \\ =\ & s_j\left(\begin{bmatrix} E_1 & O \\ O & F_1 \end{bmatrix} + \begin{bmatrix} O & F_2 \\ E_2 & O \end{bmatrix}\right) \\ \leq\ & s_j\left(\begin{bmatrix} E_1 & O \\ O & F_1 \end{bmatrix} + \begin{bmatrix} |E_2| & O \\ O & |F_2| \end{bmatrix}\right) \\ =\ & s_j((E_1 + |E_2|) \oplus (F_1 + |F_2|)) \\ =\ & s_j(E \oplus F), \end{aligned} \tag{17}$$

as required. □

Remark 2.2.

(i) Theorem 2.1 is a generalization of (12). This can be seen by letting $t = 1$ and $r = s = \frac{1}{2}$ in (16).

(ii) Letting $Y = X$, $t = 1$ and $r = s = \frac{1}{2}$ in (16) shows how Theorem 2.1 is a generalization of (10).

Theorem 2.1 immediately implies the following bound. For the rest of this paper, the notation $|||\cdot|||$ will refer to any unitarily invariant norm on $\mathbb{M}_n$.

Corollary 2.3. *Let $A, B, X, Y \in \mathbb{M}_n$ be such that $A, B \geq O$, and let $r, s \in [0, 1]$. Then*

$$2|||AX + YB||| \leq |||E \oplus F|||, \tag{18}$$

where E and F are the same as those in Theorem 2.1.

Remark 2.4. *Letting $t = 1$ and $r, s = \frac{1}{2}$ in (18), we get (14).*

In the next result, we generalize (9) and (15).

Theorem 2.5. *Let $A, B, X, Y \in \mathbb{M}_n$ be such that $A, B \geq O$ and let $r, s \in [0, 1]$. Then*

$$\|AX + YB\| \leq \frac{1}{4}(\|E_1\| + \|F_1\|) + \frac{1}{4}\sqrt{(\|E_1\| - \|F_1\|)^2 + 4\|F_2\|^2}, \tag{19}$$

where E_1, E_2, F, F_1 and F_2 are as in Theorem 2.1.

Proof. Restricting (17) to the spectral norm implies $2\|AX+YB\| \le \|H\|$, where

$$H = \begin{bmatrix} |t|^2 A^{2-2r} + \frac{1}{|t|^2} A^r |X^*|^2 A^r & \frac{\bar{t}}{t} A^{1-r} Y B^s + \frac{\bar{t}}{t} A^r X B^{1-s} \\ \frac{t}{\bar{t}} B^s Y^* A^{1-r} + \frac{t}{\bar{t}} B^{1-s} X^* A^r & \frac{1}{|t|^2} B^s |Y|^2 B^s + |t|^2 B^{2-2s} \end{bmatrix}.$$

By simple calculations, we have

$$\|H\| \le \frac{1}{2}\left(\|E_1\| + \|F_1\|\right) + \frac{1}{2}\sqrt{\left(\|E_1\| - \|F_1\|\right)^2 + 4\|F_2\|^2},$$

which completes the proof.
□

Remark 2.6.

(i) Letting $t=1$, $r=s=\frac{1}{2}$ in (19), we obtain (15).

(ii) Letting $t=1$, $r=s=\frac{1}{2}$ and $Y=X$ in (19), (9) follows.

On the other hand, a generalization of (6) may be stated as follows.

Theorem 2.7. *Let $A, B, X, Y \in \mathbb{M}_n$ be such that $A, B \ge O$ and let $r, s \in [0,1]$. Then*

$$\begin{aligned} 2|||AX+YB||| \le\ & \left|\left|\left| |t|^2 A^{2-2r} \oplus \frac{1}{|t|^2} Y B^{2s} Y^* \right|\right|\right| + \left|\left|\left| |t|^2 B^{2-2s} \oplus \frac{1}{|t|^2} X^* A^{2r} X \right|\right|\right| \\ & + \left|\left|\left| \left(\frac{\bar{t}}{t} A^{1-r} Y B^s + \frac{\bar{t}}{t} A^r X B^{1-s}\right) \oplus \left(\frac{t}{\bar{t}} B^s Y^* A^{1-r} + \frac{t}{\bar{t}} B^{1-s} X^* A^r\right) \right|\right|\right| \end{aligned} \tag{20}$$

for all nonzero complex numbers t.

