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The perturbations of two general equations in several variables on quasi-Banach spaces

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Abstract. In this paper, we study the stability of two general multi-variable functional equations on quasi-Banach spaces by the fixed point method. We also obtain generalizations of the early results on the stability of some classical equations in quasi-Banach spaces.

1. Introduction

The stability of functional equations has been interested by many authors [4], [10], [9]. This issue is related to finding approximate solutions of some equations and the size of the difference between such approximate solutions and the mappings that satisfy the equation exactly.

The stability of functional equations has been developed in a variety of spaces. Recently, the authors have been interested in the stability of functional equations in quasi-Banach spaces and their generalizations [10]. A quasi-Banach space is a generalization of Banach space and it is also a particular case of a (q_1, q_2)-quasimetric space. For further developments and open problems relating to (q_1, q_2)-quasimetric spaces, the reader may refer to [2]. The main difference is the constant of triangle inequality greater or equal to 1 on quasi-Banach spaces while that in Banach spaces is equal to 1. This implies that quasi-Banach spaces have other different properties from Banach spaces such as the discontinuity of quasi-norm, and the absence of the Hahn-Bannach theorem [11].

Recently, Cieplinski [6] dealt with the perturbations of two general equations in several variables and gave some results on the stability of the following equations in Banach spaces. Also, the author generalized the results from Banach spaces to 2-Banach spaces [5] and m-Banach spaces [7].

The first equation is

$$g(a_{11}x_{11} + a_{12}x_{12}, \ldots, a_{n1}x_{n1} + a_{n2}x_{n2}) - \sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,i_2,\ldots,i_n} g(x_{1i_1}, \ldots, x_{ni_n}) = 0. \quad (1)$$

Note that, every n-linear mapping is a solution of (1).

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The second equation is

$$\sum_{j_1,\dots,j_n\in\{-1,1\}} g(a_{1,j_1,\dots,j_n}(x_{11}+j_1x_{12}),\dots,a_{n,j_1,\dots,j_n}(x_{n1}+j_nx_{n2})) - \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} g(x_{1i_1},\dots,x_{ni_n}) = 0. \quad (2)$$

where

1. X and Y are two vector spaces over the fields $\mathbb{F}$ and $\mathbb{K}$, respectively.
2. $a_{11}, a_{12},\dots,a_{n1},a_{n2} \in \mathbb{F}$, $a_{1,j_1,\dots,j_n},\dots,a_{n,j_1,\dots,j_n} \in \mathbb{F}$ for all $j_1,\dots,j_n \in \{-1,1\}$ and $A_{i_1,i_2,\dots,i_n} \in \mathbb{K}$ for all $i_1,i_2,\dots,i_n \in \{1,2\}$ are given scalars and $g : X^n \to Y$.

Throughout the paper, let

$$A = \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,i_2,\dots,i_n},\ A \neq 0,$$

$$a_i = a_{i1} + a_{i2},\ i \in \{1,2,\dots,n\}.$$

The followings are main results of [6].

Theorem 1.1 ([6], Theorem 3). *Suppose that*

1. *X is a vector space over the field $\mathbb{F}$ and $(Y, \|.\|)$ is a Banach space over the field $\mathbb{K}$.*
2. *There exist $L \in (0,1)$ and $\xi : X^{2n} \to [0,\infty)$ satisfying for all $(x_{11},x_{12},\dots,x_{n1},x_{n2}) \in X^{2n}$,*

$$\lim_{j\to\infty} \frac{1}{|A|^j} \xi(a_1^j x_{11}, a_1^j x_{12},\dots,a_n^j x_{n1}, a_n^j x_{n2}) = 0$$

and for all $(x_{11},\dots,x_{n1}) \in X^n$,

$$\xi(a_1x_{11}, a_1x_{11},\dots,a_nx_{n1},a_nx_{n1}) \le |A|L.\xi(x_{11},x_{11},\dots,x_{n1},x_{n1}).$$

3. *The mapping $g : X^n \to Y$ satisfies for all $(x_{11},x_{12},\dots,x_{n1},x_{n2}) \in X^{2n}$,*

$$\left\| g(a_{11}x_{11}+a_{12}x_{12},\dots,a_{n1}x_{n1}+a_{n2}x_{n2}) - \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,i_2,\dots,i_n} g(x_{1i_1},\dots,x_{ni_n}) \right\| \le \xi(x_{11},x_{12},\dots,x_{n1},x_{n2}).$$

Then there exists a unique solution $G : X^n \to Y$ of equation (1) such that for all $(x_1,\dots,x_n) \in X^n$,

$$\|g(x_1,\dots,x_n) - G(x_1,\dots,x_n)\| \le \frac{1}{|A|(1-L)} \xi(x_1,x_1,\dots,x_n,x_n).$$

The mapping G is defined by

$$G(x_1,\dots,x_n) = \lim_{j\to\infty} \frac{g(a_1^j x_1,\dots,a_n^j x_n)}{A^j},\quad (x_1,\dots,x_n) \in X^n.$$

Theorem 1.2 ([6], Theorem 8). *Suppose that*

1. *X is a vector space over the field $\mathbb{F}$ and $(Y, \|.\|)$ is a Banach space over the field $\mathbb{K}$.*
2. *There exist $L \in (0,1)$ and $\xi : X^{2n} \to [0,\infty)$ satisfying for all $(x_{11},x_{12},\dots,x_{n1},x_{n2}) \in X^{2n}$,*

$$\lim_{j\to\infty} \frac{1}{|A|^j} \xi\Big((2.a_{1,1,\dots,1})^j x_{11}, (2.a_{1,1,\dots,1})^j x_{12},\dots,(2.a_{n,1,\dots,1})^j x_{n1}, (2.a_{n,1,\dots,1})^j x_{n2}\Big) = 0$$

and for all $(x_{11},\dots,x_{n1}) \in X^n$,

$$\xi(2.a_{1,1,\dots,1}x_{11}, 2.a_{1,1,\dots,1}x_{11},\dots,2.a_{n,1,\dots,1}x_{n1}, 2.a_{n,1,\dots,1}x_{n1}) \le |A|L.\xi(x_{11},x_{11},\dots,x_{n1},x_{n1}).$$

3. *The mapping* $g : X^n \to Y$ *satisfies* $g(x_1, x_2, \dots, x_n) = 0$ *for all* $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \in X^n$ *and for all* $(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in X^{2n}$,

$$\Big\| \sum_{j_1,\dots,j_n \in \{-1,1\}} g(a_{1,j_1,\dots,j_n}(x_{11} + j_1 x_{12}), \dots, a_{n,j_1,\dots,j_n}(x_{n1} + j_n x_{n2})) - \sum_{i_1,\dots,i_n \in \{1,2\}} A_{i_1,\dots i_n} g(x_{1i_1}, \dots, x_{ni_n}) \Big\| \le \xi(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}).$$

Then there exists a unique solution $G : X^n \to Y$ *of equation* (2) *such that for all* $(x_1, \dots, x_n) \in X^n$,

$$\|g(x_1, \dots, x_n) - G(x_1, \dots, x_n)\| \le \frac{1}{|A|(1-L)} \xi(x_1, x_1, \dots, x_n, x_n).$$

The mapping G is defined by

$$G(x_1, \dots, x_n) = \lim_{j\to\infty} \frac{g\Big((2.a_{1,1,\dots,1})^j x_1, \dots, (2.a_{n,1,\dots,1})^j x_n\Big)}{A^j}, \quad (x_1, \dots, x_n) \in X^n$$

and $G(x_1, x_2, \dots, x_n) = 0$ *for all* $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \in X^n$.

