Filomat 38:28 (2024), 9997–10011
https://doi.org/10.2298/FIL2428997W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A t-intersecting Hilton-Milner theorem for vector spaces for $n = 2k + 1$ and $q \geq 3$

Yunpeng Wang[a], Jizhen Yang[b,*]

[a]*Department of Mathematics and Physics, Luoyang Institute of Science and Technology, Luoyang 471023, China*
[b]*Department of Mathematics, Luoyang Normal College, Luoyang 471934, China*

Abstract. Let V be an n-dimensional vector space over $GF(q)$ and $\begin{bmatrix} V \\ k \end{bmatrix}$ denote the family of all k-dimensional subspaces of V. Suppose that $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ denotes a non-trivial t-intersecting family with $t \geq 2$. Cao et al. [2] determined the structures of $\mathcal{F}$ with maximum size for large n. Wang et al. [12] improved the applicable range to $n \geq 2k + 2$. In this paper, we determine the structures of $\mathcal{F}$ with maximum size for $n = 2k + 1$ and $q \geq 3$.

1. Introduction

The study of intersecting family is an important topic in combinatorics and has a long research history ever since Erdős, Ko, and Rado [4] determined the maximum-sized intersecting family of subset, which is usually called EKR theorem. The extremal structures of families with the maximum sizes were characterized as the family of all subsets containing a fixed element x of an n-element set X if $n \geq 2k + 1$.

Let V be an n-dimensional vector space over $GF(q)$ and $\begin{bmatrix} V \\ k \end{bmatrix}_q$ denotes the family of k-dimensional subspaces. For any complex number x and nonnegative integer k, the generalized q-binomial coefficient is defined by $\begin{bmatrix} x \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{x-i}-1}{q^{k-i}-1}$. Simple counting can prove that the size of $\begin{bmatrix} V \\ k \end{bmatrix}_q$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. Without causing confusion, the subscript q will be omitted in the following text.

The q-analogue of questions about sets and subsets are questions about vector spaces and subspaces. The study on the EKR theorem for vector spaces can be seen in [3, 5, 8, 10]. In [2, 12], for some k-space U and t-space E such that $\dim(U \bigcap E) = t - 1$ the authors defined

$$\mathcal{F}_{HM} = \left\{ W \in \begin{bmatrix} V \\ k \end{bmatrix} : E \leq W \text{ and } \dim(W \cap U) \geq t \right\} \cup \begin{bmatrix} E + U \\ k \end{bmatrix}.$$

For $k \geq t + 2$, the authors also defined

$$\mathcal{F}_{A(t+2)} = \left\{ F \in \begin{bmatrix} V \\ k \end{bmatrix} : \dim(A \cap F) \geq t + 1 \text{ for some fixed } A \in \begin{bmatrix} V \\ t+2 \end{bmatrix} \right\}.$$

2020 *Mathematics Subject Classification*. Primary 05D05; Secondary 05A30.
Keywords. Hilton-Milner theorem, t-intersecting, vector spaces.
Received: 21 February 2024; Revised: 05 April 2024; Accepted: 23 April 2024
Communicated by Paola Bonacini
Research supported by the National Natural Science Foundation of China (Grant No. 11971319, Grant No. 12271234), Young backbone teachers funding plan of Henan Province (Grant No. 2020GGJS194).
* Corresponding author: Jizhen Yang
Email addresses: yunpengwang1981@163.com (Yunpeng Wang), yangjizhe116@163.com (Jizhen Yang)

The EKR structure is commonly referred to as a trivial structure in some literature. Relatively speaking, a family is called t-intersecting and non-trivial if the intersection of any two elements of the family is not less than t and the intersection of all elements is less than t. For vector spaces, it can be expressed as a family $\mathcal{F}$ is t-intersecting and non-trivial if $\dim(F_1 \cap F_2) \geq t$ for any $F_1, F_2 \in \mathcal{F}$ and $\dim(\bigcap_{F\in\mathcal{F}} F) \leq t-1$. Hilton and Milner [7] determined the maximum size of an intersecting non-trivial family of sets and characterized extremal structures of the families with the maximum sizes. Recently, some studies have extended the Hilton-Milner theorem to vector spaces. Blokhuis et al. [1] generalized the Hilton-Milner theorem for $t = 1$ and $n \geq 2k + 1 + \delta_{2,q}$. J. Wang et al. [11] shows the proof of the case $n \geq 2k + 1$ and $t = 1$ as a corollary of a Kruskal-Katona-type theorem. M. Cao et al. [2] generalizes the theorem to t-intersection and proved that $\mathcal{F}_{HM}, \mathcal{F}_{A(t+2)}$ are the maximal non-trivial family with $n \geq 2k + t + min\{4, 2t\}$. Y. Wang et al. [12] improve this parameter to $n \geq 2k + 2$ and $t \geq 2$. The rest problem of the t-intersecting Hilton-Milner theorem for vector spaces is the case $n = 2k + 1$ and $t \geq 2$.

Due to some cases of t-intersecting Hilton-Milner theorem for $n = 2k + 1, t \geq 2$ and $q \geq 3$ that cannot be solved using the methods mentioned in the article above, this paper solves these problems by counting basis vectors. Our main result is as follows:

Theorem 1.1. *Suppose that $n = 2k + 1, q \geq 3, t \geq 2$ and $k \geq t + 2$. For any t-intersecting and non-trivial family $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$, there holds $|\mathcal{F}| \leq |\mathcal{F}_{HM}|$, if $k \geq 2t + 2$; $|\mathcal{F}| \leq |\mathcal{F}_{A(t+2)}|$, if $t + 2 \leq k \leq 2t + 1$. Equality holds if and only if*

(i) $\mathcal{F} = \mathcal{F}_{HM}$, *if $k \geq 2t + 2$;*
(ii) $\mathcal{F} = \mathcal{F}_{A(t+2)}$, *if $t + 2 \leq k \leq 2t + 1$.*

In the next section, we introduce commonly used symbols. Some preliminary results will be given in Section 3. The proof of the main result is in Section 4.

2. Notation

Let $A, B, E, L \leq V$. We have the following notation.

• $A + B$ denote the sum of A and B. In particular, if $A \cap B = 0$, we write their sum as $A \oplus B$, the direct sum of A and B.

• Let $\mathcal{F}$ be a t-intersecting family of k-spaces and L be an ℓ-space t-intersecting each $F \in \mathcal{F}$ with minimum dimension and let

$$\mathcal{L}_t = \{H \leq V : \dim(H \cap L) = t, \dim(H \cap F) \geq t \textit{ for any } F \in \mathcal{F}\},$$
$$\mathcal{F}_0 = \{F \in \mathcal{F} : \dim(F \cap L) = t\},$$
$$\mathcal{F}_1 = \{F \in \mathcal{F} : \dim(F \cap L) \geq t + 1\},$$
$$\mathcal{F}(i, t, l, k; H, L) = \{F \in \mathcal{F} : H \in \mathcal{L}_t \textit{ and } \dim(F \cap L \cap H) = i\}.$$

Then $|\mathcal{F}| = \sum_{i=0}^{t} |\mathcal{F}(i, t, \ell, k; H, L)| = |\mathcal{F}_0| + |\mathcal{F}_1|$.

• Let i, λ be nonnegative integers and t, ℓ, k be positive integers. Define

$$f(i, \ell, k, \lambda) = \begin{bmatrix} t \\ i \end{bmatrix} \begin{bmatrix} \ell - t \\ t - i \end{bmatrix} \begin{bmatrix} \ell - t + \lambda \\ t - i \end{bmatrix} \begin{bmatrix} k - t + 1 \\ 1 \end{bmatrix}^{\ell - 2t + i + \lambda} \begin{bmatrix} n - \ell - \lambda \\ k - \ell - \lambda \end{bmatrix} q^{2(t-i)^2}.$$

and $S(a, \ell, k, \lambda) = \sum_{i=a}^{t} f(i, \ell, k, \lambda)$. Let $H \in \mathcal{L}_t$ such that $\dim(H) = \ell + \lambda$. If H is the vector space with minimum dimension in $\mathcal{L}_t$, then $f(i, \ell, k, \lambda)$ is an upper bound of the number of vector spaces that t-intersect each $F \in \mathcal{F}$ and exactly i-intersect $H \cap L$. Therefore, $S(a, \ell, k, \lambda)$ is an upper bound of the number of vector spaces that t-intersect each $F \in \mathcal{F}$ and a-intersect $H \cap L$. In particular, $S(\max\{0, 2t - \ell\}, \ell, k, \lambda)$ is an upper bound of $|\mathcal{F}|$ under this assumption.

• For any family, the covering number $\tau_t(\mathcal{F})$ is the minimum dimension of a vector space that t-intersects all elements of $\mathcal{F}$.

• For any family $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$, define $\mathcal{F}_M = \{F \in \mathcal{F} : M \leq F\}$.

3. Lemmas

In this paper, we let q be a prime power and $\delta_{i,j}$ denote the Kronecker delta. To prove Theorem 1.1, we apply the following lemmas.

