Filomat 38:28 (2024), 9947–9960
https://doi.org/10.2298/FIL2428947E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a class of Kirchhoff problem involving Choquard nonlinearity with real parameter

Hamza El-Houari[a,*], Elhoussain Arhrrabi[b]

[a] *AMNEA Group, Department of Mathematics, Faculty of Sciences and Techniques Errachidia, University Moulay Ismail, Meknes, Morocco*
[b] *Laboratory of Systems, Control and Decision (LSCD), School of New Engineering Sciences (ENSI), Tangier, Morocco*

Abstract. In this paper, we focus on investigating the existence and multiplicity of nontrivial weak solutions for a class of Choquard–Kirchhoff type equations. These equations involve a variable $s(x,\cdot)$-order fractional $(\tau_1(x,\cdot),\tau_2(x,\cdot))$-Laplacian operator with a real parameter and a continuous variable parameter. The main challenges lie in the Choquard nonlinearities and Kirchhoff functions, along with the presence of double Laplace operators involving two variable parameters.

1. Introduction

In recent times, considerable emphasis has been directed towards examining issues related to the non-local fractional Laplacian or more broadly, integro-differential operators. These operators naturally emerge in various applications, including continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces, and game theory. They represent a common result of the stochastic stabilization of Lévy processes, see [7, 8, 25]. Moreover, recent research has delved into the regional fractional Laplacian, limiting the operator's scope to a variable region near each point. For example, the authors in [22], investigated the presence, symmetry features, and concentration phenomena of solutions in the nonlinear Schrödinger equation with non-local regional diffusion. The distinct characteristics of these regional operators make them particularly interesting in the mathematical theory of non-local operators. Also finds diverse applications and research applications in physical systems, particularly in non-homogeneous Kirchhoff-type equations [29], where the authors studied a Schrödinger-Kirchhoff-type equation with fractional p-Laplacian in $\mathbb{R}^n$ of the form:

$$M\left(\int\int_{\mathbb{R}^{2N}} \frac{|\kappa(x)-\kappa(y)|^{\tau}}{|x-y|^{N+s\tau}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^s_\tau\kappa + V(x)|\kappa|^{\tau-2}u = f(x,\kappa)+g(x) \qquad x\in\mathbb{R}^N,$$

where $0<s<1<\tau<\infty$, $\tau s<n$, f and g are two continuous functions.

In the setting of Kirchhoff problem involving the variable-order fractional $\tau(\cdot)$-Laplacian operator, the authors in [31] established the existence of at least two distinct solutions for the aforementioned problem

2020 *Mathematics Subject Classification*. Primary 35R11 mandatory; Secondary 35A15, 47G20 optionally.
Keywords. $s(x,\cdot)$-order fractional $(\tau_1(x,\cdot),\tau_2(x,\cdot))$-Laplacian operator, Kirchhoff type equations, Choquard nonlinearity, variational methods.
Received: 18 January 2024; Revised: 15 June 2024; Accepted: 17 June 2024
Communicated by Maria Alessandra Ragusa
* Corresponding author: Hamza El-Houari
Email addresses: h.elhouari94@gmail.com (Hamza El-Houari), arhrrabi.elhoussain@gmail.com (Elhoussain Arhrrabi)

using the generalized abstract critical point theorem. Furthermore, they demonstrated the existence of one solution and an infinite number of solutions by employing the mountain pass lemma and the Fountain Theorem. The equation is as the form:

$$\begin{cases} M\left(\int\int_{\mathfrak{Q}^2} \frac{|\kappa(x)-\kappa(y)|^{\tau(x,y)}}{\tau(x,y)|x-y|^{N+s(x,y)\tau(x,y)}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^{s(\cdot)}_{\tau(\cdot)}\kappa + |\kappa|^{\overline{\tau}(x)-2}\kappa = \mu g(x,\kappa) \quad x \in \mathbb{R}^N, \\ \kappa \in W^{s(\cdot),\tau(\cdot)}(\mathbb{R}^N). \end{cases}$$

where $N > p(x,y)s(x,y)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, M represents a continuous Kirchhoff-type function, $g(x,u)$ is a Carathéodory function, and $\mu > 0$ serves as a parameter. In the context of Kirchhoff-Choquard equations incorporating the variable-order fractional $\tau(\cdot)$-Laplacian, reference [5] details the authors' work establishing the existence of weak solutions, ground state solutions utilizing the Nehari manifold, and the existence of infinitely many solutions using the Fountain Theorem and Dual Fountain Theorem. This investigation concentrated on a class of doubly nonlocal Kirchhoff-Choquard type equations, where the variable-order fractional $p(\cdot)$-Laplacian operator takes a specific form.

$$\begin{cases} m\left(\int\int_{\mathfrak{Q}^2} \frac{|\kappa(x)-\kappa(y)|^{\tau(x,y)}}{\tau(x,y)|x-y|^{N+s(x,y)\tau(x,y)}}\mathrm{d}x\mathrm{d}y + \int_{\mathfrak{Q}} V(x)\frac{|\kappa|^{\overline{\tau}(x)}}{\overline{\tau}(x)}\right) \\ \qquad \times\left[(-\Delta)^{s(\cdot)}_{\tau(\cdot)}\kappa + V(x)|\kappa|^{\overline{\kappa}(x)-2}\kappa\right] = \left(\int_{\mathfrak{Q}} \frac{F(y,\kappa(y))}{|x-y|^{\mu(x,y)}}dy\right) f(x,\kappa) \quad x \in \mathfrak{Q}, \\ u = 0 \qquad \mathbb{R}^n \backslash \mathfrak{Q}. \end{cases}$$

where $m : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, $V : \mathfrak{Q} \to \mathbb{R}^+_0$, $\mu : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,N)$, $f : \mathfrak{Q} \times \mathbb{R} \to \mathbb{R}$, $\tau : \mathbb{R}^N \times \mathbb{R}^N \to (1,\infty)$, $s : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ are continuous functions, $\overline{\tau} := \tau(x,x)$, $\mathfrak{Q} \subset \mathbb{R}^N$ is a smooth bounded domain and F is the primitive of f.
After this, many authors have looked into the problem using Laplacian, p-Laplacian, and fractional N-Laplacian operators. They used either the technique presented in this paper or critical point methods. Some notable references include: [1, 3, 9–18, 26–28, 30].

Inspired by the aforementioned studies, our objective in this article is to examine the Choquard–Kirchhoff type equations. These equations involve the variable $s(x,\cdot)$-order fractional $(\tau_1(x,\cdot),\tau_2(x,\cdot))$-Laplacian operator with both a real parameter and a continuous variable parameter:

$$\begin{cases} \mathscr{K}_1\left([\kappa]_{s(\cdot),\tau_1(\cdot)}\right)(-\Delta)^{s(\cdot)}_{\tau_1(\cdot)}\kappa(x) + \mathscr{K}_2\left([\kappa]_{s(\cdot),\tau_2(\cdot)}\right)(-\Delta)^{s(\cdot)}_{\tau_2(\cdot)}\kappa(x) \\ = \xi|\kappa(x)|^{\mu(x)-2}\kappa(x) + \left(\int_{\mathfrak{Q}} \frac{H(y,\kappa(y))}{|x-y|^{\alpha(x,y)}}dy\right)h(x,\kappa(x)) + g(x), \qquad \text{in } \mathfrak{Q}, \\ \qquad \kappa = 0 \qquad \text{in } \mathbb{R}^N\backslash\mathfrak{Q}, \end{cases} \tag{1}$$

where

$$[\kappa]_{s(\cdot),\tau_i(\cdot)} := \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_i(x,y)}}{\tau_i(x,y)|x-y|^{N+s(x,y)\tau_i(x,y)}}\mathrm{d}x\mathrm{d}y, \quad i = 1,2 \tag{2}$$

and $\mathfrak{Q}$ represents a bounded smooth domain in $\mathbb{R}^N$, $\mathscr{K}_1$ and $\mathscr{K}_2$ denote models of Kirchhoff coefficients, ξ is a real parameter, and $s, \tau_1, \tau_2, \mu, \alpha, H, h, g$ are typically continuous functions. The specific assumptions for these functions will be introduced in the subsequent discussion. The operators $(-\Delta)^{s(\cdot)}_{\tau_i(\cdot)}$ are referred to as variable $s(\cdot)$-order $\tau_i(\cdot)$-fractional Laplacians, where $\tau_i(\cdot) : \bar{\mathfrak{Q}} \times \bar{\mathfrak{Q}} \longrightarrow (1,+\infty)$ for $i = 1,2$, and $s(\cdot) : \bar{\mathfrak{Q}} \times \bar{\mathfrak{Q}} \longrightarrow (0,1)$ with $N > s(x,y)\tau_i(x,y)$ for all $(x,y) \in \bar{\mathfrak{Q}} \times \bar{\mathfrak{Q}}$. These operators can be defined as

$$(-\Delta)^{s(\cdot)}_{\tau_i(\cdot)}\kappa(x) = p.v. \int_{\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_i(x,y)-2}(\kappa(x)-\kappa(y))}{|x-y|^{N+s(x,y)\tau_i(x,y)}}dy, \quad \forall x \in \mathfrak{Q}, \tag{3}$$

where $\kappa \in C_0^\infty(\mathfrak{Q})$ and p.p. stands for the Cauchy principal value.

