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# Hom-Lie structures on the algebra $\mathfrak{L}_{\lambda,\mu}$

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**Abstract.** For parameters $\lambda, \mu \in \mathbb{C}$, the algebra $\mathfrak{L}_{\lambda,\mu}$ is the semi-direct product of the Witt algebra and its tensor density module. In this paper, we determine all (multiplicative) Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$. As a result, we prove that any Hom-Lie structure on $\mathfrak{L}_{\lambda,\mu}$ is the direct sum of some special Hom-Lie structures, and there exist non-trivial multiplicative Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$ if and only if $\lambda = 0$ or 1. Moreover, all Hom-structures on $\mathfrak{L}_{\lambda,\mu}$ form a Jordan algebra in the usual way.

## 1. Introduction

The concept of a Hom-Lie algebra was introduced by Hartwig, Larsson and Silvestrov in order to describe the structures on certain deformations of the Witt algebra and the Virasoro algebra in [9]. More results concerning the Hom-Lie algebra refer to [1, 2, 8, 10, 16]. Recall that a Hom-Lie algebra is a vector space $\mathfrak{g}$ equipped with a skew-symmetry bilinear multiplication $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and a linear map $\sigma : \mathfrak{g} \to \mathfrak{g}$ satisfying the Hom-Jacobi identity

$$[\sigma(x), [y, z]] + [\sigma(y), [z, x]] + [\sigma(z), [x, y]] = 0, \ \forall \ x, y, z \in \mathfrak{g}. \tag{1}$$

A Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \sigma)$, in which $\sigma$ is an endomorphism (resp. automorphism) of $(\mathfrak{g}, [\cdot,\cdot])$, is called multiplicative (resp. regular). Moreover, note that when $\sigma = \mathrm{id}$, the Hom-Lie algebra degenerates to a corresponding Lie algebra.

Hence, determining all possible Hom-Lie structures on a given Lie algebra is an interesting research topic. Earlier, this work was done on finite-dimensional Lie algebras. In [11, 18], the authors showed that the Hom-Lie structures on finite-dimensional simple Lie algebras except for $\mathfrak{sl}_2$ are trivial. The study of Hom-Lie structures is gradually extended to infinite-dimensional Lie algebras. In [19], the authors characterized Hom-Lie structures on simple graded Lie algebras of finite growth, which are isomorphic to finite-dimensional simple Lie algebras, or loop algebras, or Cartan algebras, or the Virasoro algebra. In [14], the authors showed that the space of Hom-Lie structures on an affine Kac-Moody algebra is linearly spanned by central Hom-Lie structures and the identity map. Moreover, regular Hom-Lie structures on Borel subalgebras of finite-dimensional simple Lie algebras[4], nilpotent Lie algebras of strictly upper triangular matrices [5], and incidence algebras [6] were determined.

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For $\lambda, \mu \in \mathbb{C}$, the algebra $\mathfrak{L}_{\lambda,\mu} = \mathrm{span}_{\mathbb{C}}\{L_n, W_n \mid n \in \mathbb{Z}\}$ is an infinite-dimensional Lie algebra and satisfies the following brackets

$$[L_n, L_m] = (m-n)\,L_{m+n}, \quad [L_n, W_m] = (m+\mu+\lambda n)\,W_{m+n}, \quad [W_n, W_m] = 0, \ \forall\ m, n \in \mathbb{Z}.$$

It is also called $W(\mu, \lambda)$ [7, 17], which can be seen as the semi-direct product of the Witt algebra $\mathcal{W}$ and the tensor density module of $\mathcal{W}$ [15]. In addition, this algebra includes some famous algebras as subclass. For example, $\mathfrak{L}_{0,0}$ is the Heisenberg-Virasoro algebra with the one-dimensional center and $\mathfrak{L}_{-1,0}$ is the centerless $W$-algebra $W(2,2)$. More results on $\mathfrak{L}_{\lambda,\mu}$ refer to [7, 13, 17].

The aim of this paper is to determine all (multiplicative) Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$. Note that multiplicative Hom-Lie structures on $\mathfrak{L}_{0,0}$ [12], and $\mathfrak{L}_{-1,0}$ [3] were characterized. We use a different method to recover and generalize these results. The paper is organized as follows. In Section 2, we recall some basic definitions and useful lemmas. In Section 3, we determine all (multiplicative) Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$ by dividing parameters $\lambda, \mu$ into various cases. We find that any Hom-Lie structure on $\mathfrak{L}_{\lambda,\mu}$ can be written in the form of a direct sum of some special Hom-Lie structures, and there exist non-trivial multiplicative Hom-Lie structures only on $\mathfrak{L}_{0,\mu}$ and $\mathfrak{L}_{1,\mu}$.

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{Z}^*$ the sets of complex numbers, integers, and nonzero integers, respectively. All algebras and vector spaces are considered over $\mathbb{C}$.

## 2. Preliminaries

In this section, we recall some basic definitions and known results used later.

**Definition 2.1.** [11] *Given a Lie algebra $\mathfrak{g}$, a linear map $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ is called a Hom-Lie structure on $\mathfrak{g}$ if the Hom-Jacobi identity (1) holds. In particular, a scalar map $\sigma = \alpha \cdot id$ is called a trivial Hom-Lie structure, where $\alpha \in \mathbb{C}$.*

**Definition 2.2.** [11] *Given a Lie algebra $\mathfrak{g}$, a Hom-Lie structure $\sigma$ on $\mathfrak{g}$ is called multiplicative (resp. regular) if the map $\sigma$ is an endomorphism (resp. automorphism) of $\mathfrak{g}$.*

Note that a trivial Hom-Lie structure on $\mathfrak{g}$ is multiplicative if and only if $\alpha \in \{0, 1\}$. We denote by $\mathrm{HS}(\mathfrak{g})$ and $\mathrm{MHS}(\mathfrak{g})$ the sets of Hom-Lie structures and multiplicative Hom-Lie structures on $\mathfrak{g}$, respectively. It should be pointed out that $\mathrm{HS}(\mathfrak{g})$ is a subspace of $\mathrm{End}(\mathfrak{g})$ and $\mathrm{MHS}(\mathfrak{g})$ is merely a subset of $\mathrm{HS}(\mathfrak{g})$. Obviously,

$$\mathrm{HS}(\mathfrak{g}) \supseteq \mathbb{C}\mathrm{id}_{\mathfrak{g}}, \ \mathrm{MHS}(\mathfrak{g}) \supseteq \{0, \mathrm{id}_{\mathfrak{g}}\}.$$

Moreover, $\mathrm{HS}(\mathfrak{g})$ is also a representation of $\mathfrak{g}$, which has been proved in [14].

**Lemma 2.3.** [14] Given a Lie algebra $\mathfrak{g}$, $\mathrm{HS}(\mathfrak{g})$ forms a $\mathfrak{g}$-submodule of $\mathrm{End}(\mathfrak{g})$.

**Definition 2.4.** [18] *Given a Lie algebra $\mathfrak{g}$, a nonempty subset $\mathfrak{s}$ of $\mathfrak{g}$ is called a commutant generating set if $\{[x, y] \mid x, y \in \mathfrak{s}\}$ generates the whole $\mathfrak{g}$.*

**Lemma 2.5.** [19] Assume that $\mathfrak{g}$ is a centerless $\mathbb{Z}$-graded Lie algebra with a finite commutant generating set $\mathfrak{s}$. Then, $\mathrm{HS}(\mathfrak{g})$ is a $\mathbb{Z}$-graded subspace of $\mathrm{End}(\mathfrak{g}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{End}(\mathfrak{g})_n$, where $\mathrm{End}(\mathfrak{g})_n = \{\tau \in \mathrm{End}(\mathfrak{g}) \mid \tau(\mathfrak{g}_m) \in \mathfrak{g}_{m+n},\ \forall\ m \in \mathbb{Z}\}$.

Next, we recall some properties of $\mathfrak{L}_{\lambda,\mu}$.

**Lemma 2.6.** [7] For the Lie algebra $\mathfrak{L}_{\lambda,\mu}$, the following statements hold.

(1) $\mathfrak{L}_{\lambda,\mu} \cong \mathfrak{L}_{\lambda,\mu+k}$ for any $k \in \mathbb{Z}$.

(2) Up to isomorphism, the center of $\mathfrak{L}_{\lambda,\mu}$ is $Z(L_{\lambda,\mu}) = \begin{cases} \mathbb{C}W_0, & \text{if } (\lambda,\mu) = (0,0), \\ 0, & \text{otherwise} . \end{cases}$

(3) $\mathfrak{L}_{\lambda,\mu}$ has a natural $\mathbb{Z}$-grading defined by $\mathfrak{L}_{\lambda,\mu} = \bigoplus_{n\in\mathbb{Z}} (\mathfrak{L}_{\lambda,\mu})_n$, where $(\mathfrak{L}_{\lambda,\mu})_n = \mathbb{C}L_n \oplus \mathbb{C}W_n$.

By Lemma 2.6 (1), we may assume $\mu = 0$ when $\mu$ is an integer in this paper.

## 3. (Multiplicative) Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$

In this section, we will determine all (multiplicative) Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$. First, we will show any Hom-Lie structure on $\mathfrak{L}_{\lambda,\mu}$ is $\mathbb{Z}$-graded. Next, by dividing parameters $\lambda,\mu$ into different cases, all (multiplicative) Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$ are determined. Finally, we summarize the main results in this paper and consider whether all Hom-structures on $\mathfrak{L}_{\lambda,\mu}$ constitute a Jordan algebra.

**Lemma 3.1.** *$HS(\mathfrak{L}_{\lambda,\mu})$ is $\mathbb{Z}$-graded, that is,*

$$HS(\mathfrak{L}_{\lambda,\mu}) = \bigoplus_{k\in\mathbb{Z}} HS(\mathfrak{L}_{\lambda,\mu})_k,$$

*where $HS(\mathfrak{L}_{\lambda,\mu})_k = \{\tau \in HS(\mathfrak{L}_{\lambda,\mu}) \,|\, \tau((\mathfrak{L}_{\lambda,\mu})_m) \subseteq (\mathfrak{L}_{\lambda,\mu})_{k+m},\ \forall\ m \in \mathbb{Z}\}$.*

*Proof.* If $\mu \neq 0$, note that $\mathfrak{s} = \{L_{-2}, L_{-1}, L_0, L_1, L_2, W_0\}$ is a commutant generating set of $\mathfrak{L}_{\lambda,\mu}$. By Lemmas 2.5 and 2.6 (2), it is obvious that $\mathrm{HS}(\mathfrak{L}_{\lambda,\mu})$ is $\mathbb{Z}$-graded.

