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$f_\lambda-$statistical convergence in topological groups

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Abstract. In this paper, the concept of $f_\lambda-$statistical convergence is generalized to topological groups with the help of unbounded modulus function. Furthermore, the relationship between the notions of $f-$statistically convergent and $f_\lambda-$statistically convergent is investigated and some related inclusion results are proved in topological groups using properties of the modulus function. Additionally, this relationship was given in different modulus functions.

1. Introduction

The notion of statistical convergence was previously denoted as "almost convergence" by Zygmund in the initial version of his renowned monograph issued in 1935, (see [43]). Statistical convergence depends on the concept of "natural density" of subsets of the set of $\mathbb{N}$ natural numbers. The notion was initially introduced by Steinhaus [39]. In 1951, for the first time, the concept of statistical convergence in real and complex sequences and its properties were given by Fast [17]. Later on, it was reintroduced by Schoenberg [38] who investigated statistical convergence as a summability method and also outlined some fundamental characteristics of statistical convergence. Later, statistical convergence turned out to be a dynamic field of investigation in summability theory, subsequent to the publications of Borgohain and Savaş [4], Çolak [6], Fridy [18], Maddox [26] and Salat [36]. Statistical convergence is particularly relevant to probability theory. Statistical convergence has been examined in the fields of Fourier analysis, fuzzy mathematics, number theory, measure theory, ergodic theory and turnpike theory by different researchers.

Prullage ([31],[32],[33],[34]) was the first to study the notion of summability in topological groups. This concept was later studied by Çakalli and Thorpe ([9],[10]). Then Çakalli ([11],[12]) gave the concept of statistical convergence in topological groups. Çakalli and Khan [13] studied the relations between topological groups and summability. Recently, the relationship between statistical convergence and topology has been studied by mathematicians (see [5],[14],[16],[40],[41],[42]).

Leindler [24] defined the generalized de la Vallée-Poussin mean. Later, the notions of $\lambda-$density and $\lambda-$statistical convergence were introduced by Mursaleen [27]. For more details on $\lambda-$statistical convergence and its relationship to topological groups, we refer to (see [22],[37]).

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The concept of the modulus function was given by Nakano [29] in 1953. Later, some topological properties of this function were studied by Ruckle [35]. Connor [8] gave the relationship between statistically convergent and strongly summable. Many mathematicians (see [2],[15],[21],[23],[25],[28]) have defined some new sequence spaces using modulus functions and studied the topological properties of these sequence spaces.

Aizpuru [1] defined and studied the concepts of $f-$density and $f-$statistical convergence. Later, Çolak [7] defined statistical convergence with the help of two unbounded modulus functions. Garcia-Pacheco [19] worked on the properties of topological modules. Subsequently, García-Pacheco and Kama [20] defined statistical convergence in topological modules as a continuation of this study.

In this paper we have given some new definitions and theorems from the literature. In this work, firstly, we introduce the concept of $f_\lambda-$statistical convergence in topological groups. We also examined some inclusion theorems in topological groups and applied some inclusion theorems for different modulus functions previously given in the literature to our work.

2. Definitions and preliminaries

In this study, the following definitions and notions will be needed.

Definition 2.1. (Density) The density of $A \subset \mathbb{N}$ is defined, whenever the following limit exists, as

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k).$$

Here χ_A represents the characteristic function of A. It is obvious $\delta(\mathbb{N}) = 1$ and $\delta(A) = 0$ if $A \subset \mathbb{N}$ is a finite set and $\delta(\mathbb{N} \setminus A) = 1 - \delta(A)$ (see Fridy [18], Salat [36]).

Definition 2.2. (Statistical convergence) Using the notion of density for subsets of $\mathbb{N}$, statistical convergence can be defined. A sequence $z = (z_k)$ of real numbers is said to be staticsally convergent to z_0 if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |z_k - z_0| \geq \varepsilon\}) = 0,$$

(see Fridy [18], Salat [36]). In this case, we write $S - \lim(zk) = z_0$ and S denotes the set of all statistically convergent sequences.

