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Fractional Poisson process on quantum time scale with applications to practical data

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Abstract. A $q-$fractional generalization of the Poisson process has been provided by replacing the first time derivative in the relaxation equation of the survival probability with a $q-$fractional derivative of order $\nu\,(0 < \nu \leq 1)$. For $0 < \nu < 1$, $1/q-$renewal processes have been obtained where, the $1/q-$exponential probability densities, typical for the $1/q-$Poisson processes, are replaced by functions of $1/q-$Mittag–Leffler type, that decay in a power law manner with an exponent related to ν. The distributions obtained by considering the $1/q-$sum of k independent identically distributed random variables distributed according to the $1/q-$Mittag-Leffler law provide the $q-$fractional generalization of the corresponding $1/q-$Erlang distributions.

Two fitting scenarios are built on a data set including the records of serious earthquakes in Turkey. The first fitting scenario compares the $1/q-$Poisson distribution with the Poisson and the negative binomial distributions. In the second fitting scenario, the $1/q-$Erlang and the Erlang distributions are discussed. The presented results suggest that the $1/q-$Poisson and the $1/q-$Erlang are more suitable for the observed data compared to the other distributions considered. The parameters of the $1/q-$Poisson and the $1/q-$Erlang distributions are estimated by the maximum likelihood method.

1. Introduction

The concept of renewal process has been developed as a stochastic model for describing the class of counting processes for which the times between successive events are independent identically distributed (i.i.d.) non-negative random variables, obeying a given probability law. These times are referred to as waiting times or inter-arrival times. In the context of renewal processes, the inter-arrival times play a crucial role in understanding the behavior of the counting process. By characterizing the distribution of these inter-event times, we can gain insights into the long-term properties and fluctuations of the process. This framework has found applications in various fields, including reliability theory, queuing theory, and stochastic modeling of phenomena such as customer arrivals, equipment failures, and more. For more details see e.g. [12], [14], [18], [21], and [23].

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Time scale calculus [7, 8, 20] offers a unified framework for continuous and discrete calculus, extending it to any closed subset, known as a *time scale*, of the real line $\mathbb{T}$. The case $\mathbb{T} = \mathbb{R}$ corresponds to real analysis and the case $\mathbb{T} = \mathbb{Z}$ to discrete analysis. After the appearance of the time scale calculus, the use of its techniques in *fractional calculus* was started. Reference [26] extended fractional Riemann-Liouville calculations for arbitrary time scales. Fractional Riemann-Liouville calculus extends the m^{th} Riemann integration of a function (by Cauchy formula, which, for repeated integration, allows for compressing m integrations of a function into a single integral) to fractional orders. Quantum fractional calculus ($q-$*fractional calculus*) is a special case of fractional calculus on quantum time scale $\mathbb{T}_q^\alpha = \{q^{n+\alpha};\ n \in \mathbb{N},\ \alpha \in \mathbb{R}^+\} \cup \{0\}$.

As detailed in references [15–17], the fractional paradigm for probability theory appears in the nature of fractional moments in classifying them as Taylor monomials associated by some different phases based on time scale theory. In these works, some interesting probability distributions constructed by this method are studied. In this paper, a generalization of renewal theory called $1/q-$renewal theory is presented by applying $q-$fractional calculus. The $1/q-$Poisson process has a fundamental role in $1/q-$renewal theory. The use of the $1/q-$Mittag-Leffler functions allows us to provide a generalization of the process and also construct interesting subordinated stochastic processes of $q-$fractional diffusion.