If $\mu = 0$, from Lemma 2.6 (3), we have $\mathfrak{L}_{\lambda,0} = \bigoplus_{k\in\mathbb{Z}}(\mathfrak{L}_{\lambda,0})_k$, where $(\mathfrak{L}_{\lambda,0})_k = \mathbb{C}L_k \oplus \mathbb{C}W_k$. By the definition of $\mathfrak{L}_{\lambda,0}$, we find that $\mathbb{C}L_0$ is the Cartan subalgebra of $\mathfrak{L}_{\lambda,0}$. Then, $\mathfrak{L}_{\lambda,0} = (\mathfrak{L}_{\lambda,0})_0 \oplus \bigoplus_{k\in\mathbb{Z}^*}(\mathfrak{L}_{\lambda,0})_k$ can be seen as the root space decomposition with respect to the action of $L_0$. Hence, according to Lemma 2.3, this action induces a semisimple action on $\mathrm{HS}(\mathfrak{L}_{\lambda,0})$, with the corresponding root space decomposition $\mathrm{HS}(\mathfrak{L}_{\lambda,0}) = \bigoplus_{k\in(\mathbb{Z}^*-\mathbb{Z}^*)} \mathrm{HS}(\mathfrak{L}_{\lambda,0})_k$, where $\mathbb{Z}^* - \mathbb{Z}^*$ denotes the set of all differences between two roots in $\mathbb{Z}^*$, and $\mathrm{HS}(\mathfrak{L}_{\lambda,0})_k = \{\tau \in \mathrm{HS}(\mathfrak{L}_{\lambda,0}) \,|\, \tau((\mathfrak{L}_{\lambda,0})_m) \subseteq (\mathfrak{L}_{\lambda,0})_{k+m},\ \forall\ m \in \mathbb{Z}^* \cup \{0\}\}$. It is obvious that $\mathbb{Z}^* - \mathbb{Z}^* = \mathbb{Z}$. Hence, $\mathrm{HS}(\mathfrak{L}_{\lambda,0})$ is also $\mathbb{Z}$-graded. We complete the proof. □

**Lemma 3.2.** *For $k \in \mathbb{Z}$, suppose that $c_{k,m} \in \mathbb{C}$ satisfies*

$$c_{k,m}(n-p)(p+n-k-m) = 0,\ \forall\ m,n,p \in \mathbb{Z}. \tag{2}$$

*Then, $c_{k,m} = 0$ for any $m \in \mathbb{Z}$.*

*Proof.* Let $n = -1$ and $p = 1$ in Eq. (2). Then, we have

$$c_{k,m}(k+m) = 0,\ \forall\ m \in \mathbb{Z}. \tag{3}$$

Again taking $n = 0$ and $p = 1$ in Eq. (2) gives us that

$$c_{k,m}(-1+k+m) = 0,\ \forall\ m \in \mathbb{Z}. \tag{4}$$

By Eqs. (3)-(4), we have $c_{k,m} = 0$ for any $m \in \mathbb{Z}$. □

To describe all (multiplicative) Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$ conveniently, we give some special Hom-Lie structures on this algebra by Lemmas 3.3-3.5.

**Lemma 3.3.** *Suppose that $\rho : \mathfrak{L}_{0,0} \to \mathbb{C}$ and $\sigma_\rho : \mathfrak{L}_{0,0} \to \mathbb{C}W_0$ are two linear maps. If $\sigma_\rho(x) = \rho(x)W_0,\ \forall\ x \in \mathfrak{L}_{0,0}$, then $\sigma_\rho$ is a Hom-Lie structure on $\mathfrak{L}_{0,0}$, called the central Hom-Lie structure.*

*Proof.* For any $x, y, z \in \mathfrak{L}_{0,0}$, it follows from Lemma 2.6 (2) that

$$[\sigma_\rho(x), [y,z]] + [\sigma_\rho(y), [z,x]] + [\sigma_\rho(z), [x,y]] = [\rho(x)W_0, [y,z]] + [\rho(y)W_0, [z,x]] + [\rho(z)W_0, [x,y]] = 0.$$

This completes the proof. □

We denote by $\mathrm{CHS}(\mathfrak{L}_{0,0})$ the set of all central Hom-Lie structures on $\mathfrak{L}_{0,0}$.

**Lemma 3.4.** *For any $k, m \in \mathbb{Z}$, let $\sigma_k^1$, $\sigma_k^2$ and $\sigma_k^3$ be the linear maps of $\mathfrak{L}_{\lambda,\mu}$ such that*

$$\sigma_k^1(L_m) = L_{k+m}, \quad \sigma_k^1(W_m) = W_{k+m},$$
$$\sigma_k^2(L_m) = W_{k+m}, \quad \sigma_k^2(W_m) = 0,$$
$$\sigma_k^3(L_m) = mW_{k+m}, \quad \sigma_k^3(W_m) = 0.$$

*Then,*

(1) *$\sigma_k^1$ and $\sigma_k^2$ are Hom-Lie structures on $\mathfrak{L}_{-1,\mu}$,*
(2) *$\sigma_k^1$, $\sigma_k^2$ and $\sigma_k^3$ are Hom-Lie structures on $\mathfrak{L}_{1,\mu}$,*
(3) *$\sigma_0^1$, $\sigma_k^2$ are Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$ ($\lambda \neq 0, \pm 1$) and $\mathfrak{L}_{0,0}$.*

*Proof.* We only prove (2) as an example, and the proofs of (1) and (3) are similar. For any $k, m, n, p \in \mathbb{Z}$, we easily have

$$\begin{aligned}
&[\sigma_k^1(L_m), [L_n, L_p]] + [\sigma_k^1(L_n), [L_p, L_m]] + [\sigma_k^1(L_p), [L_m, L_n]]\\
=&[L_{k+m}, (p-n)L_{n+p}] + [L_{k+n}, (m-p)L_{p+m}] + [L_{k+p}, (n-m)L_{m+n}]\\
=&((p-n)(n+p-k-m) + (m-p)(p+m-k-n) + (n-m)(m+n-k-p))L_{k+m+n+p} = 0,\\
&[\sigma_k^1(L_m), [L_n, W_p]] + [\sigma_k^1(L_n), [W_p, L_m]] + [\sigma_k^1(W_p), [L_m, L_n]]\\
=&[L_{k+m}, (p+\mu+n)W_{n+p}] + [L_{k+n}, -(p+\mu+m)W_{p+m}] + [W_{k+p}, (n-m)L_{m+n}],\\
=&(n+p+\mu+k+m)((p+\mu+n) - (p+\mu+m) - (n-m))W_{k+m+n+p} = 0,\\
&[\sigma_k^1(L_m), [W_n, W_p]] + [\sigma_k^1(W_n), [W_p, L_m]] + [\sigma_k^1(W_p), [L_m, W_n]]\\
=&[W_{k+n}, -(p+\mu+m)W_{p+m}] + [W_{k+p}, (n+\mu+m)W_{m+n}] = 0,\\
&[\sigma_k^1(W_m), [W_n, W_p]] + [\sigma_k^1(W_n), [W_p, W_m]] + [\sigma_k^1(W_p), [W_m, W_n]] = 0.
\end{aligned}$$

So, $\sigma_k^1$ is a Hom-Lie structure on $\mathfrak{L}_{1,\mu}$.

For any $k, m, n, p \in \mathbb{Z}$,

$$\begin{aligned}
&[\sigma_k^2(L_m), [W_n, W_p]] + [\sigma_k^2(W_n), [W_p, L_m]] + [\sigma_k^2(W_p), [L_m, W_n]] = 0,\\
&[\sigma_k^2(W_m), [W_n, W_p]] + [\sigma_k^2(W_n), [W_p, W_m]] + [\sigma_k^2(W_p), [W_m, W_n]] = 0
\end{aligned}$$

are obvious. Moreover, we obtain

$$\begin{aligned}
&[\sigma_k^2(L_m), [L_n, L_p]] + [\sigma_k^2(L_n), [L_p, L_m]] + [\sigma_k^2(L_p), [L_m, L_n]]\\
=&[W_{k+m}, (p-n)L_{n+p}] + [W_{k+n}, (m-p)L_{p+m}] + [W_{k+p}, (n-m)L_{m+n}]\\
=&-(k+m+\mu+n+p)((p-n) + (m-p) + (n-m))W_{k+m+n+p} = 0,\\
&[\sigma_k^2(L_m), [L_n, W_p]] + [\sigma_k^2(L_n), [W_p, L_m]] + [\sigma_k^2(W_p), [L_m, L_n]]\\
=&[W_{k+m}, (p+\mu+n)W_{n+p}] + [W_{k+n}, -(p+\mu+m)W_{p+m}] = 0.
\end{aligned}$$

From these equations above, $\sigma_k^2$ is a Hom-Lie structure on $\mathfrak{L}_{1,\mu}$.

By a easy calculation, we have for any $k, m, n, p \in \mathbb{Z}$,

$$\begin{aligned}
&[\sigma_k^3(L_m), [L_n, L_p]] + [\sigma_k^3(L_n), [L_p, L_m]] + [\sigma_k^3(L_p), [L_m, L_n]]\\
=&[mW_{k+m}, (p-n)L_{n+p}] + [nW_{k+n}, (m-p)L_{p+m}] + [pW_{k+p}, (n-m)L_{m+n}]\\
=&-(k+m+\mu+n+p)(m(p-n) + n(m-p) + p(n-m))W_{k+m+n+p} = 0,\\
&[\sigma_k^3(L_m), [L_n, W_p]] + [\sigma_k^3(L_n), [W_p, L_m]] + [\sigma_k^3(W_p), [L_m, L_n]]\\
=&[mW_{k+m}, (p+\mu+n)W_{n+p}] + [nW_{k+n}, -(p+\mu+m)W_{p+m}] = 0,\\
&[\sigma_k^3(L_m), [W_n, W_p]] + [\sigma_k^3(W_n), [W_p, L_m]] + [\sigma_k^3(W_p), [L_m, W_n]] = 0,\\
&[\sigma_k^3(W_m), [W_n, W_p]] + [\sigma_k^3(W_n), [W_p, W_m]] + [\sigma_k^3(W_p), [W_m, W_n]] = 0.
\end{aligned}$$

Thus, $\sigma_k^3$ is a Hom-Lie structure on $\mathfrak{L}_{1,\mu}$. The proof is finished. □

**Lemma 3.5.** *For any $k, m \in \mathbb{Z}$, let $\sigma_k^4$, $\sigma_k^5$ and $\sigma_k^6$ be the linear maps of $\mathfrak{L}_{0,\mu}$ ($\mu \notin \mathbb{Z}$) such that*

$$\sigma_k^4(L_m) = L_{k+m}, \quad \sigma_k^4(W_m) = \frac{m+\mu}{k+m+\mu} W_{k+m},$$
$$\sigma_k^5(L_m) = \frac{1}{k+m+\mu} W_{k+m}, \quad \sigma_k^5(W_m) = 0,$$
$$\sigma_k^6(L_m) = \frac{m}{k+m+\mu} W_{k+m}, \quad \sigma_k^6(W_m) = 0.$$