Definition 2.3. (λ-density) Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \ \lambda_1 = 1.$$

The generalized de la Vallee-Poussin mean of a sequence $z = (z_k)$ is defined by

$$t_n(z) = \frac{1}{\lambda_n} \sum_{k\in I_n} z_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $z = (z_k)$ is said to be $(V, \lambda)-$summable to a number z_0 if

$$t_n(z) \to z_0, \text{ as } n \to \infty,$$

(see Leindler [24]). In this case, we write $\lambda - \lim z_k = z_0$. If $\lambda_n = n$, then $(V, \lambda)-$summability reduces to $(C, 1)-$summability. We write

$$[C, 1] = \left\{ z = (z_k) : \exists z_0 \in \mathbb{R}, \ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |z_k - z_0| = 0 \right\}$$

and

$$[V,\lambda] = \left\{ z = (z_k) : \exists z_0 \in \mathbb{R},\ \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |z_k - z_0| = 0 \right\}$$

for the sets of sequences $z = (z_k)$ which are strongly Cesaro summable and strongly $(V,\lambda)-$summable to z_0, i.e. $(z_k) \xrightarrow{[C,1]} z_0$ and $(z_k) \xrightarrow{[V,\lambda]} z_0$, respectively.

Let $A \subseteq \mathbb{N}$ be a set of positive integers. Then

$$\delta_\lambda(A) = \lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \le k \le n : k \in A\}|$$

is said to be $\lambda-$density of A provided the limit exists, (see Mursaleen [27]).

Definition 2.4. ($\lambda-$statistical convergence) A sequence $z = (z_k)$ is said to be $\lambda-$statistically convergent or $S_\lambda-$convergent to z_0 if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |z_k - z_0| \ge \varepsilon\}| = 0,$$

or equivalently a sequence $z = (z_k)$ is said to be $\lambda-$statistically convergent to z_0 or $S_\lambda-$convergent to z_0 if for every $\varepsilon > 0$ the set $A_\varepsilon = \{k \in \mathbb{N} : |z_k - z_0| \ge \varepsilon\}$ has $\lambda-$density 0, (see Mursaleen [27]). In this case we write $S_\lambda - \lim z_k = z_0$ and

$$S_\lambda = \{z = (z_k) : \exists z_0 \in \mathbb{R},\ S_\lambda - \lim z_k = z_0\}.$$

It is clear that if $\lambda_n = n$, then S_λ is same as S.

Definition 2.5. (Modulus function) A modulus function is a function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ which satisfies:
i) $f(z) = 0 \Leftrightarrow z = 0$,
ii) $f(z_1 + z_2) \le f(z_1) + f(z_2)$ for every $z_1, z_2 \in \mathbb{R}^+$,
iii) f is increasing,
iv) f is continuous from the right at 0, (see Nakano [29], Ruckle [35]).

It is clear that every such function is continuous. A modulus may be unbounded or bounded. Some examples of modulus functions are $f_1(z) = \frac{z}{z+1}$, $f_2(z) = z^p$ and $f_3(z) = \log(z+1)$ with $0 < p \le 1$. Here, while f_1 and f_2 are bounded modulus functions, f_3 is an unbounded modulus function.

García-Pacheco and Kama [20] defined the notion of $f-$ statistical convergence in uniform spaces using any modulus function.

In this paper, we will consider Z as an abelian topological Hausdoorff group and additively. Furthermore, let Z satisfy the first axiom of countability. We take f as an unbounded modulus function and give the following definition.

Definition 2.6. ($f-$statistical convergence) A sequence (z_k) in Z is said to be $f-$statistically convergent or $S_f(Z)-$convergent to an element z_0 of Z if for each neighbourhood U of 0,

$$\lim_{n\to\infty} \frac{1}{f(n)} f\left(|\{k \le n : z_k - z_0 \notin U\}|\right) = 0$$

where f is unbounded modulus function. In this case we write $S_f(Z) - \lim_{k\to\infty} z_k = z_0$. The set of $f-$statistically convergent in topological groups will be denoted by $S_f(Z)$. In case of $z_0 = 0$, we shall write $S_{f,0}(Z)$.

Lemma 2.7. *Let f be a modulus function and let $0 < \delta < 1$. Then, for each $z \ge \delta$, we have $f(z) \le 2f(1)\,\delta^{-1} z$, (see [30]).*

3. Main results

In this section, we will define the concept $f_\lambda-$statistical convergence in topological groups using an unbounded modulus function. We have also given inclusion theorems using the unbounded modulus function to examine the relationship between $f-$statistical convergence and $f_\lambda-$statistical convergence.