In this paper, we study the stability of two general multi-variable functional equations on quasi-Banach spaces by the fixed point method. Furthermore, we also obtain generalizations of the early results on the stability of classical equations in quasi-Banach spaces.

First, we recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.3 ([1], Definition 3; [12], pages 6-7). *Let X be a vector space over the field* $\mathbb{K}$, $\kappa \ge 1$ *and* $\|.\| : X \to [0, \infty)$ *be a function such that for all* $x, y \in X$, $\lambda \in \mathbb{K}$,

1. $\|x\| = 0$ *if and only if* $x = 0$.
2. $\|\lambda x\| = |\lambda|.\|x\|$.
3. $\|x + y\| \le \kappa\Big(\|x\| + \|y\|\Big)$.

Then we have

1. $\|.\|$ *is called a quasi-norm on X, the possible smallest* κ *is called the modulus of concavity, and* $(X, \|.\|, \kappa)$ *is called a quasi-normed space. For a quasi-normed space* $(X, \|.\|, \kappa)$, *without loss of the generality we can assume that* κ *is the modulus of concavity.*
2. $\|.\|$ *is called a p-norm, and* $(X, \|.\|, \kappa)$ *is called a p-normed space if there exists* $p \in (0, 1]$ *such that for all* $x, y \in X$,
$$\|x + y\|^p \le \|x\|^p + \|y\|^p.$$
3. *The sequence* $\{x_n\}$ *is called convergent to x if* $\lim_{n\to\infty} \|x_n - x\| = 0$, *written by* $\lim_{n\to\infty} x_n = x$.
4. *The sequence* $\{x_n\}$ *is called Cauchy if* $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$.
5. *The quasi-normed space* $(X, \|.\|, \kappa)$ *is called a quasi-Banach space if each Cauchy sequence is convergent.*
6. *The quasi-normed space* $(X, \|.\|, \kappa)$ *is called a p-Banach space if it is p-normed and quasi-Banach.*

The following is the well-known theorem called the Aoki-Rolewicz theorem. It shows that each quasi-norm is uniformly equivalent to some p-norm.

Theorem 1.4 ([14], Theorem 1). *Let* $(X, \|.\|, \kappa)$ *be a quasi-normed space,* $p = \log_{2\kappa} 2$ *and for all* $x \in X$,

$$|||x||| = \inf\Big\{\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}} : x = \sum_{i=1}^{n} x_i, x_i \in X, n \ge 1\Big\}.$$

Then $|||.|||$ *is a quasi-norm on* X *satisfying for all* $x, y \in X$,

$$|||x+y|||^p \leq |||x|||^p + |||y|||^p$$

and

$$\frac{1}{2\kappa}||x|| \leq |||x||| \leq ||x||.$$

In particular, the quasi-norm $|||.|||$ *is a p-norm, and if* $||.||$ *is a norm then* $p = 1$ *and* $|||.||| = ||.||$.

In 2018, Aydi *et al.* [3] introduced the generalized b-metric space. This notion is a generalization of the b-metric space [8] and the generalized metric space [13].

Definition 1.5 ([3], page 1). *Let* X *be a nonempty set,* $\kappa \geq 1$ *and* $d : X \to [0, \infty]$ *be a function such that for all* $x, y, z \in X$,

1. $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq \kappa\Big(d(x, z) + d(z, y)\Big)$.

Then we have

1. d *is called a generalized b-metric on* X*, the possible smallest* κ *is called the modulus of concavity, and* (X, d, κ) *is called a generalized b-metric space.*
2. *The sequence* $\{x_n\}$ *is called convergent to* x *if* $\lim\limits_{n\to\infty} d(x_n, x) = 0$*, written by* $\lim\limits_{n\to\infty} x_n = x$.
3. *The sequence* $\{x_n\}$ *is called Cauchy if* $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
4. *The generalized b-metric space* (X, d, κ) *is called a complete generalized b-metric space if each Cauchy sequence is convergent.*

The authors also presented the following fixed point theorem for nonlinear contractions on generalized b-metric spaces.

Theorem 1.6 ([3], Theorem 3.1). *Suppose that*

1. (X, d, κ) *is a complete generalized b-metric space.*
2. *The function* $\varphi : [0, \infty) \to [0, \infty)$ *is non-decreasing and for all* $t > 0$,
$$\lim_{n\to\infty} \varphi^n(t) = 0.$$
3. *The mapping* $T : X \to X$ *satisfies for all* $x, y \in X$ *and* $d(x, y) < \infty$,
$$d(Tx, Ty) \leq \varphi(d(x, y)).$$

Then for each $x \in X$*, we have*

(1) *either for every* $n \in \mathbb{N} \cup \{0\}$, $d(T^n x, T^{n+1} x) = \infty$
(2) *or there exists* $n_0 \in \mathbb{N} \cup \{0\}$ *such that* $d(T^{n_0} x, T^{n_0+1} x) < \infty$*. In this case, the following hold*
  (a) $\lim\limits_{n\to\infty} T^n x = x^*$*, where* x^* *is a fixed point of* T.
  (b) x^* *is a unique fixed point of* T *in the set* $X^* = \{y \in X : d(T^{n_0} x, y) < \infty\}$.
  (c) *For all* $y \in X^*$, $\lim\limits_{n\to\infty} T^n y = x^*$.

Moreover, if d *is continuous with respect to one variable and for all* $t > 0$,

$$\sum_{n=1}^{\infty} \kappa^n \varphi^n(t) < \infty,$$

then for all $y \in X^*$, $n \in \mathbb{N} \cup \{0\}$,

$$d(T^n y, x^*) \leq \sum_{k=0}^{\infty} \kappa^{k+1} \varphi^{n+k}(d(y, Ty)).$$

Recently, Sintunavarat *et al.* [15] also proved the fixed point theorem in a particular form and proposed an approximation where d is not necessarily continuous. We must say that, from the proof of [15, Theorem 2.2], the value L in [15, Theorem 2.2.(2).(b)] is exactly L^p as in (3) as follows.

Theorem 1.7 ([15], Theorem 2.2). *Suppose that*

1. (X, d, κ) *is a complete generalized b-metric space.*
2. *The mapping* $T : X \to X$ *satisfies for all* $x, y \in X$ *and some* $L \in [0, 1)$,

$$d(Tx, Ty) \le L.d(x, y).$$

Then for each $x \in X$, *we have*

(1) *either* $d(T^n x, T^{n+1}x) = \infty$ *for* $n \in \mathbb{N} \cup \{0\}$.

(2) *There exists* N *such that for all* $n > N$, $p = \log_{2\kappa} 2$, *then*

$$d(T^n x, x^*) \le (\frac{4}{1 - L^p})^{\frac{1}{p}} d(T^N x, T^{N+1}x) \tag{3}$$

where x^* *is a fixed point of* T *and* $x^* = \lim_{n\to\infty} T^n x$.

This implies that the approximation at the end of the proof of [15, Theorem 2.2] is as follows.
For all $n > N$, then

$$d(x, x^*) \le (\frac{4}{1 - L^p})^{\frac{1}{p}} d(x, Tx), \ x \in X.$$

2. Main results

First, we present three lemmas utilized to prove the main theorems.