*Then, $\sigma_k^4$, $\sigma_k^5$ and $\sigma_k^6$ are Hom-Lie structures on $\mathfrak{L}_{0,\mu}$.*

*Proof.* For any $k, m, n, p \in \mathbb{Z}$, it is obvious that

$$[\sigma_k^4(W_m), [W_n, W_p]] + [\sigma_k^4(W_n), [W_p, W_m]] + [\sigma_k^4(W_p), [W_m, W_n]] = 0.$$

By a calculation, we also have

$$\begin{aligned}
&[\sigma_k^4(L_m), [L_n, L_p]] + [\sigma_k^4(L_n), [L_p, L_m]] + [\sigma_k^4(L_p), [L_m, L_n]] \\
=&[L_{k+m}, (p-n)L_{n+p}] + [L_{k+n}, (m-p)L_{p+m}] + [L_{k+p}, (n-m)L_{m+n}] \\
=&((p-n)(n+p-k-m) + (m-p)(p+m-k-n) + (n-m)(m+n-k-p))L_{k+m+n+p} = 0, \\
&[\sigma_k^4(L_m), [L_n, W_p]] + [\sigma_k^4(L_n), [W_p, L_m]] + [\sigma_k^4(W_p), [L_m, L_n]] \\
=&[L_{k+m}, (p+\mu)W_{n+p}] + [L_{k+n}, -(p+\mu)W_{p+m}] + [\frac{p+\mu}{k+p+\mu} W_{k+p}, (n-m)L_{m+n}] \\
=&((p+\mu)(n+p+\mu) - (p+\mu)(p+m+\mu) - (n-m)(p+\mu))W_{k+m+n+p} = 0, \\
&[\sigma_k^4(L_m), [W_n, W_p]] + [\sigma_k^4(W_n), [W_p, L_m]] + [\sigma_k^4(W_p), [L_m, W_n]] \\
=&[\frac{n+\mu}{k+n+\mu} W_{k+n}, -(p+\mu)W_{p+m}] + [\frac{p+\mu}{k+p+\mu} W_{k+p}, (n+\mu)W_{m+n}] = 0,
\end{aligned}$$

So, $\sigma_k^4$ is a Hom-Lie structure on $\mathfrak{L}_{0,\mu}$.

For any $k, m, n, p \in \mathbb{Z}$, we get

$$\begin{aligned}
&[\sigma_k^5(L_m), [L_n, L_p]] + [\sigma_k^5(L_n), [L_p, L_m]] + [\sigma_k^5(L_p), [L_m, L_n]] \\
=&\frac{p-n}{k+m+\mu}[W_{k+m}, L_{n+p}] + \frac{m-p}{k+n+\mu}[W_{k+n}, L_{p+m}] + \frac{n-m}{k+p+\mu}[W_{k+p}, L_{m+n}] \\
=&-((p-n) + (m-p) + (n-m))W_{k+m+n+p} = 0, \\
&[\sigma_k^5(L_m), [L_n, W_p]] + [\sigma_k^5(L_n), [W_p, L_m]] + [\sigma_k^5(W_p), [L_m, L_n]] \\
=&[\frac{1}{k+m+\mu} W_{k+m}, (p+\mu)W_{n+p}] + [\frac{1}{k+n+\mu} W_{k+n}, -(p+\mu)W_{p+m}] = 0.
\end{aligned}$$

These equations together with

$$\begin{aligned}
&[\sigma_k^5(L_m), [W_n, W_p]] + [\sigma_k^5(W_n), [W_p, L_m]] + [\sigma_k^5(W_p), [L_m, W_n]] = 0, \\
&[\sigma_k^5(W_m), [W_n, W_p]] + [\sigma_k^5(W_n), [W_p, W_m]] + [\sigma_k^5(W_p), [W_m, W_n]] = 0
\end{aligned}$$

show that $\sigma_k^5$ is a Hom-Lie structure on $\mathfrak{L}_{0,\mu}$.

In addition, for any $k, m, n, p \in \mathbb{Z}$, it follows from

$$\begin{aligned}
&[\sigma_k^6(L_m), [L_n, L_p]] + [\sigma_k^6(L_n), [L_p, L_m]] + [\sigma_k^6(L_p), [L_m, L_n]] \\
=&\frac{m(p-n)}{k+m+\mu}[W_{k+m}, L_{n+p}] + \frac{n(m-p)}{k+n+\mu}[W_{k+n}, L_{p+m}] + \frac{p(n-m)}{k+p+\mu}[W_{k+p}, L_{m+n}] \\
=&-(m(p-n) + n(m-p) + p(n-m))W_{k+m+n+p} = 0, \\
&[\sigma_k^6(L_m), [L_n, W_p]] + [\sigma_k^6(L_n), [W_p, L_m]] + [\sigma_k^6(W_p), [L_m, L_n]] \\
=&[\frac{m}{k+m+\mu}W_{k+m}, (p+\mu)W_{n+p}] + [\frac{n}{k+n+\mu}W_{k+n}, -(p+\mu)W_{p+m}] = 0. \\
&[\sigma_k^6(L_m), [W_n, W_p]] + [\sigma_k^6(W_n), [W_p, L_m]] + [\sigma_k^6(W_p), [L_m, W_n]] = 0, \\
&[\sigma_k^6(W_m), [W_n, W_p]] + [\sigma_k^6(W_n), [W_p, W_m]] + [\sigma_k^6(W_p), [W_m, W_n]] = 0
\end{aligned}$$

that $\sigma_k^6$ is a Hom-Lie structure on $\mathfrak{L}_{0,\mu}$. We complete the proof. □

Note that $\sigma_0^1 = \mathrm{id}_{\mathfrak{L}_{\lambda,\mu}}$ and $\sigma_0^4 = \mathrm{id}_{\mathfrak{L}_{0,\mu}}$. Next, by dividing parameters $\lambda, \mu$ into five cases, we determine all (multiplicative) Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$.

### *3.1. The case for $\lambda = -1$*

By the definition, $\mathfrak{L}_{-1,\mu}$ has the following Lie brackets

$$[L_n, L_m] = (m-n)\, L_{m+n},\ [L_n, W_m] = (m+\mu-n)W_{m+n},\ [W_n, W_m] = 0,\ \forall\ m, n \in \mathbb{Z}.$$

**Lemma 3.6.** *For any $k, m \in \mathbb{Z}$, suppose that $\tau_k$ is a Hom-Lie structure on $\mathfrak{L}_{-1,\mu}$ satisfying $\tau_k((\mathfrak{L}_{-1,\mu})_m) \subseteq (\mathfrak{L}_{-1,\mu})_{k+m}$. Then, $\tau_k = a_k\sigma_k^1 + b_k\sigma_k^2$, where $a_k, b_k \in \mathbb{C}$ and the action of $\sigma_k^i$ $(i = 1, 2)$ on $\mathfrak{L}_{-1,\mu}$ is defined in Lemma 3.4.*

*Proof.* For any $m \in \mathbb{Z}$, assume that

$$\tau_k(L_m) = a_{k,m}L_{k+m} + b_{k,m}W_{k+m}, \quad \tau_k(W_m) = c_{k,m}L_{k+m} + d_{k,m}W_{k+m},$$

where $a_{k,m}, b_{k,m}, c_{k,m}, d_{k,m} \in \mathbb{C}$.

**Claim 1**. $a_{k,m} = a_{k,0},\ \ b_{k,m} = b_{k,0},\ \forall\ m \in \mathbb{Z}$.

For any $m, n, p \in \mathbb{Z}$, let $x = L_m$, $y = L_n$, $z = L_p$ and $\sigma = \tau_k$ in Eq. (1). Then, we obtain

$$a_{k,m}(p-n)(k+m-n-p) + a_{k,n}(m-p)(k+n-p-m) + a_{k,p}(n-m)(k+p-m-n) = 0, \tag{5}$$

$$b_{k,m}(p-n)(k+m+\mu-n-p) + b_{k,n}(m-p)(k+n+\mu-p-m) + b_{k,p}(n-m)(k+p+\mu-m-n) = 0. \tag{6}$$

Note that Eq. (5) is a special case of Eq. (6), i.e., when $\mu = 0$, Eq. (6) becomes Eq. (5). Hence, we only need to consider Eq. (6). Taking $n = 1$ and $p = 0$ in Eq. (6) gives us that for any $m \in \mathbb{Z}$,

$$-b_{k,m}(k+m+\mu-1) + b_{k,1}m(k-m+\mu+1) + b_{k,0}(1-m)(k-m+\mu-1) = 0. \tag{7}$$

For any $m \in \mathbb{Z}$, let $n = m+1$ and $p = m+2$ in Eq. (6). Then, we have

$$b_{k,m}(k-m+\mu-3) - 2b_{k,m+1}(k-m+\mu-1) + b_{k,m+2}(k-m+\mu+1) = 0. \tag{8}$$

**Case 1.** $\mu = 0$.

Obviously, from Eq. (7), we have

$$b_{k,m} = \frac{b_{k,1}m(k-m+1) - b_{k,0}(m-1)(k-m-1)}{k+m-1},\ m \neq -k+1. \tag{9}$$

Again taking $m = -k+1$ in Eq. (7), we have $(b_{k,1} - b_{k,0})k(k-1) = 0$.

**Subcase 1.** $k \neq 0, 1$.

It is easy to see that $b_{k,1} = b_{k,0}$. Then, Eq. (9) becomes $b_{k,m} = b_{k,0},\ m \neq -k+1$. Taking $m = -k$ in Eq. (8) gives that $b_{k,-k+1} = b_{k,0}$. Hence, $b_{k,m} = b_{k,0},\ \forall\ m \in \mathbb{Z}$.

**Subcase 2.** $k = 0$.

Eq. (9) can be written as $b_{0,m} = -b_{0,1}m + b_{0,0}(m+1),\ m \neq 1$. Putting $m = 1$ into Eq. (8), we obtain $b_{0,1} = b_{0,0}$ easily. Hence, $b_{0,m} = b_{0,0},\ \forall\ m \in \mathbb{Z}$.

**Subcase 3.** $k = 1$.

It is obvious that Eq. (9) becomes $b_{1,m} = b_{1,1}(2-m) + b_{1,0}(m-1),\ m \neq 0$. Let $m = 1$ in Eq. (8). Then, we have $b_{1,1} = b_{1,0}$. Hence, $b_{1,m} = b_{1,0},\ \forall\ m \in \mathbb{Z}$.