We will use Z in this article to denote an abelian topological Hausdorff group written additively. It satisfies the first axiom of countability. The identity element of Z will be denoted by 0.

Definition 3.1. A sequence (z_k) in Z is said to be $f_\lambda-$statistically convergent or $S_{f_\lambda}(Z)-$ convergent to an element z_0 of Z if for each neighbourhood U of 0,

$$\delta_{f_\lambda}\left(\{k \in I_n : z_k - z_0 \notin U\}\right) = 0$$

i.e.

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_k - z_0 \notin U\}|\right) = 0$$

where f is unbounded modulus function. In this case, we write $S_{f_\lambda}(Z) - \lim_{k\to\infty} z_k = z_0$ or $(z_k) \xrightarrow{S_{f_\lambda}} z_0$ and we define

$$S_{f_\lambda}(Z) = \left\{(z_k) : \text{for some } z_0,\ S_{f_\lambda}(Z) - \lim_{k\to\infty} z_k = z_0\right\}.$$

In particular,

$$S_{f_\lambda,0}(Z) = \left\{(z_k) : S_{f_\lambda}(Z) - \lim_{k\to\infty} z_k = 0\right\}.$$

Definition 3.2. A sequence (z_k) in Z is said to be $S_{f_\lambda}(Z)-$Cauchy sequences in Z if for each neighbourhood U of 0, there is an integer $t(V)$ such that

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(\left|\left\{k \in I_n : z_k - z_{t(V)} \notin U\right\}\right|\right) = 0.$$

Theorem 3.3. *Let f be an unbounded modulus function. If a sequence $z = (z_k)$ is $S_{f_\lambda}(Z)-$convergent in Z, then $S_{f_\lambda}(Z) - \lim_{k\to\infty} z_k$ is unique.*

Proof. Let $z = (z_k)$ be $S_{f_\lambda}(Z)-$convergent in Z. Suppose that (z_k) has two different $S_{f_\lambda}(Z)$ limits, z_1, z_2 say. Since Z is Hausdorff space there exists a neighbourhood U of 0 such that $z_1 - z_2 \notin U$. Also for each neighbourhood U of 0 there exists a symmetric neighbourhood V of 0 such that $V + V \subset U$, (see Arnautov *et al.* [3]). Write $x_k = z_1 - z_2$ for all $k \in \mathbb{N}$. Therefore for all $n \in \mathbb{N}$,

$$\{k \in I_n : x_k \notin U\} \subset \{k \in I_n : z_1 - z_k \notin V\} \cup \{k \in I_n : z_k - z_2 \notin V\}.$$

Now it follows from this inclusion that, for all $n \in \mathbb{N}$ and since f is increasing,

$$\begin{aligned} \tfrac{1}{f(\lambda_n)} f\left(|\{k \in I_n : x_k \notin U\}|\right) \quad \leq \quad & \tfrac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_1 - z_k \notin V\}|\right) \\ & + \tfrac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_k - z_2 \notin V\}|\right). \end{aligned}$$

Since $S_{f_\lambda}(Z) - \lim_{k\to\infty} z_k = z_1$ and $S_{f_\lambda}(Z) - \lim_{k\to\infty} z_k = z_2$, we get

$$\begin{aligned} \lim_{n\to\infty} \tfrac{1}{f(\lambda_n)} f\left(|\{k \in I_n : x_k \notin U\}|\right) \quad \leq \quad & \lim_{n\to\infty} \tfrac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_1 - z_k \notin V\}|\right) \\ & + \lim_{n\to\infty} \tfrac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_k - z_2 \notin V\}|\right). \end{aligned}$$

Hence $1 \leq 0 + 0 = 0$. This contradiction shows that $z_1 = z_2$. □

Theorem 3.4. *Let f be an unbounded modulus function. If $\lim_{k\to\infty} z_k = z_0$ and $S_{f_\lambda}(Z) - \lim_{k\to\infty} x_k = 0$, then $S_{f_\lambda}(Z) - \lim_{k\to\infty}(z_k + x_k) = z_0$.*

Proof. Let U be any neighbourhood of 0. Then we may choose a symmetric neighbourhood V of 0 such that $V + V \subset U$. Since $\lim_{k\to\infty} z_k = z_0$, then there exists an integer $k \geq n_0$ implies that $z_k - z_0 \in V$. Hence

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_k - z_0 \notin V\}|\right) = 0.$$