In contrast, Fiscella and Valdinoci presented the initial concept of a stationary Kirchhoff variational equation in [23]. This equation captures the non-local influence of tension resulting from fractional length measurements of the string. In fact, represents a fractional adaptation of the Kirchhoff equation model introduced by Kirchhoff in [24]. To be more specific, Kirchhoff formulated a model described by the following problem:

$$\varrho \frac{\partial^2 \mathfrak{Q}}{\partial \gamma^2} - \Big(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \Big| \frac{\partial_o m}{\partial x} \Big|^2 \Big) \frac{\partial^2 \mathfrak{Q}}{\partial x^2} = 0, \tag{4}$$

which extends D' Alembert's wave equation. One notable feature of model (4) is that it contains a nonlocal term $\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial_o m}{\partial x} \right|^2$. The parameters L, h, E, m, P_0 in model (4) represent different physical meanings, which we will not cover here.

In this paper, we make the assumption that $s(\cdot)$ and $\tau_i(\cdot)$ are uniformly continuous functions, while $\alpha : \bar{\mathfrak{Q}} \times \bar{\mathfrak{Q}} \longrightarrow (0, N)$ is a continuous function, and satisfies the following assupmtions:

$$(A_1): \quad 0 < s^- \leq s^+ < 1, \quad 1 < \tau_1^- \leq \tau_1^+, \quad 1 < \tau_2^- \leq \tau_2^+, \quad 0 < \alpha^- < \alpha^+ < N,$$

where

$$s^- := \inf_{(x,y) \in \bar{\mathfrak{Q}} \times \bar{\mathfrak{Q}}} s(x,y), \quad s^+ := \sup_{(x,y) \in \bar{\mathfrak{Q}} \times \bar{\mathfrak{Q}}} s(x,y),$$

$$\tau_i^- := \inf_{(x,y) \in \bar{\mathfrak{Q}} \times \bar{\mathfrak{Q}}} \tau_i(x,y) \text{ and } \tau_i^+ := \sup_{(x,y) \in \bar{\mathfrak{Q}} \times \bar{\mathfrak{Q}}} \tau_i(x,y) \quad \text{for } i = 1, 2.$$

Let denote

$$\tau_{max}(\cdot) := \max\{\tau_1(\cdot), \tau_2(\cdot)\}, \ \tau_{min}(\cdot) := \min\{\tau_1(\cdot), \tau_2(\cdot)\}, \ \bar{s}(x) = s(x,x), \ \bar{\tau}_i(x) = \tau_i(x,x)$$

and

$$\tau_{s(\cdot)}^\star(x) := \frac{N \tau_{max}(x)}{N - \bar{s}(x) \bar{\tau}_{max}(x)}.$$

(A_2) : $s(\cdot)$, $\tau_i(\cdot)$ and $\alpha(\cdot)$are symmetric function, i.e. $s(x,y) = s(y,x)$, $\tau_i(x,y) = \tau_i(y,x)$ and $\alpha(x,y) = \alpha(y,x)$ for any $(x,y) \in \mathfrak{Q} \times \mathfrak{Q}$.

(A_3): $h : \bar{\mathfrak{Q}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that $\exists C_1 > 0$ and $r \in C_+ \left(\mathbb{R}^N \right) \cap \mathscr{P}$ with $\tau_{max}^+ < r^- < \tau_{s(\cdot)}^\star(x)$such that

$$|h(x,t)| \leq C_1 |t|^{r(x)-1}, \quad \text{for any } (x,t) \in \mathfrak{Q} \times \mathbb{R},$$

where $\mathscr{P} = \left\{ r(x) : \bar{\tau}_{\max}(x) \leq r(x) q^- \leq r(x) q^+ < \tau_{s(\cdot)}^\star(x) \text{ for all } x \in \mathbb{R}^N \right\}$, with function $q \in C_+ (\mathfrak{Q} \times \mathfrak{Q})$ such that

$$\frac{2}{q(x,y)} + \frac{\alpha(x,y)}{N} = 2 \quad \text{for any } (x,y) \in \mathfrak{Q} \times \mathfrak{Q},$$

(A_4): There exists $\lambda > \tau_{\max}^+$ such that $0 < \lambda H(x,t) \leq 2 t h(x,t)$, for any $t \in \mathbb{R} \backslash \{0\}$ and for any $x \in \mathfrak{Q}$.

(A_5): The Kirchhoff functions $\mathscr{K}_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ $(i = 1, 2)$ are continuous such that there are positive constants $\beta_i \in \left[1, \frac{\tau_{s(\cdot)}^\star}{\tau_{\max}^+} \right)$, $\beta = \max\{\beta_1, \beta_2\}$ and $k_i = k_i(\upsilon) > 0$ for all $\upsilon > 0$ such that

$$t \mathscr{K}_i(t) \leq \beta_i \widetilde{\mathscr{K}}_i(t), \quad \text{for any} \quad t \in \mathbb{R}^+,$$

$$k_i^\star \geq \mathscr{K}_i(t) \geq k_i, \quad \text{for all} \quad t > \upsilon,$$

where $\tilde{\mathscr{K}}_i(t) = \int_0^t \mathscr{K}_i(\upsilon)d\upsilon$.
(A_6): For any $\upsilon > 0$, there are two positive constants k_i and $k_i^\star = k_i(\upsilon) > 0$ such that

$$k_i \leq \mathscr{K}_i(t) \leq k_i^\star, \quad \text{for all} \quad t > \upsilon,$$

(A_7): The perturbation g satisfies $g \in L^2(\mathfrak{Q})$ and $g > 0$ a.e. in $\mathfrak{Q}$,

All these assumptions lead us to consider the following two main results.

Theorem 1.1. *Assuming that* (A_1)-(A_7) *are satisfied, and considering* $\mathfrak{Q}$ *as a bounded smooth domain in* $\mathbb{R}^N$ *with* $\mu \in C_+(\bar{\mathfrak{Q}})$ *such that* $1 < \mu^+ < \tau_{\max}^-$*, then there exist* ξ^* *and* $g_0 > 0$ *such that for any* $\xi \in (-\infty, \xi^*]$ *and if* $\|g\|_2 \leq g_0$*, problem* (1) *possesses at least one positive nontrivial weak solution.*

Theorem 1.2. *Assuming that* (A_1)-(A_7) *are satisfied, and considering* $\mathfrak{Q}$ *as a bounded smooth domain in* $\mathbb{R}^N$ *with* $\mu \in C_+(\bar{\mathfrak{Q}})$ *such that* $1 < \mu^+ < \tau_{\max}^-$*, then there exist* $\xi^* > 0$ *and* $g_0 > 0$ *such that for any* $\xi \in (0, \xi^*]$ *and if* $\|g\|_2 \leq g_0$*, problem* (1) *possesses at least one negative nontrivial weak solution.*

This paper is organized like this: In the second section, we discuss familiar properties and results related to fractional Sobolev spaces with variable exponents. In the third section, we provide proofs for our two existence results.

2. Preliminaries and Functional Analytic Framework

We begin by revisiting fundamental properties of variable exponent Lebesgue spaces and variable order fractional Sobolev spaces.

Variable Exponents Lebesgue Spaces.

For any $m \in C_+(\bar{\mathfrak{Q}})$, let's revisit the variable exponent Lebesgue space

$$L^{m(\cdot)}(\mathfrak{Q}) = \left\{ \kappa : \mathfrak{Q} \longrightarrow \mathbb{R} : \kappa \text{ is measurable and } \int_{\mathfrak{Q}} |\kappa(x)|^{m(x)} \mathrm{d}x < \infty \right\},$$

equipped with the Luxemburg norm

$$\|\kappa\|_{m(\cdot)} = \inf \left\{ \varsigma > 0 : \int_{\mathfrak{Q}} \left| \frac{\kappa(x)}{\varsigma} \right|^{m(x)} \mathrm{d}x \leq 1 \right\}.$$

It is well known that $\left(L^{m(\cdot)}(\mathfrak{Q}), \|\cdot\|_{m(\cdot)}\right)$ is a separable reflexive Banach space, see [19]. Moreover, let $m' \in C_+(\bar{\mathfrak{Q}})$ be the conjugate exponent of m, that is

$$\frac{1}{m(x)} + \frac{1}{m'(x)} = 1, \quad \text{for any } x \in \bar{\mathfrak{Q}}$$

Lemma 2.1. *[19] Assume that* $\kappa \in L^{m(\cdot)}(\mathfrak{Q})$ *and* $\upsilon \in L^{m'(\cdot)}(\mathfrak{Q})$*. Then*

$$\left| \int_{\mathfrak{Q}} \kappa\upsilon \, \mathrm{d}x \right| \leq \left(\frac{1}{m^-} + \frac{1}{(m')^-} \right) \|\kappa\|_{m(\cdot)} \|\upsilon\|_{m'(\cdot)} \leq 2 \|\kappa\|_{m(\cdot)} \|\upsilon\|_{m'(\cdot)}.$$

Defining the modular function $\sigma_{m(\cdot)} : L^{m(\cdot)}(\mathfrak{Q}) \longrightarrow \mathbb{R}$, by

$$\sigma_{m(\cdot)}(\kappa) = \int_{\mathfrak{Q}} |\kappa(x)|^{m(x)} \mathrm{d}x.$$

Proposition 2.2. *[20][21] Assume that* $\kappa \in L^{m(\cdot)}(\mathfrak{Q})$ *and* $\{\kappa_n\}_{n\in\mathbb{N}} \subset L^{m(\cdot)}(\mathfrak{Q})$. *Then*
(i) $\|\kappa\|_{m(\cdot)} < 1(\text{ resp. } = 1, > 1) \Leftrightarrow \sigma_{m(\cdot)}(\kappa) < 1(\text{ resp. } = 1, > 1)$,
(ii) $\|\kappa\|_{m(\cdot)} < 1 \longrightarrow \|\kappa\|^{m^+}_{m(\cdot)} \leq \sigma_{m(\cdot)}(\kappa) \leq \|\kappa\|^{m^-}_{m(\cdot)}$,
(iii) $\|\kappa\|_{m(\cdot)} > 1 \longrightarrow \|\kappa\|^{m^-}_{m(\cdot)} \leq \sigma_{m(\cdot)}(\kappa) \leq \|\kappa\|^{m^+}_{m(\cdot)}$,
(iv) $\lim_{n\longrightarrow\infty} \|\kappa_n\|_{m(\cdot)} = 0 \ (\text{resp. } \infty) \Leftrightarrow \lim_{n\longrightarrow\infty} \sigma_{m(\cdot)}(\kappa_n) = 0 \ (\text{resp. } \infty)$,
(v) $\lim_{n\longrightarrow\infty} \|\kappa_n - \kappa\|_{m(\cdot)} = 0 \Leftrightarrow \lim_{j\longrightarrow\infty} \sigma_{m(\cdot)}(\kappa_n - \kappa) = 0$.