**Case 2.** $\mu \notin \mathbb{Z}$.

Eq. (7) becomes

$$b_{k,m} = \frac{b_{k,1}m(k-m+\mu+1) - b_{k,0}(m-1)(k-m+\mu-1)}{k+m+\mu-1},\ \forall\ m \in \mathbb{Z}.$$

So, one immediately gets

$$b_{k,2} = \frac{2b_{k,1}(k+\mu-1) - b_{k,0}(k+\mu-3)}{k+\mu+1}, \quad b_{k,3} = \frac{3b_{k,1}(k+\mu-2) - 2b_{k,0}(k+\mu-4)}{k+\mu+2}.$$

Let $m = 1$ in Eq. (8). Then, we obtain $b_{k,1}(k+\mu-4) - 2b_{k,2}(k+\mu-2) + b_{k,3}(k+\mu) = 0$. Put the expressions of $b_{k,2}$ and $b_{k,3}$ into the equation above. By a simple calculation, we get $b_{k,1} = b_{k,0}$. Hence, we have $b_{k,m} = b_{k,0},\ \forall\ m \in \mathbb{Z}$.

**Claim 2**. $c_{k,m} = 0,\ d_{k,m} = a_{k,0},\ \forall\ m, n \in \mathbb{Z}$.

For any $m, n, p \in \mathbb{Z}$, take $x = L_p$, $y = L_n$, $z = W_m$ and $\sigma = \tau_k$ in Eq. (1). Then, by Claim 1, we have

$$\begin{aligned} &c_{k,m}(n-p)(p+n-k-m) = 0, \\ &(d_{k,m} - a_{k,0})(n-p)(k+m+\mu-p-n) = 0. \end{aligned} \tag{10}$$

From Lemma 3.2, we have $c_{k,m} = 0,\ \forall\ m \in \mathbb{Z}$. Let $n = -1$ and $p = 1$ in Eq. (10). Then, we have

$$(d_{k,m} - a_{k,0})(k+m+\mu) = 0,\ \forall\ m \in \mathbb{Z}. \tag{11}$$

Again let $n = 0$ and $p = 1$ in Eq. (10). So,

$$(d_{k,m} - a_{k,0})(k+m+\mu-1) = 0,\ \forall\ m \in \mathbb{Z}. \tag{12}$$

It is obvious from Eqs. (11)-(12) that $d_{k,m} = a_{k,0}$ for any $m \in \mathbb{Z}$.

From Claims 1-2, we have

$$\tau_k(L_m) = a_{k,0}L_{k+m} + b_{k,0}W_{k+m}, \quad \tau_k(W_m) = a_{k,0}W_{k+m},$$

where $a_{k,0}, b_{k,0} \in \mathbb{C}$. Let $a_k = a_{k,0}$ and $b_k = b_{k,0}$. Therefore, by Lemma 3.4, we obtain $\tau_k = a_k\sigma_k^1 + b_k\sigma_k^2$. □

**Theorem 3.7.** *For the linear map $\sigma_k^i$ on $\mathfrak{L}_{-1,\mu}$ defined in Lemma 3.4, where $i = 1, 2$, we have*

$$HS(\mathfrak{L}_{-1,\mu}) = \bigoplus_{k\in\mathbb{Z}}(\mathbb{C}\sigma_k^1 \oplus \mathbb{C}\sigma_k^2).$$

*Proof.* From Lemmas 3.1, 3.4 and 3.6, this theorem is obvious. □

**Corollary 3.8.** *Any multiplicative Hom-Lie structure on $\mathfrak{L}_{-1,\mu}$ is trivial, that is $MHS(\mathfrak{L}_{-1,\mu}) = \{0, id_{\mathfrak{L}_{-1,\mu}}\}$.*

*Proof.* Obviously, $\{0, \mathrm{id}_{\mathfrak{L}_{-1,\mu}}\} \subseteq \mathrm{MHS}(\mathfrak{L}_{-1,\mu})$. Conversely, from Theorem 3.7, we may assume that for any $\tau \in \mathrm{MHS}(\mathfrak{L}_{-1,\mu}) \subseteq \mathrm{HS}(\mathfrak{L}_{-1,\mu})$,

$$\tau = \sum_{k\in\mathbb{Z}}(a_k\sigma_k^1 + b_k\sigma_k^2),$$

where $a_k, b_k \in \mathbb{C}$.

If $\{k \in \mathbb{Z} \,|\, a_k \neq 0\} = \emptyset$, we have $b_k = 0$ for any $k \in \mathbb{Z}$, that is, $\tau = 0$. In fact, for any $n \in \mathbb{Z}^*$, we have

$$\tau([L_0, L_n]) = \sum_{k\in\mathbb{Z}} b_k\sigma_k^2([L_0, L_n]) = n\sum_{k\in\mathbb{Z}} b_k\sigma_k^2(L_n) = n\sum_{k\in\mathbb{Z}} b_k W_{k+n},$$

$$[\tau(L_0), \tau(L_n)] = [\sum_{k\in\mathbb{Z}} b_k\sigma_k^2(L_0), \sum_{k'\in\mathbb{Z}} b_{k'}\sigma_{k'}^2(L_n)] = [\sum_{k\in\mathbb{Z}} b_k W_k, \sum_{k'\in\mathbb{Z}} b_{k'} W_{k'+n}] = 0.$$

Hence, it is obvious that $b_k = 0$ for all $k \in \mathbb{Z}$.

If $\{k \in \mathbb{Z} \,|\, a_k \neq 0\} \neq \emptyset$, take $\underline{l}, \bar{l} \in \{k \in \mathbb{Z} \,|\, a_k \neq 0\}$ such that $\underline{l}$ is minimal and $\bar{l}$ is maximal. For any $n \in \mathbb{Z}^*$,

$$\tau([L_0, L_n]) = (\sum_{l=\underline{l}}^{\bar{l}} a_l\sigma_l^1 + \sum_{k\in\mathbb{Z}} b_k\sigma_k^2)([L_0, L_n]) = n(\sum_{l=\underline{l}}^{\bar{l}} a_l\sigma_l^1 + \sum_{k\in\mathbb{Z}} b_k\sigma_k^2)(L_n) = n(\sum_{l=\underline{l}}^{\bar{l}} a_l L_{l+n} + \sum_{k\in\mathbb{Z}} b_k W_{k+n}), \quad (13)$$

$$\begin{aligned}[\tau(L_0), \tau(L_n)] =&[(\sum_{l=\underline{l}}^{\bar{l}} a_l\sigma_l^1 + \sum_{k\in\mathbb{Z}} b_k\sigma_k^2)(L_0), (\sum_{l'=\underline{l}}^{\bar{l}} a_{l'}\sigma_{l'}^1 + \sum_{k'\in\mathbb{Z}} b_{k'}\sigma_{k'}^2)(L_n)]\\
=&[\sum_{l=\underline{l}}^{\bar{l}} a_l L_l + \sum_{k\in\mathbb{Z}} b_k W_k, \sum_{l'=\underline{l}}^{\bar{l}} a_{l'} L_{l'+n} + \sum_{k'\in\mathbb{Z}} b_{k'} W_{k'+n}]\\
=&\sum_{l=\underline{l}}^{\bar{l}}\sum_{l'=\underline{l}}^{\bar{l}} a_l a_{l'}(l' - l + n)L_{l+l'+n} + \sum_{l=\underline{l}}^{\bar{l}}\sum_{k'\in\mathbb{Z}} a_l b_{k'}(k' + n + \mu - l)W_{l+k'+n}\\
&-\sum_{l'=\underline{l}}^{\bar{l}}\sum_{k\in\mathbb{Z}} a_{l'} b_k(k + \mu - l' - n)W_{l'+k+n}. \end{aligned} \quad (14)$$

It follows from Eqs. (13)-(14) that

$$a_{\underline{l}} L_{\underline{l}+n} = a_{\underline{l}}^2 L_{2\underline{l}+n}, \quad a_{\bar{l}} L_{\bar{l}+n} = a_{\bar{l}}^2 L_{2\bar{l}+n}.$$

These imply that $\underline{l} = \bar{l} = 0$ and $a_0 = 1$. Then, it is easy to compute that $b_k = 0$ for all $k \in \mathbb{Z}$. Hence, $\tau = \sigma_0^1$, that is, $\tau = \mathrm{id}_{\mathfrak{L}_{-1,\mu}}$.

Therefore, we obtain $\mathrm{MHS}(\mathfrak{L}_{-1,\mu}) \subseteq \{0, \mathrm{id}_{\mathfrak{L}_{-1,\mu}}\}$. The proof is finished. □

**Remark 3.9.** *When $\mu = 0$, Corollary 3.8 is consistent with [3, Theorem 4.1]. The proof of [3, Theorem 4.1] is to determine all endomorphisms of $\mathfrak{L}_{-1,0}$ first, and then to characterize MHS($\mathfrak{L}_{-1,0}$). However, we start from HS($\mathfrak{L}_{-1,\mu}$) for any $\mu \in \mathbb{C}$, and then use the multiplicative property to obtain MHS($\mathfrak{L}_{-1,\mu}$).*

## 3.2. *The case for $\lambda = 1$*

By the definition, $\mathfrak{L}_{1,\mu}$ has the following Lie brackets

$$[L_n, L_m] = (m - n)L_{m+n}, \ [L_n, W_m] = (m + \mu + n)W_{m+n}, \ [W_n, W_m] = 0, \ \forall\ m, n \in \mathbb{Z}.$$

**Lemma 3.10.** *For any $k, m \in \mathbb{Z}$, suppose that $\tau_k$ is a Hom-Lie structure on $\mathfrak{L}_{1,\mu}$ satisfying $\tau_k((\mathfrak{L}_{1,\mu})_m) \subseteq (\mathfrak{L}_{1,\mu})_{k+m}$. Then, $\tau_k = a_k\sigma_k^1 + b_k\sigma_k^2 + c_k\sigma_k^3$, where $a_k, b_k, c_k \in \mathbb{C}$ and the action of $\sigma_k^i$ $(i = 1, 2, 3)$ on $\mathfrak{L}_{1,\mu}$ is defined in Lemma 3.4.*

*Proof.* For any $m \in \mathbb{Z}$, we may assume that

$$\tau_k(L_m) = a_{k,m}L_{k+m} + b_{k,m}W_{k+m}, \quad \tau_k(W_m) = c_{k,m}L_{k+m} + d_{k,m}W_{k+m},$$

where $a_{k,m}, b_{k,m}, c_{k,m}, d_{k,m} \in \mathbb{C}$.