Lemma 3.1. *Let a, b, c, d be positive integers such that $b < a$ and $d < c < a$.*

(i) *If $q \geq 3$, then*

$$q^{(a-b)b} \leq \begin{bmatrix} a \\ b \end{bmatrix} \leq 2^{1-\delta_{b,0}} q^{b(a-b)}. \tag{1}$$

(ii) *If $d \leq \min\{b, a-b\}$, then*

$$\begin{bmatrix} a \\ b \end{bmatrix} q^{(c-d)d} \geq \begin{bmatrix} c \\ d \end{bmatrix} q^{(a-b)b}. \tag{2}$$

(iii) *If $d \leq a-b$ and $b \geq 2$, then*

$$\frac{q^d - 1}{(q-1)^2 \begin{bmatrix} a \\ b \end{bmatrix}} \leq \frac{q+1}{q^{b(a-b)-d+3}} \leq \frac{1}{(q-1)q^{b(a-b)-d+1}}. \tag{3}$$

Proof. From [9, Lemma 2.1], it can be seen that $q^{(a-b)b} \leq \begin{bmatrix} a \\ b \end{bmatrix} \leq 2q^{b(a-b)}$ for $q \geq 3$. Observe that $\begin{bmatrix} a \\ 0 \end{bmatrix} = 1$. Therefore, we obtain (i). The inequality of (ii) is due to [12, Lemma 2.3].

Now we prove (iii). According to the definition of q-binomial coefficients, it follows that

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(q^{a-b+1}-1)(q^{a-b+2}-1)}{(q-1)(q^2-1)} \prod_{i=3}^{b} \frac{q^{a-b+i}-1}{q^i - 1} \geq \frac{(q^{a-b+1}-1)(q^{a-b+2}-1)q^{(a-b)(b-2)}}{(q-1)(q^2-1)}. \tag{4}$$

Recall that $d \leq a-b$. Since $(q^d-1)q^{2a-2b+3} \leq (q^{a-b+1}-1)(q^{a-b+2}-1)q^d$, then by (4) we have

$$\frac{q^d - 1}{(q-1)^2 \begin{bmatrix} a \\ b \end{bmatrix}} \leq \frac{q+1}{q^{b(a-b)-d+3}} \leq \frac{1}{(q-1)q^{b(a-b)-d+1}}. \tag{5}$$

□

Lemma 3.2. *Let $n \geq k+\ell-t+1, k \geq \ell \geq t+2$ and $a_i(\ell) = q^{\binom{i}{2}} \begin{bmatrix} \ell-t+1 \\ i \end{bmatrix} \begin{bmatrix} n-t-i \\ k-t-i \end{bmatrix}$. Then*

$$\begin{bmatrix} n-t \\ k-t \end{bmatrix} - q^{(k-t)(\ell-t+1)} \begin{bmatrix} n-\ell-1 \\ k-t \end{bmatrix} = \sum_{i=1}^{k-t} (-1)^{i-1} a_i(\ell). \tag{6}$$

Furthermore,

$$|\mathcal{F}_{HM}| > a_1(k) - a_2(k) > \left(1 - \frac{1}{(q^2-1)q^{n-2k+t-1}}\right) a_1(k). \tag{7}$$

Proof. In [12, Lemma 2.4], the authors prove this Lemma for $n \geq 2k+1$. In fact, using the same method, it can be proven that this Lemma holds for $n \geq k+\ell-t+1$. □

Lemma 3.3. *[12, Lemma 2.9] Let H, T, L be h, t, ℓ-spaces, respectively, such that $H \leq T \leq L$ and for $m \geq 2$ let*

$$\mathcal{F}_m = \{M : M \leq L,\ H = M \cap T \text{ and } \dim(M) = m\}.$$

Then $|\mathcal{F}_m| = \begin{bmatrix} \ell-t \\ m-h \end{bmatrix} q^{(t-h)(m-h)}$.

Lemma 3.4. *Let H, F_1 be $2k-t, k$-spaces, respectively. If $\dim(H \cap F_1) \leq k-1$, then the number of vector spaces in $\begin{bmatrix} H \\ k \end{bmatrix}$ t-intersecting F_1 is no more than $\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t-1 \\ k-t-1 \end{bmatrix}$.*

Proof. The number of vector spaces in $\begin{bmatrix} H \\ k \end{bmatrix}$ t-intersecting F_1 increases with $\dim(H \cap F_1)$ increases. Thus we only need to consider the case $\dim(H \cap F_1) = k-1$. Let $\mathcal{F}' = \left\{F \in \begin{bmatrix} H \\ k \end{bmatrix} : \dim(F \cap (F_1 \cap H)) \geq t\right\}$ and $\overline{\mathcal{F}'} = \{F \in \begin{bmatrix} H \\ k \end{bmatrix} : \dim(F \cap (F_1 \cap H)) = t-1\}$. Recall that $\dim(H) = 2k-t$. For each $F \in \begin{bmatrix} H \\ k \end{bmatrix}$, we have $\dim(F) + \dim(F_1 \cap H) = \dim(F \cap (F_1 \cap H)) + \dim(F + (F_1 \cap H))$. Since $F + (F_1 \cap H) \leq H$, then $\dim(F + (F_1 \cap H)) \leq 2k-t$. It follows that $\dim(F \cap (F_1 \cap H)) \geq t-1$. Hence $\mathcal{F}' \uplus \overline{\mathcal{F}'} = \begin{bmatrix} H \\ k \end{bmatrix}$, where '$\uplus$' is known for the disjoint union of two set. It follows from Lemma 3.3 that $|\overline{\mathcal{F}'}| = \begin{bmatrix} k-1 \\ t-1 \end{bmatrix} q^{(k-t+1)(k-t)}$. Then we have $|\mathcal{F}'| = \begin{bmatrix} 2k-t \\ k \end{bmatrix} - \begin{bmatrix} k-1 \\ t-1 \end{bmatrix} q^{(k-t+1)(k-t)}$. Substituting $n = 2k$ and $\ell = k$ in (6) gives that

$$\begin{bmatrix} 2k-t \\ k-t \end{bmatrix} - q^{(k-t)(k-t+1)} \begin{bmatrix} k-1 \\ k-t \end{bmatrix} = \sum_{i=1}^{k-t} (-1)^{i-1} q^{\binom{i}{2}} \begin{bmatrix} k-t+1 \\ i \end{bmatrix} \begin{bmatrix} 2k-t-i \\ k-t-i \end{bmatrix}. \tag{8}$$

Let $a_i = q^{\binom{i}{2}} \begin{bmatrix} k-t+1 \\ i \end{bmatrix} \begin{bmatrix} 2k-t-i \\ k-t-i \end{bmatrix}$. Then a calculation of q-binomial coefficients shows that

$$\frac{a_i}{a_{i+1}} = \frac{q^{\binom{i}{2}} \begin{bmatrix} k-t+1 \\ i \end{bmatrix} \begin{bmatrix} 2k-t-i \\ k-t-i \end{bmatrix}}{q^{\binom{i+1}{2}} \begin{bmatrix} k-t+1 \\ i+1 \end{bmatrix} \begin{bmatrix} 2k-t-i-1 \\ k-t-i-1 \end{bmatrix}} = \frac{(q^{i+1}-1)(q^{2k-t-i}-1)}{q^i (q^{k-t+1-i}-1)(q^{k-t-i}-1)} \geq \frac{3}{4} q^{t+i} \geq 1. \tag{9}$$

The identity (8) can be rewritten as

$$\begin{bmatrix} 2k-t \\ k-t \end{bmatrix} - q^{(k-t)(k-t+1)} \begin{bmatrix} k-1 \\ k-t \end{bmatrix} = \begin{cases} a_1 - \sum_{j=1}^{(k-t-1)/2} (a_{2j} - a_{2j+1}), & if\ 2 \nmid (k-t), \\ a_1 - \sum_{j=1}^{(k-t-2)/2} (a_{2j} - a_{2j+1}) - a_{k-t}, & if\ 2 \mid (k-t). \end{cases} \tag{10}$$

Combining (9) and (10) leads to that $|\mathcal{F}'| \leq a_1 = \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix} \begin{bmatrix} 2k-t-1 \\ k-t-1 \end{bmatrix}$. The proof is complete. □

Lemma 3.5. *Let $\mathcal{F}$ be a t-intersecting family and S be an s-subspace of V, where $t-1 \leq s \leq k-1$ and L be the minimum dimensional space t-intersecting each $F \in \mathcal{F}$ with $s < \dim(L) = \ell$. Then $|\mathcal{F}_S| \leq \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}^{\ell-s} \begin{bmatrix} n-\ell \\ k-\ell \end{bmatrix}$.*

Proof. Lemma 3.5 is a spacial case of [12, Remark 2.6]. □

Lemma 3.6. *[2, Lemma 2.8] Let $n \geq 2k+1$ and $t \geq 2$. Then $|\mathcal{F}_{HM}| > |\mathcal{F}_{A(t+2)}|$, if $k \geq 2t+2$; $|\mathcal{F}_{HM}| < |\mathcal{F}_{A(t+2)}|$, if $t+2 \leq k \leq 2t+1$.*

Remark 3.7. *In [12, (1.1)] and (7), the authors shows that*

$$|\mathcal{F}_{HM}| = \begin{bmatrix} n-t \\ k-t \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix} q^{(k-t)(k-t+1)} + \begin{bmatrix} t \\ 1 \end{bmatrix} q^{k-t+1}, \tag{11}$$

$$|\mathcal{F}_{A(t+2)}| = \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \begin{bmatrix} t+2 \\ t+1 \end{bmatrix} \left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} \right). \tag{12}$$

For $q \geq 3, k \geq t+2$ and $t \geq 2$, a calculation of q-binomial coefficients yields that

$$|\mathcal{F}_{A(t+2)}| > \left(1 - \frac{1}{q^{k+1}}\right) \begin{bmatrix} t+2 \\ 1 \end{bmatrix} \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \geq \frac{242}{243} \begin{bmatrix} t+2 \\ 1 \end{bmatrix} \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}. \tag{13}$$