By assumption $S_{f_\lambda}(Z) - \lim_{k\to\infty} x_k = 0$, then we have

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(|\{k \in I_n : x_k \notin V\}|\right) = 0.$$

Thus

$$\begin{aligned}\lim_{n\to\infty} \tfrac{1}{f(\lambda_n)} f\left(|\{k \in I_n : (z_k - z_0) + x_k \notin U\}|\right) \quad &\leq \quad \lim_{n\to\infty} \tfrac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_k - z_0 \notin V\}|\right) \\ &\quad + \lim_{n\to\infty} \tfrac{1}{f(\lambda_n)} f\left(|\{k \in I_n : x_k \notin V\}|\right).\end{aligned}$$

This implies that $S_{f_\lambda}(Z) - \lim_{k\to\infty}(z_k + x_k) = \lim_{k\to\infty} z_k = z_0$. □

Theorem 3.5. *Let f be an unbounded modulus function. If a sequence (z_k) is $S_{f_\lambda}(Z)-$convergent in Z, then there are sequences (x_k) and (y_k) such that $z_k = x_k + y_k$, for each $k \in \mathbb{N}$, $S_{f_\lambda}(Z) - \lim_{k\to\infty} x_k = z_0$, $\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_k \neq x_k\}|\right) = 0$ and (y_k) is a $S_{f_\lambda}(Z)-$null sequence.*

Proof. Let $\{U_i\}$ be a nested base of neighbourhood of 0. Take $n_0 = 0$ and choose an increasing sequence (n_i) of positive integers such that

$$\delta_{f_\lambda}\left(\{k \in I_n : z_k - z_0 \notin U_i\}\right) < \frac{1}{i} \text{ for } k > n_i.$$

Let us define the sequences (x_k) and (y_k) as follows:

$x_k = z_k$ and $y_k = 0$, if $0 < k < n_i$

$x_k = z_k$ and $y_k = 0$, if $z_k - z_0 \in U_i$

$x_k = z_0$ and $y_k = z_k - z_0$, if $z_k - z_0 \notin U_i$.

We have to show that (i) $\lim_{k\to\infty} x_k = z_0$, (ii) (y_k) is an $S_{f_\lambda}-$null sequence.

(i) Let U be any neighbourhood of 0. We may choose a positive integer i such that $U_i \subset U$. Then $x_k - z_0 = z_k - z_0 \in U_i$, for $k > n_i$. If $z_k - z_0 \notin U_i$, then $x_k - z_0 = z_0 - z_0 = 0 \in U$. Hence $\lim_{k\to\infty} x_k = z_0$.

(ii) It is enough to show that $\delta_{f_\lambda}\left(\{k \in I_n : y_k \neq 0\}\right) = 0$. For any neighbourhood U of 0, we have

$$\delta_{f_\lambda}\left(\{k \in I_n : y_k \notin U\}\right) \leq \delta_{f_\lambda}\left(\{k \in I_n : y_k \neq 0\}\right).$$

If $n_r < k \leq n_{r+1}$, then

$$\{k \in I_n : y_k \neq 0\} \subset \{k \in I_n : z_k - z_0 \notin U_r\}.$$

If $r > i$ and $n_r < k \leq n_{r+1}$, then

$$\delta_{f_\lambda}\left(\{k \in I_n : y_k \neq 0\}\right) \leq \delta_{f_\lambda}\left(\{k \in I_n : z_k - z_0 \notin U_r\}\right) < \frac{1}{r} < \frac{1}{i} < \varepsilon.$$

This implies that $\delta_{f_\lambda}\left(\{k \in I_n : y_k \neq 0\}\right) = 0$. Hence (y_k), $S_{f_\lambda}(Z)-$null sequence. □

Theorem 3.6. *Let $z=(z_k)$ be a sequence in Z and f be an unbounded modulus function. If there is a $S_{f_\lambda}(Z)-$convergent sequence $x=(x_k)$ in Z such that $\delta_{f_\lambda}(\{k\in\mathbb{N}:x_k\neq z_k\notin U\})=0$ then z is also $S_{f_\lambda}(Z)-$convergent.*