Proof. It follows from (17) that

$$\begin{aligned} 2|||AX+YB||| &\le \left|\left|\left| \begin{bmatrix} |t|^2 A^{2-2r} + \frac{1}{|t|^2} A^r |X^*|^2 A^r & \frac{\bar{t}}{t} A^{1-r} Y B^s + \frac{\bar{t}}{t} A^r X B^{1-s} \\ \frac{t}{\bar{t}} B^s Y^* A^{1-r} + \frac{t}{\bar{t}} B^{1-s} X^* A^r & \frac{1}{|t|^2} B^s |Y|^2 B^s + |t|^2 B^{2-2s} \end{bmatrix} \right|\right|\right| \\ &= \left|\left|\left| \begin{bmatrix} |t|^2 A^{2-2r} & O \\ O & \frac{1}{|t|^2} B^s |Y|^2 B^s \end{bmatrix} + \begin{bmatrix} \frac{1}{|t|^2} A^r |X^*|^2 A^r & O \\ O & |t|^2 B^{2-2s} \end{bmatrix} + \begin{bmatrix} O & \frac{\bar{t}}{t} A^{1-r} Y B^s + \frac{\bar{t}}{t} A^r X B^{1-s} \\ \frac{t}{\bar{t}} B^s Y^* A^{1-r} + \frac{t}{\bar{t}} B^{1-s} X^* A^r & O \end{bmatrix} \right|\right|\right| \\ &\le \left|\left|\left| \begin{bmatrix} |t|^2 A^{2-2r} & O \\ O & \frac{1}{|t|^2} Y B^{2s} Y^* \end{bmatrix} \right|\right|\right| + \left|\left|\left| \begin{bmatrix} \frac{1}{|t|^2} X^* A^{2r} X & O \\ O & |t|^2 B^{2-2s} \end{bmatrix} \right|\right|\right| \\ &\quad + \left|\left|\left| \begin{bmatrix} O & \frac{\bar{t}}{t} A^{1-r} Y B^s + \frac{\bar{t}}{t} A^r X B^{1-s} \\ \frac{t}{\bar{t}} B^s Y^* A^{1-r} + \frac{t}{\bar{t}} B^{1-s} X^* A^r & O \end{bmatrix} \right|\right|\right|, \end{aligned}$$

which is equivalent to (20). □

Remark 2.8.

(i) Letting $Y=X$, $t=1$, and $r=s=\frac{1}{2}$ in (20), we get, as a special case, (6).

(ii) Letting $Y=X$, $t=1$ and $r=s=\frac{1}{2}$ in (20) and specifying the inequality to the spectral norm and the Schatten $p-$norms, respectively, we obtain (7) and (8).

The following particular case of (20) is a generalization of (2).

Corollary 2.9. *Let $A, B, X, Y \in \mathbb{M}_n$ be such that $A, B \geq O$ and let $r, s \in [0,1]$. Then*

$$2|||A+B||| \leq \left|\left|\left| |t|^2 A \oplus \frac{1}{|t|^2} B \right|\right|\right| + \left|\left|\left| |t|^2 B \oplus \frac{1}{|t|^2} A \right|\right|\right| + \left|\left|\left| 2\frac{\bar{t}}{t} A^{1/2}B^{1/2} \oplus 2\frac{t}{\bar{t}} B^{1/2}A^{1/2} \right|\right|\right| \tag{21}$$

for all nonzero complex numbers t.

Proof. This follows from (20) by letting $X = Y = I$, and $r = s = \frac{1}{2}$. □

Remark 2.10. *Specifying (21) to the spectral norm and the Schatten p− norms, we get the following inequalities, which are generalizations of (3) and (4), respectively.*

$$2\|A+B\| \leq \max\{|t|^2 \|A\|, \frac{1}{|t|^2}\|B\|\} + \max\{|t|^2 \|B\|, \frac{1}{|t|^2}\|A\|\} + 2\left\|A^{1/2}B^{1/2}\right\|$$

and

$$2\|A+B\|_p \leq \left(|t|^2 \|A\|_p^p + \frac{1}{|t|^2}\|B\|_p^p\right)^{1/p} + \left(|t|^2 \|B\|_p^p + \frac{1}{|t|^2}\|B\|_p^p\right)^{1/p} + 2^{1+1/p}\left\|A^{1/2}B^{1/2}\right\|_p.$$

The next theorem is a generalization of (11) and (13).

Theorem 2.11. *Let $A, B, X, Y \in \mathbb{M}_n$ be such that $A, B \geq O$ and let $r, s \in [0,1]$. Then for $j = 1, 2, ..., n$,*

$$2s_j(AX - YB) \leq s_j(G \oplus H) \tag{22}$$

where

$$G_1 = |t|^2 A^{2-2r} + \frac{1}{|t|^2} A^r |X^*|^2 A^r,\ G_2 = \frac{t}{\bar{t}} B^s Y^* A^{1-r} - \frac{t}{\bar{t}} B^{1-s} X^* A^r,\ G = G_1 + |G_2|,$$

$$H_1 = |t|^2 B^{2-2s} + \frac{1}{|t|^2} B^s |Y|^2 B^s,\ H_2 = \frac{\bar{t}}{t} A^{1-r} Y B^s - \frac{\bar{t}}{t} A^r X B^{1-s},\ H = H_1 + |H_2|,$$

and t is any nonzero complex number.

Remark 2.12.

(i) Letting $t = 1$ and $r = s = \frac{1}{2}$ in (22), we have (13).

(ii) Letting $Y = X$, $t = 1$ and $r = s = \frac{1}{2}$ in (22) implies (11).

3. The second singular value bound with its variants and applications

This section presents another upper bound for $s_j(AX + YB)$, which is easier than that in Theorem 2.1. Its consequences will be discussed, too.

Theorem 3.1. *Let $A, X, Y, B \in M_n$ be such that $A, B \geq O$. Then for any $0 \leq r, s \leq 1$, and any nonzero complex number t,*

$$s_j(AX + YB) \leq \left\|\frac{1}{|t|^2} Y B^{2s} Y^* + |t|^2 A^{2(1-r)}\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\left(\frac{1}{|t|^2} X^* A^{2r} X + |t|^2 B^{2(1-s)}\right),$$

where $j = 1, 2, \ldots, n$.