For any $m, n, p \in \mathbb{Z}$, let $x = L_m$, $y = L_n$, $z = L_p$ and $\sigma = \tau_k$ in Eq. (1). Then, we obtain

$$a_{k,m}(p-n)(k+m-n-p) + a_{k,n}(m-p)(k+n-p-m) + a_{k,p}(n-m)(k+p-m-n) = 0,$$
$$(b_{k,m}(p-n) + b_{k,n}(m-p) + b_{k,p}(n-m))(k+m+\mu+n+p) = 0. \tag{15}$$

From the proof of Lemma 3.6, we have $a_{k,m} = a_{k,0}$ for any $m \in \mathbb{Z}$.

Taking $n = 1$ and $p = 0$ in Eq. (15) gives us that for any $m \in \mathbb{Z}$,

$$(-b_{k,m} + b_{k,1}m + b_{k,0}(1-m))(k+m+\mu+1) = 0. \tag{16}$$

If $\mu \notin \mathbb{Z}$, it is obvious that $b_{k,m} = b_{k,1}m + b_{k,0}(1-m) = (b_{k,1} - b_{k,0})m + b_{k,0},\ \forall\ m \in \mathbb{Z}$. If $\mu \in \mathbb{Z}$, Eq. (16) can be written as $b_{k,m} = b_{k,1}m + b_{k,0}(1-m),\ m \neq -k-1$. So, we have

$$b_{k,-k-2} = -b_{k,1}(k+2) + b_{k,0}(k+3), \quad b_{k,-k} = -b_{k,1}k + b_{k,0}(k+1).$$

Let $m = -k-2$, $n = -k-1$, $p = -k$ and put two equations above into Eq. (15). Then, $b_{k,-k-1} = -b_{k,1}(k+1) + b_{k,0}(k+2)$. Hence, we have

$$b_{k,m} = b_{k,1}m + b_{k,0}(1-m) = (b_{k,1} - b_{k,0})m + b_{k,0},\ \forall\ m \in \mathbb{Z}.$$

Similar to the proof of Lemma 3.6, we easily get $c_{k,m} = 0,\ d_{k,m} = a_{k,0},\ \forall\ m \in \mathbb{Z}$. Then, we have

$$\tau_k(L_m) = a_{k,0}L_{k+m} + ((b_{k,1} - b_{k,0})m + b_{k,0})W_{k+m}, \quad \tau_k(W_m) = a_{k,0}W_{k+m},$$

where $a_{k,0}, b_{k,0}, b_{k,1} \in \mathbb{C}$. Let $a_k = a_{k,0}$, $b_k = b_{k,0}$ and $c_k = b_{k,1} - b_{k,0}$. Therefore, by Lemma 3.4, we obtain $\tau_k = a_k\sigma_k^1 + b_k\sigma_k^2 + c_k\sigma_k^3$. □

From Lemmas 3.1, 3.4 and 3.10, we easily get the following theorem.

**Theorem 3.11.** *For the linear map $\sigma_k^i$ on $\mathfrak{L}_{1,\mu}$ defined in Lemma 3.4, where $i = 1, 2, 3$, we have*

$$HS(\mathfrak{L}_{1,\mu}) = \bigoplus_{k\in\mathbb{Z}} (\mathbb{C}\sigma_k^1 \oplus \mathbb{C}\sigma_k^2 \oplus \mathbb{C}\sigma_k^3).$$

**Corollary 3.12.** *Suppose that $\tau$ is a linear map of $\mathfrak{L}_{1,\mu}$. Then $\tau \in MHS(\mathfrak{L}_{1,\mu})$ if and only if $\tau$ is 0 or possesses the following form $\tau = id_{\mathfrak{L}_{1,\mu}} + \sum_{k\in\mathbb{Z}} c_k((k+\mu)\sigma_k^2 + \sigma_k^3)$, where $c_k \in \mathbb{C}$ and the actions of $\sigma_k^2$ and $\sigma_k^3$ on $\mathfrak{L}_{1,\mu}$ are defined in Lemma 3.4.*

*Proof.* It is obvious that the linear maps 0 and $\mathrm{id}_{\mathfrak{L}_{1,\mu}} + \sum_{k\in\mathbb{Z}} c_k((k+\mu)\sigma_k^2 + \sigma_k^3)$ are multiplicative Hom-Lie structures on $\mathfrak{L}_{1,\mu}$, where $c_k \in \mathbb{C}$. Conversely, by Theorem 3.11, we may assume that for any $\tau \in \mathrm{MHS}(\mathfrak{L}_{1,\mu}) \subseteq \mathrm{HS}(\mathfrak{L}_{1,\mu})$,

$$\tau = \sum_{k\in\mathbb{Z}} (a_k\sigma_k^1 + b_k\sigma_k^2 + c_k\sigma_k^3),$$

where $a_k, b_k, c_k \in \mathbb{C}$.

If $\{k \in \mathbb{Z} \mid a_k \neq 0\} = \emptyset$, then $b_k = c_k = 0$ for all $k \in \mathbb{Z}$, i.e., $\tau = 0$. Indeed, for any $n \in \mathbb{Z}^*$, we have

$$\tau([L_0, L_n]) = n \sum_{k\in\mathbb{Z}} (b_k + nc_k)W_{k+n},$$

$$[\tau(L_0), \tau(L_n)] = [\sum_{k\in\mathbb{Z}} b_kW_k, \sum_{k'\in\mathbb{Z}} (b_{k'} + nc_{k'})W_{k'+n}] = 0.$$

We easily get $b_k + nc_k = 0$ for all $n \in \mathbb{Z}^*$, which deduces that $b_k = c_k = 0$.

If $\{k \in \mathbb{Z} \mid a_k \neq 0\} \neq \emptyset$, take $\underline{l}, \bar{l} \in \{k \in \mathbb{Z} \mid a_k \neq 0\}$ such that $\underline{l}$ is minimal and $\bar{l}$ is maximal. Similar to the proof of Corollary 3.8, it is easy to see $\underline{l} = \bar{l} = 0$ and $a_0 = 1$. So, for any $n \in \mathbb{Z}^*$, we have

$$\tau([L_0, L_n]) = n(L_n + \sum_{k\in\mathbb{Z}} (b_k + nc_k)W_{k+n}),$$

$$[\tau(L_0), \tau(L_n)] = n(L_n + \sum_{k\in\mathbb{Z}} (k + n + \mu)c_k W_{k+n}).$$

We obtain $b_k = (k + \mu)c_k$. Then, $\tau = \mathrm{id}_{\mathfrak{L}_{1,\mu}} + \sum_{k\in\mathbb{Z}} c_k((k + \mu)\sigma_k^2 + \sigma_k^3)$. The proof is completed. □

*3.3. The case for $\lambda \neq 0, \pm 1$*

**Lemma 3.13.** *For any $k, m \in \mathbb{Z}$, suppose that the actions of $\sigma_0^1$, $\sigma_k^2$ on $\mathfrak{L}_{\lambda,\mu}$ are defined in Lemma 3.4 and $\tau_k$ is a Hom-Lie structure on $\mathfrak{L}_{\lambda,\mu}$ satisfying $\tau_k((\mathfrak{L}_{\lambda,\mu})_m) \subseteq (\mathfrak{L}_{\lambda,\mu})_{k+m}$. Then, $\tau_k = \delta_{k,0}a_0\sigma_0^1 + b_k\sigma_k^2$, where $a_0, b_k \in \mathbb{C}$ and $\delta_{k,m}$ is the Kronecker delta.*

*Proof.* For any $m \in \mathbb{Z}$, assume that

$$\tau_k(L_m) = a_{k,m}L_{k+m} + b_{k,m}W_{k+m}, \quad \tau_k(W_m) = c_{k,m}L_{k+m} + d_{k,m}W_{k+m},$$

where $a_{k,m}, b_{k,m}, c_{k,m}, d_{k,m} \in \mathbb{C}$.

**Claim 1**. $a_{k,m} = a_{k,0}$, $b_{k,m} = b_{k,0}$, $\forall\ m \in \mathbb{Z}$.

From the proof of Lemma 3.6, we have $a_{k,m} = a_{k,0}$ for any $m \in \mathbb{Z}$. For any $m, n, p \in \mathbb{Z}$, let $x = L_m$, $y = L_n$, $z = L_p$ and $\sigma = \tau_k$ in Eq. (1). Then, we obtain

$$b_{k,m}(p-n)(k+m+\mu+\lambda(n+p)) + b_{k,n}(m-p)(k+n+\mu+\lambda(p+m)) + b_{k,p}(n-m)(k+p+\mu+\lambda(m+n)) = 0. \quad (17)$$

For any $m \in \mathbb{Z}$, let $n = m + 1$ and $p = m + 2$ in Eq. (17). Then, we have

$$b_{k,m}(k+m+\mu+\lambda(2m+3)) - 2b_{k,m+1}(k+m+1+\mu+\lambda(2m+2)) + b_{k,m+2}(k+m+2+\mu+\lambda(2m+1)) = 0. \quad (18)$$

**Case 1.** $\mu \notin \mathbb{Z}$ and there exists $n_0 \in \mathbb{Z}$ such that $\mu + \lambda n_0 \in \mathbb{Z}$.

It is easy to see $n_0 \neq 0$ and $\lambda \notin \mathbb{Z}$. Taking $n = n_0$ and $p = 0$ in Eq. (17) gives us that for any $m \in \mathbb{Z}$,

$$-b_{k,m}n_0(k + m + \mu + \lambda n_0) + b_{k,n_0}m(k + n_0 + \mu + \lambda m) + b_{k,0}(n_0 - m)(k + \mu + \lambda(m + n_0)) = 0. \quad (19)$$

This implies that

$$b_{k,m} = \frac{b_{k,n_0}m(k + n_0 + \mu + \lambda m) + b_{k,0}(n_0 - m)(k + \mu + \lambda(m + n_0))}{n_0(k + m + \mu + \lambda n_0)},\ m \neq -k - \mu - \lambda n_0. \quad (20)$$

Let $m = -k - \mu - \lambda n_0$ in Eq. (19). Then,

$$(b_{k,n_0} - b_{k,0})(\lambda - 1)(k + \mu + n_0 + \lambda n_0)(k + \mu + \lambda n_0) = 0. \quad (21)$$

**Subcase 1.** $k \neq -\mu - n_0 - \lambda n_0, -\mu - \lambda n_0$.

From Eq. (21), we have $b_{k,n_0} = b_{k,0}$. Eq. (20) becomes $b_{k,m} = b_{k,0}$, $m \neq -k - \mu - \lambda n_0$. Let $m = -k - \mu - \lambda n_0 - 1$ and $m = -k - \mu - \lambda n_0$ in Eq. (18), respectively. Then, we have

$$(b_{k,-k-\mu-\lambda n_0} - b_{k,0})(n_0 + 2(k + \mu + \lambda n_0)) = 0, \quad (b_{k,-k-\mu-\lambda n_0} - b_{k,0})(n_0 + 2(k + \mu + \lambda n_0) - 3) = 0.$$

These two equations above show that $b_{k,-k-\mu-\lambda n_0} = b_{k,0}$. Hence, $b_{k,m} = b_{k,0}$, $\forall\ m \in \mathbb{Z}$.