Proof. Suppose that $\delta_{f_\lambda}(\{k\in I_n:x_k\neq z_k\notin U\})=0$ and $S_{f_\lambda}(Z)-\lim x_k=z_0$. Then for every neighbourhood U of 0, we have

$$\delta_{f_\lambda}(\{k\in I_n:x_k-z_0\notin U\})=0.$$

Now,

$$\{k\in I_n:z_k-z_0\notin U\}\subseteq\{k\in\mathbb{N}:x_k\neq z_k\notin U\}\cup\{k\in I_n:x_k-z_0\notin U\}$$

$$\Rightarrow \delta_{f_\lambda}(\{k\in I_n:z_k-z_0\notin U\})\leq\delta_{f_\lambda}(\{k\in I_n:x_k\neq z_k\notin U\})+\delta_{f_\lambda}(\{k\in I_n:x_k-z_0\notin U\}).$$

Therefore we have

$$\delta_{f_\lambda}(\{k\in I_n:z_k-z_0\notin U\})=0.$$

This completes the proof of the theorem. □

Theorem 3.7. *Let $z=(z_k)$ be a sequence in Z and f be an unbounded modulus function. If*

$$\lim_{n\to\infty}\inf\frac{n}{f(\lambda_n)}<\infty,$$

then $S_{f_\lambda}(Z)-$convergent sequences are $S_f(Z)-$convergent sequence, i.e. $S_f(Z)\subset S_{f_\lambda}(Z)$.

Proof. Suppose that $\lim\inf_{n\to\infty}\frac{n}{f(\lambda_n)}<\infty$ and $z=(z_k)$ be $S_f(Z)-$convergent sequence. Let U be any neighbourhood of 0. Then for all $n\in\mathbb{N}$,

$$\{k\leq n:z_k-z_0\notin U\}\supset\{k\in I_n:z_k-z_0\notin U\}.$$

Therefore,

$$\begin{aligned}\frac{1}{f(\lambda_n)}f(|\{k\in I_n:z_k-z_0\notin U\}|)&\leq\frac{1}{f(\lambda_n)}f(|\{k\leq n:z_k-z_0\notin U\}|)\\&\leq\frac{n}{f(\lambda_n)}\frac{f(n)}{n}\frac{1}{f(n)}f(|\{k\leq n:z_k-z_0\notin U\}|)\\&\leq\frac{n}{f(\lambda_n)}f(1)\frac{1}{f(n)}f(|\{k\leq n:z_k-z_0\notin U\}|).\end{aligned}$$

Taking the limit as $n\to\infty$, we get $S_f(Z)\subset S_{f_\lambda}(Z)$. □

Theorem 3.8. *Let $z=(z_k)$ be a sequence in Z and f be an unbounded modulus function. If*

$$\lim_{n\to\infty}\frac{n}{f(\lambda_n)}=1 \text{ and } \lim_{n\to\infty}\frac{f(n-\lambda_n)}{n}=0,$$

then $S_f(Z)-$convergent sequence is $S_{f_\lambda}(Z)-$convergent sequence, i.e. $S_{f_\lambda}(Z)\subset S_f(Z)$.

Proof. Suppose that $\lim_{n\to\infty}\frac{n}{f(\lambda_n)}=1$ and $\lim_{n\to\infty}\frac{f(n-\lambda_n)}{n}=0$. Let U be any neighbourhood of 0. Since $\lambda_n\leq n\Rightarrow f(\lambda_n)\leq f(n)$, we get

$$\begin{aligned}\frac{1}{f(n)}f(|\{k\leq n:z_k-z_0\notin U\}|)&=\frac{1}{f(n)}f(|\{k\leq n-\lambda_n:z_k-z_0\notin U\}|)\\&\quad+\frac{1}{f(n)}f(|\{k\in I_n:z_k-z_0\notin U\}|)\\&\leq\frac{f(n-\lambda_n)}{f(n)}+\frac{1}{f(n)}f(|\{k\in I_n:z_k-z_0\notin U\}|)\\&\leq\frac{f(n-\lambda_n)}{n}\frac{n}{f(\lambda_n)}\frac{f(\lambda_n)}{f(n)}+\frac{1}{f(\lambda_n)}f(|\{k\in I_n:z_k-z_0\notin U\}|).\end{aligned}$$

Taking limit as $n\to\infty$, we get $S_{f_\lambda}(Z)\subset S_f(Z)$. □

Theorem 3.9. *Let $z = (z_k)$ be a sequence in Z and $\lambda = (\lambda_n)$ and $\varphi = (\varphi_n)$ be two sequences of positive numbers such that $\lambda_n \leq \varphi_n$ for all $n \in \mathbb{N}_{n_0}$.*

(i) if $\liminf_n \frac{f(\lambda_n)}{f(\varphi_n)} > 0$, then $S_{f_\varphi}(Z) \subset S_{f_\lambda}(Z)$,

(ii) if $\lim_{n\to\infty} \frac{\varphi_n}{\lambda_n} = 1$, then $S_{f_\lambda}(Z) \subset S_{f_\varphi}(Z)$,

where f is unbounded modulus function.

