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Monotonicity, boundedness, and convergence for sequences of fuzzy numbers

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Abstract. We present the concepts of ideal monotonicity and ideal boundedness in the context of sequences involving fuzzy numbers. Additionally, we formulate an equivalent to the monotone convergence theorem and provide the necessary proofs for decomposition theorems specifically designed for this category of sequences. Our exploration extends to the analysis of I^*-convergent sequences of fuzzy numbers, concluding with the demonstration that these sequences have a unique limit point.

1. Introduction

Fast [11] and Schonenberg [30] were pioneers in introducing the notion of statistical convergence, primarily focusing on real number sequences. Subsequently, Fridy [12], Salat [31], and many others extensively explored this concept within the context of sequence spaces, often linking it with summability theory. The realm of mathematical analysis and number theory has witnessed numerous applications based on statistical convergence, some of which are documented in [5, 6, 23, 23, 28]. The premise of this concept is built upon the concept of the natural density of subsets within the set of positive integers, denoted as $\mathbb{N}$. The natural density of a subset A of $\mathbb{N}$ is formally expressed as $\delta(\mathrm{A})$, and it is defined as follows:

$$\delta(\mathrm{A}) = \lim_{n\to\infty} \frac{1}{n} |\{k < n : k \in \mathrm{A}\}|.$$

An extension of statistical convergence, termed I-convergence, was initially proposed by Kostyrko et al. [17], who defined this concept by utilizing the notion of an ideal I comprising subsets of the set $\mathbb{N}$. Ideal convergence of sequences is a concept in mathematics that generalizes the classical notion of convergence. Researchers have explored this concept in recent papers in various spaces, including partial metric spaces [13], generalized metric spaces [16], and neutrosophic normed spaces [7, 15]. For an in-depth exploration of this topic, we recommend referring to [10, 17, 19, 20]. On a separate note, Matloka [24] was the pioneer to introduce the notion of ordinary convergence for fuzzy number sequences, establishing fundamental theorems for such sequences. Nanda's study of fuzzy number sequences [25] revealed that

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the collection of all convergent fuzzy number sequences forms a complete metric space. In a more recent development, Nuray and Savas [26] established the definitions of statistical convergence and statistically Cauchy sequences specifically within the context of fuzzy numbers. They also provided an iff criterion for a statistically convergent sequence to be statistically Cauchy. Savas [32] delved into the conditions under which a fuzzy number sequence is statistically Cauchy. He introduced cluster points and statistical limit points for fuzzy number sequences, building upon Nuray et al. [1] definition of statistical convergence. Furthermore, he explored the relationships amid the set of ordinary limit points, statistical limit points and statistical cluster points of a fuzzy number sequence. The study also included the exploration of statistical boundedness and statistical monotonicity of sequences of fuzzy numbers, leading to the derivation of decomposition-type and monotone theorems.

Aytar et al.[2] introduced the concepts of statistical limit superior and statistical limit inferior concerning statistically bounded sequences of fuzzy numbers. The authors not only defined these notions but also delved into a thorough exploration of their inherent properties.

Ideal convergence(I-convergence) of sequences of fuzzy numbers has been studied, revealing it to be a more general form of convergence than statistical convergence. While monotonicity, boundedness, and convergence have been explored statistically, ideal monotonicity and ideal boundedness have not yet been studied. This gap has motivated us to investigate the ideal monotonicity(I-monotonicity) and ideal boundedness(I-boundedness) of sequences of fuzzy numbers. Many fundamental theorems in analysis, such as the Bolzano-Weierstrass theorem, depend on the uniqueness of limits. Although I^*-convergence of sequences of fuzzy numbers has been studied with some general theorems established, the uniqueness of the limit of sequences in the I^*-convergence sense and its conditions have not yet been proved. This has further motivated us to explore and establish the uniqueness of the limit of sequences of fuzzy numbers in the context of I^*-convergence.

This article delves into the concepts of I-boundedness and I-monotonicity for sequences of fuzzy numbers, which represent significant extensions of I-convergence. This study addresses the lacuna in the theory of I-convergence of sequence of fuzzy numbers and explores the transformation of classical results for I-boundedness and I-monotonicity in the context of fuzzy numbers. Additionally, the investigation continues with an exploration of I^*-convergence.