Lemma 2.1. *Suppose that*

1. (X, d, κ) *is a complete generalized b-metric space.*
2. $T : X \to X$ *is a mapping satisfying for all* $x, y \in X$ *and some* $L \in [0, 1)$,

$$d(Tx, Ty) \le L.d(x, y).$$

3. *There exist* $n_0 \in \mathbb{N} \cup \{0\}$ *and* $x_0 \in X$ *such that* $d(T^{n_0+1}x_0, T^{n_0}x_0) < \infty$.

Then we have

(1) T *has a unique fixed point* x^* *in the set* $X^* = \{x \in X : d(T^{n_0}x_0, x) < \infty\}$.

(2) $\lim_{n\to\infty} T^n x_0 = x^*$.

(3) *For each* $x \in X^*$, $p = \log_{2\kappa} 2$,

$$d(x, x^*) \le (\frac{4}{1 - L^p})^{\frac{1}{p}} d(x, Tx).$$

Proof. It follows directly from Theorem 1.6 and Theorem 1.7 by choosing $\varphi(t) = L.t$ for all $t \in [0, \infty)$ and some $L \in [0, 1)$. □

Lemma 2.2. *Suppose that*

(1) X *is a vector space over the field* $\mathbb{F}$ *and* $(Y, \|.\|, \kappa)$ *is a quasi-Banach space over the field* $\mathbb{K}$.

(2) $S = \{g : X^n \to Y\}$ *is a set of all mappings from* X^n *to* Y, $\xi : X^{2n} \to [0, \infty)$ *and* $d : S \to [0, \infty]$ *are two functions such that for all* $g, h \in S$,

$$\begin{aligned} d(g, h) \;\; = \;\; & \inf\{a \in [0, \infty] : \|g(x_1, \dots, x_n) - h(x_1, \dots, x_n)\| \\ & \le a.\xi(x_1, x_1, \dots, x_n, x_n), (x_1, \dots, x_n) \in X^n\} \end{aligned}$$

where $\inf \emptyset = \infty$.

Then d is a generalized b-metric on S and (S,d,κ) is a complete generalized b-metric space.

Proof. For all $g,h \in S$, we have

$$d(g,h) \geq 0,$$
$$d(g,h) = 0 \text{ iff } g = h,$$
$$d(g,h) = d(h,g).$$

We claim that the triangle inequality (3) in Definition 1.5 holds. For all $g,h \in S$, we have

$$\|g(x_1,\ldots,x_n) - h(x_1,\ldots,x_n)\| \leq d(g,h)\xi(x_1,x_1,\ldots,x_n,x_n). \tag{4}$$

By using the triangle inequality in the notion of quasi-Banach space and (4), for all $g,h,u \in S,(x_1,\ldots,x_n) \in X^n$, we obtain

$$\begin{aligned}
&\|g(x_1,\ldots,x_n) - h(x_1,\ldots,x_n)\| \\
\leq\ & \kappa\Big[\|g(x_1,\ldots,x_n) - u(x_1,\ldots,x_n)\| + \|u(x_1,\ldots,x_n) - h(x_1,\ldots,x_n)\|\Big] \\
\leq\ & \kappa\Big[d(g,u)\xi(x_1,x_1,\ldots,x_n,x_n) + d(u,h)\xi(x_1,x_1,\ldots,x_n,x_n)\Big]. \\
\leq\ & \kappa\Big[d(g,u) + d(u,h)\Big]\xi(x_1,x_1,\ldots,x_n,x_n).
\end{aligned}$$

This implies that $d(g,h) \leq \kappa\Big[d(g,u) + d(u,h)\Big]$. Hence, d is a generalized b-metric on S.

Now, suppose that $\{g_k\}$ is a Cauchy sequence in S. Then $\lim\limits_{m,k\to\infty} d(g_k,g_m) = 0$. By using (4), we obtain $\{g_k(x_1,\ldots,x_n)\}$ is a Cauchy sequence in quasi-Banach Y for all $(x_1,\ldots,x_n) \in X^n$. Hence, there exists $g(x_1,\ldots,x_n) \in Y$ such that

$$\lim_{k\to\infty} g_k(x_1,\ldots,x_n) = g(x_1,\ldots,x_n),\ (x_1,\ldots,x_n) \in X^n. \tag{5}$$

For each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(g_k,g_m) \leq \dfrac{\varepsilon}{2\kappa}$ for $k,m \geq n_0$. By using Theorem 1.4 and (4), we obtain

$$\begin{aligned}
|||g_k(x_1,\ldots,x_n) - g_m(x_1,\ldots,x_n)||| &\leq \|g_k(x_1,\ldots,x_n) - g_m(x_1,\ldots,x_n)\| \\
&\leq d(g_k,g_m)\xi(x_1,x_1,\ldots,x_n,x_n) \\
&\leq \frac{\varepsilon}{2\kappa}\xi(x_1,x_1,\ldots,x_n,x_n).
\end{aligned}$$

Taking the limit as $m \to \infty$ in the inequality above and using the continuity of $|||.|||$, we have

$$|||g_k(x_1,\ldots,x_n) - g(x_1,\ldots,x_n)||| \leq \frac{\varepsilon}{2\kappa}\xi(x_1,x_1,\ldots,x_n,x_n)$$

and then

$$\begin{aligned}
\|g_k(x_1,\ldots,x_n) - g(x_1,\ldots,x_n)\| &\leq 2\kappa|||g_k(x_1,\ldots,x_n) - g(x_1,\ldots,x_n)||| \\
&\leq \varepsilon\xi(x_1,x_1,\ldots,x_n,x_n).
\end{aligned}$$

This implies that $d(g_k,g) \leq \varepsilon$ for all $k \geq n_0$. This proves that $\lim\limits_{k\to\infty} g_k = g$ in (S,d,κ). Therefore, (S,d,κ) is complete. □

It follows from (5) that if the Cauchy sequence $\{g_k\}$ in S has the property $g_k(x_1,x_2,\ldots,x_n) = 0$ for all $(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) \in X^n, k \in \mathbb{N}$ then $g_(x_1,x_2,\ldots,x_n) = 0$ for all $(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) \in X^n$. We obtain the following lemma by proving similarly to Lemma 2.2.

Lemma 2.3. *Suppose that*

(1) *X is a vector space over the field $\mathbb{F}$ and $(Y, \|.\|, \kappa)$ is a quasi-Banach space over the field $\mathbb{K}$.*

(2) *$\xi : X^{2n} \to [0, \infty)$ is a function, S is a set all mappings such that*

$$S = \{g : X^n \to Y : g(x_1, x_2, \dots, x_n) = 0 \text{ for all } (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \in X^n\}$$

and $d : S \to [0, \infty]$ is a function defined by for all $g, h \in S$,

$$\begin{aligned} d(g, h) &= \inf\{a \in [0, \infty] : \|g(x_1, \dots, x_n) - h(x_1, \dots, x_n)\| \\ &\leq a.\xi(x_1, x_1, \dots, x_n, x_n), (x_1, \dots, x_n) \in X^n\} \end{aligned}$$

where $\inf \emptyset = \infty$.

Then d is a generalized b-metric on S and (S, d, κ) is a complete generalized b-metric space.

In the following, we use Theorem 1.4, Lemma 2.1, Lemma 2.2 and Lemma 2.3 to investigate the stability of the multi-variable functional equations on quasi-Banach spaces. Firstly, we study the stability of the functional equation (1).