Proposition 2.3. *[4][2] Let* $\alpha(\cdot)$ *satisfy* $(A_1) - (A_2)$. *Let* $m_1, m_2 \in C_+(\mathfrak{Q} \times \mathfrak{Q})$ *verify*

$$\frac{1}{m_1(x,y)} + \frac{\alpha(x,y)}{N} + \frac{1}{m_2(x,y)} = 2 \quad \textit{for any } (x,y) \in \mathfrak{Q} \times \mathfrak{Q}.$$

Then, for $\kappa \in L^{m_1^+}(\mathbb{R}^N) \cap L^{m_1^-}(\mathbb{R}^N)$ *and* $\upsilon \in L^{m_2^+}(\mathbb{R}^N) \cap L^{m_2^-}(\mathbb{R}^N)$ *we have*

$$\left| \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{\kappa(x)\upsilon(y)}{|x-y|^{\alpha(x,y)}} \mathrm{d}x\, \mathrm{d}y \right| \leq C\left(\|\kappa\|_{m_1^+}\|\upsilon\|_{m_2^+} + \|\kappa\|_{m_1^-}\|\upsilon\|_{m_2^-} \right),$$

for a suitable positive constant C independent of κ *and* υ.

Variable-Order Fractional Sobolev Spaces.

Henceforth, we will provide a concise overview of fundamental properties concerning fractional Sobolev spaces with variable order. Additionally, we will introduce essential lemmas and propositions that will serve as tools in proving our main results.
The fractional Sobolev space with variable order and variable exponent is given by

$$W^{s(\cdot)}_{m(\cdot)}(\mathfrak{Q}) = \left\{ \kappa \in L^{\bar{m}(\cdot)}(\mathfrak{Q}) : \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{m(x,y)}}{\varsigma^{m(x,y)}|x-y|^{N+m(x,y)s(x,y)}} \mathrm{d}x\, \mathrm{d}y < \infty, \text{ for some } \chi > 0 \right\}$$

with the norm $\|\kappa\|_{s(\cdot),m(\cdot)} = \|\kappa\|_{\bar{m}(\cdot)} + |\kappa|_{s(\cdot),m(\cdot)}$, where

$$|\kappa|_{s(\cdot),m(\cdot)} = \inf\left\{ \varsigma > 0 : \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{m(x,y)}}{\varsigma^{m(x,y)}|x-y|^{N+m(x,y)s(x,y)}} \mathrm{d}x\, \mathrm{d}y < 1 \right\}.$$

Let $W_0 := \left\{ \kappa \in W^{s(\cdot)}_{m(\cdot)}(\mathfrak{Q}) \ : \ \kappa = 0 \text{ on } \partial\mathfrak{Q} \right\}$ endowed with the norm $\|\kappa\|_{W_0} = |\kappa|_{s(\cdot),m(\cdot)}$.

Remark 2.4. *Note that* $(W_0, \|\kappa\|_{W_0})$ *is a reflexive Banach space and* W_0^* *denotes the dual spaces of* W_0.

Now, we introduce a compact embedding theorem for $W^{s(\cdot),m(\cdot)}(\mathfrak{Q})$.

Lemma 2.5. *[4] Let* $s(\cdot), \tau_1(\cdot)$ *and* $\tau_2(\cdot)$ *satisfy* $(A_1) - (A_2)$. *Then, for any* $m \in C_+(\bar{\mathfrak{Q}})$ *with* $1 < m(x) < \bar{\tau}^*_{s(\cdot)}(x)$ *for any* $x \in \bar{\mathfrak{Q}}$, *there exists a positive constant* $C_m = C_m(N, s, \tau_1, \tau_2, m, \mathfrak{Q})$ *such that*

$$\|\kappa\|_{m(\cdot)} \leq C_m \|\kappa\|_{W_0},$$

for any $\kappa \in W_0$. *Furthermore, the embedding* $W_0 \hookrightarrow L^{m(\cdot)}(\mathfrak{Q})$ *is compact.*

Lemma 2.6. *[5] (Hardy–Littlewood–Sobolev type inequality) Let* $s(\cdot), \tau_1(\cdot), \tau_2(\cdot)$ *and* $\mu(\cdot)$ *satisfy* $(A_1) - (A_2)$, *with* $N > m^+s^+$. *Let* $q \in C_+(\mathfrak{Q} \times \mathfrak{Q})$ *be as in* (A_3). *Let* $r \in C_+(\mathbb{R}^N) \cap \mathscr{P}$, *where* $\mathscr{P}$ *is defined in* (A_3). *Then, for any* $\kappa \in W_0$ *we have* $|\kappa|^{r(\cdot)} \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$ *with*

$$\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)|^{r(x)}|\kappa(y)|^{r(x)}}{|x-y|^{\alpha(x,y)}} \mathrm{d}x\, \mathrm{d}y \leq C\left(\left\||\kappa|^{r(\cdot)}\right\|^2_{L^{q^+}(\mathfrak{Q})} + \left\||\kappa|^{r(\cdot)}\right\|^2_{L^{q^-}(\mathfrak{Q})} \right),$$

where C is a positive constant and independent of κ.

We define the fractional modular function $\sigma^{s(\cdot)}_{m(\cdot)} : W_0 \longrightarrow \mathbb{R}$, by

$$\sigma^{s(\cdot)}_{m(\cdot)}(\kappa) = \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{m(x,y)}}{|x-y|^{N+m(x,y)s(x,y)}} \mathrm{d}x\, \mathrm{d}y.$$

Proposition 2.7. *[4] Suppose that $\kappa \in W_0$ and $\{\kappa_n\}_{n\in\mathbb{N}} \subset W_0$. Then*
(i) $\|\kappa\|_{W_0} < 1(\text{ resp. } = 1, > 1) \Leftrightarrow \sigma^{s(\cdot)}_{m(\cdot)}(\kappa) < 1(\text{ resp. } = 1, > 1)$,
(ii) $\|\kappa\|_{W_0} < 1 \longrightarrow \|\kappa\|^{m^+}_{W_0} \leq \sigma^{s(\cdot)}_{m(\cdot)}(\kappa) \leq \|\kappa\|^{m^-}_{W_0}$,
(iii) $\|\kappa\|_{W_0} > 1 \longrightarrow \|\kappa\|^{m^-}_{W_0} \leq \sigma^{s(\cdot)}_{m(\cdot)}(\kappa) \leq \|\kappa\|^{m^+}_{W_0}$,
(iv) $\lim_{n\longrightarrow\infty} \|\kappa_n\|_{W_0} = 0$ *(resp ∞)* $\Leftrightarrow \lim_{n\longrightarrow\infty} \sigma^{s(\cdot)}_{m(\cdot)}(\kappa_n) = 0$ *(resp ∞)*,
(v) $\lim_{n\longrightarrow\infty} \|\kappa_n - \kappa\|_{W_0} = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} \sigma^{s(\cdot)}_{m(\cdot)}(\kappa_n - \kappa) = 0$.

Definition 2.8. *We say that a function $\kappa \in W_0$ is a weak solution of 1, if*

$$\begin{aligned}
&\mathscr{K}_1\Big([\kappa]_{s(\cdot),\tau_1(\cdot)}\Big) \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_1(x,y)-2}(\kappa(x)-\kappa(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+\tau_1(x,y)s(x,y)}} \mathrm{d}x\, \mathrm{d}y \\
&+ \mathscr{K}_2\Big([\kappa]_{s(\cdot),\tau_2(\cdot)}\Big) \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_2(x,y)-2}(\kappa(x)-\kappa(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+\tau_2(x,y)s(x,y)}} \mathrm{d}x\, \mathrm{d}y \\
&= \xi \int_{\mathfrak{Q}} |\kappa(x)|^{\mu(x)-2}\kappa(x)\varphi(x)\mathrm{d}x + \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{H(x,\kappa(x))H(y,\kappa(y))\varphi(y)}{|x-y|^{\alpha(x,y)}} \mathrm{d}x\, \mathrm{d}y + \int_{\mathfrak{Q}} g(x)\varphi(x)\mathrm{d}x,
\end{aligned}$$

for any $\varphi \in W_0$.