**Subcase 2.** $k = -\mu - n_0 - \lambda n_0$.

Eq. (20) becomes

$$b_{-\mu-n_0-\lambda n_0,m} = \frac{b_{-\mu-n_0-\lambda n_0,n_0}\lambda m - b_{-\mu-n_0-\lambda n_0,0}(\lambda m - n_0)}{n_0},\ m \neq n_0.$$

Taking $m = n_0 + 1$ in Eq. (18) gives us that $\lambda(\lambda-1)(b_{-\mu-n_0-\lambda n_0,n_0} - b_{-\mu-n_0-\lambda n_0,0}) = 0$, which implies that $b_{-\mu-n_0-\lambda n_0,n_0} = b_{-\mu-n_0-\lambda n_0,0}$. Hence, $b_{-\mu-n_0-\lambda n_0,m} = b_{-\mu-n_0-\lambda n_0,0}$, $\forall\ m \in \mathbb{Z}$.

**Subcase 3.** $k = -\mu - \lambda n_0$.

Eq. (20) becomes

$$b_{-\mu-\lambda n_0,m} = \frac{b_{-\mu-\lambda n_0,n_0}(-\lambda n_0 + n_0 + \lambda m) + b_{-\mu-\lambda n_0,0}\lambda(n_0 - m)}{n_0},\ m \neq 0.$$

Let $m = 1$ in Eq. (18). Then, we obtain $\lambda(\lambda-1)(b_{-\mu-\lambda n_0,n_0} - b_{-\mu-\lambda n_0,0}) = 0$, thus $b_{-\mu-\lambda n_0,n_0} = b_{-\mu-\lambda n_0,0}$. Hence, $b_{-\mu-\lambda n_0,m} = b_{-\mu-\lambda n_0,0}$, $\forall\ m \in \mathbb{Z}$.

**Case 2.** $\mu \notin \mathbb{Z}$ and $\mu + \lambda n \notin \mathbb{Z}$ for any $n \in \mathbb{Z}$.

Taking $n = 1$ and $p = 0$ in Eq. (17) shows that for any $m \in \mathbb{Z}$,

$$-b_{k,m}(k + m + \mu + \lambda) + b_{k,1}m(k + 1 + \mu + \lambda m) + b_{k,0}(1 - m)(k + \mu + \lambda(m + 1)) = 0.$$

Then, we have

$$b_{k,m} = \frac{b_{k,1}m(k + 1 + \mu + \lambda m) - b_{k,0}(m - 1)(k + \mu + \lambda(m + 1))}{k + m + \mu + \lambda},\ \forall\ m \in \mathbb{Z}.$$

So, one can immediately get

$$b_{k,2} = \frac{2b_{k,1}(k + 1 + \mu + 2\lambda) - b_{k,0}(k + \mu + 3\lambda)}{k + 2 + \mu + \lambda},\quad b_{k,3} = \frac{3b_{k,1}(k + 1 + \mu + 3\lambda) - 2b_{k,0}(k + \mu + 4\lambda)}{k + 3 + \mu + \lambda}.$$

Let $m = 1$ in Eq. (18). Then, we obtain $b_{k,1}(k+1+\mu+5\lambda) - 2b_{k,2}(k+2+\mu+4\lambda) + b_{k,3}(k+3+\mu+3\lambda) = 0$. Put the expressions of $b_{k,2}$ and $b_{k,3}$ into the equation above. By a simple calculation, we get $b_{k,1} = b_{k,0}$. Hence, we have $b_{k,m} = b_{k,0}$, $\forall\ m \in \mathbb{Z}$.

**Case 3.** $\mu = 0$ and $\lambda \in \mathbb{Z} \setminus \{0, \pm 1\}$.

Taking $n = 1$ and $p = 0$ in Eq. (17) gives us that for any $m \in \mathbb{Z}$,

$$-b_{k,m}(k + m + \lambda) + b_{k,1}m(k + 1 + \lambda m) + b_{k,0}(1 - m)(k + \lambda(m + 1)) = 0, \tag{22}$$

which implies that

$$b_{k,m} = \frac{b_{k,1}m(k + 1 + \lambda m) - b_{k,0}(m - 1)(k + \lambda(m + 1))}{k + m + \lambda},\ m \neq -k - \lambda. \tag{23}$$

Taking $m = -k - \lambda$ in Eq. (22), we have $(b_{k,1} - b_{k,0})(\lambda - 1)(k + \lambda + 1)(k + \lambda) = 0$.

**Subcase 1.** $k \neq -\lambda - 1, -\lambda$.

It is easy to see that $b_{k,1} = b_{k,0}$. Then, Eq. (23) becomes $b_{k,m} = b_{k,0}$, $m \neq -k - \lambda$. Taking $m = -k - \lambda - 1$ in Eq. (18) gives that $b_{k,-k-\lambda} = b_{k,0}$. Hence, $b_{k,m} = b_{k,0}$, $\forall\ m \in \mathbb{Z}$.

**Subcase 2.** $k = -\lambda - 1$.

Eq. (23) can be written as $b_{-\lambda-1,m} = b_{-\lambda-1,1}\lambda m - b_{-\lambda-1,0}(\lambda m - 1)$, $m \neq 1$. Put $m = 1$ into Eq. (18). It is easy to compute that $b_{-\lambda-1,1} = b_{-\lambda-1,0}$. Hence, we obtain $b_{-\lambda-1,m} = b_{-\lambda-1,0}$, $\forall\ m \in \mathbb{Z}$.

**Subcase 3.** $k = -\lambda$.

It is obvious that Eq. (23) becomes $b_{-\lambda,m} = b_{-\lambda,1}(-\lambda + 1 + \lambda m) - b_{-\lambda,0}\lambda(m - 1)$, $m \neq 0$. Let $m = 1$ in Eq. (18). Then, by a simple calculation, we have $b_{-\lambda,1} = b_{-\lambda,0}$. Hence, $b_{-\lambda,m} = b_{-\lambda,0}$, $\forall\ m \in \mathbb{Z}$.

**Case 4.** $\mu = 0$ and $\lambda \notin \mathbb{Z}$.

Take $n = 1$ and $p = 0$ in Eq. (17). For any $m \in \mathbb{Z}$, we have

$$-b_{k,m}(k + m + \lambda) + b_{k,1}m(k + 1 + \lambda m) + b_{k,0}(1 - m)(k + \lambda(m + 1)) = 0,$$

which implies that

$$b_{k,m} = \frac{b_{k,1}m(k + 1 + \lambda m) - b_{k,0}(m - 1)(k + \lambda(m + 1))}{k + m + \lambda},\ \forall\ m \in \mathbb{Z}.$$

Similar the proof of Case 2, we easily get $b_{k,m} = b_{k,0}$, $\forall\ m \in \mathbb{Z}$.

**Claim 2**. $c_{k,m} = 0$, $d_{k,m} = a_{k,0} = \delta_{k,0}a_{0,0}$, $\forall\ m \in \mathbb{Z}$.

For any $m, n, p \in \mathbb{Z}$, take $x = L_p$, $y = L_n$, $z = W_m$ and $\sigma = \tau_k$ in Eq. (1). Then, we have

$$c_{k,m}(n-p)(p+n-k-m) = 0,$$
$$(n-p)(d_{k,m}(k+m+\mu+\lambda(p+n)) - a_{k,0}(\lambda^2 k + m + \mu + \lambda(p+n))) = 0. \quad (24)$$

From Lemma 3.2, we have $c_{k,m} = 0$, $\forall\ m \in \mathbb{Z}$. Let $n = -1$ and $p = 1$ in Eq. (24). It is obvious that

$$d_{k,m}(k+m+\mu) - a_{k,0}(\lambda^2 k + m + \mu) = 0,\ \forall\ m \in \mathbb{Z}. \quad (25)$$

Again let $n = 0$ and $p = 1$ in Eq. (24). So,

$$d_{k,m}(k+m+\mu+\lambda) - a_{k,0}(\lambda^2 k + m + \mu + \lambda) = 0,\ \forall\ m \in \mathbb{Z}. \quad (26)$$

From Eqs. (25)-(26), it is obvious that $(d_{k,m} - a_{k,0})\lambda = 0$, which implies that $d_{k,m} = a_{k,0}$ for any $m \in \mathbb{Z}$. Moreover, note that Eq. (25) or Eq. (26) can be written as $d_{k,m}k(\lambda-1)(\lambda+1) = 0$, $\forall\ m \in \mathbb{Z}$. Hence, if $k \neq 0$, we have $d_{k,m} = a_{k,0} = 0$.

From Claims 1-2, we get

$$\tau_k(L_m) = \delta_{k,0}a_{0,0}L_{k+m} + b_{k,0}W_{k+m}, \quad \tau_k(W_m) = \delta_{k,0}a_{0,0}W_{k+m},$$

where $a_{0,0}, b_{k,0} \in \mathbb{C}$. Let $a_0 = a_{0,0}$ and $b_k = b_{k,0}$. Therefore, by Lemma 3.4, we obtain $\tau_k = \delta_{k,0}a_0\sigma_0^1 + b_k\sigma_k^2$. □

From Lemmas 3.1, 3.4 and 3.13, the following theorem is easy to obtain.

**Theorem 3.14.** *For the linear maps $\sigma_0^1$ and $\sigma_k^2$ on $\mathfrak{L}_{\lambda,\mu}$ defined in Lemma 3.4, we have*

$$HS(\mathfrak{L}_{\lambda,\mu}) = \bigoplus_{k\in\mathbb{Z}}(\mathbb{C}\delta_{k,0}\sigma_0^1 \oplus \mathbb{C}\sigma_k^2).$$

**Corollary 3.15.** *Any multiplicative Hom-Lie structure on $\mathfrak{L}_{\lambda,\mu}$ is trivial, that is $MHS(\mathfrak{L}_{\lambda,\mu}) = \{0, id_{\mathfrak{L}_{\lambda,\mu}}\}$.*

*Proof.* It is clear that $\{0, \mathrm{id}_{\mathfrak{L}_{\lambda,\mu}}\} \subseteq \mathrm{MHS}(\mathfrak{L}_{\lambda,\mu})$. Now, for any $\tau \in \mathrm{MHS}(\mathfrak{L}_{\lambda,\mu}) \subseteq \mathrm{HS}(\mathfrak{L}_{\lambda,\mu})$, from Theorem 3.14, we may assume $\tau = a_0\sigma_0^1 + \sum_{k\in\mathbb{Z}} b_k\sigma_k^2$, where $a_0, b_k \in \mathbb{C}$.