Proof. Let $x, y \in \mathbb{C}^n$ be two unit vectors. Then, for K, L as in Theorem 2.1,

$$\begin{aligned}
&\left|\left\langle (AX+YB)\,x, y\right\rangle\right| \\
&= \left|\left\langle KL^* x, y\right\rangle\right| \\
&= \left|\left\langle L^* x, K^* y\right\rangle\right| \\
&\leq \|L^* x\| \left\|K^* y\right\| \\
&= \sqrt{\langle LL^* x, x\rangle \langle KK^* y, y\rangle} \\
&= \sqrt{\left\langle \begin{bmatrix} \frac{1}{|t|^2} X^* A^{2r} X + |t|^2 B^{2(1-s)} & O \\ O & O \end{bmatrix} x, x \right\rangle \left\langle \begin{bmatrix} \frac{1}{|t|^2} Y B^{2s} Y^* + |t|^2 A^{2(1-r)} & O \\ O & O \end{bmatrix} y, y \right\rangle}.
\end{aligned}$$

That is,

$$\begin{aligned}
&\left|\left\langle (AX+YB)\,x, y\right\rangle\right| \\
&\leq \sqrt{\left\langle \begin{bmatrix} \frac{1}{|t|^2} X^* A^{2r} X + |t|^2 B^{2(1-s)} & O \\ O & O \end{bmatrix} x, x \right\rangle \left\langle \begin{bmatrix} \frac{1}{|t|^2} Y B^{2s} Y^* + |t|^2 A^{2(1-r)} & O \\ O & O \end{bmatrix} y, y \right\rangle}.
\end{aligned}$$

Now, by taking supremum over $y \in \mathbb{C}^n$ with $\left\|y\right\| = 1$, we infer that

$$\begin{aligned}
&\|(AX+YB)\,x\| \\
&\leq \sqrt{\left\| \frac{1}{|t|^2} Y B^{2s} Y^* + |t|^2 A^{2(1-r)} \right\| \left\langle \begin{bmatrix} \frac{1}{|t|^2} X^* A^{2r} X + |t|^2 B^{2(1-s)} & O \\ O & O \end{bmatrix} x, x \right\rangle} \\
&\leq \sqrt{\left\| \frac{1}{|t|^2} Y B^{2s} Y^* + |t|^2 A^{2(1-r)} \right\| \left\| \begin{bmatrix} \frac{1}{|t|^2} X^* A^{2r} X + |t|^2 B^{2(1-s)} & O \\ O & O \end{bmatrix} x \right\|}.
\end{aligned}$$

Thus, using the min-max principle

$$\begin{aligned}
&s_j\,(AX+YB) \\
&= \max_{\dim M = j} \min_{\substack{x\in M \\ \|x\|=1}} \|(AX+YB)\,x\| \\
&\leq \left\| \frac{1}{|t|^2} Y B^{2s} Y^* + |t|^2 A^{2(1-r)} \right\|^{\frac{1}{2}} \max_{\dim M = j} \min_{\substack{x\in M \\ \|x\|=1}} \left\| \begin{bmatrix} \frac{1}{|t|^2} X^* A^{2r} X + |t|^2 B^{2(1-s)} & O \\ O & O \end{bmatrix} x \right\|^{\frac{1}{2}} \\
&= \left\| \frac{1}{|t|^2} Y B^{2s} Y^* + |t|^2 A^{2(1-r)} \right\|^{\frac{1}{2}} \max_{\dim M = j} \min_{\substack{x\in M \\ \|x\|=1}} \left\| \begin{bmatrix} \frac{1}{|t|^2} X^* A^{2r} X + |t|^2 B^{2(1-s)} & O \\ O & O \end{bmatrix} x \right\|^{\frac{1}{2}} \\
&= \left\| \frac{1}{|t|^2} Y B^{2s} Y^* + |t|^2 A^{2(1-r)} \right\|^{\frac{1}{2}} s_j^{\frac{1}{2}} \left(\left(\frac{1}{|t|^2} X^* A^{2r} X + |t|^2 B^{2(1-s)} \right) \oplus O \right),
\end{aligned}$$

as required. □

Let $T_1 = U\,|T_1|$ and $T_2 = V\,|T_2|$ be the polar decompositions of T_1 and T_2, respectively. Substituting $A = |T_1|^{\frac{1}{2}}$, $X = |T_1|^{\frac{1}{2}} U^*$, $B = |T_2|^{\frac{1}{2}}$, and $Y = V|T_2|^{\frac{1}{2}}$, in Theorem 3.1, we obtain the following corollary, whose application can be seen in Section 5 below.

Corollary 3.2. *Let $T_1, T_2 \in M_n$. Then for any $0 \le r, s \le 1$ and any nonzero complex number t,*

$$s_j\left(T_1^* + T_2\right) \le \left\| \frac{1}{|t|^2}\left|T_2^*\right|^{1+s} + |t|^2 |T_1|^{1-r} \right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\left(\frac{1}{|t|^2}\left|T_1^*\right|^{1+r} + |t|^2|T_2|^{1-s}\right)$$

where $j = 1, 2, \ldots, n$.