2. Preliminaries

Let's start by revisiting some fundamental notations within the realm of fuzzy numbers. Consider any interval, denoted as A, with its endpoints represented as $\underline{\mathrm{A}}$ and $\overline{\mathrm{A}}$. Now, let's introduce the set D, which encompasses all closed bounded intervals on the real number line, formally defined as:

$$\mathrm{D} = \{\mathrm{A} \subset \mathbb{R} : \mathrm{A} = [\underline{\mathrm{A}}, \overline{\mathrm{A}}]\}.$$

For any two given intervals A and B in D, we say $\mathrm{A} \leqslant \mathrm{B}$ iff $\underline{\mathrm{A}} \leqslant \underline{\mathrm{B}}$ and $\overline{\mathrm{A}} \leqslant \overline{\mathrm{B}}$.
Furthermore, we define the distance function between A and B, denoted as $d(\mathrm{A}, \mathrm{B})$, as:

$$d(\mathrm{A}, \mathrm{B}) = \max\{|\underline{\mathrm{A}} - \underline{\mathrm{B}}|, |\overline{\mathrm{A}} - \overline{\mathrm{B}}|\}.$$

Significantly, the distance metric denoted as d plays a crucial role by establishing a Hausdorff metric on the set D, thereby making (D, d) a complete metric space. Moreover, the partial order $\leqslant$ on D enhances the mathematical framework.

Now, we proceed to define a fuzzy number as follows, elaborating on its essential characteristics.

Definition 2.1. *[21] A fuzzy number is a function* X *from* $\mathbb{R}$ *to* $[0, 1]$, *which satisfying the following conditions*
(i) X *is normal, i.e., there exists an* $x_0 \in \mathbb{R}$ *such that* $\mathrm{X}(x_0) = 1$;
(ii) X *is fuzzy convex, i.e., for any* $x, y \in \mathbb{R}$ *and* $\lambda \in [0, 1]$, $\mathrm{X}(\lambda x + (1 - \lambda)y) \geqslant \min\{\mathrm{X}(x), \mathrm{X}(y)\}$;
(iii) X *is upper semi-continuous;*
(iv) the closure of the set $\{x \in \mathbb{R} : \mathrm{X}(x) > 0\}$, *denoted by* X^0 *is compact.*

The properties (i)-(iv) imply that for each $\alpha \in (0,1]$, the α-level set,

$$X^\alpha = \{x \in \mathbb{R} : X(x) \geqslant \alpha\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right].$$

Here, X^α represents a non-empty, compact, and convex subset of the real numbers $\mathbb{R}$.
The entire collection of fuzzy numbers is denoted by $L(\mathbb{R})$. We establish a mapping, denoted as $\overline{d}$, from the set of fuzzy numbers $L(\mathbb{R}) \times L(\mathbb{R})$ to the real numbers $\mathbb{R}$ as follows:

$$\overline{d}(X, Y) = \sup_{\alpha \in [0,1]} d\left(X^\alpha, Y^\alpha\right).$$

In this context, $\overline{d}(X, Y)$ calculates the supremum of the distance, d, between the α-level sets of the fuzzy numbers X and Y across all values of α within the interval $[0,1]$.

Puri and Ralescu [29] established that the space $(L(\mathbb{R}), \bar{d})$ forms a complete metric space. We define $X \leqslant Y$ for $X, Y \in L(\mathbb{R})$ iff $\underline{X}^\alpha \leqslant \underline{Y}^\alpha$ and $\bar{X}^\alpha \leqslant \bar{Y}^\alpha$ for each $\alpha \in [0,1]$. If $X \leqslant Y$ and $\alpha_0 \in [0,1]$ exist, we say that $X < Y$ if $X \leqslant Y$ and $\alpha_0 \in [0,1]$ exist such that$\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}$. X and Y are said to be incomparable fuzzy numbers if neither $X \leqslant Y$ nor $Y \leqslant X$.

Within the metric space $L(\mathbb{R})$, we can define addition $X + Y$ and scalar multiplication λX, where λ is a real number, in terms of α-level sets as $[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$ for each $\alpha \in [0,1]$ and $[\lambda X]^\alpha = \lambda[X]^\alpha$ for each $\alpha \in [0,1]$ respectively.

In the context of fuzzy integers within a subset S of $L(\mathbb{R})$, when there exists a fuzzy integer denoted by μ such that $X \leqslant \mu$ holds for every X in the subset S, we label S as having an upper bound, with μ serving as the upper bound for the set. Moreover, if μ qualifies as an upper bound for S and, importantly, for all other upper bounds μ' of S, it is established that $\mu \leqslant \mu'$, then we define μ as the least upper bound (sup) of S.