If $a_0 = 0$, it is easy to compute that $\tau = 0$.

If $a_0 \neq 0$, we have $\tau = \mathrm{id}_{\mathfrak{L}_{\lambda,\mu}}$. In fact, for any $n \in \mathbb{Z}^*$, we have

$$\tau([L_0, L_n]) = n(a_0L_n + \sum_{k\in\mathbb{Z}} b_kW_{k+n}),$$

$$[\tau(L_0), \tau(L_n)] = n(a_0^2L_n + \sum_{k\in\mathbb{Z}} a_0b_k(1-\lambda)W_{k+n}).$$

We have $a_0 = 1$ and $b_k = 0$ for all $k \in \mathbb{Z}$, i.e., $\tau = \mathrm{id}_{\mathfrak{L}_{\lambda,\mu}}$. Hence, we obtain $\mathrm{MHS}(\mathfrak{L}_{\lambda,\mu}) \subseteq \{0, \mathrm{id}_{\mathfrak{L}_{\lambda,\mu}}\}$. The proof is completed. □

*3.4. The case for $(\lambda, \mu) = (0, 0)$*

By the definition, $\mathfrak{L}_{0,0}$ has the following Lie brackets

$$[L_n, L_m] = (m-n)L_{m+n},\ [L_n, W_m] = mW_{m+n},\ [W_n, W_m] = 0,\ \forall\ m, n \in \mathbb{Z}.$$

**Lemma 3.16.** *For any $k \in \mathbb{Z}$, suppose that the actions of $\sigma_0^1$, $\sigma_k^2$ on $\mathfrak{L}_{0,0}$ are defined in Lemma 3.4. Let $\tau_k$ be a Hom-Lie structure on $\mathfrak{L}_{0,0}$ satisfying $\tau_k((\mathfrak{L}_{0,0})_m) \subseteq (\mathfrak{L}_{0,0})_{k+m}$ for any $m \in \mathbb{Z}$. Then,*

$$\tau_k = \delta_{k,0}a_0\sigma_0^1 + b_k\sigma_k^2 + \textit{up to a central Hom-Lie structure},$$

*where $a_0, b_k \in \mathbb{C}$ and $\delta_{k,m}$ is the Kronecker delta.*

*Proof.* For any $m \in \mathbb{Z}$, one may assume that

$$\tau_k(L_m) = a_{k,m}L_{k+m} + b_{k,m}W_{k+m}, \quad \tau_k(W_m) = c_{k,m}L_{k+m} + d_{k,m}W_{k+m},$$

where $a_{k,m}, b_{k,m}, c_{k,m}, d_{k,m} \in \mathbb{C}$. From the proof of Lemma 3.6, we have $a_{k,m} = a_{k,0}$ for any $m \in \mathbb{Z}$.

For any $m, n \in \mathbb{Z}$, let $x = L_m$, $y = L_n$, $z = L_{-k}$ and $\sigma = \tau_k$ in Eq. (1). Then, we obtain $(b_{k,m}-b_{k,n})(k+m)(k+n) = 0$, which deduces that $b_{k,m} = b_{k,n}$, $m \neq -k$, $n \neq -k$. Then, we have $b_{k,m} = b_{k,-k+1} + \delta_{m,-k}(b_{k,-k} - b_{k,-k+1})$, $\forall\ m \in \mathbb{Z}$.

For any $m, n, p \in \mathbb{Z}$, let $x = L_p$, $y = L_n$, $z = W_m$ and $\sigma = \tau_k$ in Eq. (1). Then,

$$c_{k,m}(n-p)(p+n-k-m) = 0,$$

$$(n-p)(d_{k,m}(k+m) - a_{k,0}m) = 0. \tag{27}$$

From Lemma 3.2, it is easy to see $c_{k,m} = 0$, $\forall\ m \in \mathbb{Z}$. For any $m \in \mathbb{Z}$, let $n = 1$, $p = -1$ in Eq. (27). Then, $(k+m)d_{k,m} = a_{k,0}m$, $\forall\ m \in \mathbb{Z}$. If $k = 0$, we have $d_{0,m} = a_{0,0} + \delta_{m,0}(d_{0,0} - a_{0,0})$. If $k \neq 0$, it is easy to compute that $a_{k,0} = 0$ and $d_{k,m} = \delta_{m,-k}d_{k,-k}$.

Hence, we obtain

$$\tau_0(L_m) = a_{0,0}L_m + b_{0,1}W_m + \delta_{m,0}(b_{0,0} - b_{0,1})W_0, \quad \tau_0(W_m) = a_{0,0}W_m + \delta_{m,0}(d_{0,0} - a_{0,0})W_0,$$

and for $k \neq 0$

$$\tau_k(L_m) = b_{k,-k+1}W_{k+m} + \delta_{m,-k}(b_{k,-k} - b_{k,-k+1})W_0, \quad \tau_k(W_m) = \delta_{m,-k}d_{k,-k}W_0.$$

Now, for any $k \in \mathbb{Z}$, define the linear map $\rho_k : \mathfrak{L}_{0,0} \to \mathbb{C}$ such that

$$\rho_0(L_m) = \delta_{m,0}(b_{0,0} - b_{0,1}), \quad \rho_0(W_m) = \delta_{m,0}(d_{0,0} - a_{0,0}),$$

and for $k \neq 0$

$$\rho_k(L_m) = \delta_{m,-k}(b_{k,-k} - b_{k,-k+1}), \quad \rho_k(W_m) = \delta_{m,-k}d_{k,-k}.$$

Obviously, $\sigma_{\rho_k}$ is a central Hom-Lie structure on $\mathfrak{L}_{0,0}$. Let $a_0 = a_{0,0}$ and $b_k = b_{k,-k+1}$. Then, $\tau_k = \delta_{k,0}a_0\sigma_0^1 + b_k\sigma_k^2 + \sigma_{\rho_k}$. □

**Theorem 3.17.** *For any $k \in \mathbb{Z}$ and the linear maps $\sigma_0^1$, $\sigma_k^2$ defined in Lemma 3.4, we have*

$$HS(\mathfrak{L}_{0,0}) = \bigoplus_{k\in\mathbb{Z}}(\mathbb{C}\delta_{k,0}\sigma_0^1 \oplus \mathbb{C}\sigma_k^2) \bigoplus CHS(\mathfrak{L}_{0,0}).$$

*Proof.* By Lemmas 3.1, 3.3-3.4 and 3.16, this theorem is obvious. □

**Corollary 3.18.** *For any $k \in \mathbb{Z}$, suppose that the action of $\sigma_k^2$ on $\mathfrak{L}_{0,0}$ is defined in Lemma 3.4. Then, $\tau \in MHS(\mathfrak{L}_{0,0})$ if and only if $\tau$ is 0 or possesses the following form $\tau = id_{\mathfrak{L}_{0,0}} + \sum_{k\in\mathbb{Z}} b_k\sigma_k^2$. where $b_k \in \mathbb{C}$.*

*Proof.* Obviously, the linear maps 0 and $\mathrm{id}_{\mathfrak{L}_{0,0}} + \sum_{k\in\mathbb{Z}} b_k\sigma_k^2$ are multiplicative Hom-Lie structures on $\mathfrak{L}_{0,0}$, where $b_k \in \mathbb{C}$. In addition, from Theorem 3.17, we may assume that for any $\tau \in \mathrm{MHS}(\mathfrak{L}_{0,0}) \subseteq \mathrm{HS}(\mathfrak{L}_{0,0})$,

$$\tau = a_0\sigma_0^1 + \sum_{k\in\mathbb{Z}} b_k\sigma_k^2 + \sigma_\rho,$$

where $a_0, b_k \in \mathbb{C}$ and $\sigma_\rho$ is a central Hom-Lie structure on $\mathfrak{L}_{0,0}$ defined in Lemma 3.3.

If $a_0 = 0$, we have $\tau = 0$. More specifically, for any $n \in \mathbb{Z}^*$,

$$\tau([L_0, L_n]) = n(\sum_{k\in\mathbb{Z}} b_kW_{k+n} + \rho(L_n)W_0),$$

$$[\tau(L_0),\tau(L_n)] = [\sum_{k\in\mathbb{Z}} b_k W_k + \rho(L_0)W_0, \sum_{k'\in\mathbb{Z}} b_{k'} W_{k'+n} + \rho(L_n)W_0] = 0.$$

We have $\sum_{k\in\mathbb{Z}} b_k W_{k+n} + \rho(L_n)W_0 = 0$ for any $n \in \mathbb{Z}^*$. If we take $n_1, n_2 \in \mathbb{Z}^*$ such that $n_1 \neq n_2$, then

$$b_k = 0,\ k \neq -n_1,$$

$$b_k = 0,\ k \neq -n_2.$$

Hence, $b_k = 0$ for any $k \in \mathbb{Z}$. This deduces that $\rho(L_n) = 0$ for any $n \in \mathbb{Z}^*$. By the multiplicative property, it is easy to see that $\rho(x) = 0$ for any $x \in \mathfrak{L}_{0,0}$. This shows that $\sigma_\rho = 0$. Hence, $\tau = 0$.

If $a_0 \neq 0$, for any $n \in \mathbb{Z}^*$, we have

$$\tau([L_0, L_n]) = n(a_0 L_n + \sum_{k\in\mathbb{Z}} b_k W_{k+n} + \rho(L_n)W_0), \quad [\tau(L_0),\tau(L_n)] = n(a_0^2 L_n + a_0 \sum_{k\in\mathbb{Z}} b_k W_{k+n}).$$

We have $a_0 = 1$ and $\rho(L_n) = 0$ for any $n \in \mathbb{Z}^*$. Because of the multiplicative property, it is easy to compute that $\rho(x) = 0$ for any $x \in \mathfrak{L}_{0,0}$, which deduces that $\sigma_\rho = 0$. So, $\tau = \mathrm{id}_{\mathfrak{L}_{0,0}} + \sum_{k\in\mathbb{Z}} b_k \sigma_k^2$. We complete the proof. □

**Remark 3.19.** *Using the endomorphism of $\mathfrak{L}_{0,0}$, multiplicative Hom-Lie structures on this algebra are also determined in [12, Theorem 4.1]. Corollary 3.18 contains the result in [12, Theorem 4.1] and gives a more complete characterization of MHS($\mathfrak{L}_{0,0}$).*

Now, we give two examples of multiplicative Hom-Lie structures on $\mathfrak{L}_{0,0}$.