It follows from (11) that

$$|\mathcal{F}_{HM}| > \begin{bmatrix} n-t \\ k-t \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k-t \end{bmatrix} q^{(k-t)(k-t+1)}.$$

Substituting $n = 2k+1$ *and* $\ell = k$ *into* (6) *yields that*

$$\begin{bmatrix} 2k+1-t \\ k-t \end{bmatrix} - q^{(k-t)(k-t+1)} \begin{bmatrix} k-1 \\ t-1 \end{bmatrix} = \sum_{i=1}^{k-t} (-1)^{i-1} a_i(k) \geq \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix} \begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix} - \begin{bmatrix} k-t+1 \\ 2 \end{bmatrix} \begin{bmatrix} 2k-t-1 \\ k-t-2 \end{bmatrix} q.$$

Combining the two equalities above, we can get

$$|\mathcal{F}_{HM}| > \left(1 - \frac{(q^{k-t}-1)(q^{k-t-1}-1)q}{(q^2-1)(q^{2k-t}-1)}\right) \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix} \begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix} \geq \left(1 - \frac{1}{(q^2-1)q^t}\right) \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix} \begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}.$$

For $q \geq 3, k \geq t+2$ *and* $t \geq 2$, *there holds*

$$|\mathcal{F}_{HM}| > \frac{71}{72} \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix} \begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}. \tag{14}$$

For simplicity, let

$$|\mathcal{F}^*_{HM}| = \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}, \quad |\mathcal{F}^*_{A(t+2)}| = \begin{bmatrix} t+2 \\ 1 \end{bmatrix} \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}.$$

By (13) *and* (14), *we will prove that* $|\mathcal{F}| \leq 0.986 \max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$ *instead of* $|\mathcal{F}| < max\{|\mathcal{F}_{HM}|, |\mathcal{F}_{A(t+2)}|\}$ *in most cases.*

Lemma 3.8. *Let* t, ℓ, k, a, λ *be integers satisfying* $4 \leq t+2 \leq \ell \leq \min\{k-\lambda, k-1\}$ *and* $a \geq \max\{1, 2t+1-\ell\}$ *and* $q \geq 3$. *Then*

$$\frac{S(a,\ell,k,\lambda)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{8\varphi(a,\ell,t,\lambda)}{q^{k-\ell+t-1+a^2+(\ell-2t+\lambda-2)a}(q-1)^{\ell-2t+\lambda+a-1}} + \frac{1}{q^{(\ell-t+\lambda-1)t}(q-1)^{\ell-t+\lambda-1}}. \tag{15}$$

where $S(a,t,\ell,k)$ *is defined in Section* 2 *and*

$$\varphi(a,\ell,t,\lambda) = \frac{q^{2a+\ell-2t+\lambda-1}(q-1)}{q^{2a+\ell-2t+\lambda-1}(q-1)-1}. \tag{16}$$

Proof. Since $t-i \leq \ell-t \leq k-t-1$, it follows from Lemma 3.1(ii) that

$$\frac{\left[{\ell-t \atop t-i}\right]}{\left[{2k-t \atop k-t-1}\right]} \leq \frac{q^{(\ell-2t+i)(t-i)}}{q^{(k+1)(k-t-1)}}. \tag{17}$$

By Lemma 3.1(i), we can get

$$\begin{bmatrix} \ell-t \\ t-i \end{bmatrix} = \begin{bmatrix} \ell-t \\ l-2t+i \end{bmatrix} \leq 2^{1-\delta_{\ell-t,t-i}} q^{(\ell-t)(t-i)} \quad and \quad \begin{bmatrix} 2k+1-\ell-\lambda \\ k-\ell-\lambda \end{bmatrix} \leq 2q^{(k+1)(k-\ell-\lambda)}. \tag{18}$$

Combining (17) and (18), we can obtain

$$\frac{f(i,\ell,k,\lambda)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{2^{3-\delta_{i,0}-\delta_{\ell-t,t-i}} q^{i(t-i)+(2\ell-2t+\lambda)(t-i)+(k-t+1)(\ell-2t+\lambda+i-1)+(k+1)(k-\ell-\lambda)}}{q^{(k+1)(k-t-1)}(q-1)^{\ell-2t+\lambda+i-1}}.$$

Simplifying the right-hand side of the inequality above leads to

$$\frac{f(i,\ell,k,\lambda)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{2^{3-\delta_{i,0}-\delta_{\ell-t,t-i}}}{q^{(k-\ell)(t-i)+i^2+(\ell-2t+\lambda-1)i}(q-1)^{\ell-2t+\lambda+i-1}}. \tag{19}$$

Assume that $i \leq t-1$. Since $(k-\ell)(t-i)+i^2+(\ell-2t+\lambda-1)i = k-\ell+t-1+(k-\ell-1)(t-i-1)+i^2+(\ell-2t+\lambda-2)i$ and $(k-\ell-1)(t-i-1) \geq 0$, it follows that

$$\frac{f(i,\ell,k,\lambda)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{2^{3-\delta_{i,0}-\delta_{\ell-t,t-i}}}{q^{k-\ell+t-1+i^2+(\ell-2t+\lambda-2)i}(q-1)^{\ell-2t+\lambda+i-1}}. \tag{20}$$

Recall that $a \geq \{1, 2t+1-\ell\}$. It is easy to see that $(a+j)^2 \geq a^2+(2a+1)j$ for $j \geq 1$. Therefore, it can be seen from the formula for the summations formula of geometric series that

$$\frac{\sum_{i=a}^{t-1} f(i,\ell,k,\lambda)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{8}{q^{k-\ell+t-1+a^2+(\ell-2t+\lambda-2)a}(q-1)^{\ell-2t+\lambda+a-1}} \times \frac{q^{2a+\ell-2t+\lambda-1}(q-1)}{q^{2a+\ell-2t+\lambda-1}(q-1)-1}. \tag{21}$$

In view of Lemma 3.1 (ii), we see that $\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix} \geq \begin{bmatrix} 2k+1-\ell-\lambda \\ k-\ell-\lambda \end{bmatrix} q^{(k+1)(\ell-t+\lambda-1)}$ for $\ell+\lambda \geq t+1$. Then

$$\frac{f(t,\ell,k,\lambda)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}} = \frac{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}^{\ell-t+\lambda-1}\begin{bmatrix} 2k+1-\ell-\lambda \\ k-\ell-\lambda \end{bmatrix}}{\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{1}{q^{(\ell-t+\lambda-1)t}(q-1)^{\ell-t+\lambda-1}}. \tag{22}$$

Combining (21) and (22) yields (15). The proof is complete. □

Lemma 3.9. *Let ℓ be an integer such that $t+2 \leq \ell \leq k$ and L be the ℓ-space with minimum dimension that t-intersects each $F \in \mathcal{F}$. If $q \geq 3$ and $\dim(L \cap F) \geq t+1$ for any $F \in \mathcal{F}$, then $|\mathcal{F}| < \frac{2}{9}|\mathcal{F}^*_{HM}|$.*

Proof. Select a $(t+1)$-space on L. The number of choices is $\begin{bmatrix} \ell \\ t+1 \end{bmatrix}$. Expand this $(t+1)$-space to ℓ-spaces and by Lemma 3.5 we see that the number of the spaces $(t+1)$-intersecting L is no more than $\begin{bmatrix} \ell \\ t+1 \end{bmatrix}\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}^{\ell-t-1}\begin{bmatrix} 2k+1-\ell \\ k-\ell \end{bmatrix}$. By Lemma 3.1 (i) and (ii), we can get $\begin{bmatrix} \ell \\ t+1 \end{bmatrix} \leq 2q^{(\ell-t-1)(t+1)}$ and $\begin{bmatrix} 2k+1-\ell \\ k-\ell \end{bmatrix} q^{(k+1)(k-t-1)} \leq \begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix} q^{(k+1)(k-\ell)}$, respectively. A calculation of q-binomial coefficients shows that

$$\frac{\begin{bmatrix} \ell \\ t+1 \end{bmatrix}\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}^{\ell-t-1}\begin{bmatrix} 2k+1-\ell \\ k-\ell \end{bmatrix}}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{2q^{(\ell-t-1)(t+1)+(k-t+1)(\ell-t-2)}}{(q-1)^{\ell-t-2}q^{(k+1)(\ell-t-1)}} = \frac{2}{q^{k-\ell+2}(q-1)^{\ell-t-2}} \leq \frac{2}{9}. \tag{23}$$

The proof is complete. □

4. Proof of Theorem 1.1

Proof. In this section, we always assume that $q \geq 3$. Let V be a $(2k+1)$-dimensional space and $\mathcal{F} \in \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximum-sized t-intersecting non-trivial family. We divide the proof into three cases according to the value of $\tau_t(\mathcal{F})$. Since the ratio of $|\mathcal{F}^*_{HM}|$ to $|\mathcal{F}_{HM}|$ and the ratio of $|\mathcal{F}^*_{A(t+2)}|$ to $|\mathcal{F}_{A(t+2)}|$ are easy to estimate, then the trick of the proof is to compare the upper bound of $|\mathcal{F}|$ with $\max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$.

4.1. $\tau_t(\mathcal{F}) = t+1$

In this subsection, we first estimate upper bounds of $|\mathcal{F}|$ and then compare them with $\max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$.