Similarly, we can define lower bounds and the greatest lower bound (inf). If a set E is both bounded below and bounded above, it is termed a bounded set.
For each $\alpha \in [0,1]$, if we define $\overline{Z}^\alpha := \overline{X}^\alpha + \overline{Y}^\alpha$ and $\underline{Z}^\alpha := \underline{X}^\alpha + \underline{Y}^\alpha$, we can express Z as the sum of X and Y, denoted as $Z = X + Y$. Likewise, following a similar pattern, we represent Z as the difference of X and Y, expressed as $Z = X - Y$, iff $\overline{Z}^\alpha := \overline{X}^\alpha - \overline{Y}^\alpha$ and $\underline{Z}^\alpha := \overline{X}^\alpha - \underline{Y}^\alpha$ for each $\alpha \in [0,1]$.

Definition 2.2. *[21] A sequence $X = (X_n)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if, for every $\varepsilon > 0$, there exists a positive integer m such that $\overline{d}(X_n, X_0) < \varepsilon$ for every $n \geqslant m$. The fuzzy number X_0 is referred to as the ordinary limit of the sequence (X_n), denoted as $\lim_{n\to\infty} X_n = X_0$".*

This definition characterizes the behavior of sequences of fuzzy numbers as they approach a limiting value, much like conventional sequences of real numbers.

Definition 2.3. *[21] A sequence $X = (X_n)$ of fuzzy numbers is regarded as a Cauchy sequence if, for every $\varepsilon > 0$, there exists a positive integer n_0 such that $\overline{d}(X_n, X_m) < \varepsilon$ for all $n, m \geqslant n_0$.*

This definition signifies that in a Cauchy sequence of fuzzy numbers, the elements of the sequence become arbitrarily close to each other as the sequence progresses, much like the concept of Cauchy sequences in conventional real numbers.

This concept is essential in the study of the convergence properties of sequences of fuzzy numbers.

Definition 2.4. *[21] A sequence $X = (X_n)$ of fuzzy numbers is categorized as a bounded sequence if the set $\{X_n : n \in \mathbb{N}\}$, comprising all the fuzzy numbers in the sequence, is itself a bounded set of fuzzy numbers.*

In simpler terms, a sequence of fuzzy numbers is considered bounded if the fuzzy numbers within the sequence do not exhibit unbounded behavior, meaning they remain within a certain range or bound.

We denote the set of all bounded sequences of fuzzy numbers as ℓ_∞. This set consists of sequences of fuzzy numbers that do not grow without bound and are confined within a certain range.

The concept of bounded sequences and the set ℓ_∞ is essential in analysing the convergence and studying the limit of sequences of fuzzy numbers, much like their significance in the theory of real numbers.

Statistical convergence is an extension of the classical notion of convergence. The concept was originally introduced for real and complex sequences by Fast and Schoenberg separately. Subsequently, Nuray and Savas [26] expanded the concepts of "statistically Cauchy sequence" and "statistical convergence" to include fuzzy numbers and provided their definitions. This approach has proven to be relevant and applicable in diverse mathematical disciplines, enhancing the understanding of sequences comprised of fuzzy numbers.

Definition 2.5. *[21] A sequence $X = (X_n)$ of fuzzy numbers is considered to be statistically convergent to a fuzzy number X_0 if, for any $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : \overline{d}(X_n, X_0) \geqslant \varepsilon\}$ exhibits a natural density of zero. In this context, the natural density of a set refers to the proportion of natural numbers within the set concerning the whole set of natural numbers. The fuzzy number X_0 is termed the statistical limit of the sequence (X_n), denoted as $st - \lim_{n\to\infty} X_n = X_0$.*

This definition characterizes statistical convergence, where the elements of the sequence become arbitrarily close to the fuzzy number X_0 in a statistical sense, akin to conventional convergence but considering the natural density of terms within a given range.

Definition 2.6. *[21] A sequence $X = (X_n)$ of fuzzy numbers is termed statistically Cauchy if, for any $\varepsilon > 0$, there exists a positive integer $m = m(\varepsilon)$ such that the set $\{n \in \mathbb{N} : \overline{d}(X_n, X_m) \geqslant \varepsilon\}$ has a natural density of zero. In this context, the term "natural density" pertains to the proportion of natural numbers within the set concerning the entire set of natural numbers.*

This definition characterizes statistical Cauchy sequences, where the terms within the sequence eventually become arbitrarily close to each other in a statistical sense, akin to the behavior of Cauchy sequences, but considering the natural density of terms within a given range.

Throughout this paper, we will use $\mathbb{R}$ and $\mathbb{N}$ to represent, respectively, the set of positive integers and real numbers. We will denote the power set of any set X as P(X), and the complement of the set A will be denoted as A^c.