Theorem 2.4. *Suppose that*

(1) *X is a vector space over the field $\mathbb{F}$ and $(Y, \|.\|, \kappa)$ is a quasi-Banach space over the field $\mathbb{K}$.*

(2) *There exist $L \in [0, 1)$ and $\xi : X^{2n} \to [0, \infty)$ satisfying for all $(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in X^{2n}$,*

$$\lim_{j\to\infty} \frac{1}{|A|^j} \xi(a_1^j x_{11}, a_1^j x_{12}, \dots, a_n^j x_{n1}, a_n^j x_{n2}) = 0 \tag{6}$$

and for all $(x_{11}, \dots, x_{n1}) \in X^n$,

$$\xi(a_1 x_{11}, a_1 x_{11}, \dots, a_n x_{n1}, a_n x_{n1}) \leq |A| L.\xi(x_{11}, x_{11}, \dots, x_{n1}, x_{n1}). \tag{7}$$

(3) *The mapping $g : X^n \to Y$ satisfies for all $(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in X^{2n}$,*

$$\begin{aligned} &\left\| g(a_{11}x_{11} + a_{12}x_{12}, \dots, a_{n1}x_{n1} + a_{n2}x_{n2}) - \sum_{i_1,\dots,i_n \in \{1,2\}} A_{i_1,i_2,\dots,i_n} g(x_{1i_1}, \dots, x_{ni_n}) \right\| \\ \leq\ & \xi(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}). \end{aligned} \tag{8}$$

Then there exists a unique solution $G : X^n \to Y$ of equation (1) such that for each $(x_1, \dots, x_n) \in X^n$ and $p = \log_{2\kappa} 2$,

$$\|g(x_1, \dots, x_n) - G(x_1, \dots, x_n)\| \leq \frac{1}{|A|} \left(\frac{4}{1 - L^p} \right)^{\frac{1}{p}} \xi(x_1, x_1, \dots, x_n, x_n).$$

The mapping G is defined by

$$G(x_1, \dots, x_n) = \lim_{j\to\infty} \frac{g(a_1^j x_1, \dots, a_n^j x_n)}{A^j},\ (x_1, \dots, x_n) \in X^n.$$

Proof. Let $S = \{g : X^n \to Y\}$ and

$$\begin{aligned} d(g, h) &= \inf\{a \in [0, \infty] : \|g(x_1, \dots, x_n) - h(x_1, \dots, x_n)\| \\ &\leq a\xi(x_1, x_1, \dots, x_n, x_n), (x_1, \dots, x_n) \in X^n\} \end{aligned}$$

for all $g, h \in S$, where $\inf \emptyset = \infty$. It follows from Lemma 2.2 that d is a generalized b-metric on S and (S, d, κ) is a complete generalized b-metric space.

Let $T : S \to S$ be defined by

$$Tg(x_1, \dots, x_n) = \frac{1}{|A|} g(a_1 x_1, \dots, a_n x_n),\ g \in S, (x_1, \dots, x_n) \in X^n. \tag{9}$$

Choosing $x_{i2} = x_{i1}, i \in \{1, 2, \dots, n\}$ in (8), we obtain

$$\begin{aligned}&\Big\| g\Big((a_{11}+a_{12})x_{11},\dots,(a_{n1}+a_{n2})x_{n1}\Big) - \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,i_2,\dots,i_n} g(x_{11},\dots,x_{n1})\Big\| \\ \leq\ & \xi(x_{11},x_{11},\dots,x_{n1},x_{n1}). \end{aligned} \tag{10}$$

Replacing $a_i = a_{i1} + a_{i2},\ i \in \{1, 2, \dots, n\}$ and $A = \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,i_2,\dots,i_n},\ A \neq 0$ to (10), we have

$$\Big\| g\Big(a_1x_{11},\dots,a_nx_{n1}\Big) - Ag(x_{11},\dots,x_{n1})\Big\| \leq \xi(x_{11},x_{11},\dots,x_{n1},x_{n1}).$$

This implies that

$$\Big\|\frac{1}{A}g(a_1x_{11},\dots,a_nx_{n1}) - g(x_{11},\dots,x_{n1})\Big\| \leq \frac{1}{|A|}\xi(x_{11},x_{11},\dots,x_{n1},x_{n1}). \tag{11}$$

Applying (9) to (11), we have

$$\|Tg(x_1,\dots,x_n) - g(x_1,\dots,x_n)\| \leq \frac{1}{|A|}\xi(x_1,x_1\dots,x_n,x_n),\ (x_1,\dots,x_n)\in X^n.$$

This implies that

$$d(Tg,g) \leq \frac{1}{|A|} < \infty.$$

We claim that $d(Tg, Th) \leq L.d(g, h)$ for all $g, h \in S$. Indeed, by using definition of T, (4) and (7), we have

$$\begin{aligned}\|Tg(x_1,\dots,x_n) - Th(x_1,\dots,x_n)\| &= \frac{1}{|A|}\Big\|g(a_1x_1,\dots,a_nx_n) - h(a_1x_1,\dots,a_nx_n)\Big\| \\ &\leq \frac{1}{|A|}d(g,h)\xi(a_1x_1,a_1x_1,\dots,a_nx_n,a_nx_n) \\ &\leq L.d(g,h)\xi(x_1,x_1,\dots,x_n,x_n).\end{aligned}$$

This implies that

$$d(Tg,Th) \leq L.d(g,h).$$

Hence, all the assumptions of Lemma 2.1 are satisfied. This implies that

$$\lim_{k\to\infty} T^k g(x_1,\dots,x_n) = G(x_1,\dots,x_n), \tag{12}$$

where G is a fixed point of T, $p = \log_{2\kappa} 2$ and

$$d(g,G) \leq \Big(\frac{4}{1-L^p}\Big)^{\frac{1}{p}} d(g,Tg) \leq \frac{1}{|A|}\Big(\frac{4}{1-L^p}\Big)^{\frac{1}{p}}. \tag{13}$$

Using the definition of d and (13), we obtain

$$\begin{aligned}\|G(x_1,\dots,x_n) - g(x_1,\dots,x_n)\| &\leq d(g,G)\xi(x_1,x_1,\dots,x_n,x_n) \\ &\leq \frac{1}{|A|}\Big(\frac{4}{1-L^p}\Big)^{\frac{1}{p}}\xi(x_1,x_1,\dots,x_n,x_n).\end{aligned} \tag{14}$$

Now, we prove the following formulation by induction,

$$T^j g(x_1,\dots,x_n) = \frac{1}{A^j} g(a_1^j x_1,\dots,a_n^j x_n) \tag{15}$$

for all $(x_1, \dots, x_n) \in X^n$, $j \in \mathbb{N}$. The formulation holds with $j = 1$ by the definition of T. Suppose that (15) holds with some $j \geq 1$. We prove (15) holds with $j + 1$. Indeed,

$$\begin{aligned} T^{j+1}g(x_1, \dots, x_n) &= \frac{1}{A} T^j g(a_1x_1, \dots, a_nx_n) \\ &= \frac{1}{A} . \frac{1}{A^j} g(a_1^j a_1 x_1, \dots, a_n^j a_n x_n) \\ &= \frac{1}{A^{j+1}} g(a_1^{j+1} x_1, \dots, a_n^{j+1} x_n). \end{aligned}$$

Hence, (15) holds for all $j \in \mathbb{N}$. Then taking the limit as $j \to \infty$ in (15) and using (12), we obtain

$$G(x_1, \dots, x_n) = \lim_{j\to\infty} \frac{g(a_1^j x_1, \dots, a_n^j x_n)}{A^j}, \ (x_1, \dots, x_n) \in X^n. \tag{16}$$