Clearly, the weak solutions of 1 are exactly the critical points of the Euler Lagrange functional $\mathfrak{E}_\xi : W_0 \longrightarrow \mathbb{R}$, given as follow

$$\begin{aligned}
\mathfrak{E}_\xi(\kappa) = \tilde{\mathscr{K}}_1\Big([\kappa]_{s(\cdot),\tau_1(\cdot)}\Big) + \tilde{\mathscr{K}}_2\Big([\kappa]_{s(\cdot),\tau_2(\cdot)}\Big) - \xi \int_{\mathfrak{Q}} \frac{1}{\mu(x)}|\kappa(x)|^{\mu(x)}\mathrm{d}x \\
- \frac{1}{2}\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{H(x,\kappa(x))H(y,\kappa(y))}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\, \mathrm{d}y - \int_{\mathfrak{Q}} g(x)\kappa(x)\mathrm{d}x,
\end{aligned}$$

where $\tilde{\mathscr{K}}_i(t) = \int_0^t \mathscr{K}_i(v)dv$, $i = 1, 2$. Note that $\mathfrak{E}_\xi \in C^1(W_0, \mathbb{R})$ and its G$\hat{a}$teaux derivative at the point $\varphi \in W_0$ is the functional

$$\begin{aligned}
\langle \mathfrak{E}'(\kappa), \varphi\rangle &= \mathscr{K}_1\Big([\kappa]_{s(\cdot),\tau_1(\cdot)}\Big) \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_1(x,y)-2}(\kappa(x)-\kappa(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+\tau_1(x,y)s(x,y)}} \mathrm{d}x\, \mathrm{d}y \\
&+ \mathscr{K}_2\Big([\kappa]_{s(\cdot),\tau_2(\cdot)}\Big) \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_2(x,y)-2}(\kappa(x)-\kappa(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+\tau_2(x,y)s(x,y)}} \mathrm{d}x\, \mathrm{d}y \\
&= \xi \int_{\mathfrak{Q}} |\kappa(x)|^{\mu(x)-2}\kappa(x)\varphi(x)\mathrm{d}x + \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{H(x,\kappa(x))H(y,\kappa(y))\varphi(y)}{|x-y|^{\alpha(x,y)}} \mathrm{d}x\, \mathrm{d}y + \int_{\mathfrak{Q}} g(x)\varphi(x)\mathrm{d}x,
\end{aligned}$$

3. Main result

Palais-Smale compactness condition.
The objective of this section is to demonstrate the fulfillment of the Palais-Smale condition, denoted briefly as (PS). To achieve this, we revisit the fact that $\mathfrak{E}_\xi$ satisfies the (PS) condition at the level $c \in \mathbb{R}$, if every sequence $\{\kappa_n\}_{n\in\mathbb{N}} \subset W_0$ that satisfies:

$$\mathfrak{E}_\xi(\kappa_n) \longrightarrow c \quad \text{and} \quad \mathfrak{E}'_\xi(\kappa_n) \longrightarrow 0 \quad \text{in } W_0^* \quad \text{as } n \longrightarrow \infty, \tag{5}$$

has a convergent subsequence in W_0.

Lemma 3.1. *Assuming that* $(A_3)-(A_4)$ *hold. Then*
(i) *if* $1<\mu^+<\tau^-_{\max}$*, then for any* $\xi\in\mathbb{R}$, $\mathfrak{E}_\xi$ *fulfills the (PS) condition,*
(ii) *if* $\lambda<\mu^-<\mu^+<(\bar{\tau}^\star_s)^-$*, then for any* $\xi>0$, $\mathfrak{E}_\xi$ *fulfills the (PS) condition.*

Proof. Let $\xi\in\mathbb{R}$ and let $\{\kappa_n\}_{n\in\mathbb{N}}\subset W_0$ be a sequence satisfying (5).
Step 1: Let prove the boundedness of the sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ by employing a proof by contradiction. Subsequently, by passing to a subsequence, which is still denoted as $\{\kappa_n\}_{n\in\mathbb{N}}$, we have

$$\lim_{n\longrightarrow\infty}\|\kappa_n\|_{W_0}=\infty \quad\text{and}\quad \|\kappa_n\|_{W_0}\geq 1,$$

for all $n\in\mathbb{N}$. From (A_2) we have for any $\kappa\in W_0$

$$\frac{|\kappa(x)-\kappa(y)|^{\tau_1(x,y)}}{|x-y|^{N+\tau_1(x,y)s(x,y)}}+\frac{|\kappa(x)-\kappa(y)|^{\tau_2(x,y)}}{|x-y|^{N+\tau_2(x,y)s(x,y)}}\geq\frac{|\kappa(x)-\kappa(y)|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\quad\text{for a.e. } x,y\in\mathbb{R}^N. \tag{6}$$

Therefore, using the Hölder inequality, Propositions 2.2 and 2.7, (A_4), (A_7), (5), (6), Lemma 2.5, $\exists\theta_\xi>0$ such that as $n\longrightarrow\infty$

$$\begin{aligned}
c+\theta_\xi\|\kappa_n\|_{W_0}+o(1)&\geq\lambda\mathfrak{E}_\xi(\kappa_n)-\left\langle\mathfrak{E}'_\xi(\kappa_n),\kappa_n\right\rangle\\
&=\lambda\widetilde{\mathscr{K}}_1\Big([\kappa_n]_{s(\cdot),\tau_1(\cdot)}\Big)+\lambda\widetilde{\mathscr{K}}_2\Big([\kappa_n]_{s(\cdot),\tau_2(\cdot)}\Big)-\lambda\xi\int_{\mathfrak{Q}}\frac{1}{\mu(x)}|\kappa_n(x)|^{\mu(x)}\mathrm{d}x\\
&-\frac{\lambda}{2}\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{H(x,\kappa_n(x))H(y,\kappa_n(y))}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y-\lambda\int_{\mathfrak{Q}}g(x)\kappa_n(x)\mathrm{d}x\\
&-\mathscr{K}_1\Big([\kappa_n]_{s(\cdot),\tau_1(\cdot)}\Big)\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{|\kappa_n(x)-\kappa_n(y)|^{\tau_1(x,y)}}{|x-y|^{N+\tau_1(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y\\
&-\mathscr{K}_2\Big([\kappa_n]_{s(\cdot),\tau_2(\cdot)}\Big)\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{|\kappa_n(x)-\kappa_n(y)|^{\tau_2(x,y)}}{|x-y|^{N+\tau_2(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y+\int_{\mathfrak{Q}}g(x)\kappa_n(x)\mathrm{d}x\\
&+\xi\int_{\mathfrak{Q}}|\kappa_n(x)|^{\mu(x)}\mathrm{d}x+\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{H(x,\kappa_n(x))H(y,\kappa_n(y))\varphi(y)}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y\\
&\geq k_1\left(\frac{\lambda}{\beta_1\tau_1^+}-1\right)\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_1(x,y)}}{|x-y|^{N+\tau_1(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y\\
&+k_2\left(\frac{\lambda}{\beta_2\tau_2^+}-1\right)\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_2(x,y)}}{|x-y|^{N+\tau_2(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y\\
&-\xi\int_{\mathfrak{Q}}\left(\frac{\lambda}{\mu(x)}-1\right)|\kappa_n(x)|^{\mu(x)}\,\mathrm{d}x\\
&-\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{H(x,\kappa_n(x))\left[\frac{\lambda}{2}H(y,\kappa_n(y))-h(y,\kappa_n(y))\kappa_n(y)\right]}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y\\
&-(\lambda-1)\int_{\mathfrak{Q}}g(x)\kappa_n(x)\mathrm{d}x\\
&\geq\min\{k_1,k_2\}\left(\frac{\lambda}{\beta\tau^+_{\max}}-1\right)\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y\\
&-\xi^+\left(\frac{\lambda}{\mu^-}-1\right)\sigma_{\mu(\cdot)}(\kappa_n)-(\lambda-1)\|g\|_2\,\|\kappa_n\|_2\\
&\geq\min\{k_1,k_2\}\left(\frac{\lambda}{\beta\tau^+_{\max}}-1\right)\|\kappa_n\|^{\tau^-_{\max}}_{W_0}-\xi^+\left(\frac{\lambda}{\mu^-}-1\right)C^{\mu^+}_\mu\|\kappa_n\|^{\mu^+}_{W_0}-C_g\|\kappa_n\|_{W_0},
\end{aligned}$$

where $\xi^+ = \max\{\xi, 0\}$ and $C_g > 0$. From this, we can discern two situations:
Case 1: if $1 < \mu^+ < \tau^-_{\max}$, then since $\lambda > \beta\tau^+_{\max}$ and by (A_4), from the aforementioned estimate, we promptly encounter a contradiction.
Case 2: if $\lambda < \mu^- < \mu^+ < (\bar{\tau}^\star_s)^-$, we consider $\xi > 0$ then, by the above estimate we have as $n \longrightarrow \infty$

$$c + \theta_\xi \|\kappa_n\|_{W_0} + o(1) \geq \min\{k_1, k_2\}\left(\frac{\lambda}{\beta\tau^+_{\max}} - 1\right)\|\kappa_n\|^{\tau^-_{\max}}_{W_0} - C_g \|\kappa_n\|_{W_0},$$

which still gives a contradiction, since $\lambda > \beta\tau^+_{\max} > \beta\tau^-_{\max} > 1$. Thus $\{\kappa_n\}_{n\in\mathbb{N}}$ is bounded in W_0.
Step 2: $\{\kappa_n\}_{n\in\mathbb{N}}$ has a strong convergent subsequence. Indeed, from Lemma 2.5, combined with the reflexivity of W_0, there exist a subsequence, still denoted by $\{\kappa_n\}_{n\in\mathbb{N}}$, and $\kappa \in W_0$ such that

$$\begin{cases} \kappa_n \rightharpoonup \kappa \text{ in } W_0, \\ \kappa_n \longrightarrow \kappa \text{ in } L^{\beta(\cdot)}(\mathfrak{Q}), \\ \kappa_n(x) \longrightarrow \kappa(x) \text{ a.e. in } \mathfrak{Q}. \end{cases} \tag{7}$$