The benefit of using specific probability distributions for corresponding stochastic processes lies in the ability to accurately model and analyze the behavior of the processes. Different distributions have different properties and characteristics, which can be useful for understanding and making predictions about the stochastic process in question. The Poisson distribution plays a crucial role in the analysis and simulation of stochastic processes where events occur at a constant rate and independently of the time since the last event. It is often used to model the number of events that occur in a fixed interval of time or space, like the arrival of customers in a queue, the number of phone calls received in a call center, the number of radioactive particles emitted from a radioactive source, and many other similar processes. In the context of stochastic processes, the Erlang distribution can be used to model the time between events in a process that occurs at a constant rate, such as the arrival of customers at a service center or the occurrence of radioactive decay. By fitting an Erlang distribution to the inter-arrival times of events in a stochastic process, one can better understand and predict the behavior of the process. Here, we explore whether the $1/q-$Poisson and the $1/q-$Erlang distributions exhibit similar behavior in $1/q-$processes.

The paper is organized in the following way. In Sections 2 and 3, we provide a set of definitions and related results, which are essential and will be used in the proceeding discussions. In Section 4, we provide via fractional calculus on quantum time scales the generalization of the Poisson process, and then define the $1/q-$renewal theory including its fundamental concepts, like waiting time between events and the survival probability. If the waiting time is $1/q-$exponentially distributed we have the $1/q-$Poisson process, this is the topic of Subsection 4.1. However, other waiting time distributions are also relevant in applications, in particular such ones with a fat tail caused by a power law decay of their density. In Subsection 4.2 we analyze the $1/q-$renewal processes with waiting time distributions described by functions of $1/q-$Mittag-Leffler, that exhibit a similar power law decay. It depends on a parameter $\nu \in (0, 1)$ related to the common exponent in the power law. In the limit $\nu = 1$ that becomes the Poisson process. Section 5 is devoted to the estimation of the unknown parameters, where the maximum likelihood method is tested on simulated data. In Section 6, we examine a dataset containing records of major earthquakes in Turkey during the 20th century. The observed data are modeled through two fitting schemes, where the $1/q-$Poisson and the $1/q-$Erlang distributions are used. The section presents the obtained results highlighting the superiority of the $1/q-$Poisson distribution over the Poisson and negative binomial distributions, as well as the advantage of $1/q-$Erlang over the Erlang distribution. The concluding remarks are given in Section 7.

2. Definitions and essential lemmas

The observation $\lim_{q\to 1^-} \frac{1-q^x}{1-q} = x$ plays a basic role in the theory of $q-$calculus, where $x, q \in \mathbb{C}$. We define $[x]_q = \frac{1-q^x}{1-q}$ as the $q-$number of x. Also, the factorial of the $q-$number $[x]_q$ of order k, which is defined by

$$[x]_{k,q} = [x]_q[x-1]_q \ldots [x-k+1]_q.$$

Clearly, $\lim_{q\to 1^-}[x]_q = x$. The $q-$factorial of n is given by $[n]_q! = [1]_q[2]_q...[n]_q$ and the $q-$Gauss binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For the exponential function, it has given two $q-$analogues as $e_q(x) = \sum_{n\geq 0} \frac{x^n}{[n]_q!}$ and $E_q(x) = \sum_{n\geq 0} \frac{q^{\binom{n}{2}} x^n}{[n]_q!}$, where the series converges for $| x |< \frac{1}{1-q}$ and $x \in \mathbb{C}$, respectively. In this work, we call them the $1/q-$exponential and $q-$exponential functions, respectively. The $q-$derivative of an arbitrary function $f(x)$ (see [13]) is defined by

$$\nabla_q(f(x)) = \frac{f(qx) - f(x)}{x(q-1)},$$

where $x \neq 0$ and the definite Jackson $q-$integral is given by

$$\int_0^x f(t)\nabla_q t = (1-q)\sum_{a=0}^{\infty} f(q^a x)xq^a.$$

Obviously, if the function f is differentiable then $\lim_{q\to 1^-} \nabla_q(f(x)) = \frac{d}{dx} f(x)$. Clearly, $\nabla_q(e_q(ax)) = ae_q(ax)$ and $\nabla_q(E_q(ax)) = aE_q(aqx)$ for $| ax |< \frac{1}{1-q}$ and $a,\ x \in \mathbb{C}$, respectively. The $q-$extension of gamma function is defined by