Proposition 4.1. *[2, Lemma 3.7] Assume that $\tau_t(\mathcal{F}) = t+1$ and define $\mathcal{T}$ to be the family of $(t+1)$-subspaces of V that t-intersect all subspaces in $\mathcal{F}$. One of the three possibilities holds:*

(i) $|\mathcal{T}| = 1$ *and*

$$|\mathcal{F}| \leq \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \begin{bmatrix} k-t \\ 1 \end{bmatrix}\begin{bmatrix} t+1 \\ 1 \end{bmatrix}\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}q.$$

(ii) $|\mathcal{T}| > 1, \tau(\mathcal{T}) = t$ *and there is an* ℓ*-subspace* W $(t+2 \le \ell \le k+1)$*, and a* t*-space* E*, such that* $\mathcal{T} = \{M : E \le M \le W, \dim M = t+1\}$*. In this case,*

$$|\mathcal{F}| \le \begin{bmatrix} \ell - t \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + \begin{bmatrix} t \\ 1 \end{bmatrix}\begin{bmatrix} n-\ell \\ k-\ell+1 \end{bmatrix} q^{k-\ell+1} + \begin{bmatrix} k-\ell+1 \\ 1 \end{bmatrix}\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} q^{\ell-t}. \tag{24}$$

(iii) $\mathcal{F} = \mathcal{F}_{A(t+2)}$*. In this case,*

$$|\mathcal{F}| = \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix} + \begin{bmatrix} t+2 \\ t+1 \end{bmatrix}\left(\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} - \begin{bmatrix} n-t-2 \\ k-t-2 \end{bmatrix}\right).$$

According to Proposition 4.1, we only need compare the upper bounds of the first two cases with $\max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$. Record the upper bounds of the first two cases in Proposition 4.1 as $|\mathcal{F}^{(i)}_{upper}|$ and $|\mathcal{F}^{(ii)}_{upper}|$, respectively. Recall that $n = 2k+1$. By Lemma 3.1(ii), we have $\begin{bmatrix} 2k-t-1 \\ k-t-2 \end{bmatrix} q^{k+1} \le \begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}$. Then

$$\frac{|\mathcal{F}^{(i)}_{upper}|}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}} \le \frac{q-1}{q^{k-t+1}-1} + \frac{(q^{k-t}-1)(q^{t+1}-1)q}{q^{k+1}(q-1)^2} \le \frac{q-1}{q^{k-t+1}-1} + \frac{q}{(q-1)^2}.$$

Recall that $q \ge 3$ and $k \ge t+2$. It is easy to see that the right-hand side of the inequality above achieves its maximum value when $q = 3$ and $k = t+2$. Hence $|\mathcal{F}^{(i)}_{upper}| \le 0.827|\mathcal{F}^*_{HM}|$.

Now we consider $|\mathcal{F}^{(ii)}_{upper}|$. Let ℓ be defined as in Proposition 4.1 (ii). In this case, if $\ell = k+1$, then $\mathcal{F} = \mathcal{F}_{HM}$ by [2, Lemma 3.4]. Assume that $t+2 \le \ell \le k$. It follows from Lemma 3.1(ii) that $\begin{bmatrix} 2k+1-\ell \\ k-\ell+1 \end{bmatrix} q^{(k+1)(k-t-1)} \le \begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix} q^{k(k-\ell+1)}$. Then

$$\begin{aligned} \frac{|\mathcal{F}^{(ii)}_{upper}|}{\max\left\{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}, \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\right\}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}} &\le \frac{q^{\ell-t}-1}{q^{k-t+1}-1} + \frac{q^{(k+1)(k-\ell+1)}(q^t-1)}{q^{(k+1)(k-t-1)}(q^{t+2}-1)} + \frac{(q^{k-\ell+1}-1)q^{\ell-t}}{q^{k+1}(q-1)} \\ &\le \frac{1}{q^{k-\ell+1}} + \frac{1}{q^{(k+1)(\ell-t-2)+2}} + \frac{1}{q^t(q-1)}. \end{aligned}$$

A simple argument shows that the right-hand side of the inequality above achieves its maximum value at $(k, \ell, t, q) = (4, 4, 2, 3)$. It follows that $|\mathcal{F}^{(ii)}_{upper}| \le 0.5 \max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$.

4.2. $t+2 \le \tau_t(\mathcal{F}) = \ell \le k-1$

In this subsection, we assume that $t+2 \le \tau_t(\mathcal{F}) = \ell \le k-1$ and L is the vector space with minimum dimension that t-intersects each $F \in \mathcal{F}$. Recall the definition of $\mathcal{L}_t$. We categorize the discussion by the dimension of the vector space in $\mathcal{L}_t$. By Lemma 3.9, $|\mathcal{F}| < \frac{2}{9}|\mathcal{F}^*_{HM}|$ if $\dim(F \cap L) \ge t+1$ for each $F \in \mathcal{F}$. In the following we may assume that there exists an $F \in \mathcal{F}$ such that $\dim(F \cap L) = t$. Hence $\mathcal{L}_t \ne \emptyset$, if $|\mathcal{F}| > 0.986 \max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}|$.

Let $H \in \mathcal{L}_t$. Our proof process is mainly divided into two parts. Firstly, we assume $\dim(H) = t$ and prove $|\mathcal{F}| \le 0.986 \max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$. Secondly, if $|\mathcal{F}| > 0.986 \max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$, then we have $\dim(H) \ge t+1$. From this we can get $|\mathcal{F}| \le 0.986 \max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$, which is a contradiction.

Proposition 4.2. *Let* $H \in \mathcal{L}_t$ *and* $\dim(H) = \ell$*. Then* $|\mathcal{F}| \le 0.986 \max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$*.*

Proof. It follows from [12, Lemma 2.12] that $|\mathcal{F}| < S(\max\{0, 2t-\ell\}, \ell, k, 0)$. A calculation of q-binomial coefficients yields that

$$\frac{f(t-2, t+2, k, 0)}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}} = \frac{\begin{bmatrix} t \\ 2 \end{bmatrix}\begin{bmatrix} 2k-t-1 \\ k-t-2 \end{bmatrix} q^8}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} = \frac{(q^t-1)(q^{t-1}-1)(q^{k-t-1}-1)q^8}{(q^{2k-t}-1)(q^2-1)(q^{t+2}-1)}, \tag{25}$$

$$\frac{f(t-1, t+2, k, 0)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}} = \frac{\begin{bmatrix} t \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}^2\begin{bmatrix} 2k-t-1 \\ k-t-2 \end{bmatrix} q^2}{\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} = \frac{(q^t-1)(q+1)^2(q^{k-t-1}-1)q^2}{(q^{2k-t}-1)(q-1)}. \tag{26}$$

The following proof process needs to be discussed in detail.

Case 1. $\ell \geq 2t+1$. Substituting $a=2, \lambda=0$ into (16), we have $\varphi(2,\ell,k,0) \leq \frac{162}{161}$. It follows from (15) that

$$\frac{S(2,\ell,k,0)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{8}{q^{k+\ell-3t-1}(q-1)^{\ell-2t+1}} \times \frac{162}{161} + \frac{1}{q^{(\ell-t-1)t}(q-1)^{\ell-t-1}}. \tag{27}$$

The right-hand side of (27) obtains its maximum value when $(k,\ell,t,q)=(6,5,2,3)$. Hence

$$\frac{S(2,\ell,k,0)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq 0.028. \tag{28}$$

Substituting $i=0, \lambda=0$ and $i=1, \lambda=0$ into (20) respectively yields that

$$\frac{f(0,\ell,k,0)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{4}{q^{k-\ell+t-1}(q-1)^{\ell-2t-1}} \leq \frac{4}{9}, \tag{29}$$

$$\frac{f(1,\ell,k,0)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{8}{q^{k-t-2}(q-1)^{\ell-2t}} \leq \frac{4}{9}. \tag{30}$$

Combining the (28), (29) and (30), we see that $S(0,\ell,k,\lambda) \leq 0.917|\mathcal{F}^*_{HM}|$.

Case 2. $\ell = t+2$. If $t=2$ and $\ell=4$, then a simple argument shows that the right-hand sides of both inequalities (25) and (26) reach their maximums when $(q,k)=(3,5)$, respectively. That is

$$\frac{f(0,4,k,0)}{\left[{4 \atop 1}\right]\left[{2k-2 \atop k-3}\right]} \leq \frac{(q-1)(q^2-1)q^8}{(q^4-1)(q^8-1)} \leq 0.201, \tag{31}$$

$$\frac{f(1,4,k,0)}{\left[{k-1 \atop 1}\right]\left[{2k-2 \atop k-3}\right]} \leq \frac{(q^2-1)^4q^2}{(q^8-1)(q-1)^3} \leq 0.703. \tag{32}$$

According to (22), it follows that

$$\frac{f(2,4,k,0)}{\left[{k-1 \atop 1}\right]\left[{2k-2 \atop k-3}\right]} \leq \frac{1}{q^2(q-1)} \leq 0.056. \tag{33}$$

Combining (31), (32) and (33), we see that $|\mathcal{F}| \leq 0.96\max\{|\mathcal{F}^*_{A(t+2)}|, |\mathcal{F}^*_{HM}|\}$.