Moreover, we have

$$|\langle \mathfrak{E}'_\xi(\kappa_n), (\kappa_n - \kappa)\rangle| \leq \|\mathfrak{E}'_\xi(\kappa_n)\|\Big(\|\kappa_n\|_{W_0} + \|\kappa\|_{W_0}\Big) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

As $\{\kappa_n\}_{n\in\mathbb{N}}$ is bounded in W_0 and $\mathfrak{E}'_\xi(\kappa_n) \longrightarrow 0$, one has

$$\langle \mathfrak{E}'_\xi(\kappa_n), (\kappa_n - \kappa)\rangle \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

For simplicity, let us denote by $\mathbf{L}_{\tau_i} : W_0 \longrightarrow W_0^*$ as

$$\langle \mathbf{L}_{\tau_i}(\kappa), v\rangle = \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x) - \kappa(y)|^{\tau_i(x,y)-2}(\kappa(x) - \kappa(y))(v(x) - v(y))}{|x - y|^{N+\tau_i(x,y)s(x,y)}} \mathrm{d}x\, \mathrm{d}y,$$

for any $\kappa, v \in W_0$ and $i = 1, 2$. By combining the boundedness of $\{\kappa_n\}_{n\in\mathbb{N}}$ in W_0 and (7), we derive that

$$\begin{aligned} o(1) &= \langle \mathfrak{E}'_\xi(\kappa_n), (\kappa_n - \kappa)\rangle \\ &= \mathscr{K}_1\Big([\kappa_n]_{s(\cdot),\tau_1(\cdot)}\Big)\langle \mathbf{L}_{\tau_1}(\kappa_n), (\kappa_n - \kappa)\rangle + \mathscr{K}_2\Big([\kappa_n]_{s(\cdot),\tau_2(\cdot)}\Big)\langle \mathbf{L}_{\tau_2}(\kappa_n), (\kappa_n - \kappa)\rangle \\ &- \xi \int_{\mathfrak{Q}} |\kappa_n(x)|^{\mu(x)-2}\kappa_n(x)(\kappa_n - \kappa)(x)\mathrm{d}x \\ &- \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{H(x, \kappa_n(x))\, h(y, \kappa_n(y))(\kappa_n - \kappa)(y)}{|x - y|^{\alpha(x,y)}} \mathrm{d}x\, \mathrm{d}y - \int_{\mathfrak{Q}} g(x)(\kappa_n - \kappa)(x)\mathrm{d}x \end{aligned} \tag{8}$$

By employing Lemma 2.1 and (7), we get

$$\left|\int_{\mathfrak{Q}} |\kappa_n(x)|^{\mu(x)-2}\kappa_n(x)(\kappa_n - \kappa)(x)\mathrm{d}x\right| \leq 2\Big\||\kappa_n|^{\mu(\cdot)-1}\Big\|_{\mu'(\cdot)}\Big\|\kappa_n - \kappa\Big\|_{\mu(\cdot)} \longrightarrow 0,$$

as $n \longrightarrow \infty$. Therefore,

$$\lim_{n\longrightarrow\infty} \int_{\mathfrak{Q}} |\kappa_n(x)|^{\mu(x)-2}\kappa_n(x)(\kappa_n - \kappa)(x)\mathrm{d}x = 0. \tag{9}$$

Moreover, from (A_7) and Hölder's inequality, we deduce that:

$$\left|\int_{\mathfrak{Q}} g(x)(\kappa_n - \kappa)(x)\mathrm{d}x\right| \leq \|g\|_2\|\kappa_n - \kappa\|_2 \longrightarrow 0,$$

as $n \longrightarrow \infty$. Therefore,

$$\lim_{n\longrightarrow\infty}\int_{\mathfrak{Q}} g(x)\,(\kappa_n-\kappa)\,(x)\mathrm{d}x = 0. \tag{10}$$

Now, let's estimate the fourth term on the right-hand side of (8). So, using (A_3), Proposition 2.2 and Lemma 2.5, for any $\kappa \in W_0$ one has

$$\begin{aligned}\|H(\cdot,\kappa_n(\cdot))\|_{q^+} &\le C_1\left(\int_{\mathfrak{Q}}|\kappa_n(x)|^{r(x)q^+}\mathrm{d}x\right)^{1/q^+} \le C_1\max\left\{\|\kappa_n\|^{r^-}_{r(\cdot)q^+},\|\kappa_n\|^{r^+}_{r(\cdot)q^+}\right\}\\ &\le C_1\max\left\{C^{r^-}_{r(\cdot)q^+}\|\kappa_n\|^{r^-}_{W_0},C^{r^+}_{r(\cdot)q^+}\|\kappa_n\|^{r^+}_{W_0}\right\},\end{aligned} \tag{11}$$

that is $H(\cdot,\kappa_n(\cdot)) \in L^{q^+}(\mathfrak{Q})$, and similarly

$$\|H(\cdot,\kappa_n(\cdot))\|_{q^-} \le C_1\max\left\{C^{r^-}_{r(\cdot)q^-}\|\kappa_n\|^{r^-}_{W_0},C^{r^+}_{r(\cdot)q^-}\|\kappa_n\|^{r^+}_{W_0}\right\}. \tag{12}$$

Therefore, with reference to (11), (12), and Proposition 2.3, we have

$$\begin{aligned}&\left|\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{H\,(x,\kappa_n(x))\,h\,(y,\kappa_n(y))\,(\kappa_n-\kappa)\,(y)}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y\right|\\ &\le C\left(\|H\,(\cdot,\kappa_n(\cdot))\|_{q^+}\,\|h\,(\cdot,\kappa_n(\cdot))\,(\kappa_n(\cdot)-\kappa(\cdot))\|_{q^+} + \|H\,(\cdot,\kappa_n(\cdot))\|_{q^-}\,\|h\,(\cdot,\kappa_n(\cdot))\,(\kappa_n(\cdot)-\kappa(\cdot))\|_{q^-}\right)\\ &\le C_2\max\left\{C^{r^-}_{r(\cdot)q^+}\|\kappa_n\|^{r^-}_{W_0},C^{r^+}_{r(\cdot)q^+}\|\kappa_n\|^{r^+}_{W_0}\right\}\|h\,(\cdot,\kappa_n(\cdot))\,(\kappa_n(\cdot)-\kappa(\cdot))\|_{q^+}\\ &+C_2\max\left\{C^{r^-}_{r(\cdot)q^-}\|\kappa_n\|^{r^-}_{W_0},C^{r^+}_{r(\cdot)q^-}\|\kappa_n\|^{r^+}_{W_0}\right\}\|h\,(\cdot,\kappa_n(\cdot))\,(\kappa_n(\cdot)-\kappa(\cdot))\|_{q^-}.\end{aligned} \tag{13}$$

Moreover, by using (A_3), Lemma 2.1, Lemma and (7) we obtain as $n \longrightarrow \infty$

$$\begin{aligned}\|h\,(\cdot,\kappa_n(\cdot))\,(\kappa_n(\cdot)-\kappa(\cdot))\|^{q^+}_{q^+} &\le C_1^{q^+}\int_{\mathfrak{Q}}|\kappa_n(x)|^{(r(x)-1)q^+}\,|(\kappa_n(x)-\kappa(x))|^{q^+}\,\mathrm{d}x\\ &\le C_3\left\|\kappa_n^{(r(\cdot)-1)q^+}\right\|_{r'(\cdot)}\left\|(\kappa_n-\kappa)^{q^+}\right\|_{r(\cdot)}\\ &\le C_4\left(\|\kappa_n\|^{(r^+-1)q^+}_{r(\cdot)q^+}+\|\kappa_n\|^{(r^--1)q^+}_{r(\cdot)q^+}\right)\|(\kappa_n-\kappa)\|^{q^+}_{r(\cdot)q^+}\\ &\le C_6\,\|(\kappa_n-\kappa)\|^{q^+}_{r(\cdot)q^+} = o(1),\end{aligned} \tag{14}$$

and similarly as $n \longrightarrow \infty$

$$\|h\,(\cdot,\kappa_n(\cdot))\,(\kappa_n(\cdot)-\kappa(\cdot))\|^{q^-}_{q^-} = o(1). \tag{15}$$

Therefore, combining with (13), (14) and (15), we obtain

$$\left|\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{H\,(x,\kappa_n(x))\,h\,(y,\kappa_n(y))\,(\kappa_n-\kappa)\,(y)}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y\right| \longrightarrow 0, \quad \text{as} \quad n\longrightarrow\infty. \tag{16}$$

Hence, according to (8)-(9) and (16), one has

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\langle\mathfrak{E}'_\xi(\kappa_n),(\kappa_n-\kappa)\rangle\\ &=\lim_{n\longrightarrow\infty}\left[\mathscr{K}_1\Big([\kappa_n]_{s(\cdot),\tau_1(\cdot)}\Big)\big\langle\mathbf{L}_{\tau_1}(\kappa_n),(\kappa_n-\kappa)\big\rangle+\mathscr{K}_2\Big([\kappa_n]_{s(\cdot),\tau_2(\cdot)}\Big)\big\langle\mathbf{L}_{\tau_2}(\kappa_n),(\kappa_n-\kappa)\big\rangle\right]=0.\end{aligned}$$

