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Fully cycle extendibility of Quasi-Claw-free graphs

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Abstract. A graph G is Quasi-Claw-free if for any two vertices x and y with distance 2, there exists a vertex $u \in N(x) \cap N(y)$ such that $N(u) \subseteq N[x] \cup N[y]$. In [1], Ainouche conjectured that every connected, locally connected Quasi-Claw-free graph is vertex pancyclic. In this paper, we prove that if G is connected, locally connected Quasi-Claw-free then G is Fully cycle extendible. Several results exist to respond to Ainouche's conjecture [1], but our result enhances them presenting a significant improvement. Moreover, it is a pioneer result in fully cycle extendibility dealing with this class of graphs.

1. Introduction

We consider only finite, undirected graphs, without loops, multiple edges and isolated vertices. We use [3], for terminologies and notations, not defined here.

Let G be a graph, and let n be the order of the graph G. We write $G = (V, E)$, $V(G)$ is the set of vertices and $E(G)$ the set of edges. Let $S \subset V(G)$, the neighbourhood of S and the subgraph of G induced by S, are denoted by $N(S)$ and $< S >$ respectively. For a subgraph H of G and $S \subset V(G) \backslash V(H)$, $N_H(S) = N(S) \cap V(H)$. If $S = \{s\}$, then $N_H(S)$ is written $N_H(s)$. The set $N_G(S)$ and $N_G(s)$ are denoted $N(S)$ and $N(s)$ respectively, and $N[s] = N(s) \cup \{s\}$.

Let C be a cycle of G with a given orientation, for two distinct vertices u and v, in C, uCv is the path of C joining u to v in the given orientation. We denote $u\bar{C}v$ the path of C joining u and v in the reverse order. For $u \in C$, u^+ and u^- denote respectively the successor and the predecessor of u in C. For an integer l, $3 \leq l \leq |V(G)|$, a l-cycle is a 2-regular connected graph with l edges. A $|V(G)|$-cycle is called a Hamiltonian cycle. If G contains a Hamiltonian cycle then G is called a Hamiltonian graph.

If every vertex of G is contained in cycles of all lengths l, $3 \leq l \leq |V(G)|$, then G is vertex pancyclic.

A graph G is fully cycle extendible, if every vertex of G lies on a 3-cycle, and for every cycle C of length r, $3 \leq r < |V(G)|$, there is a cycle C' of length $r + 1$, such that $\left|V(C) \cap V(C')\right| = r$, and we call such a cycle C' an extension of C.
If a graph G is fully cycle extendible then G is vertex pancyclic, the contrary is not necessary true (See figure 1).

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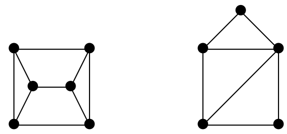


Vertex pancyclic

Fully cycle extendible

Figure 1

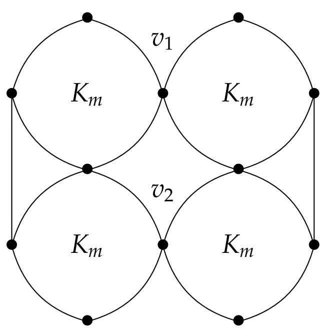
A vertex v in G is locally connected if $\langle N(v)\rangle$ is connected. If each vertex in G is locally connected, then G is called a locally connected graph.

A graph G is quasi locally connected if every vertex cut of G contains a locally connected vertex.

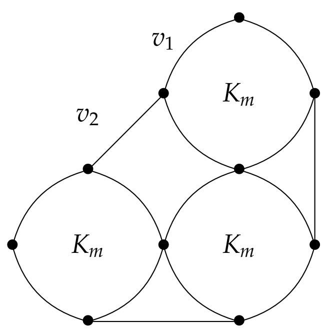
Let G be a graph, and let $B(G) = \{v \in V(G) : \langle N(v)\rangle \textit{ is not connected}\}$. The graph G is called almost locally connected if for every vertex $v \in B(G)$, there exists a vertex $u \in V(G) - \{v\}$ such that $\langle N(v) \cup \{u\}\rangle$ is connected.

A graph G is triangularly connected, if for every pair of edges $e_1, e_2 \in E(G)$, G has a sequence of triangles $T_1, T_2, ...T_l$ such that $e_1 \in T_1, e_2 \in T_l$ and $E(T_i) \cap E(T_{i+1}) \neq \emptyset$, for $1 \leq i \leq l-1$.