The introduction of the concept of statistical convergence and the exploration of various convergence properties pave the way for the introduction of the notion of *I*-convergence of sequences.

Definition 2.7. *[21]Let X be a non-empty set, then a collection of subsets I contained in the power set of X denoted as P(X) is said to be ideal iff it satisfies the following conditions:*

1. *The empty set belongs to I i.e., $\emptyset \in I$.*
2. *For any set A and B belonging to I, $A \cup B$ also belongs to I.*
3. *If $A \in I$ and $B \subset A$ then $B \in I$.*

Definition 2.8. *Let X be a non-empty set. A non-empty family of sets F contained within the power set P(X) is denoted as a filter on X iff it adheres to the following criteria:*

1. *The empty set $\emptyset$ is not an element of the filter, meaning $\emptyset \notin F$.*
2. *For any two sets A and B hat belong to the filter, their intersection denoted as $A \cap B$, is also a part of the filter, formally expressed as $A \cap B \in F$.*
3. *If a set A is a member of the filter and B is a superset of A, then B is also an element of the filter, i.e., $B \in F$.*

These conditions (i), (ii), and (iii) jointly define the properties of a filter on set X.

An ideal I is termed non-trivial if it satisfies two conditions: it is not an empty set ($I \neq \emptyset$), and it does not contain the entire set X ($X \notin I$). Notably, a non-trivial ideal $I \subset P(X)$ corresponds to a filter, denoted as $F(I)$, which is formed by taking the set complement of each element of I with respect to the entire set X. The filter $F = F(I)$ is referred to as the filter associated with the ideal I.

An ideal I in X is considered admissible iff it includes all singleton sets i.e.,$\{\{x\} : x \in X\}$.

Definition 2.9. *[21] Suppose $I \subset P(\mathbb{N})$ be a non-trivial ideal. We define a sequence $X = (X_n)$ of fuzzy numbers as I-convergent to a fuzzy number X_0 if, for any ϵ, the set $A(\epsilon) = \{n \in \mathbb{N} : \overline{d}(X_n, X_0) \geqslant \epsilon\} \in I$.*

The fuzzy number X_0 is then referred to as the I-limit of the sequence (X_n), and this is denoted as $\lim_{n\to\infty} X_n = X_0$.

The set of fuzzy numbers squences that are both convergent and I-convergent be denoted by ℓ_1. These sequences exhibit both conventional convergence and convergence according to the ideal I, providing a rich framework for the study of their convergence properties.

3. I-Monotonic and I-bounded Sequences of Fuzzy Numbers

A sequence, denoted as $\mathrm{X} = (\mathrm{X}_k)$, is characterized as monotonically increasing if, for each natural number k in the set of positive integers, it holds that X_k is less than or equal to X_{k+1}.

Guangquan, as documented in the work by Guangquan [14], introduced a monotone convergence theorem applicable to a sequence of fuzzy numbers. This theorem shares similarities with the classical monotone convergence theorem for real numbers. The classical monotone convergence theorem for real numbers is expressed as follows:

In the context of real numbers, the monotone convergence theorem states that if you have a monotonically increasing sequence (X_k) of real numbers, meaning that X_k is less than or equal to X_{k+1} for all natural numbers k, then there exists a limit L such that the sequence converges to L. In other words, as k goes to infinity, X_k approaches the limit L, and this limit is the supremum of the sequence (X_k).

Guangquan's proposed monotone convergence theorem for a sequence of fuzzy numbers likely mirrors this concept in the realm of fuzzy numbers, wherein a monotonically increasing sequence of fuzzy numbers converges to a specific limit value, analogous to the classical real number case.

Note that the similarity between Guangquan's theorem and the classical monotone convergence theorem for real numbers is drawn to provide context and understanding, and it is important to refer to the source[14] for the precise details and formulation of Guangquan's theorem for fuzzy numbers.

Theorem 3.1. *[14] Let $\mathrm{X} = (X_n) \subset L(\mathbb{R})$. If $\mathrm{X} = (\mathrm{X}_n)$ is a monotone increasing sequence and has an upper bound $\mu \in L(\mathbb{R})$, then it is convergent.*

Theorem 3.2. *[21] For any sequence $\mathrm{X} = (\mathrm{X}_n)$ of fuzzy numbers, the following are equivalent:*
(a) $\mathrm{X} = (\mathrm{X}_n)$ is I-convergent to X_0.
(b) There exists a subsequence $\mathrm{K} = (\mathrm{K}_n)$ of $\mathbb{N}$ such that $\mathrm{K} \in \mathrm{F}(I)$ and $\overline{d}\,(\mathrm{X}_{m_k}, \mathrm{X}_0) \to 0$ as $n \to \infty$.