Proof. (*i*) Suppose that $\lambda_n \leq \varphi_n$ for all $n \in \mathbb{N}_{n_0}$ which implies that $f(\lambda_n) \leq f(\varphi_n)$. Let U be any neighbourhood of 0. Since $\lambda_n \leq \varphi_n$ for all $n \in \mathbb{N}_{n_0}$, so $I_n \subset J_n$, where $I_n = [n - \lambda_n + 1, n]$ and $J_n = [n - \varphi_n + 1, n]$.

Now, we can write

$$\{k \in J_n : z_k - z_0 \notin U\} \supset \{k \in I_n : z_k - z_0 \notin U\}$$

and so

$$\frac{1}{f(\varphi_n)} f(|\{k \in J_n : z_k - z_0 \notin U\}|) \geq \frac{f(\lambda_n)}{f(\varphi_n)} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : z_k - z_0 \notin U\}|),$$

for all $n \in \mathbb{N}_{n_0}$. Since $\liminf_n \frac{f(\lambda_n)}{f(\varphi_n)} > 0$ and also taking as $n \to \infty$, we get $S_{f_\varphi}(Z) \subset S_{f_\lambda}(Z)$.

(*ii*) Let U be any neighbourhood of 0 and $S_{f_\lambda}(Z) - \lim_{k\to\infty} z_k = z_0$. Since $I_n \subset J_n$, we write

$$\begin{aligned}
\frac{1}{\varphi_n} |\{k \in J_n : z_k - z_0 \notin U\}| &= \frac{1}{\varphi_n} \left|\{n - \varphi_n + 1 \leq k \leq n - \lambda_n : z_k - z_0 \notin U\}\right| + \frac{1}{\varphi_n} |\{k \in I_n : z_k - z_0 \notin U\}| \\
&\leq \frac{(\varphi_n - \lambda_n)}{\varphi_n} + \frac{1}{\varphi_n} |\{k \in I_n : z_k - z_0 \notin U\}| \\
&\leq \frac{\varphi_n - \lambda_n}{\lambda_n} + \frac{1}{\varphi_n} |\{k \in I_n : z_k - z_0 \notin U\}| \\
&\leq \left(\frac{\varphi_n}{\lambda_n} - 1\right) + \frac{1}{\lambda_n} |\{k \in I_n : z_k - z_0 \notin U\}|,
\end{aligned}$$

for all $n \in \mathbb{N}_{n_0}$. We use the definition of the modulus function and Lemma 2.7, we have

$$\begin{aligned}
&\frac{1}{f(\varphi_n)} f(|\{k \in J_n : z_k - z_0 \notin U\}|) \\
&\leq f\left(\frac{\varphi_n}{\lambda_n} - 1\right) + \frac{1}{f(\lambda_n)} f(|\{k \in I_n : z_k - z_0 \notin U\}|) \\
&\leq 2f(1)\,\delta^{-1}\left(\frac{\varphi_n}{\lambda_n} - 1\right) + \frac{1}{f(\lambda_n)} f(|\{k \in I_n : z_k - z_0 \notin U\}|).
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{\varphi_n}{\lambda_n} = 1$ by *(ii)* the first term goes to 0 since $S_{f_\lambda}(Z) - \lim_{k\to\infty} z_k = z_0$, the second term of right hand side of above inequality tends to 0 as $n \to \infty$ (note that $\frac{\varphi_n}{\lambda_n} - 1 \geq 0$). This implies that $S_{f_\lambda}(Z) \subset S_{f_\varphi}(Z)$. □

Theorem 3.10. *Let $z = (z_k)$ be a sequence in Z and f and g be two unbounded modulus functions. Then*

(i) if $\lim_{t\to\infty} \frac{f(t)}{g(t)} > 0$, then a sequence (z_k) is $f_\lambda-$statistically convergent in Z if it is $g_\lambda-$statistically convergent in Z, that is $S_{g_\lambda}(Z) \subset S_{f_\lambda}(Z)$.