We claim that G is a solution of equation (1). Indeed, using Theorem 1.4 and the assumption (8), we obtain for all $(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in X^{2n}$ and $j \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} &\Bigg|\Bigg|\Bigg| \frac{1}{A^j} g(a_1^j(a_{11}x_{11} + a_{12}x_{12}), \dots, a_n^j(a_{n1}x_{n1} + a_{n2}x_{n2})) \\ &\qquad - \frac{1}{A^j} \sum_{i_1,\dots,i_n \in \{1,2\}} A_{i_1,i_2,\dots,i_n} g(a_1^j x_{1i_1}, \dots, a_n^j x_{ni_n}) \Bigg|\Bigg|\Bigg| \\ \leq\ & \Bigg|\Bigg| \frac{1}{A^j} g(a_1^j(a_{11}x_{11} + a_{12}x_{12}), \dots, a_n^j(a_{n1}x_{n1} + a_{n2}x_{n2})) \\ &\qquad - \frac{1}{A^j} \sum_{i_1,\dots,i_n \in \{1,2\}} A_{i_1,i_2,\dots,i_n} g(a_1^j x_{1i_1}, \dots, a_n^j x_{ni_n}) \Bigg|\Bigg| \\ \leq\ & \frac{1}{|A|^j} \xi(a_1^j x_{11}, a_1^j x_{12}, \dots, a_n^j x_{n1}, a_n^j x_{n2}). \end{aligned} \tag{17}$$

Taking the limit as $j \to \infty$ in (17), using (16), (6) and the continuity of $|||.|||$, we obtain

$$\begin{aligned} &\Bigg|\Bigg|\Bigg| G((a_{11}x_{11} + a_{12}x_{12}), \dots, (a_{n1}x_{n1} + a_n x_{n2})) \\ &\qquad - \sum_{i_1,\dots,i_n \in \{1,2\}} A_{i_1,i_2,\dots,i_n} G(x_{1i_1}, \dots, x_{ni_n}) \Bigg|\Bigg|\Bigg| = 0. \end{aligned}$$

Then G satisfies the equation (1).

Suppose that H is also a solution of the equation (1) and satisfies (14). Since H is a solution of the equation (1), we get

$$\frac{1}{A} H(a_1x_1, \dots, a_nx_n) = H(x_1, \dots, x_n),$$

that is, H is a fixed point of T. Since H satisfies (14) we also obtain

$$d(g, H) \leq \frac{1}{|A|} \Big(\frac{4}{1 - L^p} \Big)^{\frac{1}{p}} < \infty.$$

This implies that $H \in X^* = \{h \in S : d(g, h) < \infty\}$. By Lemma 2.1, we have $H = G$. Hence, G is a unique solution of equation (1). □

Choosing $a_{11} = a_{12} = \cdots = a_{n1} = a_{n2} = 1$ and $A_{i_1,i_2,\ldots,i_n} = 1$ for all $i_1, \ldots, i_n \in \{1,2\}$ in Theorem 2.4, we obtain the generalization of [6, Corollary 4] on the generalized Ulam stability of functional equation (18) on quasi-Banach spaces.

$$g(x_{11} + x_{12}, \ldots, x_{n1} + x_{n2}) = \sum_{i_1,\ldots,i_n \in \{1,2\}} g(x_{1i_1}, \ldots, x_{ni_n}). \tag{18}$$

Corollary 2.5. *Suppose that*

(1) *X is a vector space over the field* $\mathbb{F}$ *and* $(Y, \|.\|, \kappa)$ *is a quasi-Banach space over the field* $\mathbb{K}$.

(2) *There exist* $L \in [0,1)$ *and* $\xi : X^{2n} \to [0,\infty)$ *satisfying for all* $(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in X^{2n}$,

$$\lim_{j\to\infty} \frac{1}{2^{jn}} \xi(2^j.x_{11}, 2^j.x_{12}, \ldots, 2^j.x_{n1}, 2^j.x_{n2}) = 0$$

and for all $(x_{11}, \ldots, x_{n1}) \in X^n$,

$$\xi(2.x_{11}, 2.x_{11}, \ldots, 2.x_{n1}, 2.x_{n1}) \leq 2^n L.\xi(x_{11}, x_{11}, \ldots, x_{n1}, x_{n1}).$$

(3) *The mapping* $g : X^n \to Y$ *satisfies for all* $(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in X^{2n}$,

$$\left\| g(x_{11} + x_{12}, \ldots, x_{n1} + x_{n2}) - \sum_{i_1,\ldots,i_n \in \{1,2\}} g(x_{1i_1}, \ldots, x_{ni_n}) \right\| \leq \xi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}).$$

Then there exists a unique solution $G : X^n \to Y$ *of equation* (18) *such that for all* $(x_1, \ldots, x_n) \in X^n$, *and* $p = \log_{2\kappa} 2$,

$$\|g(x_1, \ldots, x_n) - G(x_1, \ldots, x_n)\| \leq \frac{1}{2^n} \left(\frac{4}{1 - L^p} \right)^{\frac{1}{p}} \xi(x_1, x_1, \ldots, x_n, x_n).$$

The mapping G is defined by

$$G(x_1, \ldots, x_n) = \lim_{j\to\infty} \frac{f(2^j.x_1, \ldots, 2^j.x_n)}{2^{jn}}, \qquad (x_1, \ldots, x_n) \in X^n.$$

Remark 2.6. 1. *Choosing* $\xi = \varepsilon > 0$ *and* $L = \frac{1}{|A|}$, $|A| > 1$ *in Theorem 2.4, we obtain a generalization of [6, Corollary 5] on the classical Ulam stability of the functional equation* (1) *on quasi-Banach spaces.*

2. *Choosing* $\xi = \varepsilon > 0$ *and* $L = \frac{1}{2^n}$ *in Corollary 2.5, we obtain a generalization of [6, Corollary 6] on the classical Ulam stability of functional equation* (18).

In the following theorem, we address the stability of the functional equation (2).

Theorem 2.7. *Suppose that*

(1) *X is a vector space over the field* $\mathbb{F}$ *and* $(Y, \|.\|, \kappa)$ *is a quasi-Banach space over the field* $\mathbb{K}$.

(2) *There exist* $L \in [0,1)$ *and* $\xi : X^{2n} \to [0,\infty)$ *satisfying for all* $(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in X^{2n}$,

$$\lim_{j\to\infty} \frac{1}{|A|^j} \xi\Big((2.a_{1,1,\ldots,1})^j x_{11}, (2.a_{1,1,\ldots,1})^j x_{12}, \ldots, (2.a_{n,1,\ldots,1})^j x_{n1}, (2.a_{n,1,\ldots,1})^j x_{n2}\Big) = 0 \tag{19}$$

and for all $(x_{11}, \ldots, x_{n1}) \in X^n$,

$$\xi(2.a_{1,1,\ldots,1} x_{11}, 2.a_{1,1,\ldots,1} x_{11}, \ldots, 2.a_{n,1,\ldots,1} x_{n1}, 2.a_{n,1,\ldots,1} x_{n1}) \leq |A| L.\xi(x_{11}, x_{11}, \ldots, x_{n1}, x_{n1}). \tag{20}$$