$$\Gamma_q(t) = \int_0^{\infty} x^{t-1}E_q(-qx)\nabla_q x, \ \ t > 0 \tag{1}$$

and by Eq. (1) we obtain $[n]_q! = \Gamma_q(n+1)$ and $\Gamma_q(t+1) = [t]_q\Gamma_q(t)$. The $q-$factorial function for $n \in \mathbb{N}$ is defined by $(t-s)_q^n = \prod_{i=0}^{n-1}(t - sq^i)$. When α is a non-positive integer, the $q-$factorial is of the form

$$(t-s)_q^{\alpha} = t^{\alpha}\prod_{i=0}^{\infty}\frac{1-\frac{s}{t}q^i}{1-\frac{s}{t}q^{i+\alpha}}.$$

For $t, s > 0$, the $q-$beta function is given by

$$\beta_q(t,s) = \int_0^{\infty} x^{t-1}(1-qx)_q^{s-1}\nabla_q x,$$

with $\beta_q(t,s) = \Gamma_q(t)\Gamma_q(s)/\Gamma_q(t+s)$.

Reference [9] introduced a $q-$analogue of the Laplace transform, referred to as the $q-$Laplace transform and throughout this paper, we will denote by $\tilde{f}(s)$ the $q-$Laplace transform of a sufficiently function $f(t)$ according to

$$\mathcal{L}_q\{f\}(s) = \tilde{f}(s) = \int_0^{\infty} E_q(-qst)f(t)\nabla_q t, \ \ s > 0.$$

Lemma 2.1 ([9]). *For any $\alpha \in \mathbb{R}$ with $\alpha > -1$, we have $\mathcal{L}_q\{t^{\alpha}\}(s) = \frac{\Gamma_q(\alpha+1)}{s^{\alpha+1}}$.*

Definition 2.2 ([1]). *Let $\alpha > 0$. If $\alpha \notin \mathbb{N}$, then the $\alpha-$order Caputo left $q-$fractional derivative of a function f is defined by*

$${}_qC_a^{\alpha} f(t) = {}_q I_a^{(n-\alpha)}\nabla_q^n f(t) = \frac{1}{\Gamma_q(n-\alpha)}\int_a^t (t-qs)_q^{n-\alpha-1}\nabla_q^n f(s)\nabla_q s,$$

where $n = [\alpha] + 1$ *and*

$$ {}_qI_a^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f(s)\nabla_q s $$

is the left $q-$*fractional integral of order* α. *If* $\alpha = n \in \mathbb{N}$, *then*

$$ {}_qC_a^\alpha f(t) = \nabla_q^n f(t). $$

Example 2.3. *For* $0 < \alpha \leq 1$ *the* $\alpha-$*order Caputo left* $q-$*fractional difference of a function* f *defined on* $\mathbb{R}_q^+$ *is*

$$ \begin{aligned} {}_qC_0^\alpha f(t) &= {}_q I_0^{1-\alpha}\nabla_q f(t) \\ &= \frac{1}{\Gamma_q(1-\alpha)} \int_0^t (t-qs)_q^{-\alpha}\nabla_q f(s)\nabla_q s, \end{aligned} $$

where ${}_qI_0^\alpha = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)_q^{\alpha-1} f(s)\nabla_q s$ *is the left* $q-$*fractional integral of order* α.

The definition for $1/q-$Mittag-Leffler functions is given by [1] while they named it "modified $q-$Mittag-Leffler functions". We rewrite their definition by introducing this modification.