(ii) if $\lim_{t\to\infty} \frac{f(t)}{g(t)} < \infty$, then a sequence (z_k) is $g_\lambda-$statistically convergent in Z if and only if it is $f_\lambda-$statistically convergent in Z, that is $S_{f_\lambda}(Z) = S_{g_\lambda}(Z)$.

Proof. Let f and g be two unbounded modulus functions.

(*i*) Suppose $\lim_{t\to\infty} \frac{f(t)}{g(t)} = \delta > 0$, then $\lim_{n\to\infty} \frac{f(\lambda_n)}{g(\lambda_n)} = \delta > 0$ and (z_k) is $g_\lambda-$statistically convergent to z, that is $S_{g_\lambda}(Z) - \lim_{k\to\infty} z_k = z_0$. Then given any $\varepsilon > 0$ there exist a real number n_0 such that $(\delta - \varepsilon)\, g(\lambda_n) < f(\lambda_n) < (\delta + \varepsilon)\, g(\lambda_n)$ if $n > n_0$. Therefore we have the inequality $f(\lambda_n) < 2\delta g(\lambda_n)$ if $n > n_0$. Now we may write the inequality the for each neighbourhood U of 0

$$\frac{1}{g(\lambda_n)} g(|\{k \in I_n : z_k - z_0 \notin U\}|) \geq \frac{1}{2\delta} \frac{f(|\{k \in I_n : z_k - z_0 \notin U\}|)}{f(\lambda_n)} \frac{f(\lambda_n)}{g(\lambda_n)}$$

if $|\{k \in I_n : z_k - z_0 \notin U\}| > n_0$. Since $\lim_{n\to\infty} \frac{f(\lambda_n)}{g(\lambda_n)} = \delta > 0$ from the above inequality we obtain

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_k - z_0 \notin U\}|\right) = 0$$

if

$$\lim_{n\to\infty} \frac{1}{g(\lambda_n)} g\left(|\{k \in I_n : z_k - z_0 \notin U\}|\right) = 0.$$

Therefore $S_{g_\lambda}(Z) \subset S_{f_\lambda}(Z)$.

(*ii*) We may write the following equality the for each neighbourhood U of 0

$$\frac{1}{g(\lambda_n)} g\left(|\{k \in I_n : z_k - z_0 \notin U\}|\right) = \frac{g\left(|\{k \in I_n : z_k - z_0 \notin U\}|\right)}{f\left(|\{k \in I_n : z_k - z_0 \notin U\}|\right)} \frac{f\left(|\{k \in I_n : z_k - z_0 \notin U\}|\right)}{f(\lambda_n)} \frac{f(\lambda_n)}{g(\lambda_n)}.$$

Suppose $0 < \lim_{n\to\infty} \frac{f(\lambda_n)}{g(\lambda_n)} = \delta < \infty$ and so that $\lim_{n\to\infty} \frac{g(\lambda_n)}{f(\lambda_n)} = \frac{1}{\delta}$. Using this fact from the above equality we obtain

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(|\{k \in I_n : z_k - z_0 \notin U\}|\right) = 0$$

if and only if

$$\lim_{n\to\infty} \frac{1}{g(\lambda_n)} g\left(|\{k \in I_n : z_k - z_0 \notin U\}|\right) = 0.$$

Therefore $S_{f_\lambda}(Z) = S_{g_\lambda}(Z)$. □

Theorem 3.11. *Let $z = (z_k)$ be a sequence in Z and f and g be two unbounded modulus functions. Then*

(*i*) *if $\lim_{t\to\infty} \frac{f(t)}{g(t)} > 0$, then a sequence (z_k) is $S_{g_\lambda}(Z)-$Cauchy sequences in Z if it is $S_{f_\lambda}(Z)-$Cauchy sequences in Z.*

(*ii*) *if $\lim_{t\to\infty} \frac{f(t)}{g(t)} < \infty$, then a sequence (z_k) is $S_{f_\lambda}(Z)-$Cauchy sequences in Z if and only if it is $S_{g_\lambda}(Z)-$Cauchy sequences in Z.*

Proof. The proof can be similarly to the proof of Theorem 3.10. □

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