Theorem 3.3. *[21] Let $\mathrm{X} = (\mathrm{X}_n)$, $\mathrm{Y} = (\mathrm{Y}_n) \subset L(\mathbb{R})$. If $I - lim \mathrm{X}_n = \mathrm{X}_0$ and $I - lim \mathrm{Y}_n = \mathrm{Y}_0$, then $I - lim(\mathrm{X}_n + \mathrm{Y}_n) = \mathrm{X}_0 + \mathrm{Y}_0$.*

Definition 3.4. *A sequence of fuzzy numbers $\mathrm{X} = (X_n)$ is said to be ideally monotone increasing if there exists a subset $\mathrm{K} = \{k_1 < k_2 < k_3 < \dots\} \subset \mathbb{N}$ such that $\mathrm{K} \in \mathrm{F}(I)$ and $\mathrm{X}_{k_n} \leq \mathrm{X}_{k_{n+1}}$ for every $n \in \mathbb{N}$. An ideally monotone decreasing sequence can be defined similarly.*

A sequence can exhibit ideal monotonicity in two possible ways, either as an ideally monotone increasing sequence or as an ideally monotone decreasing sequence. It is important to note that a sequence can be ideally monotonic even if it is not strictly monotonic increasing or decreasing however the reverse of this statement is not always true. In other words, a sequence may be ideally monotonic without strictly adhering to the conditions of being monotonically increasing or decreasing.

For instance, an ideal monotone increasing sequence possesses a unique property where its index set, comprising those elements that are non-increasing and incomparable, belongs to the ideal.

Proposition 3.5. *Let $\mathrm{X} = (\mathrm{X}_k) \subset L(\mathbb{R})$ (the set of fuzzy numbers):*

1. *If $\mathrm{X} = (\mathrm{X}_k)$ is I-monotone increasing sequence, then $\{k \in \mathbb{N} : \mathrm{X}_k \nleqslant \mathrm{X}_{k+1}\} \in I$.*
2. *If $\mathrm{X} = (\mathrm{X}_k)$ is I-monotone decreasing sequence, then we have $\{k \in \mathbb{N} : \mathrm{X}_k \ngeqslant \mathrm{X}_{k+1}\} \in I$.*

Proof. 1. Let $\mathrm{X} = (\mathrm{X}_k)$ be I-monotone increasing sequence. Then according to the definition of I-monotone increasing sequence there exists a subset $\mathrm{K} = \{k_1 < k_2 < k_3 < \dots\} \subset \mathbb{N}$ such that $\mathrm{K} \in \mathrm{F}(I)$ and $\{\mathrm{X}_{k_n}\}$ is monotonic increasing. Since $\{k \in \mathbb{N} : \mathrm{X}_k \leqslant \mathrm{X}_{k+1}\} \supseteq \{k \in K : \mathrm{X}_k \leqslant \mathrm{X}_{k+1}\}$ and $\{k \in K : \mathrm{X}_k \leqslant \mathrm{X}_{k+1}\} \in \mathrm{F}(I)$. $\{k \in \mathbb{N} : \mathrm{X}_k \leqslant \mathrm{X}_{k+1}\} \in \mathrm{F}(I)$ as a result. Thus, $\{k \in \mathbb{N} : \mathrm{X}_k \nleqslant \mathrm{X}_{k+1}\} \in I$.

2. Let $X = (X_k)$ be I-monotone decreasing sequence. Then according to the definition of I-monotone decreasing sequence there exists a subset $K = \{k_1 < k_2 < k_3 < \dots\} \subset \mathbb{N}$ such that $K \in F(I)$ and $\{X_{k_n}\}$ is monotonic decreasing. Since $\{k \in \mathbb{N} : X_k \geqslant X_{k+1}\} \supseteq \{k \in K : X_k \geqslant X_{k+1}\}$ and $\{k \in K : X_k \geqslant X_{k+1}\} \in F(I)$. $\{k \in \mathbb{N} : X_k \geqslant X_{k+1}\} \in F(I)$ as a result. Thus, $\{k \in \mathbb{N} : X_k \ngeqslant X_{k+1}\} \in I$.
   □

Debnath et al. [9], as presented in their work, introduced innovative sequence spaces in which they introduced the concept of an "I-bounded sequence of fuzzy numbers". They also established a sequence space encompassing all such I-bounded sequences of fuzzy numbers. The concept of "I-boundedness" for a sequence of fuzzy numbers is defined as follows:

Definition 3.6. *The sequence $X = \{X_k\}$ is said to be I-bounded above if there exists a fuzzy number μ such that $\{k \in \mathbb{N} : X_k > \mu\} \cup \{k \in . \mathbb{N} : X_k \nsim \mu\} \in I$. Similarly, we can define I-boundedness of the sequence. A sequence which is I-bounded above and below is called I-bounded.*

It is easy to see that a bounded sequence is also I-bounded. In general, the converse is not true.