From this, employing a similar argument as in (6) and using (A_5), we have as $n \longrightarrow \infty$

$$\begin{aligned} o(1) &= \mathscr{K}_1\Big([\kappa_n]_{s(\cdot),\tau_1(\cdot)}\Big)\left\langle \mathbf{L}_{\tau_1}(\kappa_n),(\kappa_n-\kappa)\right\rangle + \mathscr{K}_2\Big([\kappa_n]_{s(\cdot),\tau_2(\cdot)}\Big)\left\langle \mathbf{L}_{\tau_2}(\kappa_n),(\kappa_n-\kappa)\right\rangle \\ &\geq \min\{K_1,K_2\}\left\langle \mathbf{L}_{\tau_{\max}}(\kappa_n),(\kappa_n-\kappa)\right\rangle. \end{aligned} \tag{17}$$

For any given $(x,y)\in\mathfrak{Q}\times\mathfrak{Q}$, applying the Young inequality yields

$$\begin{aligned} &|\kappa_n(x)-\kappa_n(y)|^{\tau_{\max}(x,y)-1}|\kappa(x)-\kappa(y)| \\ &\quad\leq \frac{1}{\tau'(x,y)}\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_{\max}(x,y)} + \frac{1}{\tau(x,y)}|\kappa(x)-\kappa(y)|^{\tau_{\max}(x,y)} \\ &\quad\leq \frac{1}{(\tau')^-}\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_{\max}(x,y)} + \frac{1}{\tau^-}|\kappa(x)-\kappa(y)|^{\tau_{\max}(x,y)}, \end{aligned} \tag{18}$$

such that

$$\begin{aligned} \min\{K_1,K_2\}\left\langle \mathbf{L}_{\tau_{\max}}(\kappa_n),(\kappa_n-\kappa)\right\rangle \geq &\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y \\ &-\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_{\max}(x,y)-1}|\kappa(x)-\kappa(y)|}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y \\ \geq &C_7\Bigg(\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y \\ &-\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y\Bigg). \end{aligned} \tag{19}$$

By utilizing (7) and the Fatou lemma,

$$\liminf_{n\longrightarrow\infty}\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y \geq \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y,$$

this, combined with (17) and (19), results in

$$\lim_{n\longrightarrow\infty}\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y = \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y. \tag{20}$$

Nevertheless, employing (7) and the Brézis-Lieb type lemma for variable exponents, we obtain as $n \longrightarrow \infty$

$$\begin{aligned} &\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{\left|\kappa_n(x)-\kappa_n(y)\right|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y - \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\kappa(x)-\kappa(y)|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y \\ &\qquad = \int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{\left|\kappa_n(x)-\kappa(x)-\kappa_n(y)+\kappa(y)\right|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y + o(1). \end{aligned}$$

This, along with (20), results in

$$\lim_{n\longrightarrow\infty}\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{\left|\kappa_n(x)-\kappa(x)-\kappa_n(y)+\kappa(y)\right|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y = \lim_{n\longrightarrow\infty}\sigma^{s(\cdot)}_{\tau_{\max}(\cdot)}(\kappa_n-\kappa) = 0.$$

Based on this and Proposition 2.7, we can conclude that $\kappa_n \longrightarrow \kappa$ in W_0 as $n \longrightarrow \infty$. This concludes the proof. □

Mountain pass structure.

Lemma 3.2. *Assume that* $(A_1)-(A_3)$ *holds. Let* $\mu \in C_+(\bar{\mathfrak{Q}})$ *such that* $1<\mu^+<\tau^-_{\max}$. *Then, there exist a positive constants* ν, $\xi^*=\xi^*(\nu)$, $\eta_0=\eta_0(\nu)$ *and* $\alpha=\alpha(\nu)$ *such that* $\mathfrak{E}_\xi(\kappa)\geq\alpha>0$ *for any* $\kappa\in W_0$ *with* $\|\kappa\|_{W_0}=\nu$, *whenever* $\xi\in(0,\xi^*]$ *and* $\|\eta\|_2\leq\eta_0$.

Proof. Let $\kappa\in W_0$ be such that $\|\kappa\|_{W_0}=\nu$ (where $\nu\in(0,\min\{1,1/C_\mu\})$, with C_μ given in Lemma 2.5). using (6), (11), (12), Lemmas 2.5-2.6, Propositions 2.2 and 2.7, one has

$$
\begin{aligned}
\mathfrak{E}_\xi(\kappa) &= \tilde{\mathscr{K}}_1\Big([\kappa]_{s(\cdot),\tau_1(\cdot)}\Big)+\tilde{\mathscr{K}}_2\Big([\kappa]_{s(\cdot),\tau_2(\cdot)}\Big)-\xi\int_{\mathfrak{Q}}\frac{1}{\mu(x)}|\kappa(x)|^{\mu(x)}\mathrm{d}x\\
&-\frac{1}{2}\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{H(x,\kappa(x))H(y,\kappa(y))}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y-\int_{\mathfrak{Q}}g(x)\kappa(x)\mathrm{d}x\\
&\geq\frac{1}{\beta_1\tau_1^+}\mathscr{K}_1\Big([\kappa]_{s(\cdot),\tau_1(\cdot)}\Big)\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{|\kappa(x)-\kappa(y)|^{\tau_1(x,y)}}{|x-y|^{N+\tau_1(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y\\
&+\frac{1}{\beta_2\tau_2^+}\mathscr{K}_2\Big([\kappa]_{s(\cdot),\tau_2(\cdot)}\Big)\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{|\kappa(x)-\kappa(y)|^{\tau_2(x,y)}}{|x-y|^{N+\tau_2(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y\\
&-\xi\int_{\mathfrak{Q}}\frac{1}{\mu(x)}|\kappa(x)|^{\mu(x)}\mathrm{d}x-\frac{1}{2}\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{H(x,\kappa(x))H(y,\kappa(y))}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y-\int_{\mathfrak{Q}}g(x)\kappa(x)\mathrm{d}x\\
&\geq\frac{K_{min}}{\beta\tau^+_{\max}}\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{|\kappa(x)-\kappa(y)|^{\tau_{\max}(x,y)}}{|x-y|^{N+\tau_{\max}(x,y)s(x,y)}}\mathrm{d}x\,\mathrm{d}y-\xi\int_{\mathfrak{Q}}\frac{1}{\mu(x)}|\kappa(x)|^{\mu(x)}\mathrm{d}x\\
&-\frac{1}{2}\int_{\mathfrak{Q}\times\mathfrak{Q}}\frac{H(x,\kappa(x))H(y,\kappa(y))}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y-\int_{\mathfrak{Q}}g(x)\kappa(x)\mathrm{d}x\\
&\geq\frac{K_{min}}{\beta\tau^+_{\max}}\sigma^{s(\cdot)}_{\tau_{\max}(\cdot)}-\frac{\xi^+}{\mu^-}\sigma_{\mu(\cdot)}-\frac{C}{2}\left(\|H(\cdot,\kappa(\cdot))\|^2_{q^+}+\|H(\cdot,\kappa(\cdot))\|^2_{q^-}\right)-\|g\|_2\|\kappa\|_2\\
&\geq\frac{K_{min}}{\beta\tau^+_{\max}}\|\kappa\|^{\tau^+_{\max}}_{W_0}-\frac{\xi+}{\mu^-}\|\kappa\|^{\mu^-}_{\mu(\cdot)}-C_8\|\kappa\|^{2r^-}_{W_0}-C_9\|g\|_2\|\kappa\|_{W_0}\\
&\geq\frac{K_{min}}{\beta\tau^+_{\max}}\|\kappa\|^{\tau^+_{\max}}_{W_0}-\frac{\xi^+C_{10}}{\mu^-}\|\kappa\|^{\mu^-}_{W_0}-C_8\|\kappa\|^{2r^-}_{W_0}-C_9\|g\|_2\|\kappa\|_{W_0}\\
&=\nu^{\tau^+_{\max}}\left(\frac{K_{min}}{\beta\tau^+_{\max}}-C_8\nu^{2r^--\tau^+_{\max}}\right)-\frac{\xi^+C_{10}}{\mu^-}\nu^{\mu^-}-C_9\|g\|_2\nu,
\end{aligned}
\tag{21}
$$

with $\xi^+=\max\{\xi,0\}$ and $K_{min}=\min\{K_1,K_2\}$. We have $\tau^+_{\max}<2r^-$ and choosing $\nu\in\Big(0,\min\Big\{1,1/C_\mu\,,\,[K_{min}/2\beta\tau^+_{\max}C_8]^{1/(2r^--\tau^+_{\max})}\Big\}\Big)$, we get, for any $\kappa\in W_0$ with $\|\kappa\|_{W_0}=\nu$, that

$$
\mathfrak{E}_\xi(\kappa)\geq\frac{\nu^{\tau^+_{\max}}K_{min}}{2\beta\tau^+_{\max}}-\frac{\xi^+C_{10}}{\mu^-}\nu^{\mu^-}-C_9\|g\|_2\nu. \tag{22}
$$

Moreover, set $\xi^*=\nu^{\tau^+_{\max}-\mu^-}K_{min}\mu^-/4\beta\tau^+_{\max}C_{10}$. Thus, for any $\xi\in(0,\xi^*]$, according to (22) one has

$$
\mathfrak{E}_\xi(\kappa)\geq\frac{\nu^{\tau^+_{\max}}K_{min}}{4\beta\tau^+_{\max}}=\alpha>0.
$$

This complete the proof. □

Lemma 3.3. *Assume that* (A_1), (A_2), (A_4), (A_5) *and* (A_7) *holds. Let* $\mu\in C_+(\bar{\mathfrak{Q}})$ *be such that* $1<\mu^+<\tau^-_{\max}$. *Then, for any* $\xi\in\mathbb{R}$ $\exists\varphi\in W_0$ *with* $\|\varphi\|_{W_0}>\nu$, *where* $\nu>0$ *is given in Lemma 3.2, such that* $\mathfrak{E}_\xi(\varphi)<0$.