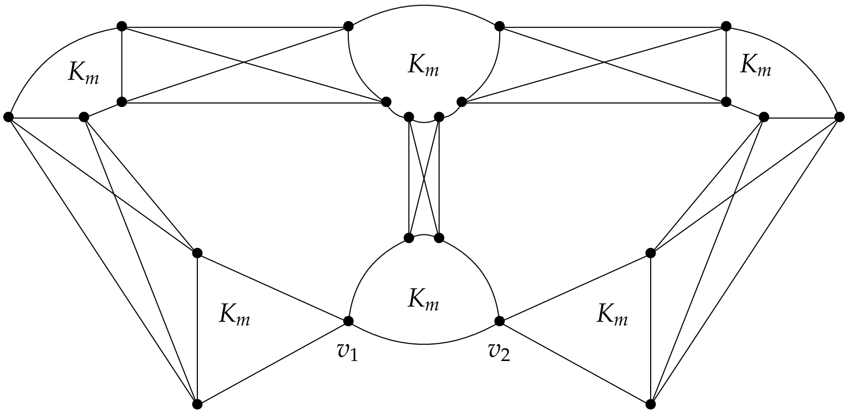
Obviously every connected, locally connected graph is quasi locally connected, almost locally connected and triangularly connected. In the graphs G_1, G_2 and G_3 [9], the vertices v_1 and v_2 are not locally connected, that implies that the graphs G_1, G_2 and G_3 are not locally connected, but the graph G_1 is almost locally connected, the graph G_2 is quasi locally connected and the graph G_3 is triangularly connected. For more details see [9].



G_1 : Almost locally connected graph



G_2 : Quasi locally connected graph



G_3 : Triangularly connected

Figure 2

A graph G is called Claw-free if no induced subgraph of G is isomorphic to $K_{1,3}$ (See figure 3). A graph G is called Quasi-Claw-free if for any two vertices x and y with distance 2, there exists a vertex $u \in N(x) \cap N(y)$ such that $N(u) \subseteq N[x] \cup N[y]$. This definition of Quasi-Claw-free graphs given by Ainouche in [1], generalizes the concept of Claw-free graphs. Obviously, every Claw-free graph is Quasi-Claw-free, but not every Quasi-Claw-free is Claw-free. (See figure 4)

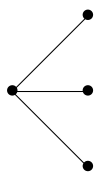


Figure 3

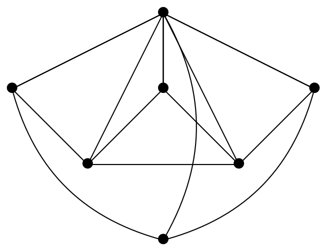


Figure 4

For each pair of vertices $\{a, b\}$ such that $d(a, b) = 2$, we associate the set $J(a, b) = \{u \in N_G(a) \cap N_G(b) : N(u) \subseteq N_G[a] \cup N_G[b]\}$. So, a Quasi-Claw-free graph is defined as a graph such that for every pair $\{x, y\}$ of vertices with distance 2, $J(x, y) \neq \emptyset$.

2. Local connectivity conditions for the existence of cycles in Claw-free graphs

Many results for the existence of cycles in Claw-free graphs have been given [5]. Here, we are interested in local connectivity conditions.

The first result was obtained by Oberly and Summer. They proved in [8]:

Theorem 1. *[8]. Every connected, locally connected Claw-free graph on at least three vertices is Hamiltonian.*

Later, Clark [4], Shi [11] and Zhang [13] independently, improved the latest result by proving:

Theorem 2. *[4], [11], [13]. Every connected, locally connected claw-free graph on at least three vertices is vertex pancyclic.*

Theorem 2 has been improved in several ways. The authors proved that we can reach the same result under lower conditions.

Theorem 3. *[2]. Every connected, Quasi locally connected Claw-free graph with* $|V(G)| \geq 3$ *is vertex pancyclic.*

Theorem 4. *[7]. Every triangularly connected claw-free graph G with* $|E(G)| \geq 3$ *is vertex pancyclic.*

One of the strongest result was provided by Hendry in [6], he proved:

Theorem 5. *[6]. If G is a connected, locally connected Claw-free graph on at least three vertices, then G is fully cycle extendible.*

3. Local connectivity conditions for the existence of cycles in Quasi-Claw-free graphs

In [1] Ainouche proved that :

Theorem 6. *[1]. Every connected, locally connected, Quasi-Claw-free graph on at least three vertices is pancyclic.*

And, in the same paper, he conjectured:

Conjecture 1. *[1]. Every connected, locally connected, Quasi-Claw-free graph on at least three vertices is vertex pancyclic.*

Motivated by the conjecture of Ainouche, several results concerning local connectivity are reached in this new class of graphs.

This conjecture was established in [12], and in [10], independently.