Example 3.7. *[2] Let*

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} 0, & \text{if } x \in (-\infty, n-1) \cup (n+1, \infty) \\ x-(k-1), & \text{if } x \in [n-1, n] \\ -x+(k+1), & \text{if } x \in (n, n+1] \end{array}\right\}, & \text{if } k \text{ is an even square} \\ \left.\begin{array}{ll} 0, & \text{if } x \in (-\infty, -n-1) \cup (-n+1, \infty) \\ x+k+1, & \text{if } x \in [-n-1, -n] \\ -x-k+1, & \text{if } x \in (-n, -n+1] \end{array}\right\}, & \text{if } k \text{ is an odd square} \\ \mu_1(x), & \text{if } k \text{ is an odd nonsquare} \\ \mu_2(x), & \text{if } k \text{ is an even nonsquare} \end{cases}$$

where

$$\mu_1(x) := \begin{cases} 0, & \text{if } x \in (-\infty, 0) \cup (2, \infty) \\ x, & \text{if } x \in [0, 1] \\ -x+2, & \text{if } x \in (1, 2] \end{cases}$$

and

$$\mu_2(x) := \begin{cases} 0, & \text{if } x \in (-\infty, 3) \cup (5, \infty) \\ x-3, & \text{if } x \in [3, 4] \\ -x+5, & \text{if } x \in (4, 5] \end{cases}.$$

since

$$\{n \in \mathbb{N} : X_n < \mu_1\} \cup \{n \in \mathbb{N} : X_n \nsim \mu_1\} = \{1, 9, 25, \dots\} \cup \{\emptyset\} \in I$$

and

$$\{n \in \mathbb{N} : X_n > \mu_2\} \cup \{n \in \mathbb{N} : X_n \nsim \mu_2\} = \{16, 36, 64, \dots\} \cup \{\emptyset\} \in I.$$

$X = (X_n)$ *is a I-bounded sequence. Since there isn't a fuzzy number u such that* $X_n \leqslant u$ *for each* $n \in \mathbb{N}$, *therefore, this sequence isn't bounded.*

The second decomposition theorem states that an I-bounded fuzzy number sequence can be decomposed into a bounded sequence and an ideal null sequence.

Theorem 3.8. *If $X = (X_n)$ is an I-bounded sequence, $Y = (Y_n)$ is bounded and $\{n \in \mathbb{N} : Z_n \neq 0_1\} \in I$ then we have $X = Y + Z$.*

Proof. Let $X = (X_n)$ be an I-bounded sequence. For large enough $M > 0$, the set $\mathcal{N} := \left\{n \in \mathbb{N} : \overline{d}(X_n, 0_1) > M\right\} \subset \mathbb{N}$ and $\mathcal{N} \in I$. Define

$$Y_n := \begin{cases} X_n, & \text{if } n \in F(I) \\ 0_1, & \text{otherwise} \end{cases},$$

$$Z_n := \begin{cases} X_n, & \text{if } n \in \mathcal{N} \\ 0_1, & \text{otherwise} \end{cases}$$

Clearly, $Y = (Y_n)$ is bounded and $\{n \in \mathbb{N} : Z_n \neq 0_1\} \in I$. Thus, based on the construction of Y and Z it implies that $X = Y + Z$. □

Lemma 3.9. *A sequence $X = (X_n)$ is I-bounded iff there exists a subset $\mathcal{N} = \{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ such that $\mathcal{N} \in F(I)$, and (X_{n_k}) is bounded.*

Proof. The proof of this is trivial from Definition (3.6) and from Theorem (3.8). □

Theorem 3.10. *An I-monotone sequence is I-convergent if and only if it is I-bounded.*

Proof. The necessary part is clear. *Sufficient part*. Let $X = (X_n)$ be an I-monotonically increasing sequence which is I-bounded. According to the Definition (3.4) there exists a subset $\mathcal{N} = \{n_1 < n_2 < \dots\} \subset \mathbb{N}$ such that $\mathcal{N} \in F(I)$ and (X_{n_k}) is a monotonically increasing subsequence. Moreover, $X = (X_n)$ is I-bounded, then from Lemma (3.9) there exists a subset $\mathcal{L} = \{l_1 < l_2 < \cdots\} \subset \mathbb{N}$ such that $\mathcal{L} \in F(I)$, and $\{X_{l_n}\}$ is bounded. Let $\mathcal{P} := \mathcal{K} \cap \mathcal{L}$ where $\mathcal{P} = \{p_1 < p_2 < \cdots\} \subset \mathbb{N}$. Then $\mathcal{P} \in F(I)$ because $\mathcal{K} \in F(I)$ and $\mathcal{L} \in F(I)$.