Proof. Let $\xi \in \mathbb{R}$. According to (A_4) there exist two numbers $z_1, z_2 > 0$ such that

$$H(x,t) \geq z_1|t|^{\lambda/2}, \quad \text{for any } x \in \mathfrak{Q} \text{ and } |t| \geq z_2. \tag{23}$$

By utilizing the condition (A_5), one has

$$\tilde{\mathscr{K}}_i(t) \leq \tilde{\mathscr{K}}_i(1)t^{\beta}, \quad \text{for any} \quad t \geq 1. \tag{24}$$

Take $\phi \in C_0^{\infty}\left(\mathbb{R}^N\right)$, with $\phi > 0$ and let $t \in \mathbb{R}$ such that $t\varphi^- \geq d_2$. From (23) and (24) we obtain

$$\begin{aligned}
\mathfrak{E}_{\xi}(t\phi) &= \tilde{\mathscr{K}}_1\Big([t\phi]_{s(\cdot),\tau_1(\cdot)}\Big) + \tilde{\mathscr{K}}_2\Big([t\phi]_{s(\cdot),\tau_2(\cdot)}\Big) - \xi \int_{\mathfrak{Q}} \frac{1}{\mu(x)}|t\phi(x)|^{\mu(x)}\mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{H(x,t\phi(x))H(y,t\phi(y))}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y - \int_{\mathfrak{Q}} g(x)t\phi(x)\mathrm{d}x \\
&\leq \tilde{\mathscr{K}}_1(1)([t\phi]_{s(\cdot),\tau_1(\cdot)})^{\beta} + \tilde{\mathscr{K}}_2(1)([t\phi]_{s(\cdot),\tau_2(\cdot)})^{\beta} - \frac{\xi}{\mu^-}t^{\mu^+}\int_{\mathfrak{Q}}|\phi(x)|^{\mu(x)}\mathrm{d}x \\
&\quad - \frac{z_1^2t^{\lambda}}{2}\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\phi(x)|^{\lambda/2}|\phi(y)|^{\lambda/2}}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y - t\int_{\mathfrak{Q}} g(x)\phi(x)\mathrm{d}x \\
&\leq \frac{\tilde{\mathscr{K}}_1(1)}{(\tau_{min}^-)^{\beta}}\left(\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|t\phi(x)-t\phi(y)|^{\tau_1(x,y)}}{|x-y|^{N+s(x,y)\tau_1(x,y)}}\right)^{\beta}\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{\tilde{\mathscr{K}}_2(1)}{(\tau_{min}^-)^{\beta}}\left(\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|t\phi(x)-t\phi(y)|^{\tau_2(x,y)}}{|x-y|^{N+s(x,y)\tau_2(x,y)}}\right)^{\beta}\mathrm{d}x\,\mathrm{d}y - \frac{\xi}{\mu^-}t^{\mu^+}\int_{\mathfrak{Q}}|\phi(x)|^{\mu(x)}\mathrm{d}x \\
&\quad - \frac{z_1^2t^{\lambda}}{2}\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\phi(x)|^{\lambda/2}|\phi(y)|^{\lambda/2}}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y - t\int_{\mathfrak{Q}} g(x)\phi(x)\mathrm{d}x \\
&\leq \frac{t^{\beta\tau_{\max}^+}}{(\tau_{min}^-)^{\beta}}\max\{\tilde{\mathscr{K}}_1(1),\tilde{\mathscr{K}}_2(1)\}\left(\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\phi(x)-\phi(y)|^{\tau_{\max}(x,y)}}{|x-y|^{N+s(x,y)\tau_{\max}(x,y)}}\right)^{\beta}\mathrm{d}x\,\mathrm{d}y \\
&\quad - \frac{z_1^2t^{\lambda}}{2}\int_{\mathfrak{Q}\times\mathfrak{Q}} \frac{|\phi(x)|^{\lambda/2}|\phi(y)|^{\lambda/2}}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y - t\int_{\mathfrak{Q}} g(x)\phi(x)\mathrm{d}x
\end{aligned} \tag{25}$$

Since $\lambda > \beta\tau_{\max}^+$ we obtain that $\mathfrak{E}_{\xi}(t\phi) \longrightarrow -\infty$ as $t \longrightarrow \infty$. Thus, given sufficiently high t, we may obtain $\varphi = t\phi$ such that $\|\varphi\|_{W_0} > \nu$ and $\mathfrak{E}_{\xi}(\varphi) < 0$. □

Proof of Theorem 1.1:
It can be deduced from Lemma 3.1 (i) and Lemmas 3.2-3.3 and taking into account that $\mathfrak{E}_{\xi}(0) = 0$, because of (A_3), that for any $\xi < \xi^*$, $\mathfrak{E}_{\xi}$ meets all Mountain Pass Theorem requirements. Consequently, there exists a Palais–Smale subsequence $\{\kappa_n\}_{n\in\mathbb{N}}$ such that $\kappa_n \longrightarrow \kappa_0$ as $n \longrightarrow \infty$ in W_0. Thus, κ_0 serves as a nontrivial solution to problem (1) with positive energy $\mathfrak{E}_{\xi}(\kappa_0) > 0$.
To prove Theorem 1.2 we need the following lemma.

Proposition 3.4. *Assume that (A_1), (A_2), (A_4), (A_5) and (A_7) holds. Then, for any $\xi \in (0, \xi^*]$, there exists $\kappa_1 \in W_0$ such that*

$$-\infty < \mathfrak{E}_{\xi}(\kappa_1) = \inf_{\kappa\in\bar{\mathbb{B}}_{\nu}} \mathfrak{E}_{\xi}(\kappa) < 0,$$

where $\bar{\mathbb{B}}_{\nu} := \{\kappa \in W_0 : \|\kappa\|_{W_0} \leq \nu\}$ and ξ^* and ν are given by Lemma 3.3

Proof. Let $\omega \in C_0^{\infty}\left(\mathbb{R}^N\right)$ with $\omega > 0$, and let $t > 0$ sufficiently small such that $\|t\omega\|_{W_0} < 1$. Therefore, according to (A_5) and (23), we have for any $\xi > 0$

$$\begin{aligned}
\mathfrak{E}_\xi(t\omega) &= \tilde{\mathscr{K}}_1\Big([t\omega]_{s(\cdot),\tau_1(\cdot)}\Big) + \tilde{\mathscr{K}}_2\Big([t\omega]_{s(\cdot),\tau_2(\cdot)}\Big) - \xi\int_\Omega \frac{1}{\mu(x)}|t\omega(x)|^{\mu(x)}\mathrm{d}x \\
&- \frac{1}{2}\int_{\Omega\times\Omega} \frac{H(x,t\omega(x))H(y,t\omega(y))}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y - \int_\Omega g(x)t\omega(x)\mathrm{d}x \\
&\le k_1^*([t\omega]_{s(\cdot),\tau_1(\cdot)}) + k_2^*([t\omega]_{s(\cdot),\tau_1(\cdot)}) - \frac{\xi}{\mu^-}t^{\mu^+}\int_\Omega |\omega(x)|^{\mu(x)}\mathrm{d}x \\
&- \frac{z_1^2 t^\lambda}{2}\int_{\Omega\times\Omega} \frac{|\omega(x)|^{\lambda/2}|\omega(y)|^{\lambda/2}}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y - t\int_\Omega g(x)\omega(x)\mathrm{d}x \\
&\le \frac{k^*}{\tau_{min}^-}\left(\int_{\Omega\times\Omega} \frac{|t\omega(x)-t\omega(y)|^{\tau_1(x,y)}}{|x-y|^{N+s(x,y)\tau_1(x,y)}}\right)\mathrm{d}x\,\mathrm{d}y + \frac{k^*}{\tau_{min}^-}\left(\int_{\Omega\times\Omega} \frac{|t\omega(x)-t\omega(y)|^{\tau_2(x,y)}}{|x-y|^{N+s(x,y)\tau_2(x,y)}}\right)\mathrm{d}x\,\mathrm{d}y \\
&- \frac{\xi}{\mu^-}t^{\mu^+}\int_\Omega |\omega(x)|^{\mu(x)}\mathrm{d}x - \frac{z_1^2 t^\lambda}{2}\int_{\Omega\times\Omega} \frac{|\omega(x)|^{\lambda/2}|\omega(y)|^{\lambda/2}}{|x-y|^{\alpha(x,y)}}\mathrm{d}x\,\mathrm{d}y - t\int_\Omega g(x)\omega(x)\mathrm{d}x \\
&\le \frac{k^* t^{\tau_{\min}^-}}{\tau_{min}^-}\left(\int_{\Omega\times\Omega} \frac{|\omega(x)-\omega(y)|^{\tau_{\max}(x,y)}}{|x-y|^{N+s(x,y)\tau_{\max}(x,y)}}\right)\mathrm{d}x\,\mathrm{d}y - \frac{\xi}{\mu^-}t^{\mu^+}\int_\Omega |\omega(x)|^{\mu(x)}\mathrm{d}x - t\int_\Omega g(x)\omega(x)\mathrm{d}x,
\end{aligned}$$

where $k^* = \max\{k_1^*, k_2^*\}$. By (A_1), we deduce that $\mathfrak{E}_\xi(t\omega) < 0$ as $t \longrightarrow 0^+$. Hence, we get

$$-\infty < \mathfrak{E}_\xi(\kappa_1) = \inf_{\kappa\in\bar{\mathbb{B}}_\nu} \mathfrak{E}_\xi(u) < 0. \tag{26}$$