**Example 3.20.** *Let $\sigma_0^2$ be a linear map on $\mathfrak{L}_{0,0}$, where the action of $\sigma_0^2$ is defined in Lemma 3.4, i.e.,*

$$\sigma_0^2(L_m) = W_m, \quad \sigma_0^2(W_m) = 0,\ \forall\ m \in \mathbb{Z}.$$

*Obviously, $\overline{\tau} = id_{\mathfrak{L}_{0,0}} + \sigma_0^2$ is a multiplicative Hom-Lie structure on $\mathfrak{L}_{0,0}$, which is contained in both [12, Theorem 4.1] and Corollary 3.18.*

**Example 3.21.** *Assume that $\sigma_{-1}^2$, $\sigma_1^2$ are linear maps on $\mathfrak{L}_{0,0}$, where the actions of $\sigma_{-1}^2$ and $\sigma_1^2$ are defined in Lemma 3.4, i.e.,*

$$\sigma_{-1}^2(L_m) = W_{m-1}, \quad \sigma_{-1}^2(W_m) = 0,$$
$$\sigma_1^2(L_m) = W_{m+1}, \quad \sigma_1^2(W_m) = 0,\ \forall\ m \in \mathbb{Z}.$$

*It is easy to verify that $\widetilde{\tau} = id_{\mathfrak{L}_{0,0}} + \sigma_{-1}^2 + \sigma_1^2$ is a multiplicative Hom-Lie structure on $\mathfrak{L}_{0,0}$, which is included in Corollary 3.18, not in [12, Theorem 4.1].*

### *3.5. The case for $\lambda = 0, \mu \notin \mathbb{Z}$*

By the definition, $\mathfrak{L}_{0,\mu}$ has the following Lie brackets

$$[L_n, L_m] = (m-n)L_{m+n},\ [L_n, W_m] = (m+\mu)W_{m+n},\ [W_n, W_m] = 0,\ \forall\ m, n \in \mathbb{Z}.$$

**Lemma 3.22.** *For any $k, m \in \mathbb{Z}$, suppose that $\tau_k$ is a Hom-Lie structure on $\mathfrak{L}_{0,\mu}$ satisfying $\tau_k((\mathfrak{L}_{0,\mu})_m) \subseteq (\mathfrak{L}_{0,\mu})_{k+m}$. Then, $\tau_k = a_k\sigma_k^4 + b_k\sigma_k^5 + c_k\sigma_k^6$, where $a_k, b_k, c_k \in \mathbb{C}$ and the action of $\sigma_k^i$ $(i = 4, 5, 6)$ on $\mathfrak{L}_{0,\mu}$ is defined in Lemma 3.5.*

*Proof.* For any $m \in \mathbb{Z}$, assume that

$$\tau_k(L_m) = a_{k,m}L_{k+m} + b_{k,m}W_{k+m}, \quad \tau_k(W_m) = c_{k,m}L_{k+m} + d_{k,m}W_{k+m},$$

where $a_{k,m}, b_{k,m}, c_{k,m}, d_{k,m} \in \mathbb{C}$.

From the proof of Lemma 3.6, we have $a_{k,m} = a_{k,0}$ for any $m \in \mathbb{Z}$. For any $m, n, p \in \mathbb{Z}$, let $x = L_m$, $y = L_n$, $z = L_p$ and $\sigma = \tau_k$ in Eq. (1). Then, we obtain

$$b_{k,m}(p-n)(k+m+\mu) + b_{k,n}(m-p)(k+n+\mu) + b_{k,p}(n-m)(k+p+\mu) = 0. \tag{28}$$

Taking $n = 1$ and $p = 0$ in Eq. (28) gives us that for any $m \in \mathbb{Z}$,

$$b_{k,m} = \frac{b_{k,1}m(k+1+\mu) + b_{k,0}(1-m)(k+\mu)}{k+m+\mu} = \frac{(b_{k,1}(k+1+\mu) - b_{k,0}(k+\mu))m + b_{k,0}(k+\mu)}{k+m+\mu}.$$

For any $m, n, p \in \mathbb{Z}$, let $x = L_p$, $y = L_n$, $z = W_m$ and $\sigma = \tau_k$ in Eq. (1). Then, for any $m \in \mathbb{Z}$, we have

$$c_{k,m}(n-p)(p+n-k-m) = 0,$$

$$(n-p)(d_{k,m}(k+m+\mu) - a_{k,0}(m+\mu)) = 0. \tag{29}$$

By Lemma 3.2, $c_{k,m} = 0$, $\forall\ m \in \mathbb{Z}$. Taking $n = 1$ and $p = 0$ in Eq. (29) shows that

$$d_{k,m} = \frac{m+\mu}{k+m+\mu} a_{k,0},\ \forall\ m \in \mathbb{Z}.$$

Hence, we have

$$\tau_k(L_m) = a_{k,0}L_{k+m} + \frac{(b_{k,1}(k+1+\mu) - b_{k,0}(k+\mu))m + b_{k,0}(k+\mu)}{k+m+\mu} W_{k+m}, \quad \tau_k(W_m) = \frac{m+\mu}{k+m+\mu} a_{k,0} W_{k+m},$$

where $a_{k,0}, b_{k,0}, b_{k,1} \in \mathbb{C}$. Let $a_k = a_{k,0}$, $b_k = b_{k,0}(k+\mu)$ and $c_k = b_{k,1}(k+1+\mu) - b_{k,0}(k+\mu)$. So, $\tau_k = a_k\sigma_k^4 + b_k\sigma_k^5 + c_k\sigma_k^6$. □

From Lemmas 3.1, 3.5 and 3.22, we determine all Hom-Lie structures on $\mathfrak{L}_{0,\mu}$.

**Theorem 3.23.** *For the linear map $\sigma_k^i$ on $\mathfrak{L}_{0,\mu}$ defined in Lemma 3.5, where $i = 4, 5, 6$, we have*

$$HS(\mathfrak{L}_{0,\mu}) = \bigoplus_{k\in\mathbb{Z}} (\mathbb{C}\sigma_k^4 \oplus \mathbb{C}\sigma_k^5 \oplus \mathbb{C}\sigma_k^6).$$

**Corollary 3.24.** *Suppose that the actions of $\sigma_k^5$, $\sigma_k^6$ on $\mathfrak{L}_{0,\mu}$ are defined in Lemma 3.5. Then $\tau \in MHS(\mathfrak{L}_{0,\mu})$ if and only if $\tau$ is 0 or possesses the following form $\tau = id_{\mathfrak{L}_{0,\mu}} + \sum_{k\in\mathbb{Z}} c_k((k+\mu)\sigma_k^5 + \sigma_k^6)$, where $c_k \in \mathbb{C}$.*

*Proof.* The proof is similar to Corollary 3.12. □

We summarize the results concerning (multiplicative) Hom-Lie structures on $\mathfrak{L}_{\lambda,\mu}$ ($\lambda, \mu \in \mathbb{C}$) in the following theorem and corollary.

**Theorem 3.25.** *For $k \in \mathbb{Z}$, assume that the actions of $\sigma_\rho$, $\sigma_k^i$ ($i = 1, 2, 3$) and $\sigma_k^j$ ($j = 4, 5, 6$) on $\mathfrak{L}_{\lambda,\mu}$ are defined in Lemmas 3.3, 3.4 and 3.5, respectively. Then*

$$HS(\mathfrak{L}_{\lambda,\mu}) = \begin{cases} \bigoplus_{k\in\mathbb{Z}} (\mathbb{C}\sigma_k^1 \oplus \mathbb{C}\sigma_k^2), & \text{if } \lambda = -1, \\ \bigoplus_{k\in\mathbb{Z}} (\mathbb{C}\sigma_k^1 \oplus \mathbb{C}\sigma_k^2 \oplus \mathbb{C}\sigma_k^3), & \text{if } \lambda = 1, \\ \bigoplus_{k\in\mathbb{Z}} (\mathbb{C}\delta_{k,0} id_{\mathfrak{L}_{\lambda,\mu}} \oplus \mathbb{C}\sigma_k^2), & \text{if } \lambda \neq 0, \pm 1, \\ \bigoplus_{k\in\mathbb{Z}} (\mathbb{C}\delta_{k,0} id_{\mathfrak{L}_{0,0}} \oplus \mathbb{C}\sigma_k^2) \bigoplus CHS(\mathfrak{L}_{0,0}), & \text{if } \lambda = 0, \mu = 0, \\ \bigoplus_{k\in\mathbb{Z}} (\mathbb{C}\sigma_k^4 \oplus \mathbb{C}\sigma_k^5 \oplus \mathbb{C}\sigma_k^6), & \text{if } \lambda = 0, \mu \notin \mathbb{Z}, \end{cases}$$

*where $\delta_{k,m}$ is the Kronecker delta.*

**Corollary 3.26.** *If $\lambda \neq 0, 1$, any multiplicative Hom-Lie structure on $\mathfrak{L}_{\lambda,\mu}$ is trivial. Moreover, for any $k \in \mathbb{Z}$, suppose that $\sigma_k^i$ $(i = 2, 3)$ and $\sigma_k^j$ $(j = 5, 6)$ are defined in Lemmas 3.4 and 3.5, respectively. Then, any $\tau \in MHS(\mathfrak{L}_{0,\mu})$ or $MHS(\mathfrak{L}_{1,\mu})$ if and only if $\tau$ is 0 or possesses the following form*

$$\tau = \begin{cases} id_{\mathfrak{L}_{0,0}} + \sum_{k\in\mathbb{Z}} b_k\sigma_k^2 & \text{if } \lambda = 0, \mu = 0, \\ id_{\mathfrak{L}_{0,\mu}} + \sum_{k\in\mathbb{Z}} c_k((k+\mu)\sigma_k^5 + \sigma_k^6) & \text{if } \lambda = 0, \mu \notin \mathbb{Z}, \\ id_{\mathfrak{L}_{1,\mu}} + \sum_{k\in\mathbb{Z}} c_k((k+\mu)\sigma_k^2 + \sigma_k^3), & \text{if } \lambda = 1, \end{cases}$$

*where $b_k, c_k \in \mathbb{C}$.*

Observing [14, 18, 19], there is a common phenomenon that Hom-Lie structures on these given Lie algebras $\mathfrak{g}$ are closed with respect to Jordan product. More specifically, HS($\mathfrak{g}$) forms a Jordan algebra with respect to the usual multiplication $\sigma \circ \tau = \frac{1}{2}(\sigma\tau + \tau\sigma)$, where $\sigma, \tau \in$ HS($\mathfrak{g}$). It should be noted that in [20], when Hom-Lie structures form a Jordan algebra was studied, which is also valid for the algebra $\mathfrak{L}_{\lambda,\mu}$, as the following conclusion.

**Corollary 3.27.** *$HS(\mathfrak{L}_{\lambda,\mu})$ is a Jordan algebra with respect to the multiplication $\sigma \circ \tau = \frac{1}{2}(\sigma\tau + \tau\sigma)$, where $\sigma, \tau \in HS(\mathfrak{L}_{\lambda,\mu})$.*

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