Consequently, the subsequence (X_{p_n}) is both bounded and exhibits monotonic growth. In accordance with Theorem (3.1), we assert the convergence of (X_{p_n}). As a result, applying Theorem (3.2), we deduce that the sequence $X = (X_n)$ converges in the sense of I-convergence. □

4. I^*-convergence of fuzzy number sequences

Salat [31] established the iff condition for a sequence of real numbers to exhibit statistical convergence. He defined that a sequence $X = (X_n)$ of real numbers converges statistically to η iff there exists a subset $K = \{m_1 < m_2 < m_3 < \cdots < m_k \dots\} \subset \mathbb{N}$ with $\delta(K) = 1$ such that $\lim_{n\to\infty} X_n = \eta$. This same result was later proven by Savas [32] for fuzzy numbers sequence as a sequence of fuzzy numbers is statistically convergent to X_0 iff there exists a subset $K = \{m_1 < m_2 < m_3 < \cdots < m_k < \dots\} \subset \mathbb{N}$ with $\delta(K) = 1$ such that $\overline{d}(X_m, X_0) \to 0$ as $n \to \infty$. Using this result, the notion of I^*-convergence is introduced. This convergence is closely related to I-convergence. The I^*-convergence was firstly introduced by Koystroko et al. [18] in metric space after that many researchers studied it and extended the theory. An extensive view can be found in [8, 19, 21].

Definition 4.1. *[21] A sequence $X = (X_n)$ of fuzzy numbers is said to be I^*-convergent to a fuzzy number X_0 iff there exists a set $K = \{m_1 < m_2 < m_3 < \cdots < m_k < \dots\} \subset \mathbb{N}$ such that $K \in F(I)$ and $\overline{d}(X_{m_k}, X_0) \to 0$ as $n \to \infty$.*

Theorem 4.2. *If I be an admissible ideal then I^*-convergent sequence of fuzzy numbers converges to a unique fuzzy number.*

Proof. Let $X = (X_n)$ be an I^*-convergent to two distinct fuzzy numbers X_0 and Y_0. Assume that X_0 and Y_0 are comparable fuzzy numbers[21].

Assume that X_0 and Y_0 are not comparable. Consequently, there exists $\alpha_0 \in [0, 1]$ such that

$$\underline{X}_0^{\alpha_0} < \underline{Y}_0^{\alpha_0} \quad \text{and} \quad \overline{X}_0^{\alpha_0} > \overline{Y}_0^{\alpha_0} \tag{1}$$

or

$$\underline{X_0}^{\alpha_0} > \underline{Y_0}^{\alpha_0} \quad \text{and} \quad \overline{X_0}^{\alpha_0} < \overline{Y_0}^{\alpha_0}. \tag{2}$$

We will provide a proof for result (1) alone, as the argument for (2) can be demonstrated similarly. Let us assume that (1) is valid. Choose $\epsilon_1 = \underline{Y_0}^{\alpha_0} - \underline{X_0}^{\alpha_0}$ and $\epsilon_2 = \underline{X_0}^{\alpha_0} - \underline{Y_0}^{\alpha_0}$. Clearly, $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Let

$\epsilon' = min\{\epsilon_1, \epsilon_2\}$. Select ϵ such that $0 < \epsilon < \epsilon'$. Given that X_n is I^*-converges to both X_0 and Y_0 therefore we have
$M = \{m_1 < m_2 < m_3 < \dots\} \subset \mathbb{N}$ and $N = \{n_1 < n_2 < n_3 < \dots\} \subset \mathbb{N}$ such that $M \in F(I)$ and $N \in F(I)$ and
$\overline{d}(X_{m_k}, X_0) \to 0$ as $n \to \infty$ and
$\overline{d}(X_{n_k}, Y_0) \to 0$ as $n \to \infty$. (3)
Since, $F(I)$ is a filter on $\mathbb{N}$ therefore by the definition of filter $M \cap N \neq \emptyset$.
Let $m \in M \cap N$. Then by virtue of (3) there exists positive integers k_1 and k_2 such that
$\overline{d}(X_{m_k}, X_0) < \epsilon$ for every $m_k \in M$ with $m_k > k_1$ and $\overline{d}(X_{n_k}, Y_0) < \epsilon$ for every $n_k \in N$ with $n_k > k_2$. Let $k = max\{k_1, k_2\}$ then $\overline{d}(X_{m_k}, X_0) < \epsilon$ and $\overline{d}(X_{n_k}, Y_0) < \epsilon$ for $m \in M \cap N$ with $n_k, m_k > k$ for each $\alpha \in [0, 1]$. Hence, we have $\overline{d}\left(X^{\alpha_0}_{m_k}, X_0\right) < \epsilon$ and $\overline{d}\left(X^{\alpha_0}_{n_k}, Y_0\right) < \epsilon$. Let, $max\{m_k, n_k\} = m$.
Now the definition of $\bar{d}$, implies that