4. Singular value bounds for a certain block partition

We recall that a matrix $T \in \mathbb{M}_n$ has a Cartesian decomposition in the form $\Re T + \mathrm{i}\Im T$, where $\Re T$ and $\Im T$ are the real and imaginary parts of T, defined by

$$\Re T = \frac{T + T^*}{2} \text{ and } \Im T = \frac{T - T^*}{2\mathrm{i}}.$$

When $\Re T \ge O$, we say that T is an accretive matrix. If both $\Re T, \Im T \ge O$, T is said to be accretive-dissipative. Let $\mathbb{T} \in \mathbb{M}_{2n}$. Then the Cartesian decomposition of $\mathbb{T}$ can be written as

$$\mathbb{T} := \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \mathrm{i}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix};\ A_{12} = A_{21}^*, B_{12} = B_{21}^*, \tag{23}$$

where $T_{ij}, A_{ij}, B_{ij} \in \mathbb{M}_n$. This decomposition has been given in [14].

Before proceeding to the next result, we state the following lemma from [19, Lemma 1].

Lemma 4.1. *Let $A, B, C \in \mathbb{M}_n$ be such that $A, B \ge O$. Then*

$$\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \ge O \Leftrightarrow \left|\langle Cx, y\rangle\right|^2 \le \langle Ax, x\rangle \langle By, y\rangle, \forall x, y \in \mathbb{C}^n.$$

Theorem 4.2. *Let $\mathbb{T}$ be partitioned as in* (23)*. If $\mathbb{T}$ is an accretive-dissipative matrix, then*

$$s_j(T_{12}) \le \|A_{22} + B_{22}\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(A_{11} + B_{11})$$

for $j = 1, 2, \ldots, n$.

Proof. Let $x, y \in \mathbb{C}^n$ be unit vectors. Since $\mathbb{T}$ is accretive-dissipative, it follows that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \ge O.$$

Applying Lemma 4.1, we have

$$\left|\langle A_{12}x, y\rangle\right| \le \sqrt{\langle A_{11}x, x\rangle \langle A_{22}y, y\rangle} \text{ and } \left|\langle B_{12}x, y\rangle\right| \le \sqrt{\langle B_{11}x, x\rangle \langle B_{22}y, y\rangle}.$$

Consequently, applying the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|\langle T_{12}x, y\rangle\right| &= \left|\langle (A_{12} + \mathrm{i}B_{12})\, x, y\rangle\right| \\
&= \left|\langle A_{12}x, y\rangle + \mathrm{i}\,\langle B_{12}x, y\rangle\right| \\
&\le \left|\langle A_{12}x, y\rangle\right| + \left|\langle B_{12}x, y\rangle\right| \\
&\le \sqrt{\langle A_{11}x, x\rangle \langle A_{22}y, y\rangle} + \sqrt{\langle B_{11}x, x\rangle \langle B_{22}y, y\rangle} \\
&\le \sqrt{\langle A_{11}x, x\rangle + \langle B_{11}x, x\rangle}\sqrt{\langle A_{22}y, y\rangle + \langle B_{22}y, y\rangle} \\
&= \sqrt{\langle (A_{11} + B_{11})\, x, x\rangle}\sqrt{\langle (A_{22} + B_{22})\, y, y\rangle}.
\end{aligned}$$

That is,

$$\left|\langle T_{12}x, y\rangle\right| \leq \sqrt{\langle (A_{11}+B_{11})\, x, x\rangle}\sqrt{\langle (A_{22}+B_{22})\, y, y\rangle}.$$

Now, taking the supremum over $y \in \mathbb{C}^n$ with $\left\|y\right\| = 1$, we conclude that

$$\|T_{12}x\| \leq \|A_{22}+B_{22}\|^{\frac{1}{2}} \|(A_{11}+B_{11})\, x\|^{\frac{1}{2}}.$$

So, applying the min-max principle and proceeding as in the proof of Theorem 3.1, we get the desired result. □

Theorem 4.3. *Let* $\mathbb{T}$ *be partitioned as in* (23)*. Then for any* $j = 1, 2, \ldots, n$*,*

$$s_j\left(|T_{12}|^2 + \left|T_{21}^*\right|^2\right) \leq 2\left\||A_{12}|^2 + |B_{12}|^2\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\left(|A_{12}|^2 + |B_{12}|^2\right).$$

Proof. Let $x, y \in \mathbb{C}^n$ be two unit vectors. Then

$$\begin{aligned}
\left|\left\langle\left(|T_{12}|^2 + \left|T_{21}^*\right|^2\right)x, y\right\rangle\right| &= \left|\left\langle\left(|A_{12} + \mathrm{i}B_{12}|^2 + |A_{12} - \mathrm{i}B_{12}|^2\right)x, y\right\rangle\right| \\
&= 2\left|\left\langle\left(|A_{12}|^2 + |B_{12}|^2\right)x, y\right\rangle\right| \\
&= 2\left|\left\langle |A_{12}|^2 x, y\right\rangle + \left\langle |B_{12}|^2 x, y\right\rangle\right| \\
&\leq 2\left(\left|\left\langle |A_{12}|^2 x, y\right\rangle\right| + \left|\left\langle |B_{12}|^2 x, y\right\rangle\right|\right) \\
&\leq 2\left(\sqrt{\left\langle |A_{12}|^2 x, x\right\rangle\left\langle |A_{12}|^2 y, y\right\rangle} + \sqrt{\left\langle |B_{12}|^2 x, x\right\rangle\left\langle |B_{12}|^2 y, y\right\rangle}\right) \\
&\leq 2\sqrt{\left\langle |A_{12}|^2 x, x\right\rangle + \left\langle |B_{12}|^2 x, x\right\rangle}\sqrt{\left\langle |A_{12}|^2 y, y\right\rangle + \left\langle |B_{12}|^2 y, y\right\rangle} \\
&\leq 2\sqrt{\left\langle\left(|A_{12}|^2 + |B_{12}|^2\right)x, x\right\rangle}\sqrt{\left\langle\left(|A_{12}|^2 + |B_{12}|^2\right)y, y\right\rangle}.
\end{aligned}$$