If $t \geq 3$, then multiplying both sides of (26) by $\left[{k-t+1 \atop 1}\right]/\left[{t+2 \atop 1}\right]$ at the same time yields that

$$\frac{f(t-1,t+2,0)}{\left[{t+2 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} = \frac{(q^t-1)(q+1)^2(q^{k-t+1}-1)(q^{k-t-1}-1)q^2}{(q^{2k-t}-1)(q^{t+2}-1)(q-1)} \leq \frac{(q+1)^2}{q^t(q-1)} \leq \frac{8}{27}. \tag{34}$$

Observe that the right-side hand of (25) reach its maximum when $(k,q)=(t+3,3)$. Substituting $(k,q)=(t+3,3)$ into (25) and $(\ell,q)=(t+2,3)$ into (22) yields that

$$\frac{f(t-2,t+2,t+3,0)}{\left[{t+2 \atop 1}\right]\left[{t+6 \atop 2}\right]} \leq \frac{(q^t-1)(q^{t-1}-1)q^8}{(q^{t+6}-1)(q^{t+2}-1)} \leq \frac{1}{3}, \tag{35}$$

$$\frac{f(t,t+2,k,0)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{1}{q^t(q-1)} \leq \frac{1}{54}. \tag{36}$$

Combining (34), (35) and (36) yields that $|\mathcal{F}| \leq 0.649\max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$.

Case 3. $t+3 \leq \ell \leq 2t$. It is clear that $t \geq 3$. Since $t+3 \leq \ell \leq 2t$, then $2t+2-\ell \geq 2$. Therefore $\varphi(2t+2-\ell,\ell,t,0) \leq \frac{54}{53}$ by (16). It follows from (15) that

$$\frac{S(2t+2-\ell,\ell,k,0)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{8}{q^t(q-1)} \times \frac{54}{53} + \frac{1}{q^{2t}(q-1)^2} \leq 0.152. \tag{37}$$

By Lemma 3.1(i) and (ii), we can obtain $\begin{bmatrix} t \\ 2t+\ell-l \end{bmatrix} \le 2q^{(\ell-t-1)(2t+1-\ell)}$ and $\begin{bmatrix} 2k+1-\ell \\ k-\ell \end{bmatrix} q^{(k+1)(\ell-t-1)} \le \begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}$, respectively. A calculation of q-binomial coefficients shows that

$$\frac{f(2t+1-\ell,\ell,k,0)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} = \frac{\begin{bmatrix} t \\ 2t+1-\ell \end{bmatrix}\begin{bmatrix} \ell-t \\ \ell-t-1 \end{bmatrix}^2\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k+1-\ell \\ k-\ell \end{bmatrix} q^{2(\ell-t-1)^2}}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \le \frac{2q^{(\ell-t-1)(2t+1-\ell)+2\ell-2t+2(\ell-t-1)^2}}{(q-1)^2 q^{(k+1)(\ell-t-1)}}.$$

Observe that $k \ge \ell+1 \ge t+4$. Simplification of the right-hand side of the inequality above gives

$$\frac{f(2t+1-\ell,\ell,k,0)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \le \frac{2}{(q-1)^2 q^{(k-\ell)(\ell-t-1)-2}} \le \frac{1}{2}. \tag{38}$$

Again by Lemma 3.1(i), we can get $\begin{bmatrix} t \\ 2t-\ell \end{bmatrix} \le 2q^{(\ell-t)(2t-\ell)}$ and $\begin{bmatrix} t+2 \\ 1 \end{bmatrix} \ge q^{t+1}$. Then

$$\frac{f(2t-\ell,\ell,k,0)}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \le \frac{2q^{(\ell-t)(2t-\ell)+2(\ell-t)^2}}{q^{(k+1)(\ell-t-1)+t+1}} = \frac{2}{q^{(k-\ell)(\ell-t-1)}} \le \frac{2}{9}. \tag{39}$$

Combining (37), (38) and (39), we see that $|\mathcal{F}| \le S(2t-\ell,\ell,k,0) \le 0.875\max\{|\mathcal{F}^*_{A(t+2)}|, |\mathcal{F}^*_{HM}|\}$. $\square$

Let $H \in \mathcal{L}_t$ and $\dim(H) = \ell+\lambda$. If $\lambda = 0$, then $|\mathcal{F}| \le 0.986\max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$ by Proposition 4.2. Recall that $|\mathcal{F}_{A(t+2)}| > 0.995|\mathcal{F}^*_{A(t+2)}|$ and $|\mathcal{F}_{HM}| > 0.986|\mathcal{F}^*_{HM}|$ by (13) and (14), respectively. That is, $|\mathcal{F}| < \max\{|\mathcal{F}_{HM}|, |\mathcal{F}_{A(t+2)}|\}$. If $|\mathcal{F}| \ge \max\{|\mathcal{F}_{HM}|, |\mathcal{F}_{A(t+2)}|\}$, then $\dim(H) \ge \ell+1$. In the following, we may assume that $\lambda \ge 1$. Let $a \ge \max\{1, 2t+1-\ell\}$. Then $\varphi(a,\ell,k,\lambda) \le \frac{18}{17}$ by (16). It follows from (15) that

$$\frac{S(\max\{1,2t+1-\ell\},\ell,k,\lambda)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \le \frac{8}{q^{k-\ell+t-1}(q-1)} \times \frac{18}{17} + \frac{1}{q^{2t}(q-1)^2} \le 0.474. \tag{40}$$

If $2t+1 \le \ell \le k-1$, substituting $i = 0$ into (19) we can get

$$\frac{f(0,\ell,k,\lambda)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \le \frac{4}{(q-1)^{\ell-2t+\lambda-1} q^{(k-\ell)t}} \le \frac{2}{9}. \tag{41}$$

If $t+2 \le \ell \le 2t$ and $t \ge 3$, then $q^{t+2}-1 \ge \frac{242}{243}q^{t+2}$. Substituting $i = 2t-\ell$ into (19) and multiplying $\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}/\begin{bmatrix} t+2 \\ 1 \end{bmatrix}$ on both sides of (19) we can get

$$\frac{f(2t-\ell,\ell,k,\lambda)}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \le \frac{486}{121(q-1)^{\lambda-1} q^{(k-\ell)(\ell-t)+2t+1-k}} = \frac{486}{121(q-1)^{\lambda-1} q^{(k-\ell-1)(\ell-t-1)+t}} \le \frac{18}{121}. \tag{42}$$

If $t = 2$, then $\ell = 4$ and $f(0,4,k,\lambda) = \begin{bmatrix} 2+\lambda \\ 2 \end{bmatrix}\begin{bmatrix} k-1 \\ 1 \end{bmatrix}^{\lambda}\begin{bmatrix} 2k-3-\lambda \\ k-4-\lambda \end{bmatrix} q^8$. By Lemma 3.1 (ii), we have $\begin{bmatrix} 2k-3-\lambda \\ k-4-\lambda \end{bmatrix} q^{(k+1)(\lambda+1)} \le \begin{bmatrix} 2k-2 \\ k-3 \end{bmatrix}$. It follows that

$$\frac{f(0,4,k,\lambda)}{\begin{bmatrix} k-1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-2 \\ k-3 \end{bmatrix}} \le \frac{q^{2\lambda+11+(k-1)(\lambda-1)}}{(q-1)^{\lambda}(q^2-1)q^{(k+1)(\lambda+1)}} = \frac{1}{(q-1)^{\lambda}(q^2-1)q^{2k-11}} \le \frac{q}{(q-1)(q^2-1)} \le \frac{3}{16}. \tag{43}$$

Combining (40), (41) and (43) yields that

$$S(\max\{0,2t-\ell\},\ell,k,\lambda) \le \begin{cases} 0.697|\mathcal{F}^*_{HM}|, & if\ \ell \ge 2t+1, \\ 0.623\max\{|\mathcal{F}^*_{A(t+2)}|, |\mathcal{F}^*_{HM}|\}, & if\ t+2 \le \ell \le 2t\ and\ t \ge 3, \\ 0.662|\mathcal{F}^*_{HM}|, & if\ (\ell,t) = (4,2). \end{cases} \tag{44}$$

It follows from [12, Lemma 2.12] that $|\mathcal{F}| \le S(\max\{0,2t-\ell\},\ell,k,\lambda) + |\mathcal{F}_1|$. According to the definition of $\mathcal{F}_1$, any $F \in \mathcal{F}_1$ must contain a $(t+1)$-subspace E on L and the number of choices for E is $\begin{bmatrix} \ell \\ t+1 \end{bmatrix}$. Since $\tau_t(F) = \ell$, then by Lemma 3.5 we have $|\mathcal{F}_1| \le \begin{bmatrix} \ell \\ t+1 \end{bmatrix}\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}^{\ell-t-1}\begin{bmatrix} 2k+1-\ell \\ k-\ell \end{bmatrix}$. It follows from Lemma 3.1(i) and (ii) that

$$\frac{|\mathcal{F}_1|}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \le \frac{2q^{(\ell-t-1)(t+1)+(k-t+1)(\ell-t-2)}}{q^{(k+1)(\ell-t-1)}(q-1)^{\ell-t-2}} \le \frac{2}{q^{k-\ell+2}} \le \frac{2}{27}. \tag{45}$$

Combining (44) and (45) yields that $|\mathcal{F}| \le S(\max\{0,2t-\ell\},\ell,k,\lambda) + |\mathcal{F}_1| \le 0.772\max\{|\mathcal{F}^*_{A(t+2)}|, |\mathcal{F}^*_{HM}|\}$.

4.3. $\tau_t(\mathcal{F}) = k > t+1$

In this subsection, we assume that $\tau_t(\mathcal{F}) = k > t+1$. By Lemma 3.9, $|\mathcal{F}| < \frac{2}{9}|\mathcal{F}^*_{HM}|$ if $\dim(F \cap L) \geq t+1$ for each $F \in \mathcal{F}$. In the following we may assume that there exists an $L_1, L_2 \in \mathcal{F}$ such that $\dim(L_1 \cap L_2) = t$.