Definition 2.4. *For* $z, z_0 \in \mathbb{C}$ *and* $\mathcal{R}(\alpha) > o$, *the* $1/q-$*Mittag-Leffler functions are defined by*

$$ {}_{1/q}E_{\alpha,\beta}(\lambda, z-z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z-z_0)_q^{\alpha k+(\beta-1)}}{\Gamma_q(k\alpha+\beta)}. $$

In the case $\beta = 1$, *we may use*

$$ {}_{1/q}E_\alpha(\lambda, z-z_0) := {}_{1/q} E_{\alpha,1}(\lambda, z-z_0). $$

Fractional difference equations model the systems with memory and non-local effects in various fields like physics, engineering, and finance. A Caputo fractional difference equation utilizes the Caputo fractional derivative [3, 4]. Caputo $q-$fractional difference equations use the Caputo $q-$derivative operator instead of the Caputo fractional derivative [1, 2].

Example 2.5. *Let* $0 < \alpha \leq 1$ *and consider the left Caputo* $q-$*fractional difference equation*

$$ {}_qC_a^\alpha y(t) = \lambda y(t) + f(t), \quad y(a) = a_0, \ t \in \mathbb{T}_q. \tag{2} $$

The solution for Eq. (2) has the form

$$ y(t) = a_0\, {}_{1/q}E_\alpha(\lambda, t-a) + \int_a^t {}_{1/q}E_{\alpha,\alpha}(\lambda, t-qs) f(s)\nabla_q s. $$

We will show that ${}_{1/q}F_\alpha(x) = 1 - {}_{1/q} E_\alpha(-\lambda, x)$ has the $q-$Laplace transform $\mathcal{L}_q(s) = \lambda(\lambda + s^\alpha)^{-1}$ which is completely monotone for $0 < \alpha \leq 1$ and $\lambda > 0$, and therefore it is a $1/q-$distribution function. We call ${}_{1/q}F_\alpha(x)$ for $0 < \alpha \leq 1$ a $1/q-$Mittag-Leffler distribution. ${}_{1/q}F_1(x)$ is the $1/q-$exponential distribution.

Lemma 2.6 ([9]). *Let* f *be defined on* $\mathbb{R}_q^+$. *Then,*

$$ \mathcal{L}_q\{\nabla_q(f(t))\}(s) = s\mathcal{L}_q\{f(t)\}(s) - f(0). $$

Lemma 2.7. *Define the convolution* $(f * g)(t) = \int_0^t (t-q\tau)_q^{\beta-1} f(\tau)\nabla_q\tau$ *with* $g(t) = t^{\beta-1}$. *Assume that* f *is of the type* $\mathcal{L}_q\{(f * g)\} = \mathcal{L}_q\{f\}\mathcal{L}_q\{g\}$ *is valid. then for* $0 < \alpha \leq 1$, *we have*

$$ \mathcal{L}_q\{{}_qC_0^\alpha f\}(s) = s^\alpha \mathcal{L}_q\{f(t)\}(s) - s^{\alpha-1} f(0). $$

Proof. Following the discussion of Example 2.3, Lemmas 2.1, and 2.6 we can conclude that

$$\begin{aligned}\mathcal{L}_q\{{}_qC_0^{\alpha}f\}(s) =&\mathcal{L}_q\{\frac{1}{\Gamma_q(1-\alpha)}g^{-\alpha} * \nabla_q f(t)\}(s)\\ =&s^{\alpha-1}(\mathcal{L}_q\{\nabla_q f\})(s)\\ =&s^{\alpha-1}\{s\mathcal{L}_q(f(t)) - f(0)\}\\ =&s^{\alpha}\mathcal{L}_q\{f(t)\}(s) - s^{\alpha-1}f(0).\end{aligned}$$

□

In the convolution formula of the recent lemma, let $f(t) = t^{\alpha}$, then by using the definition of the $q-$beta function and Lemma 2.1, we obtain $\mathcal{L}_q\{(f * g)\} = \mathcal{L}_q\{f\}\mathcal{L}_q\{g\}$, which implies that if $f(t) = \sum_i a_i t^{\alpha_i}$, then

$$\mathcal{L}_q\{(f * g)\} = \sum_i a_i\mathcal{L}_q\{t^{\alpha_i} * g\} = \sum_i a_i\mathcal{L}_q\{t^{\alpha_i}\}\mathcal{L}_q\{g\} = \mathcal{L}_q\{\sum_i a_i t^{\alpha_i}\}\mathcal{L}_q\{g\} = \mathcal{L}_q\{f\}\mathcal{L}_q\{g\}. \tag{3}$$

We will use Eq. (3) to obtain the $q-$Laplace transform of the $1/q-$Mittag-Leffler function.