That is,

$$\left|\left\langle\left(|T_{12}|^2 + \left|T_{21}^*\right|^2\right)x, y\right\rangle\right| \leq 2\sqrt{\left\langle\left(|A_{12}|^2 + |B_{12}|^2\right)x, x\right\rangle}\sqrt{\left\langle\left(|A_{12}|^2 + |B_{12}|^2\right)y, y\right\rangle}.$$

Now, by taking supremum over $y \in \mathbb{C}^n$ with $\left\|y\right\| = 1$, we deduce that

$$\begin{aligned}
\left\|\left(|T_{12}|^2 + \left|T_{21}^*\right|^2\right)x\right\| &\leq 2\left\||A_{12}|^2 + |B_{12}|^2\right\|^{\frac{1}{2}}\left\langle\left(|A_{12}|^2 + |B_{12}|^2\right)x, x\right\rangle^{\frac{1}{2}} \\
&\leq 2\left\||A_{12}|^2 + |B_{12}|^2\right\|^{\frac{1}{2}}\left\|\left(|A_{12}|^2 + |B_{12}|^2\right)x\right\|^{\frac{1}{2}}.
\end{aligned}$$

So, applying the min-max principle and proceeding as in the proof of Theorem 3.1, we get the desired result □

5. Bounds for the real part

In this section, we follow the ideas of the previous section to obtain some bounds for the norms of the real parts of certain products. This entails some numerical radii bounds, as we shall see.

As a first result, we have the following consequence of Corollary 3.2. This bound for the numerical radius was given as one of the sharpest simple bounds in the literature in [21].

Corollary 5.1. *Let $T \in \mathbb{M}_n$. Then*

$$\omega(T) \leq \frac{1}{2} \left\| |T| + |T^*| \right\|.$$

Proof. In Corollary 3.2, let $T_1 = T_2 = T, s = r = 0, t = 1$ and $j = 1$. Then

$$2 \left\| (\Re T) \right\| \leq \left\| |T| + |T^*| \right\|.$$

Replacing T, in the above inequality, by $e^{i\theta}T$, and using the fact $\omega(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re \left(e^{i\theta} T \right) \right\|$ (see [28]) imply the desired inequality. □

Next, we state the upper bound for the norm of the real part of the product AX when $A \geq O$. Applications of this result will follow.

Proposition 5.2. *Let $A, X \in \mathbb{M}_n$ be such that $A \geq O$. Then for any $0 \leq r, s \leq 1$ and any nonzero real number t,*

$$|||\Re (AX)||| \leq \frac{1}{2} \left|\left|\left| \begin{bmatrix} \Re \left(A^r X A^{1-r} \right) & \frac{1}{2} \left(\frac{1}{t^2} A^r X X^* A^s + t^2 A^{2-(r+s)} \right) \\ \frac{1}{2} \left(t^2 A^{2-(r+s)} + \frac{1}{t^2} A^s X X^* A^r \right) & \Re \left(A^s X A^{1-s} \right) \end{bmatrix} \right|\right|\right|.$$

Proof. Let

$$K = \begin{bmatrix} tA^{1-r} & \frac{1}{t} X^* A^s \\ O & O \end{bmatrix}, \ L^* = \begin{bmatrix} \frac{1}{t} A^r X & O \\ tA^{1-s} & O \end{bmatrix}.$$

Then

$$\begin{aligned} & 2|||\Re (AX)||| \\ & = |||KL^*||| \\ & \leq \left|\left|\left| \Re \left(\begin{bmatrix} A^r X A^{1-r} & \frac{1}{t^2} A^r X X^* A^s \\ t^2 A^{2-(r+s)} & A^{1-s} X^* A^s \end{bmatrix} \right) \right|\right|\right| \\ & = \left|\left|\left| \begin{bmatrix} \Re \left(A^r X A^{1-r} \right) & \frac{1}{2} \left(\frac{1}{t^2} A^r X X^* A^s + t^2 A^{2-(r+s)} \right) \\ \frac{1}{2} \left(t^2 A^{2-(r+s)} + \frac{1}{t^2} A^s X X^* A^r \right) & \Re \left(A^s X A^{1-s} \right) \end{bmatrix} \right|\right|\right|, \end{aligned}$$

where we have used the fact that $|||KL^*||| \leq |||\Re(L^*K)|||$ since KL^* is Hermitian to obtain the first inequality in the above argument (see [10, Proposition IX. 1.2]). This completes the proof. □

Corollary 5.3. *Let $A, X \in \mathbb{M}_n$ such that $A \geq O$. Then for any $0 \leq r, s \leq 1$ and any nonzero real number t,*

$$\begin{aligned} & \|\Re (AX)\| \\ & \leq \frac{1}{4} \left(\left\| \Re \left(A^r X A^{1-r} \right) \right\| + \left\| \Re \left(A^s X A^{1-s} \right) \right\| \right) \\ & \quad + \frac{1}{4} \sqrt{ \left(\left\| \Re \left(A^r X A^{1-r} \right) \right\| - \left\| \Re \left(A^s X A^{1-s} \right) \right\| \right)^2 + \frac{1}{4} \left(\left\| \frac{1}{t^2} A^r X X^* A^s + t^2 A^{2-(r+s)} \right\| + \left\| t^2 A^{2-(r+s)} + \frac{1}{t^2} A^s X X^* A^r \right\| \right)^2 }. \end{aligned}$$