In this section, we still use the method of comparing the upper bound of $|\mathcal{F}|$ with $\max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$. However, for some special cases, $S(\max\{0, 2t-l\}, k, k, 0)$ is ineffective for this method. Therefore, we introduce a third vector space beyond L_1, L_2 to prove $|\mathcal{F}| \leq 0.986 \max\{|\mathcal{F}^*_{HM}|, |\mathcal{F}^*_{A(t+2)}|\}$ by intersection.

Observe that

$$\frac{f(i,k,k,0)}{\max\left\{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}, \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\right\}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} = \frac{\begin{bmatrix} k-t \\ t-i \end{bmatrix}}{\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \times \begin{bmatrix} t \\ i \end{bmatrix}\begin{bmatrix} k-t \\ t-i \end{bmatrix} \times \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}^{k-2t+i} q^{2(t-i)^2}. \tag{46}$$

Estimating the first term of the right-hand side of (46) by Lemma 3.1 (ii) and estimating the second term of the right-hand side of (46) by Lemma 3.1 (i) yields that

$$\frac{f(i,k,k,0)}{\max\left\{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}, \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\right\}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{2^{2-\delta_{i,0}-\delta_{k-t,t-i}} q^{i(t-i)+(2k-2t)(t-i)+(k-t+1)(k-2t+i-1)}}{q^{(k+1)(k-t-1)+\max\{0,2t+1-k\}}(q-1)^{k-2t+i-1}}.$$

After simplification, we can get

$$\frac{f(i,k,k,0)}{\max\left\{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}, \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\right\}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{2^{2-\delta_{i,0}-\delta_{k-t,t-i}}}{q^{i^2+(k-2t-1)i+\max\{0,2t+1-k\}}(q-1)^{k-2t+i-1}}. \tag{47}$$

If $i \geq 1$ and $k-t > t-i$ for $i \geq 1$, then $\delta_{i,0} = \delta_{k-t,t-i} = 0$. It follows that

$$\frac{f(i,k,k,0)}{\max\left\{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}, \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\right\}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{4}{q^{i^2+(k-2t-1)i+\max\{0,2t+1-k\}}(q-1)^{k-2t+i-1}}. \tag{48}$$

Observe that $(a+j)^2 - a^2 = 2aj + j^2 \geq (2a+1)j$. Then

$$\frac{S(a,k,k,0)}{\max\left\{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}, \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\right\}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{4}{q^{a^2+(k-2t-1)a+\max\{0,2t+1-k\}}(q-1)^{k-2t+a-1}} \times \sum_{j\geq 0} \frac{1}{q^{(k-2t+2a)j}(q-1)^j}. \tag{49}$$

Applying the formula for the summations formula of geometric series to the second term on the right-hand side of (49) yields that

$$\frac{S(a,k,k,0)}{\max\left\{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}, \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\right\}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{4}{q^{a^2+(k-2t-1)a+\max\{0,2t+1-k\}}(q-1)^{k-2t+a-1}} \times \frac{q^{(k-2t+2a)}(q-1)}{q^{(k-2t+2a)}(q-1)-1}. \tag{50}$$

We divide our proof into four cases.

Case 1. $k \geq 2t+2$. If $(k,q) = (2t+2, 3)$, then by Lemma 3.1 (iii) we see that

$$\begin{aligned}\frac{f(0,k,k,0)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} &= \frac{q^{t+3}-1}{(q-1)(q^2-1)\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \times \frac{(q^{t+1}-1)^2(q^{t+2}-1)^2 q^{2t^2}}{(q-1)^2(q^2-1)} \\ &\leq \frac{q^3}{(q-1)^2(q^2-1)} = \frac{27}{32}.\end{aligned} \tag{51}$$

If $(k,q) \neq (2t+2, 3)$, by (47), there holds

$$\frac{f(0,k,k,0)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{2}{(q-1)^{k-2t-1}} \leq \begin{cases} \frac{2}{3}, & if\ q \geq 4, \\ \frac{1}{2} & if\ k \geq 2t+3. \end{cases} \tag{52}$$

It is not difficult to see that the right-hand side of (50) decreases as k increases. Substituting $a = 1$ and $k = 2t + 2$ in (50) yields that

$$\frac{S(1,k,k,0)}{\begin{bmatrix}k-t+1\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} \le \frac{1}{9} \times \frac{q^4(q-1)}{q^4(q-1)-1} \le \frac{18}{161}. \tag{53}$$

Combining (51) and (53) yields that $|\mathcal{F}| \le f(0,k,k,0)+S(1,k,k,0) \le 0.956|\mathcal{F}^*_{HM}|$, if $(k,q) = (2t+2,3)$; Combining (52) and (53) yields that $|\mathcal{F}| \le f(0,k,k,0) + S(1,k,k,0) \le 0.779|\mathcal{F}^*_{HM}|$, if $(k,q) \neq (2t+2,3)$.

Case 2. $t+2 \le k \le 2t-1$. It is clear that $t \ge 3$. It follows from Lemma 3.1(ii) that $\begin{bmatrix}t\\2t-k\end{bmatrix}q^{(k+1)(k-t-1)} \le \begin{bmatrix}2k-t\\k-t-1\end{bmatrix}q^{(2t-k)(k-t)}$. Then

$$\frac{f(2t-k,k,k,0)}{\begin{bmatrix}t+2\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} = \frac{\begin{bmatrix}t\\2t-k\end{bmatrix}q^{2(k-t)^2}}{\begin{bmatrix}t+2\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} \le \frac{q^{(k-t)(2t-k)+2(k-t)^2}(q-1)}{(q^{t+2}-1)q^{(k+1)(k-t-1)}} = 1 - \frac{q^{t+1}-1}{q^{t+2}-1}. \tag{54}$$

Since $t \ge 3$ and $q \ge 3$, then $q^{t+1} - 1 \ge \frac{80}{81}q^{t+1}$. In view of (54), we have

$$\frac{f(2t-k,k,k,0)}{\begin{bmatrix}t+2\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} \le 1 - \frac{80}{81}\frac{1}{q}. \tag{55}$$

Again by Lemma 3.1(ii), we can obtain $\begin{bmatrix}t\\2t+1-k\end{bmatrix}q^{(k+1)(k-t-1)} \le \begin{bmatrix}2k-t\\k-t-1\end{bmatrix}q^{(2t+1-k)(k-t-1)}$. Then

$$\frac{f(2t+1-k,k,k,0)}{\begin{bmatrix}t+2\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} \le \frac{q^{(k-t-1)(2t+1-k)+2(k-t-1)^2+2k-2t}}{(q-1)^2q^{(k+1)(k-t-1)+2t+1-k}} = \frac{1}{q^{2t-1-k}(q-1)^2}. \tag{56}$$

According to $k \le 2t-1$ and $q \ge 3$, it follows from (56) that

$$\frac{f(2t+1-k,k,k,0)}{\begin{bmatrix}t+2\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} \le \frac{1}{q^{2t-1-k}(q-1)^2} \le \frac{9}{4}\frac{1}{q^2}. \tag{57}$$

Substituting $a = 2t+2-k$ in (50) yields that

$$\frac{S(2t+2-k,k,k,0)}{\begin{bmatrix}t+2\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} \le \frac{4q^{2t+4-k}(q-1)}{(q-1)q^{4t+3-2k}(q^{2t+4-k}(q-1)-1)} = \frac{4}{q^{2t-1-k}(q^{2t+4-k}(q-1)-1)}. \tag{58}$$

Since $k \le 2t-1$ and $q \ge 3$, then $q^{2t-1-k}(q^{2t+4-k}(q-1)-1) \ge q^6 - q^5 - 1 \ge 4q^4$. That is

$$\frac{S(2t+2-k,k,k,0)}{\begin{bmatrix}t+2\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} \le \frac{1}{q^4}. \tag{59}$$

Combining (55), (57) and (59), we see that

$$\frac{S(2t-k,k,k,0)}{\begin{bmatrix}t+2\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} \le 1 - \frac{80}{81}\frac{1}{q} + \frac{9}{4}\frac{1}{q^2} + \frac{1}{q^4} = 1 - \frac{1}{q^4}\left(\frac{80}{81}q^3 - \frac{9}{4}q^2 - 1\right) \le 1 - \frac{1}{q^4} \le 1 - \frac{1}{q^{k+1}}.$$

Hence $|\mathcal{F}| < S(2t-k,k,k,0) \le \left(1 - q^{-k-1}\right)|\mathcal{F}^*_{A(t+2)}| \le |\mathcal{F}_{A(t+2)}|$ by (13).