(3) *The mapping* $g : X^n \to Y$ *satisfies* $g(x_1, x_2, \ldots, x_n) = 0$ *for all* $(x_1, \ldots, x_{i-1}, 0, x_{i+1} \ldots, x_n) \in X^n$ *and for all* $(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in X^{2n}$,

$$\Big\| \sum_{j_1,\ldots,j_n \in \{-1,1\}} g(a_{1,j_1,\ldots,j_n}(x_{11} + j_1 x_{12}), \ldots, a_{n,j_1,\ldots,j_n}(x_{n1} + j_n x_{n2})) - \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots i_n} g(x_{1i_1}, \ldots, x_{ni_n}) \Big\| \leq \xi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}). \tag{21}$$

Then there exists a unique solution $G : X^n \to Y$ *of the equation* (2) *such that for all* $(x_1, \ldots, x_n) \in X^n$ *and* $p = \log_{2\kappa} 2$,

$$\|g(x_1, \ldots, x_n) - G(x_1, \ldots, x_n)\| \leq \frac{1}{|A|} \Big(\frac{4}{1 - L^p} \Big)^{\frac{1}{p}} \xi(x_1, x_1, \ldots, x_n, x_n).$$

The mapping G is defined by

$$G(x_1, \ldots, x_n) = \lim_{j \to \infty} \frac{g\Big((2.a_{1,1,\ldots,1})^j x_1, \ldots, (2.a_{n,1,\ldots,1})^j x_n\Big)}{A^j}, \ (x_1, \ldots, x_n) \in X^n$$

and $G(x_1, x_2, \ldots, x_n) = 0$ *for all* $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \in X^n$.

Proof. Let

$$S = \{g : X^n \to Y : g(x_1, x_2, \ldots, x_n) = 0 \text{ for all } (x_1, \ldots, x_{i-1}, 0, x_{i+1} \ldots, x_n) \in X^n\}$$

and d be defined by

$$d(g, h) = \inf\{a \in [0, \infty] : \|g(x_1, \ldots, x_n) - h(x_1, \ldots, x_n)\| \leq a.\xi(x_1, x_1, \ldots, x_n, x_n), (x_1, \ldots, x_n) \in X^n\}$$

for all $g, h \in S$, where $\inf \emptyset = \infty$. It follows from Lemma 2.3 that d is a generalized b-metric in S and (S, d, κ) is a complete generalized b-metric space.

Next, replacing $x_{i1} = x_{i2} = x_i, \ i \in \{1, 2, \ldots, n\}$ to (21), we obtain for all $(x_1, \ldots, x_n) \in X^n$,

$$\Big\| \sum_{j_1,\ldots,j_n \in \{-1,1\}} g(a_{1,j_1,\ldots,j_n}(x_1 + j_1 x_1), \ldots, a_{n,j_1,\ldots,j_n}(x_n + j_n x_n)) - \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots i_n} g(x_1, \ldots, x_n) \Big\| \leq \xi(x_1, x_1, \ldots, x_n, x_n). \tag{22}$$

Replacing $A = \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,i_2,\ldots,i_n}$, $A \neq 0$ to (22) and using $g(x_1, x_2, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_{i-1}, 0, x_{i+1} \ldots, x_n) \in X^n$, we have

$$\Big\| g(2.a_{1,1,\ldots,1} x_1, \ldots, 2.a_{n,1,\ldots,1} x_n) - A g(x_1, \ldots, x_n) \Big\| \leq \xi(x_1, x_1 \ldots, x_n, x_n).$$

This implies that

$$\Big\| \frac{1}{A} g(2.a_{1,1,\ldots,1} x_1, \ldots, 2.a_{n,1,\ldots,1} x_n) - g(x_1, \ldots, x_n) \Big\| \leq \frac{1}{|A|} \xi(x_1, x_1 \ldots, x_n, x_n). \tag{23}$$

Let $T : S \to S$ be defined by

$$Tg(x_1, \ldots, x_n) = \frac{1}{|A|} g(2.a_{1,1,\ldots,1} x_1, \ldots, 2.a_{n,1,\ldots,1} x_n), \ (x_1, \ldots, x_n) \in X^n.$$

Then (23) becomes

$$\|Tg(x_1,\dots,x_n)-g(x_1,\dots,x_n)\|\le\frac{1}{|A|}\xi(x_1,x_1\dots,x_n,x_n)$$

for all $(x_1,\dots,x_n)\in X^n$. This implies that

$$d(Tg,g)\le\frac{1}{|A|}<\infty.$$

Now, we claim that $d(Tg,Th)\le L.d(g,h)$ for all $g,h\in S$. Indeed, we have for all $g,h\in S$, $(x_1,\dots,x_n)\in X^n$,

$$\|g(x_1,\dots,x_n)-h(x_1,\dots,x_n)\|\le d(g,h)\xi(x_1,x_1,\dots,x_n,x_n). \tag{24}$$

Using definition of T, (20) and (24), we obtain

$$\begin{aligned}
&\|Tg(x_1,\dots,x_n)-Th(x_1,\dots,x_n)\|\\
=\;&\frac{1}{|A|}\Big\|g(2.a_{1,1,\dots,1}x_1,\dots,2.a_{n,1,\dots,1}x_n)-h(2.a_{1,1,\dots,1}x_1,\dots,2.a_{n,1,\dots,1}x_n)\Big\|\\
\le\;&\frac{1}{|A|}d(g,h)\xi(2.a_{1,1,\dots,1}x_1,2.a_{1,1,\dots,1}x_1,\dots,2.a_{n,1,\dots,1}x_n,2.a_{n,1,\dots,1}x_n)\\
\le\;&L.d(g,h)\xi(x_1,x_1,\dots,x_n,x_n).
\end{aligned}$$

This implies that

$$d(Tg,Th)\le L.d(g,h).$$

Hence, all the assumptions of Lemma 2.1 are satisfied. This implies that

$$\lim_{k\to\infty}T^kg(x_1,\dots,x_n)=G(x_1,\dots,x_n),\qquad(x_1,\dots,x_n)\in X^n, \tag{25}$$

where G is a fixed point of T and

$$d(G,g)\le\Big(\frac{4}{1-L^p}\Big)^{\frac{1}{p}}d(g,Tg)\le\frac{1}{|A|}\Big(\frac{4}{1-L^p}\Big)^{\frac{1}{p}}. \tag{26}$$

By using the definition of d and (26), we obtain for all $(x_1,\dots,x_n)\in X^n$,

$$\begin{aligned}
\|G(x_1,\dots,x_n)-g(x_1,\dots,x_n)\|&\le d(g,G)\xi(x_1,x_1\dots,x_n,x_n)\\
&\le\frac{1}{|A|}\Big(\frac{4}{1-L^p}\Big)^{\frac{1}{p}}\xi(x_1,x_1\dots,x_n,x_n).
\end{aligned} \tag{27}$$

Now, we prove the following formulation by using induction,

$$T^jg(x_1,\dots,x_n)=\frac{1}{A^j}g\Big((2.a_{1,1,\dots,1})^jx_1,\dots,(2.a_{n,1,\dots,1})^jx_n\Big) \tag{28}$$

for all $(x_1,\dots,x_n)\in X^n$, $j\in\mathbb{N}$. It follows from the definition of T that (28) holds with $j=1$. Suppose that (28) holds with some $j\ge1$, we prove (28) holds with $j+1$. Indeed, we have

$$\begin{aligned}
T^{j+1}g(x_1,\dots,x_n)&=\frac{1}{A}T^jg\Big(2.a_{1,1,\dots,1}x_1,\dots,2.a_{n,1,\dots,1}x_n\Big)\\
&=\frac{1}{A}.\frac{1}{A^j}g\Big((2.a_{1,1,\dots,1})^j2.a_{1,1,\dots,1}x_1,\dots,(2.a_{n,1,\dots,1})^j2.a_{n,1,\dots,1}x_n\Big)\\
&=\frac{1}{A^{j+1}}g((2.a_{1,1,\dots,1})^{j+1}x_1,\dots,(2.a_{n,1,\dots,1})^{j+1}x_n).
\end{aligned}$$

Hence, (28) holds for all $(x_1, \dots, x_n) \in X^n$, for all $j \in \mathbb{N}$. Then taking the limit as $j \to \infty$ in (28) and by using (25), we obtain

$$G(x_1, \dots, x_n) = \lim_{j\to\infty} \frac{g\Big((2.a_{1,1,\dots,1})^j x_1, \dots, (2.a_{n,1,\dots,1})^j x_n\Big)}{A^j}, \ (x_1, \dots, x_n) \in X^n. \tag{29}$$

Moreover, G is a fixed point of T so $G \in S$ and hence $G(x_1, x_2, \dots, x_n) = 0$ for all $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \in X^n$.