In particular, when $r = s$,

$$\left\| \Re (AX) \right\| \leq \frac{1}{2} \left\| \Re \left(A^r X A^{1-r} \right) \right\| + \frac{1}{4} \left\| \frac{1}{t^2} A^r X X^* A^r + t^2 A^{2(1-r)} \right\|. \tag{24}$$

Proof. We prove the first inequality. Applying Proposition 5.2, then direct calculations of the singular values imply

$$\begin{aligned}
&\|\Re(AX)\| \\
&\leq \frac{1}{2}\left\|\begin{bmatrix} \Re\left(A^rXA^{1-r}\right) & \frac{1}{2}\left(\frac{1}{t^2}A^rXX^*A^s + t^2A^{2-(r+s)}\right) \\ \frac{1}{2}\left(t^2A^{2-(r+s)} + \frac{1}{t^2}A^sXX^*A^r\right) & \Re\left(A^sXA^{1-s}\right) \end{bmatrix}\right\| \\
&\leq \frac{1}{4}\left(\left\|\Re\left(A^rXA^{1-r}\right)\right\| + \left\|\Re\left(A^sXA^{1-s}\right)\right\|\right) \\
&\quad + \frac{1}{4}\sqrt{\left(\left\|\Re\left(A^rXA^{1-r}\right)\right\| - \left\|\Re\left(A^sXA^{1-s}\right)\right\|\right)^2 + \frac{1}{4}\left(\left\|\frac{1}{t^2}A^rXX^*A^s + t^2A^{2-(r+s)}\right\| + \left\|t^2A^{2-(r+s)} + \frac{1}{t^2}A^sXX^*A^r\right\|\right)^2},
\end{aligned}$$

as required. □

Corollary 5.4. *Let $A, X \in \mathbb{M}_n$ such that $A \geq O$. Then for any $0 \leq r, s \leq 1$,*

$$\begin{aligned}
\left\|\Re(AX)\right\| \leq{}& \frac{1}{4}\left(\left\|\Re\left(A^rXA^{1-r}\right)\right\| + \left\|\Re\left(A^sXA^{1-s}\right)\right\|\right) \\
&+ \frac{1}{4}\sqrt{\left(\left\|\Re\left(A^rXA^{1-r}\right)\right\| - \left\|\Re\left(A^sXA^{1-s}\right)\right\|\right)^2 + 4\left\|A^rXX^*A^s\right\|\left\|A\right\|^{2-(r+s)}}.
\end{aligned}$$

In particular, when $r = s$,

$$\left\|\Re(AX)\right\| \leq \frac{1}{2}\left(\left\|\Re(XA)\right\| + \|A\|\,\|X\|\right).$$

Proof. We prove the first inequality. It follows from Corollary 5.3 that

$$\begin{aligned}
\left\|\Re(AX)\right\| \leq{}& \frac{1}{4}\left(\left\|\Re\left(A^rXA^{1-r}\right)\right\| + \left\|\Re\left(A^sXA^{1-s}\right)\right\|\right) \\
&+ \frac{1}{4}\sqrt{\left(\left\|\Re\left(A^rXA^{1-r}\right)\right\| - \left\|\Re\left(A^sXA^{1-s}\right)\right\|\right)^2 + \left(\frac{1}{t^2}\left\|A^rXX^*A^s\right\| + t^2\|A\|^{2-(r+s)}\right)^2}.
\end{aligned}$$

We infer the desired result by taking a minimum over $t > 0$. □

6. Further applications of Theorem 2.1

We conclude this work by presenting the following discussion, leading to some interesting bounds for the spectral norm. Assume that $0 < t \in \mathbb{R}$ and let $T_1 = U\,|T_1|$ and $T_2 = V\,|T_2|$ be the polar decompositions of T_1 and T_2, respectively. Letting $A = |T_1|^{\frac{1}{2}}$, $X = |T_1|^{\frac{1}{2}}U^*$, $Y = V|T_2|^{\frac{1}{2}}$, and $B = |T_2|^{\frac{1}{2}}$, in Theorem 2.1, we get

$$\begin{aligned}
&2\left\|T_1^* + T_2\right\| \\
&\leq \max\left\{\left\|t^2|T_1|^{1-r} + \frac{1}{t^2}|T_1|^{1+r} + \left||T_2|^{\frac{1+s}{2}}V^*|T_1|^{\frac{1-r}{2}} + |T_2|^{\frac{1-s}{2}}U|T_1|^{\frac{1+r}{2}}\right|\right\|\right. \\
&\quad \left., \left\|t^2|T_2|^{1-s} + \frac{1}{t^2}|T_2|^{1+s} + \left||T_1|^{\frac{1-r}{2}}V|T_2|^{\frac{1+s}{2}} + |T_1|^{\frac{1+r}{2}}U^*|T_2|^{\frac{1-s}{2}}\right|\right\|\right\}.
\end{aligned} \tag{25}$$