Case 3. $k = 2t$. Substituting $a = 2$ and $k = 2t$ in (50) yields that

$$\frac{S(2,k,k,0)}{\begin{bmatrix}t+2\\1\end{bmatrix}\begin{bmatrix}2k-t\\k-t-1\end{bmatrix}} \le \frac{4}{q^3(q-1)} \times \frac{q^4(q-1)}{q^4(q-1)-1} \le \frac{1}{q^2}. \tag{60}$$

By Lemma 3.1(i), it can be seen that $\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix} \geq q^{(k+1)(k-t-1)}$. Then

$$\frac{f(0,k,k,0)}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{q^{2t^2}}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{(q-1)q^{t+1}}{q^{t+2}-1} = 1 - \frac{q^{t+1}-1}{q^{t+2}-1} \leq 1 - \frac{1}{2q}. \tag{61}$$

If $|\mathcal{F}(1,t,k,k;L_1,L_2)| = 0$, then

$$\frac{|\mathcal{F}|}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{f(0,k,k,0) + S(2,k,k,0)}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq 1 - \frac{1}{2q} + \frac{1}{q^2} \leq 1 - \frac{1}{q^3} \leq 1 - \frac{1}{q^{k+1}},$$

which implies $|\mathcal{F}| < |\mathcal{F}_{A(t+2)}|$. Since $k = 2t$, then $\mathcal{F}(0,t,k,k;L_1,L_2) \subseteq L_1 + L_2$. If $|\mathcal{F}(1,t,k,k;L_1,L_2)| > 0$ and $\mathcal{F}(1,t,k,k;L_1,L_2) \subseteq \begin{bmatrix} L_1+L_2 \\ k \end{bmatrix}$, then $|\mathcal{F}(0,t,k,k;L_1,L_2)| + |\mathcal{F}(1,t,k,k;L_1,L_2)| \leq \begin{bmatrix} 2k-t \\ k \end{bmatrix}$. Observe that

$$\frac{\begin{bmatrix} 2k-t \\ k \end{bmatrix}}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} = \frac{(q-1)(q^{k+1}-1)}{(q^{t+2}-1)(q^{k-t}-1)} = \frac{q^{k+2}-q^{k+1}-q+1}{q^{k+2}-q^{t+2}-q^{k-t}+1}. \tag{62}$$

Since $q \geq 3$ and $t \geq 2$, then $q^{k+1} - q^{t+2} - q^{k-t} + q \geq \frac{1}{2}q^{k+1}$. It follows that

$$\frac{\begin{bmatrix} 2k-t \\ k \end{bmatrix}}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} = 1 - \frac{q^{k+1}-q^{t+2}-q^{k-t}+q}{q^{k+2}-q^{t+2}-q^{k-t}+1} \leq 1 - \frac{\frac{1}{2}q^{k+1}}{q^{k+2}} \leq 1 - \frac{1}{2q}. \tag{63}$$

Combining (60) and (63) yields that $|\mathcal{F}| \leq \left(1 - \frac{1}{2q} + \frac{1}{q^2}\right)|\mathcal{F}^*_{A(t+2)}| < (1 - q^{-k-1})|\mathcal{F}^*_{A(t+2)}| \leq |\mathcal{F}_{A(t+2)}|$. If there exists an $F_1 \in \mathcal{F}(1,t,k,k;L_1,L_2)$ such that $F_1 \nsubseteq L_1 + L_2$, then we re-estimate $|\mathcal{F}(0,t,k,k;L_1,L_2)|$. Recall that $\mathcal{F}(0,t,k,k;L_1,L_2) \subseteq \begin{bmatrix} L_1+L_2 \\ k \end{bmatrix}$ and $\dim(F_1 \cap (L_1 + L_2)) \leq k - 1$. By Lemma 3.4, the number of vector spaces in $\begin{bmatrix} L_1+L_2 \\ k \end{bmatrix}$ t-intersecting $F_1 \cap (L_1 + L_2)$ is no more than $\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t-1 \\ k-t-1 \end{bmatrix}$. Then

$$\frac{|\mathcal{F}(0,t,k,k;L_1,L_2)|}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t-1 \\ k-t-1 \end{bmatrix}}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} = \frac{(q^{t+1}-1)(q^{2t+1}-1)}{(q^{t+2}-1)(q^{3t}-1)} \leq \frac{1}{q^t}. \tag{64}$$

By (56), we can get

$$\frac{f(1,k,k,0)}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{q}{(q-1)^2} \leq \frac{3}{4}. \tag{65}$$

Combining (60), (64) and (65), we see that $|\mathcal{F}| < |\mathcal{F}(0,t,k,k;L_1,L_2)| + f(1,k,k,0) + S(2,k,k,0) \leq 0.973|\mathcal{F}^*_{A(t+2)}|$.

Case 4. The case $k = 2t + 1$. Firstly, assume that $\mathcal{F} \subseteq \begin{bmatrix} L_1+L_2 \\ k \end{bmatrix}$. Then $|\mathcal{F}| \leq \begin{bmatrix} 2k-t \\ k \end{bmatrix}$. By (63) and (13), we see that $\begin{bmatrix} 2k-t \\ k \end{bmatrix} \leq \left(1 - \frac{1}{2q}\right)|\mathcal{F}^*_{A(t+2)}| < |\mathcal{F}_{A(t+2)}|$.

Secondly, assume that $\mathcal{F}(0,t,k,k;L_1,L_2) \subseteq \begin{bmatrix} L_1+L_2 \\ k \end{bmatrix}$ and $\cup_{i=1}^{t}\mathcal{F}(i,t,k,k;L_1,L_2) \nsubseteq \begin{bmatrix} L_1+L_2 \\ k \end{bmatrix}$. Then there exists an $F_2 \in \cup_{i=1}^{t}\mathcal{F}(i,t,k,k;L_1,L_2)$ such that $\dim(F_2 \cap (L_1 + L_2)) \leq k - 1$. It is clear that $|\mathcal{F}(0,t,k,k;L_1,L_2)|$ is less than the number of the vector spaces t-intersecting F_2 in $\begin{bmatrix} L_1+L_2 \\ k \end{bmatrix}$. Then $|\mathcal{F}(0,t,k,k;L_1,L_2)| < \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t-1 \\ k-t-1 \end{bmatrix}$ by Lemma 3.4. It follows that

$$\frac{|\mathcal{F}(0,t,k,k;L_1,L_2)|}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t-1 \\ k-t-1 \end{bmatrix}}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} = \frac{q^{2t+2}-1}{q^{3t+2}-1} \leq \frac{1}{q^t}, \tag{66}$$

Substituting $a = 2$ and $k = 2t + 1$ in (50) yields that

$$\frac{S(2,k,k,0)}{\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{4}{q^4(q-1)^2} \times \frac{q^5(q-1)}{q^5(q-1)-1} \leq \frac{6}{485}. \tag{67}$$

By Lemma 3.1(iii), we can get

$$\frac{f(1,k,k,0)}{\left[{k-t+1 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{q^{t+2}-1}{(q-1)(q^2-1)\left[{2k-t \atop k-t-1}\right]} \times \frac{(q^t-1)^3(q^{t+1}-1)^2q^{2(t-1)^2}}{(q-1)^3(q^2-1)} \leq \frac{q^3}{(q-1)^3(q^2-1)} \leq \frac{27}{64}. \tag{68}$$

Combining (66), (67) and (68) yields $|\mathcal{F}| \leq |\mathcal{F}(0,t,k,k;L_1,L_2)| + f(1,k,k,0) + S(2,k,k,0) \leq 0.546|\mathcal{F}^*_{A(t+2)}|$.

Finally, assume that $\mathcal{F}(0,t,k,k;L_1,L_2) \nsubseteq \left[{L_1+L_2 \atop k}\right]$. Then there exists an $F_3 \in \mathcal{F}(0,t,k,k;L_1,L_2)$ such that $F_3 \notin \left[{L_1+L_2 \atop k}\right]$. Let $E_1 = F_3 \cap L_1$ and $E_2 = F_3 \cap L_2$. We divide the vector spaces in $\mathcal{F}(0,t,k,k;L_1,L_2)$ into three classes as follows:

$$\mathcal{F}' = \{F \in \mathcal{F}(0,t,k,k;L_1,L_1) : F \leq L_1 + L_2\},$$
$$\mathcal{F}'' = \{F \in \mathcal{F}(0,t,k,k;L_1,L_2) : F \nleq L_1 + L_2 \text{ and } \dim(F \cap E_1) + \dim(F \cap E_2) \geq 1\},$$
$$\mathcal{F}''' = \{F \in \mathcal{F}(0,t,k,k;L_1,L_2) : F \nleq L_1 + L_2 \text{ and } \dim(F \cap E_1) + \dim(F \cap E_2) = 0\}.$$

Recall that $\dim(F_3 \cap (L_1+L_2)) = 2t$ and $\dim(L_1+L_2) = 2k-t$. It follows from Lemma 3.4 that $|\mathcal{F}'| \leq \left[{k-t+1 \atop 1}\right]\left[{2k-t-1 \atop k-t-1}\right]$. By (66), we have

$$\frac{|\mathcal{F}'|}{\left[{t+2 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{1}{q^t}. \tag{69}$$

If $\dim(F \cap E_1) + \dim(F \cap E_2) \geq 1$, then select a 1-dimensional vector space A in E_1 or E_2. The number of choices is $2\left[{t \atop 1}\right]$. Without loss of generality, it is assumed that $A \leq E_1$. Let $E = L_1 \cap L_2$ where $\dim(E) = t$. Select a t-dimensional vector space containing A outside of E on L_1 and a t-dimensional vector space outside of E on L_2. The numbers of choices are $\left[{k-t-1 \atop t-1}\right]q^{t(t-1)}$ and $\left[{k-t \atop t}\right]q^{t^2}$ by Lemma 3.3, respectively. Now we have selected a $2t$-dimensional vector space. Since $\tau_t(\mathcal{F}) = k$, then there exists a k-dimensional space in $\mathcal{F}$ that is disjoint with this $2t$-dimensional space, and by Lemma 3.5 we can obtain

$$|\mathcal{F}''| \leq 2\begin{bmatrix} t \\ 1 \end{bmatrix}\begin{bmatrix} k-t-1 \\ t-1 \end{bmatrix}\begin{bmatrix} k-t \\ t \end{bmatrix}\begin{bmatrix} k-t+1 \\ 1 \end{bmatrix}q^{2t^2-t}. \tag{70}$$