Now, we claim that G is a solution of equation (2). By applying Theorem 1.4 and the assumption (21), we obtain

$$\begin{aligned}
&\Bigg|\Bigg|\Bigg| \sum_{j_1,\dots,j_n\in\{-1,1\}} \frac{1}{A^j} g\Big((2.a_{1,1,\dots,1})^j a_{1,j_1,\dots,j_n}(x_{11} + j_1 x_{12}), \dots, (2.a_{n,1,\dots,1})^j a_{n,j_1,\dots,j_n}(x_{n1} + j_n x_{n2})\Big) \\
&\qquad - \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots i_n} \frac{1}{A^j} g((2.a_{1,1,\dots,1})^j x_{1i_1}, \dots, (2.a_{n,1,\dots,1})^j x_{ni_n})\Bigg|\Bigg|\Bigg| \\
\leq\ &\Bigg\| \sum_{j_1,\dots,j_n\in\{-1,1\}} \frac{1}{A^j} g\Big((2.a_{1,1,\dots,1})^j a_{1,j_1,\dots,j_n}(x_{11} + j_1 x_{12}), \dots, (2.a_{n,1,\dots,1})^j a_{n,j_1,\dots,j_n}(x_{n1} + j_n x_{n2})\Big) \\
&\qquad - \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots i_n} \frac{1}{A^j} g((2.a_{1,1,\dots,1})^j x_{1i_1}, \dots, (2.a_{n,1,\dots,1})^j x_{ni_n})\Bigg\| \\
\leq\ &\frac{1}{|A|^j} \xi((2.a_{1,1,\dots,1})^j x_{11}, (2.a_{1,1,\dots,1})^j x_{12}, \dots, (2.a_{n,1,\dots,1})^j x_{n1}, (2.a_{n,1,\dots,1})^j x_{n2}).
\end{aligned} \tag{30}$$

Taking the limit as $j \to \infty$ in (30), using (19), (29) and the continuity of $|||.|||$, we get

$$\Bigg|\Bigg|\Bigg| G(a_{1,j_1,\dots,j_n}(x_{11} + j_1 x_{12}), \dots, a_{n,j_1,\dots,j_n}(x_{n1} + j_n x_{n2})) - \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,i_2,\dots,i_n} G(x_{1i_1}, \dots, x_{ni_n})\Bigg|\Bigg|\Bigg| = 0.$$

This implies that G satisfies the equation (2).

Suppose that H is also a solution of the equation (2) and satisfies (27). Since H is a solution of the equation (2), we claim

$$\frac{1}{A} H(2.a_{1,1,\dots,1} x_1, \dots, 2.a_{n,1,\dots,1} x_n) = H(x_1, \dots, x_n), \qquad (x_1, \dots, x_n) \in X^n,$$

that means H is a fixed point of T. Moreover, H satisfies (27), we obtain

$$d(g, H) \leq \frac{1}{|A|}\Big(\frac{4}{1 - L^p}\Big)^{\frac{1}{p}} < \infty.$$

This implies that $H \in X^* = \{h \in S : d(g, h) < \infty\}$. By Lemma 2.1, we have $H = G$. □

Choosing $a_{1,j_1,\dots,j_n} = \dots = a_{n,j_1,\dots,j_n} = 1$ for all $j_1, \dots, j_n \in \{-1, 1\}$ and $A_{i_1,\dots,i_n} = 2^n$ for all $i_1, \dots, i_n \in \{1, 2\}$ in Theorem 2.7, we obtain the following generalization of [6, Corollary 9] on the the stability of the n-quadratic functional equation (31) on quasi-Banach spaces.

$$\sum_{j_1,\dots,j_n\in\{-1,1\}} g(x_{11} + j_1 x_{12}, \dots, x_{n1} + j_n x_{n2}) - \sum_{i_1,\dots,i_n\in\{1,2\}} 2^n.g(x_{1i_1}, \dots, x_{ni_n}) = 0. \tag{31}$$

Corollary 2.8. *Suppose that*

(1) *X is a vector space over the field* $\mathbb{F}$ *and* $(Y, \|.\|, \kappa)$ *is a quasi-Banach space over the field* $\mathbb{K}$.
(2) *There exist* $L \in [0,1)$ *and* $\xi : X^{2n} \to [0,\infty)$ *satisfying for all* $(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in X^{2n}$,

$$\lim_{j\to\infty} \frac{1}{4^{jn}} \xi\Big(2^j.x_{11}, 2^j.x_{12}, \ldots, 2^j.x_{n1}, 2^j.x_{n2}\Big) = 0$$

and for all $(x_{11}, \ldots, x_{n1}) \in X^n$,

$$\xi(2x_{11}, 2x_{11}, \ldots, 2x_{n1}, 2x_{n1}) \le 4^n L.\xi(x_{11}, x_{11}, \ldots, x_{n1}, x_{n1}).$$

(3) *The mapping* $g : X^n \to Y$ *satisfies* $g(x_1, x_2, \ldots, x_n) = 0$ *for all* $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \in X^n$ *and for all* $(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in X^{2n}$,

$$\begin{aligned}&\Big\| \sum_{j_1,\ldots,j_n \in \{-1,1\}} g(x_{11} + j_1 x_{12}, \ldots, x_{n1} + j_n x_{n2}) - \sum_{i_1,\ldots,i_n \in \{1,2\}} 2^n g(x_{1i_1}, \ldots, x_{ni_n}) \Big\| \\ \le\ & \xi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}).\end{aligned}$$

Then there exists a unique solution $G : X^n \to Y$ *of equation* (31) *such that for all* $(x_1, \ldots, x_n) \in X^n$ *and* $p = \log_{2\kappa} 2$,

$$\|g(x_1, \ldots, x_n) - G(x_1, \ldots, x_n)\| \le \frac{1}{4^n}\Big(\frac{4}{1 - L^p}\Big)^{\frac{1}{p}} \xi(x_1, x_1, \ldots, x_n, x_n).$$

The mapping G is defined by

$$G(x_1, \ldots, x_n) = \lim_{j\to\infty} \frac{g(2^j.x_1, \ldots, 2^j.x_n)}{4^{jn}}, \qquad (x_1, \ldots, x_n) \in X^n$$

and $G(x_1, x_2, \ldots, x_n) = 0$ *for all* $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \in X^n$.