If we set $r = s = 1$, in (25), we infer that

$$2\left\|T_1^* + T_2\right\| \leq \max\left\{\left\|t^2I + \frac{1}{t^2}|T_1|^2 + \left|T_2^* + T_1\right|\right\|, \left\|t^2I + \frac{1}{t^2}|T_2|^2 + \left|T_2 + T_1^*\right|\right\|\right\}.$$

If we put $r = s = 0$, in (25), we obtain

$$\begin{aligned}
2\left\|T_1^* + T_2\right\| \leq \max&\left\{\left\|t^2\,|T_1| + \frac{1}{t^2}\,|T_1| + \left||T_2|^{-\frac{1}{2}}T_2^*|T_1|^{\frac{1}{2}} + |T_2|^{\frac{1}{2}}T_1|T_1|^{-\frac{1}{2}}\right|\right\|\right. \\
&\left., \left\|t^2\,|T_2| + \frac{1}{t^2}\,|T_2| + \left||T_1|^{\frac{1}{2}}T_2|T_2|^{-\frac{1}{2}} + |T_1|^{-\frac{1}{2}}T_1^*|T_2|^{\frac{1}{2}}\right|\right\|\right\}
\end{aligned} \tag{26}$$

provided that T_1, T_2 are invertible. From (26), we have

$$\begin{aligned}
&2\left\|T_1^* + T_2\right\| \\
&\le \max\left\{\left\|t^2\,|T_1| + \frac{1}{t^2}\,|T_1| + \left||T_2|^{-\frac{1}{2}}T_2^*|T_1|^{\frac{1}{2}} + |T_2|^{\frac{1}{2}}T_1|T_1|^{-\frac{1}{2}}\right|\right\|\right.\\
&\quad\left.,\left\|t^2\,|T_2| + \frac{1}{t^2}\,|T_2| + \left||T_1|^{\frac{1}{2}}T_2|T_2|^{-\frac{1}{2}} + |T_1|^{-\frac{1}{2}}T_1^*|T_2|^{\frac{1}{2}}\right|\right\|\right\}\\
&\le \max\left\{\left(t^2 + \frac{1}{t^2}\right)\|T_1\| + \left\||T_2|^{-\frac{1}{2}}T_2^*|T_1|^{\frac{1}{2}} + |T_2|^{\frac{1}{2}}T_1|T_1|^{-\frac{1}{2}}\right\|\right.\\
&\quad\left.,\left(t^2 + \frac{1}{t^2}\right)\|T_2\| + \left\||T_1|^{\frac{1}{2}}T_2|T_2|^{-\frac{1}{2}} + |T_1|^{-\frac{1}{2}}T_1^*|T_2|^{\frac{1}{2}}\right\|\right\}\\
&\le \left(t^2 + \frac{1}{t^2}\right)\max\left\{\|T_1\|, \|T_2\|\right\}\\
&\quad + \max\left\{\left\||T_2|^{-\frac{1}{2}}T_2^*|T_1|^{\frac{1}{2}} + |T_2|^{\frac{1}{2}}T_1|T_1|^{-\frac{1}{2}}\right\|, \left\||T_1|^{\frac{1}{2}}T_2|T_2|^{-\frac{1}{2}} + |T_1|^{-\frac{1}{2}}T_1^*|T_2|^{\frac{1}{2}}\right\|\right\}.
\end{aligned}$$

That is,

$$\begin{aligned}
\left\|T_1^* + T_2\right\| &\le \max\left\{\|T_1\|, \|T_2\|\right\}\\
&\quad + \frac{1}{2}\max\left\{\left\||T_2|^{-\frac{1}{2}}T_2^*|T_1|^{\frac{1}{2}} + |T_2|^{\frac{1}{2}}T_1|T_1|^{-\frac{1}{2}}\right\|, \left\||T_1|^{\frac{1}{2}}T_2|T_2|^{-\frac{1}{2}} + |T_1|^{-\frac{1}{2}}T_1^*|T_2|^{\frac{1}{2}}\right\|\right\}.
\end{aligned}$$

If $T = U|T|$ is the polar decomposition of T, Inequality (26) also implies

$$\left\|\Re T\right\| \le \frac{1}{4}\left\|t^2\,|T| + \frac{1}{t^2}\,|T| + 2\left|\Re\widetilde{T}\right|\right\|, \tag{27}$$

where $\widetilde{T}$ is the Aluthge transform of T, defined by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This transform played a key role in advancing some norm and numerical radius inequalities, as one can see in [1, 3, 25, 27, 28]. While the above argument works for invertible T_1, T_2, the conclusion in (27) holds true for any $T \in \mathbb{M}_n$ by a limit argument.

Thus, we reach the following corollary.

Corollary 6.1. *Let $T \in \mathbb{M}_n$. Then*

$$\begin{aligned}
\left\|\Re T\right\| &\le \frac{1}{4}\min_{t>0}\left\|\left(t^2 + \frac{1}{t^2}\right)|T| + 2\left|\Re\widetilde{T}\right|\right\|\\
&\le \frac{1}{4}\min_{t>0}\left(t^2 + \frac{1}{t^2}\right)\|T\| + \frac{1}{2}\left\|\Re\widetilde{T}\right\|\\
&= \frac{1}{2}\|T\| + \frac{1}{2}\left\|\Re\widetilde{T}\right\|.
\end{aligned}$$

If we substitute T by $e^{i\theta}T$ in the above corollary, we obtain $\omega(T) \le \frac{1}{2}\left(\|T\| + \omega\left(\widetilde{T}\right)\right)$; which is the celebrated bound shown in [28].