Theorem 7. *[12]. Every triangularly connected, Quasi-Claw-free graph on at least three vertices is vertex pancyclic.*

Theorem 8. *[10]. A connected, Quasi locally connected, Quasi-Claw-free graph on at least three vertices is vertex pancyclic.*

Theorem 9. *[10]. A connected, Almost locally connected, Quasi-Claw-free graph on at least three vertices is vertex pancyclic.*

In this paper we prove that under the same conditions of the Ainouche's conjecture, we obtain a stronger result :

Theorem 10. *A connected, locally connected Quasi-Claw-free graph on at least three vertices, is fully cycle extendible.*

4. Proof of theorem 10

Let G be a connected, locally connected Quasi-Claw-free graph, of order n, $n \geq 3$. It is sufficient to prove that for every cycle C in G of length r, $3 \leq r < |V(G)|$, there exists a cycle C' in G such that C' is an extension of C. Suppose on the contrary, that there is a cycle C in G, that does not admit an extension.

As G is connected, there is a vertex u, $u \in V(C)$ and there is a vertex y such that and $y \notin V(C)$, and $yu \in E(G)$. Obviously, $yu^-, yu^+ \notin E(G)$ else an extending cycle can easily be found through y. So $d(y, u^+) = d(y, u^-) = 2$. By the definition of a Quasi-Claw-free, $J(y, u^+) \neq \emptyset$ and $J(y, u^-) \neq \emptyset$.

Let $a \in J(y, u^+)$ and $b \in J(y, u^-)$ (a and b may be the same). Three cases are possible.

Case 1. a or $b \in V(C) - \{u\}$

Without loss of generality, suppose $a \in V(C)$.
As $a \in J(y, u^+)$ and by the definition of Quasi-Claw-free graphs we have $ya^+ \in E(G)$ or $u^+a^+ \in E(G)$.
Clearly $ya^+ \notin E(G)$, else, $C' = uCaya^+Cu$, is an extension of C.
$a^+u^+ \in E(G)$, in this case we obtain also a cycle $C' = u\bar{C}a^+u^+Cayu$ as an extension of C. So case 1 is impossible.

Case 2. $a, b \notin V(C)$

Obviously au and $bu \notin E(G)$ or C' can be found easily.

If $|C| = 3$. As G is locally connected, we have a is locally connected then $N(a)$ is connected, as we supposed in the beginning $yu^+ \notin E(G)$, so there must be a vertex $k \in V(G)$ and $k \in N(a)$, clearly $k \notin V(C)$ and $ku^+, ky \in E(G)$. In this case as G is locally connected so u^+ is locally connected and $N(u^+)$ is connected, so we have $ku \in E(G)$, and then $C' = u^+kuu^-u^+$ is an extension of C.

So $|C| \geq 4$, obviously u^{++} exist and $d(a, u^{++}) = 2$. So there exists a vertex z such that $z \in J(a, u^{++})$. We have the following sub-cases :

Case 2.1. $z \in V(C)$. Obviously, $z^+a \notin E(G)$, then by the definition of a Quasi-Claw-free graph, $z^+u^{++} \in E(G)$, and $C' = u^{++}Czau^+\bar{C}z^+u^{++}$ is an extension of C.

Case 2.2. $z \notin V(C)$ and $z = y$. By the definition of a Quasi-Claw-free graph, $uu^{++} \in E(G)$, because u is a neighbour of z and $z \in J(a, u^{++})$, then $ua \in E(G)$ or $uu^{++} \in E(G)$, so if $ua \in E(G)$ we find $C' = uau^+Cu$ is an extension of C. So $uu^{++} \in E(G)$.

If $u^-u^+ \in E(G)$ then $C' = u^+uyu^{++}Cu^-u^+$ is an extending cycle of C. Else $u^-u^+ \notin E(G)$ then $d(u^-, u^+) = 2$, so there exists a vertex o such that $o \in J(u^-, u^+)$ and $o \neq u$.

i. $o \in V(C)$. If $o^-u^- \in E(G)$ then $C' = u^{++}zuu^+oCu^-o^-\bar{C}u^{++}$. Else $o^-u^+ \in E(G)$ then $C' = u^{++}Co^-u^+oCuzu^{++}$ is the extending cycle of C.

ii. $o \notin V(C)$. Then $C' = u^+ou^-\bar{C}u^{++}uu^+$ is an extending cycle of C.

Case 2.3. $z \notin V(C)$ and $z \neq y$

We have $z \in J(a, u^{++})$ and $a \in J(y, u^+)$ so obviously we have the edge $zy \in E(G)$, because $zu^+ \notin E(G)$.