$$\left|\underline{X}^{\alpha_0}_m - \underline{X}^{\alpha_0}_0\right| < \varepsilon \text{ and } \left|\underline{X}^{\alpha_0}_m - \underline{Y}^{\alpha_0}_0\right| < \varepsilon,$$
$$\left|\overline{X}^{\alpha_0}_m - \overline{X}^{\alpha_0}_0\right| < \varepsilon \text{ and } \left|\overline{X}^{\alpha_0}_m - \overline{Y}^{\alpha_0}_0\right| < \varepsilon.$$

$\underline{X}^{\alpha_0}_m \in \left(\underline{X}^{\alpha_0}_0 - \varepsilon, \underline{X}^{\alpha_0}_0 + \varepsilon\right) \cap \left(\underline{Y}^{\alpha_0}_0 - \varepsilon, \underline{Y}^{\alpha_0}_0 + \varepsilon\right) = \Phi$. Thus, a contradiction arises, implying the comparability of fuzzy numbers X_0 and Y_0.
Consider $X_0 \leqslant Y_0$ and the neighborhoods $A = \left\{n \in N : \overline{d}(X_n, X_0) < \varepsilon\right\}$ and $B = \left\{n \in N : \overline{d}(X_n, Y_0) < \varepsilon\right\}$ of X_0 and Y_0 , respectively, are disjoint for $\varepsilon = \frac{\overline{d}(X_0, Y_0)}{3} > 0$. By Definition (4.1), both the sets A and B belongs to $F(I)$ so that $A \cap B \neq \phi$. A contradiction has arised that the neighborhoods of X_0 and Y_0 are disjoint.
Let $M' = \{1, 2, 3, \dots, m_1\}$ and $N' = \{1, 2, 3, \dots, n_1\}$. Since M and N belong to the ideal I , so there exists a set H_1 and $H_2 \in I$ such that $M = \mathbb{N} - H_1$ and $N = \mathbb{N} - H_2$. Now it is clear that the set
$M'(\epsilon) = \{n \in \mathbb{N} : \overline{d}(X_n, X_0) \geq \epsilon\} \in M' \cup H_1$ and $N'(\epsilon) = \{n \in \mathbb{N} : \overline{d}(X_n, Y_0)\} \in N' \cup H_2$.
This implies that $M', N' \in I$ As I is an admissible ideal. Therefore, sets A and $B \in F(I)$ so that $A \cap B \neq \phi$. This contradicts our assumption that the neighborhoods X_0 and Y_0 are disjoint sets. Hence, X_0 is determined uniquely. □

Conclusion

Our study delves into the novel concepts of ideal monotonicity and ideal boundedness within the framework of sequences comprising fuzzy numbers. By introducing an equivalent to the monotone convergence theorem and substantiating it with rigorous proofs, we contribute to the foundational understanding of sequences involving fuzzy numbers. Furthermore, our investigation extends to the realm of I^*-convergent sequences of fuzzy numbers, where we unveil that these sequences exhibit a remarkable property—a unique limit point. This finding not only enriches the theoretical foundation of fuzzy number sequences but also opens avenues for practical applications in diverse fields where the representation of uncertainty through fuzzy numbers is crucial. Our work not only advances the theoretical understanding of fuzzy number sequences but also paves the way for future research and applications in this intricate domain.
The future scope of this study includes the exploration and generalization of ideal monotonicity and boundedness to sequences of fuzzy numbers, aiming to identify unique properties and challenges in higher-dimensional contexts. Additionally, the theoretical framework developed through this research can be applied to investigate these concepts within various topological spaces. This extension will provide a broader and deeper understanding of the behavior of sequences, contributing valuable insights to theoretical mathematics and practical applications.

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