Assume that $0 < t \in \mathbb{R}$. Let $T_1 = U\,|T_1|$ and $T_2 = V\,|T_2|$ be the polar decompositions of T_1 and T_2, respectively. Letting $A = |T_1|^{\frac{1}{2}}$, $X = |T_1|^{\frac{1}{2}}U^*$, $Y = V|T_2|^{\frac{1}{2}}$, and $B = |T_2|^{\frac{1}{2}}$, in (19), we get

$$\begin{aligned}
&\left\|T_1^* + T_2\right\|\\
&\le \frac{1}{4}\left(\left\|t^2|T_1|^{1-r} + \frac{1}{t^2}|T_1|^{1+r}\right\| + \left\|t^2|T_2|^{1-s} + \frac{1}{t^2}|T_2|^{1+s}\right\|\right)\\
&\quad + \frac{1}{4}\sqrt{\left(\left\|t^2|T_1|^{1-r} + \frac{1}{t^2}|T_1|^{1+s}\right\| - \left\|t^2|T_2|^{1-s} + \frac{1}{t^2}|T_2|^{1+s}\right\|\right)^2 + 4\left\||T_1|^{\frac{1-r}{2}}V|T_2|^{\frac{1+s}{2}} + |T_1|^{\frac{1+r}{2}}U^*|T_2|^{\frac{1-s}{2}}\right\|^2}.
\end{aligned}$$

From the above inequality, one can get

$$\begin{aligned}&\left\|T_1^*+T_2\right\|\\&\le \frac{1}{4}\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+r}\right\|+\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)\\&\quad+\frac{1}{4}\sqrt{\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+s}\right\|-\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)^2+4\left(\left\||T_1|^{\frac{1-r}{2}}|T_2^*|^{\frac{1+s}{2}}\right\|+\left\||T_1^*|^{\frac{1+r}{2}}|T_2|^{\frac{1-s}{2}}\right\|\right)^2}.\end{aligned}$$

Indeed,

$$\begin{aligned}&\left\|T_1^*+T_2\right\|\\&\le \frac{1}{4}\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+r}\right\|+\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)\\&\quad+\frac{1}{4}\sqrt{\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+s}\right\|-\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)^2+4\left\||T_1|^{\frac{1-r}{2}}V|T_2|^{\frac{1+s}{2}}+|T_1|^{\frac{1+r}{2}}U^*|T_2|^{\frac{1-s}{2}}\right\|^2}\\&\le \frac{1}{4}\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+r}\right\|+\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)\\&\quad+\frac{1}{4}\sqrt{\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+s}\right\|-\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)^2+4\left(\left\||T_1|^{\frac{1-r}{2}}V|T_2|^{\frac{1+s}{2}}\right\|+\left\||T_1|^{\frac{1+r}{2}}U^*|T_2|^{\frac{1-s}{2}}\right\|\right)^2}\\&= \frac{1}{4}\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+r}\right\|+\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)\\&\quad+\frac{1}{4}\sqrt{\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+s}\right\|-\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)^2+4\left(\left\||T_1|^{\frac{1-r}{2}}V|T_2|^{\frac{1+s}{2}}V^*\right\|+\left\|U|T_1|^{\frac{1+r}{2}}U^*|T_2|^{\frac{1-s}{2}}\right\|\right)^2}\\&\le \frac{1}{4}\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+r}\right\|+\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)\\&\quad+\frac{1}{4}\sqrt{\left(\left\|t^2|T_1|^{1-r}+\frac{1}{t^2}|T_1|^{1+s}\right\|-\left\|t^2|T_2|^{1-s}+\frac{1}{t^2}|T_2|^{1+s}\right\|\right)^2+4\left(\left\||T_1|^{\frac{1-r}{2}}|T_2^*|^{\frac{1+s}{2}}\right\|+\left\||T_1^*|^{\frac{1+r}{2}}|T_2|^{\frac{1-s}{2}}\right\|\right)^2},\end{aligned}$$

where we have used the facts that $\|\cdot\|$ is unitarily invariant, and that $|T^*|^q = U|T|^qU^*$, when $T = U|T|$ is the polar decomposition of T to obtain the equality in the above argument.
In particular, we have the following.

Corollary 6.2. *Let $T \in \mathbb{M}_n$ and let $t > 0$. Then*

$$\begin{aligned}&\left\|\Re T\right\|\\&\le \frac{1}{8}\left(\left\|t^2|T|^{1-r}+\frac{1}{t^2}|T|^{1+r}\right\|+\left\|t^2|T|^{1-s}+\frac{1}{t^2}|T|^{1+s}\right\|\right)\\&\quad+\frac{1}{8}\sqrt{\left(\left\|t^2|T|^{1-r}+\frac{1}{t^2}|T|^{1+s}\right\|-\left\|t^2|T|^{1-s}+\frac{1}{t^2}|T|^{1+s}\right\|\right)^2+4\left(\left\||T|^{\frac{1-r}{2}}|T^*|^{\frac{1+s}{2}}\right\|+\left\||T^*|^{\frac{1+r}{2}}|T|^{\frac{1-s}{2}}\right\|\right)^2}.\end{aligned}$$

Declarations

- **Availability of data and materials**: Not applicable.

- **Conflict of interest**: The authors declare that they have no conflict of interest.

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