With the use of Lemma 3.1(iii), we can obtain

$$\frac{|\mathcal{F}''|}{\left[{t+2 \atop 1}\right]\left[{2k-t \atop k-t-1}\right]} \leq \frac{q^{t+1}-1}{(q-1)^2\left[{2k-t \atop k-t-1}\right]} \times \frac{2(q^t-1)^2q^{2t^2-t}}{q-1} \leq \frac{2(q+1)}{(q-1)q^2}. \tag{71}$$

Assume that $\dim(F \cap E_1) + \dim(F \cap E_2) = 0$. Since $\dim(F \cap F_3) \geq t$ and $\dim(F_3 \cap (L_1 + L_2)) = 2t$, then $\dim(F \cap (F_3 \cap (L_1 + L_2))) \geq t-1$. That is, F intersects $F_3 \cap (L_1 + L_2)$ outside E_1 and E_2 at least $(t-1)$-dimensional vector space. Observe that $\dim((F \cap L_1) \cap (F \cap L_2)) = 0$. Then $F \cap (L_1 + L_2) = ((F \cap L_1) \oplus (F \cap L_2))$. Therefore, there is a unique decomposition of the basis vectors on $F \cap (F_3 \cap (L_1 + L_2))$. Let $E = \langle e_1, e_2, \ldots, e_t\rangle$ and select a $(t-1)$-dimensional subspace T on $F \cap (F_3 \cap (L_1 + L_2))$. Then T can be written as $T = \langle e'_{1,1} + e'_{2,1}, e'_{1,2} + e'_{2,2}, \ldots, e'_{1,t-1} + e'_{2,t-1}\rangle$, where $\langle e'_{i,1}, e'_{i,2}, \ldots, e'_{i,t-1}\rangle \leq E_i$ for $i \in \{1,2\}$. We now consider the number of choices of T. Select a $(t-1)$-dimensional space on $F \cap L_1$, which can be written as $T_1 = \langle e'_{1,1} + \sum_{i=1}^t \lambda_{1,i}e_i, e'_{1,2} + \sum_{i=1}^t \lambda_{2,i}e_i, \ldots, e'_{1,t-1} + \sum_{i=1}^t \lambda_{t-1,i}e_i\rangle$, where $0 \leq \lambda_{j,i} \leq q-1$ and $\sum_{i=1}^t \lambda_{j,i}^2 \neq 0$ for $j \in \{1, 2, \ldots t-1\}$. Let $T'_1 = \langle e'_{1,1}, e'_{1,2}, \ldots, e'_{1,t-1}\rangle$. Then $\dim(T'_1) = t-1$, otherwise $\dim(T_1 \cap E) > 0$. Hence the number of the choice of T'_1 is $\left[{t \atop t-1}\right]$. For a fixed j, the number of the choices of $e_{1,j} + \sum_{i=1}^t \lambda_{j,i}e_i$ is $q^t - 1$. If $t \geq 3$, then we have $\sum_{i=1}^t \lambda_{j_1,i}e_i \neq \sum_{i=1}^t \lambda_{j_2,i}e_i$ for different $j_1, j_2 \in \{1, 2, \ldots t-1\}$. Otherwise $\dim(T_1 \cap E_1) \geq 1$. A simple counting shows that the number of choices of T_1 is no more than $\left[{t \atop t-1}\right](q^t-1)^{t-1}$. Since T is decomposed in a unique way, then we have to select $T_2 = \langle e'_{2,1} + \sum_{i=1}^t (q-\lambda_{1,i})e_i, e'_{2,2} + \sum_{i=1}^t (q-\lambda_{2,i})e_i, \ldots, e'_{2,t-1} + \sum_{i=1}^t (q-\lambda_{t-1,i})e_i\rangle$ on $F \cap L_2$. We first select a $(t-1)$-dimensional subspace from E_2, named T'_2. The number of choices of T'_2 is $\left[{t \atop t-1}\right]$. Then we select $(t-1)$ vectors from T'_2 one by one and name them $e'_{2,1}, e'_{2,2}, \ldots, e'_{2,t-1}$, respectively. Since

T_2' has $\begin{bmatrix} t-1 \\ 1 \end{bmatrix}$ vectors, then the number of the choices of $e_{2,1}', e_{2,2}', \ldots, e_{2,t-1}'$ is no more than $\begin{bmatrix} t-1 \\ 1 \end{bmatrix}^{t-1}$. Therefore the number of choices of T_2 is no more than $\begin{bmatrix} t \\ t-1 \end{bmatrix}\begin{bmatrix} t-1 \\ 1 \end{bmatrix}^{t-1}$. Now we also need to select two 1-dimensional vector spaces outside of T_1, E_1, E on L_1 and outside of T_2, E_2, E on L_2, respectively. Since we have selected $(t-1)$-dimensional vector spaces in $T_1 + E$ and $T_2 + E$, respectively, then the 1-dimensional spaces selected have to be outside of $T_1 + E$ and $T_2 + E$. Otherwise $\dim(F \cap E) \geq 1$, according to the dimension sum formula. Recall that $k = 2t + 1$ and $\dim(T_1 + E) = \dim(T_2 + E) = 2t - 1$. Then the numbers of choices of these two 1-spaces are both no more than $\begin{bmatrix} 2 \\ 1 \end{bmatrix} q^t$ by Lemma 3.3. Now we have selected $2t$-dimensional vector space. Since $\tau_t(\mathcal{F}) = k = 2t + 1$, it follows from Lemma 3.5 that

$$|\mathcal{F}'''| \leq \begin{bmatrix} 2 \\ 1 \end{bmatrix}^2 \begin{bmatrix} t \\ t-1 \end{bmatrix}^2 \begin{bmatrix} t-1 \\ 1 \end{bmatrix}^{t-1} \begin{bmatrix} k-t+1 \\ 1 \end{bmatrix} (q^t - 1)^{t-1} q^{2t}. \tag{72}$$

By the definition of q-binomial coefficient and (72), we can get

$$\frac{|\mathcal{F}'''|}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{q^t - 1}{(q-1)^2 \begin{bmatrix} 3t+2 \\ t \end{bmatrix}} \times \frac{(q+1)^2 (q^t - 1)^t (q^{t-1} - 1)^{t-1} q^{2t}}{(q-1)^{t-1}}. \tag{73}$$

Applying Lemma 3.1(iii) to the first term of the right-hand side of of inequality (73) yields that

$$\frac{|\mathcal{F}'''|}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \frac{(q+1)^3 (q^t - 1)^t (q^{t-1} - 1)^{t-1} q^{3t}}{(q-1)^{t-1} q^{2t^2+2t+3}} \leq \frac{(q+1)^3}{(q-1)^{t-1} q^{t+2}}. \tag{74}$$

By (74), we see that

$$\frac{|\mathcal{F}'''|}{\begin{bmatrix} t+2 \\ 1 \end{bmatrix}\begin{bmatrix} 2k-t \\ k-t-1 \end{bmatrix}} \leq \begin{cases} \frac{16}{243}, & if\ t \geq 3, \\ \frac{54}{625}, & if\ t = 2, q \geq 5. \end{cases} \tag{75}$$

Combining (69),(71) and (75) yields that

$$|\mathcal{F}(0, t, k, k; L_1, L_2)| = |\mathcal{F}'| + |\mathcal{F}''| + |\mathcal{F}'''| \leq \begin{cases} 0.548 |\mathcal{F}^*_{A(t+2)}|, & if\ t \geq 3, \\ 0.247 |\mathcal{F}^*_{A(t+2)}|, & if\ t = 2\ and\ q \geq 5. \end{cases} \tag{76}$$

Combining (67), (68) and (76) yields that $|\mathcal{F}| \leq |\mathcal{F}(0, t, k, k; L_1, L_2)| + f(1, k, k, 0) + S(2, k, k, 0) \leq 0.983 |\mathcal{F}^*_{A(t+2)}|$ for $(t, q) \notin \{(2, 3), (2, 4)\}$. Assume that $(t, q) \in \{(2, 3), (2, 4)\}$. By the definition of $f(1, k, k, 0)$, (70) and (72), we list the values of $f(1, k, k, 0)$ and $|\mathcal{F}^*_{A(t+2)}|$, as well as the upper bound values of $|\mathcal{F}''|$ and $|\mathcal{F}'''|$ in the following table.

(q, t, k)	$f(1, 5, 5, 0)$	$\lvert\mathcal{F}''\rvert$	$\lvert\mathcal{F}'''\rvert$	$\lvert\mathcal{F}^*_{A(t+2)}\rvert$
$(3, 2, 5)$	9734400	12130560	6635520	35850400
$(4, 2, 5)$	254898000	365568000	204000000	2028024265

Combining (67), (69) and the table above yields that $|\mathcal{F}| \leq |\mathcal{F}(0, t, k, k; L_1, L_2)| + f(1, k, k, 0) + S(2, k, k, 0) \leq 0.919 |\mathcal{F}^*_{A(t+2)}|$, if $(t, q) = (2, 3)$; $|\mathcal{F}| \leq 0.482 |\mathcal{F}^*_{A(t+2)}|$, if $(t, q) = (2, 4)$.

The proof is complete. □

5. Conclusion

For $n = 2k + 1, q \geq 3, k \geq t + 2$ and $t \geq 2$, we prove that $\mathcal{F}_{HM}$ is the maximal non-trivial t-intersecting family, if $k \geq 2t + 2$; $\mathcal{F}_{A(t+2)}$ is the maximal non-trivial t-intersecting family, if $t + 2 \leq k \leq 2t + 1$. This result improves the applicable range of parameter n to $n \geq 2k + 1 + \delta_{2,q}$ for t-intersecting Hilton-Milner theorem for vector spaces.

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