This end the proof. □

Proof of Theorem 1.2:
According to Ekeland variational principle in $\bar{\mathbb{B}}_\nu$ and Lemma 3.2, Proposition 3.4, $\exists\{\kappa_n\}_{n\in\mathbb{N}} \subset \bar{\mathbb{B}}_\nu$ such that

$$\mathfrak{E}_\xi(\kappa_1) \le \mathfrak{E}_\xi(\kappa_n) \le \mathfrak{E}_\xi(\kappa_1) + \frac{1}{n} \quad \text{and} \quad \mathfrak{E}_\xi(\kappa_n) \le \mathfrak{E}_\xi(\kappa) + \frac{1}{n}\|\kappa_n - \kappa\|_{W_0}, \tag{27}$$

for all $\kappa \in \bar{\mathbb{B}}_\nu$ and $n \in \mathbb{N}$. Let for all $v \in \partial\mathbb{B}_1 := \{\kappa \in W_0 : \|\kappa\|_{W_0} = 1\}$, and for any $\varepsilon > 0$ small enough that $\kappa_n + \varepsilon v \in \bar{\mathbb{B}}_\nu$ and using (26), one has

$$\mathfrak{E}_\xi(\kappa_n + \varepsilon v) - \mathfrak{E}_\xi(\kappa_n) \ge -\frac{\varepsilon}{n}$$

Since $\mathfrak{E}_\xi$ is Gâteaux differentiable in W_0, we have

$$\langle \mathfrak{E}'_\xi(\kappa_n), v\rangle = \lim_{\varepsilon\longrightarrow 0} \frac{\mathfrak{E}_\xi(\kappa_n + \varepsilon v) - \mathfrak{E}_\xi(\kappa_n)}{\varepsilon} \ge -\frac{1}{n},$$

for all $v \in \partial\mathbb{B}_1$. Thus

$$\left|\langle \mathfrak{E}'_\xi(\kappa_n), v\rangle\right| \le \frac{1}{n},$$

Hence, there exists a sequence $\{\kappa_n\}_{n\in\mathbb{N}} \subset \bar{\mathbb{B}}_\nu$ such that $\mathfrak{E}_\xi(\kappa_n) \longrightarrow \mathfrak{E}_\xi(\kappa_1) < 0$ and $\mathfrak{E}'_\xi(\kappa_n) \longrightarrow 0$ in W_0^* as $n \longrightarrow \infty$.
By Lemma 3.1, $\{\kappa_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in W_0, still represented by $\{\kappa_n\}_{n\in\mathbb{N}}$, such that $\kappa_n \longrightarrow \kappa_2$ in W_0 as $j \longrightarrow \infty$. Therefore, κ_2 is a solution of (1), with $\mathfrak{E}_\xi(\kappa_2) < 0$. Theorem 1.1 yields two distinct solutions: κ_1 with positive energy and κ_2 with negative energy. To summarize, the existence of the second solution relies on ξ being positive. Thus, there exists $\xi^* > 0$ such that for each $\xi \in (0, \xi^*]$, problem (1) has two separate nontrivial weak solutions. This finishes the proof.

Authors contributions: All authors of this manuscript contributed equally to this work.
Funding: Not applicable.
Data Availability: No data were used to support this study.
Conflicts of Interest: The authors declare that there are no conflicts of interest.

References

[1] Ambrosio, V. (2022). A Kirchhoff Type Equation in $\mathbb{R}^N$ Involving the fractional (p,q)-Laplacian. The Journal of Geometric Analysis, 32(4), 135.

[2] Alves CO, Tavares LS. A Hardy-Littlewood-Sobolev-type inequality for variable exponents andapplications to quasilinear Choquard equations involving variable exponent. Mediterr J Math.2019;16:Art. 55, 27 pp.

[3] Binlin, Z., Fiscella, A., & Liang, S. (2019). Infinitely many solutions for critical degenerate Kirchhoff type equations involving the fractional p–Laplacian. Applied Mathematics & Optimization, 80, 63-80.

[4] Biswas R, Tiwari S. Multiplicity and uniform estimate for a class of variable order fractional $p(x)$-Laplacian problems with concave-convex nonlinearities.

[5] Biswas, R., & Tiwari, S. (2021). On a class of Kirchhoff-Choquard equations involving variable-order fractional $p(\cdot)$-Laplacian and without Ambrosetti-Rabinowitz type condition.

[6] Chadli, L. S., El-Houari, H., & Moussa, H. (2023). Multiplicity of solutions for nonlocal parametric elliptic systems in fractional Orlicz–Sobolev spaces. Journal of Elliptic and Parabolic Equations, 9(2), 1131-1164.

[7] Caffarelli, L. (2012). Non-local diffusions, drifts and games. In Nonlinear Partial Differential Equations: The Abel Symposium 2010 (pp. 37-52). Springer Berlin Heidelberg.

[8] Caffarelli, L., & Silvestre, L. (2007). An extension problem related to the fractional Laplacian. Communications in partial differential equations, 32(8), 1245-1260.

[9] Eddine, N. C., Nguyen, A. T., & Ragusa, M. A. (2024). The dirichlet problem for a class of anisotropic Schrödinger-Kirchhoff-type equations with critical exponent. Mathematical Modelling and Analysis, 29(2), 254-267.

[10] El-Houari, H., Chadli, L. S., & Hicham, M. (2023). Nehari manifold and fibering map approach for fractional p (.)-Laplacian Schrödinger system. SeMA Journal, 1-23.

[11] El-Houari, H., CHADLI, L. S., & Moussa, H. (2021, May). Existence of solution to M-Kirchhoff system type. In 2021 7th International Conference on Optimization and Applications (ICOA) (pp. 1-6). IEEE.

[12] El-Houari, H., Chadli, L. S., & Moussa, H. (2024). Multiple solutions in fractional Orlicz–Sobolev Spaces for a class of nonlocal Kirchhoff systems. Filomat, 38(8), 2857-2875.

[13] El-Houari, H., Chadli, L. S., & Moussa, H. (2023). On a class of fractional $p(.,.)$-Kirchhoff–Schrödinger system type. Eur. J. Math. Appl.

[14] El-Houari, H., Chadli, L. S., & Moussa, H. (2022). Existence of a solution to a nonlocal Schrödinger system problem in fractional modular spaces. Advances in Operator Theory, 7(1), 1-30.

[15] El-Houari, H., Chadli, L. S., & Moussa, H. (2024). On a class of fractional $\Gamma(.)$-Kirchhoff-Schrödinger system type. Cubo (Temuco), 26(1), 53-73.

[16] El-Houari, H., & Moussa, H. (2024). On a class of generalized Choquard system in fractional Orlicz-Sobolev Spaces. Journal of Mathematical Analysis and Applications, 128563.

[17] El-Houari, H., Sabiki, H., & Moussa, H. (2024). On topological degree for pseudomonotone operators in fractional Orlicz-Sobolev spaces: study of positive solutions of non-local elliptic problems. Advances in Operator Theory, 9(2), 16.

[18] El-Houari, H. A. M. Z. A., Chadli, L. S., & Moussa, H. I. C. H. A. M. (2023). A weak solution to a non-local problem in fractional Orlicz-Sobolev spaces. Asia Pac. J. Math., 10(2).

[19] Kováčik O, Rákosník J. On spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czechoslovak Math J. 1991;41:592–618.

[20] Fan X, Zhao D. On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J Math Anal Appl. 2001;263:424–446.

[21] Fan X, Shen J, Zhao D. Sobolev embedding theorems for spaces$W^{m,p(x)}(\Omega)$. J Math Anal Appl.2001;262:749–760.

[22] Felmer, P., & Torres, C. (2015). Non-linear Schrödinger equation with non-local regional diffusion. Calculus of Variations and Partial Differential Equations, 54, 75-98.

[23] Fiscella, A., & Valdinoci, E. (2014). A critical Kirchhoff type problem involving a nonlocal operator. Nonlinear Analysis: Theory, Methods & Applications, 94, 156-170.

[24] Kirchhoff, Gustav. Vorlesungen,Huber. Mechanik, Leipzig, Teubner, 1883.

[25] Laskin, N. (2000). Fractional quantum mechanics and Lévy path integrals. Physics Letters A, 268(4-6), 298-305.

[26] Pucci, P., Xiang, M., & Zhang, B. Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian, Adv. Calc. Var., 12 (2019), 253–275.

[27] Shen, L. (2018). Multiplicity and asymptotic behavior of solutions to a class of Kirchhoff-type equations involving the fractional p-Laplacian. Journal of Inequalities and Applications, 2018, 1-19.

[28] Sun, D. (2023). Ground State Solutions of Schrödinger-Kirchhoff Equations with Potentials Vanishing at Infinity. Journal of Function Spaces, 2023(1), 8829268.

[29] Torres Ledesma, C. E. (2018). Multiplicity result for non-homogeneous fractional Schrodinger–Kirchhoff-type equations in $\mathbb{R}^n$. Advances in Nonlinear Analysis, 7(3), 247-257.

[30] Yacini, S., Allalou, C., & Hilal, K. (2024). On the weak solution for the nonlocal parabolic problem with p-Kirchhoff term via topological degree. Filomat, 38(8), 2889-2898.

[31] Zuo, J., Yang, L., & Liang, S. (2021). A variable-order fractional $p(\cdot)$-Kirchhoff type problem in $\mathbb{R}^N$. Mathematical Methods in the Applied Sciences, 44(5), 3872-3889.