Choosing $n = 2, a_{1,1,1} = a_{2,1,1} = a_{1,-1,-1} = a_{2,-1,-1} = 1, a_{1,1,-1} = a_{2,1,-1} = a_{1,-1,1} = a_{2,-1,1} = 0, A_{1,1} = A_{1,2} = 2$ and $A_{2,1} = A_{2,2} = 0$ in Theorem 2.7, we obtain the following generation of [6, Corollary 12] on the stability of the additive-quadratic functional equation (32) on quasi-Banach spaces.

$$g(x_{11} + x_{12}, x_{21} + x_{22}) + g(x_{11} - x_{12}, x_{21} - x_{22}) = 2g(x_{11}, x_{21}) + 2g(x_{12}, x_{22}). \tag{32}$$

Corollary 2.9. *Suppose that*

(1) *X is a vector space over the field* $\mathbb{F}$ *and* $(Y, \|.\|, \kappa)$ *is a quasi-Banach space over the field* $\mathbb{K}$.
(2) *There exist* $L \in [0,1)$ *and* $\xi : X^4 \to [0,\infty)$ *satisfying for all* $(x_{11}, x_{12}, x_{21}, x_{22}) \in X^4$,

$$\lim_{j\to\infty} \frac{1}{4^j} \xi\Big(2^j.x_{11}, 2^j.x_{12}, 2^j.x_{21}, 2^j.x_{22}\Big) = 0$$

and $(x_{11}, x_{21}) \in X^2$,

$$\xi(2x_{11}, 2x_{11}, 2x_{21}, 2x_{21}) \le 4L.\xi(x_{11}, x_{11}, x_{21}, x_{21}).$$

(3) *The mapping* $g : X^2 \to Y$ *satisfies* $g(x_1, 0) = 0 = g(0, x_2)$ *for all* $(x_1, x_2) \in X^2$ *and for all* $(x_{11}, x_{12}, x_{21}, x_{22}) \in X^4$,

$$\begin{aligned}&\Big\| g(x_{11} + x_{12}, x_{21} + x_{22}) + g(x_{11} - x_{12}, x_{21} - x_{22}) - 2g(x_{11}, x_{21}) - 2g(x_{12}, x_{22}) \Big\| \\ \le\ & \xi(x_{11}, x_{12}, x_{21}, x_{22}).\end{aligned}$$

Then there exists a unique solution $G : X^2 \to Y$ of the following equation (32) *and satisfying for all* $(x_1, x_2) \in X^2$ *and* $p = \log_{2\kappa} 2$,

$$\|g(x_1, x_2) - G(x_1, x_2)\| \le \frac{1}{4}\Big(\frac{4}{1 - L^p}\Big)^{\frac{1}{p}} \xi(x_1, x_1, x_2, x_2).$$

The mapping G is defined by

$$G(x_1, x_2) = \lim_{j\to\infty} \frac{g(2^j.x_1, 2^j.x_2)}{4^j},\ (x_1, x_2) \in X^2$$

and $G(x_1, 0) = 0 = G(0, x_2)$ *for all* $(x_1, x_2) \in X^2$.

Remark 2.10. 1. *Choosing* $\xi = \varepsilon > 0$ *and* $L = \frac{1}{|A|}$, $|A| > 1$ *in Theorem 2.7, we obtain a generalization of [6, Corollary 10] on the classical Ulam stability of the functional equation* (2) *on quasi-Banach spaces.*

2. *Choosing* $\xi = \varepsilon > 0$ *and* $L = \frac{1}{4^n}$ *in Corollary 2.8, we obtain a generalization of [6, Corollary 11] on the classical Ulam stability of the functional equation* (31) *on quasi-Banach spaces.*
3. *Choosing* $\xi = \varepsilon > 0$ *and* $L = \frac{1}{4}$ *in Corollary 2.9, we obtain a generalization of [6, Corollary 13] on the classical Ulam stability of the functional equation* (32) *on quasi-Banach spaces.*
4. *Choosing* $n = 1$ *in Corollary 2.8, we have the generalization of the stability of the quadratic functional equations on quasi-Banach spaces.*

The above results immediately imply the results of [6] in the Banach spaces with $\kappa = 1$. The next example gives a proper illustration on quasi-Banach spaces. This example also exemplifies a limitation of Theorem 1.1.

Example 2.11. *Suppose that*

1. $0 < p < 1, X = Y = L^p[0, 1] = \Big\{x : [0, 1] \to \mathbb{R}$ *such that* $|x|^p$ *is Lebesgue integrable*$\Big\}$, *and* $\|x\| = \Big(\int_0^1 |x(t)|^p dt\Big)^{\frac{1}{p}}$ *for all* $x \in L^p[0, 1]$.
2. *The mappings* $g : X^2 \to Y$ *and* $\xi : X^4 \to [0, \infty)$ *are defined by*

$$g(x, y)(t) = x(t) + y(t) + 1, (x, y) \in X^2, t \in [0, 1],$$

$$\xi(x, y, z, w) = 1, (x, y, z, w) \in X^4.$$

Then we have

1. *All assumptions of Theorem 1.1 hold, except for the assumption of Y being a Banach space.*
2. *Theorem 2.4 is applicable to* X, Y, g *and* ξ *but Theorem 1.1 is not.*

Proof. (1). It follows from [14, Examples 1 & 2] that Y is a quasi-Banach space with $\kappa = 2$ which is not normable. Then it is not a Banach space.

Choosing $a_{11} = a_{12} = a_{21} = a_{22} = 1, A_{11} = A_{22} = 1, A_{12} = A_{21} = 0, L = \frac{1}{2}$, we get $A = 2$ and $a_1 = a_2 = 2$. For all $(x_{11}, x_{12}, x_{21}, x_{22}) \in X^4$, we have

$$\lim_{j\to\infty} \frac{1}{|A|^j} \xi(a_1^j x_{11}, a_1^j x_{12}, a_2^j x_{21}, a_2^j x_{22}) = \lim_{j\to\infty} \frac{1}{2^j} = 0$$

and for all $(x_{11}, x_{21}) \in X^2$,

$$\xi(a_1 x_{11}, a_1 x_{11}, a_2 x_{21}, a_2 x_{21}) = 1 = |A|L.\xi(x_{11}, x_{11}, \dots, x_{n1}, x_{n1}).$$

We also have

$$\begin{aligned}
& \|g(x_{11}+x_{12},x_{21}+x_{22})-A_{11}g(x_{11},x_{21})-A_{22}g(x_{12},x_{22})-A_{12}g(x_{11},x_{22})-A_{21}g(x_{12},x_{21})\| \\
= & \Big(\int_0^1 |(x_{11}+x_{12})(t)+(x_{21}+x_{22})(t)+1-(x_{11}(t)+x_{21}(t)+1)-(x_{12}(t)+x_{22}(t)+1)|^p dt\Big)^{\frac{1}{p}} \\
= & \Big(\int_0^1 1^p dt\Big)^{\frac{1}{p}} \\
= & \ 1 \\
= & \ \xi(x_{11},x_{12},x_{21},x_{22}).
\end{aligned}$$

Therefore, all assumptions of Theorem 1.1 hold, except for the assumption of Y being a Banach space.

(2). It follows from (1) that all the assumptions of Theorem 2.4 are satisfied with $a_{11}=a_{12}=a_{21}=a_{22}=1$, $A_{11}=A_{22}=1, A_{12}=A_{21}=0$ and $L=\frac{1}{2}$. Therefore, Theorem 2.4 is applicable to X, Y, g and ξ. However, Theorem 1.1 does not apply to g and ξ since Y is not a Banach space. □

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