As G is locally connected, z is a locally connected vertex, then $< N(z) >$ is a connected sub-graph. So we have $yu^{++} \in E(G)$. If $u^-u^+ \in E(G)$ then $C' = u^+uyu^{++}Cu^-u^+$ is an extending cycle of C. Else $u^-u^+ \notin E(G)$ then $d(u^-, u^+) = 2$, so there exists a vertex o such that $o \in J(u^-, u^+)$ and $o \neq u$.

i. $o \in V(C)$. If $o^-u^- \in E(G)$ then $C' = u^{++}Co^-u^-\bar{C}ou^+uyu^{++}$. Else $o^-u^+ \in E(G)$ then $C' = u^{++}Co^-u^+oCuyu^{++}$ is the extending cycle of C.

ii. $o \notin V(C)$. As G is locally connected, u^+ is a locally connected vertex, then $< N(u^+) >$ is a connected sub-graph. So we have $uu^{++} \in E(G)$. Then $C' = u^+ou^-\bar{C}u^{++}uu^+$ is an extending cycle of C.

So Case 2 is impossible.

Case 3. $a = b = u$

As $u = a$ and $u = b$ we have obviously $u^-u^+ \in E(G)$. We have two sub-cases:

1. If $uu^{++} \in E(G)$. As G is locally connected graph, u is a locally connected vertex, then $< N(u) >$ is a connected sub-graph and we have $yu^{++} \in E(G)$. We find in this case $C' = u^{++}Cu^-u^+uyu^{++}$ is an extending cycle of C.
2. $uu^{++} \notin E(G)$. We have then $d(u, u^{++}) = 2$, so there exists a vertex w such that $w \in J(u, u^{++})$. As u is a locally connected vertex, then $yw \in E(G)$. So we have three other sub-cases :
   - $w \in V(C)$ and $w^+u^{++} \in E(G)$. We find in this case $C' = u^{++}Cw^-\ wyuu^+u^-\bar{C}w^+u^{++}$ is an extending cycle of C.
   - $w \in V(C)$ and $w^+u \in E(G)$. We find in this case $C' = u^{++}Cwyu\ w^+Cu^-u^+u^{++}$ is an extending cycle of C.
   - $w \notin V(C)$. We find in this case $C' = u^{++}Cu^-u^+uwu^{++}$ is an extending cycle of C.

So Case 3 is impossible.

In all the cases, we have contradictions, so an extending cycle of C exists. And theorem 10 is proved.□

With this result we can establish once again the previous Ainouche's Conjecture. And we have:

Corollary 1. *Every connected, locally connected, Quasi-Claw-free graph on at least three vertices is vertex pancyclic.*

We can also deduce the following results:

Corollary 2. *Every connected, locally connected Claw-free graph on at least three vertices, is Hamiltonian.*

Corollary 3. *Every connected, locally connected Claw-free graph on at least three vertices, is vertex pancyclic.*

Corollary 4. *If G is a connected, locally connected Claw-free graph on at least three vertices, then G is fully cycle extendible*

We believe that the following propositions are true:

Conjecture 2. *Every triangularly connected, Quasi-Claw-free graphs on at least three vertices, is fully cycle extendible.*

Conjecture 3. *A connected, Quasi locally connected, Quasi-Claw-free graphs on at least three vertices, is fully cycle extendible.*

Conjecture 4. *A connected, Almost locally connected, Quasi-Claw-free graphs on at least three vertices, is fully cycle extendible.*

References

[1] Ainouche, A. Quasi-claw-free graph. *Discrete Mathematics 179* (1998), 13–26.
[2] Ainouche, A., and Boresma, H. Remarks on hamiltonian properties of claw free graphs. *Ars combinat* (1990), 110–121.
[3] Boundy, J., and Murphy, U. Graph theory with application. *Macmillan & Co. London* (1976).
[4] Clark, L. Hamiltonian properties of connected, locally connected graphs. *Cong. Numer. 32* (1981), 199–204.
[5] Flandrin, E., Faudree, R., and Ryjacek, Z. Claw-flee graphs a survey. *Discrete Mathematics 164* (nov 1997), 87–147.
[6] Hendry, G. Extending cycles in graphs. *Discrete Math. 85* (1990), 59–72.
[7] Lai, H., Miao, L., Shao, Y., and Wan, L. Triangularly connected claw-free graph. *Preprint* (2004).
[8] Oberly, D., and Summer, D. Every connected, locally connected nontrivial graph with no induced claw is hamiltonian. *J. Graph Theory 3* (1979), 351–356.
[9] Qu, X., and Lin, H. Quasilocally connected, almost locally connected or triangularly connected claw-free graphs. *Discrete Geometry, Combinatorics and Graph Theory* (2007), 162–165.
[10] Qu, X., and Wang, J. Vertex pancyclicity in quasi-claw-free graphs. *Discrete Mathematics 309* (2009), 1135–1141.
[11] Shi, R. Connected and locally connected graphs with no induced claws are vertex pancyclic. *Kexue Tongbao 31* (1986), 427.
[12] Zhan, M. Vertex pancyclicity in quasi-claw-free graphs. *Discrete Mathematics 307* (2007), 1679–1683.
[13] Zhang, C. Cycles of given length in some claw-free graphs. *Discrete Math. 78* (1989), 307–313.