As follows, the $1/q-$Mittag-Leffler function ${}_{1/q}E_{\alpha,\beta}(\lambda, z - z_0)$ is a generalization of the $1/q-$exponential function $e_q(t)$, therefore, the $1/q-$exponential function is a particular case of the $1/q-$Mittag-Leffler function. We will outline here the way to obtain the $q-$Laplace transform of the $1/q-$Mittag-Leffler function with the help of the analogy between this function and the function $e_q(t)$. To do this, we obtain the $q-$Laplace transform of the function $t^k e_q(at)$ in the following way. First, let us prove that

$$\int_0^{\infty} E_q(-qt)e_q(\pm zt)\nabla_q t = \frac{1}{1 \mp z}, \quad |z| < 1. \tag{4}$$

By using the series expansion for $e_q(z)$, we obtain

$$\begin{aligned}\int_0^{\infty} E_q(-qt)e_q(\pm zt)\nabla_q t =&\frac{1}{1 \mp z} = \sum_{k=0}^{\infty}\frac{(\pm z)^k}{[k]_q!}\int_0^{\infty} E_q(-qt)t^k\nabla_q t\\ &= \sum_{k=0}^{\infty}(\pm z)^k = \frac{1}{1 \mp z}.\end{aligned}$$

Then, we $q-$differentiate both sides of Eq. (4) with respect to z. This yields the following result:

$$\int_0^{\infty} E_q(-qt)t^k e_q(\pm zt)\nabla_q t = \frac{q^{-\binom{k+1}{2}}[k]_q!}{\prod_{n=0}^{k}(q^{-n} \mp z)}.$$

After straightforward substitutions, we obtain the well-known pair of $q-$Laplace transforms of the function $t^k e_q(\pm at)$ as

$$\int_0^{\infty} E_q(-qpt)t^k e_q(\pm at)\nabla_q t = \frac{q^{-\binom{k+1}{2}}[k]_q!}{\prod_{n=0}^{k}(pq^{-n} \mp a)}.$$

Now, we consider the $1/q-$Mittag-Leffler function where the substitution of the function in the integral leads to

$$\int_0^{\infty} E_q(-qpt)_{1/q}E_{\alpha,1}(z, t)\nabla_q t = \frac{p^{\alpha-1}}{p^{\alpha} - z}. \tag{5}$$

From Eq. (5) we obtain a pair of $q-$Laplace transforms of the function $t^{\alpha k+\beta-1}\,{}_{1/q}E_{\alpha}^{\langle k\rangle}(\pm\lambda,t)$, where ${}_{1/q}E_{\alpha,\beta}^{\langle k\rangle}(y)=\frac{\nabla_q^k}{\nabla_q y^k}\,{}_{1/q}E_{\alpha,\beta}(y)$, as:

$$\int_0^\infty E_q(-qpt)t^{\alpha k+\beta-1}\,{}_{1/q}E_{\alpha}^{\langle k\rangle}(\pm\lambda,t)\nabla_q t=\frac{q^{-\binom{k+1}{2}}[k]_q!p^{\alpha-\beta}}{\prod_{n=0}^{k}(p^\alpha q^{-n}\mp\lambda)}.$$

For the case $\beta=1$, we have,

$$\int_0^\infty E_q(-qpt)t^{\alpha k}\,{}_{1/q}E_{\alpha}^{\langle k\rangle}(\pm\lambda,t)\nabla_q t=\frac{q^{-\binom{k+1}{2}}[k]_q!p^{\alpha-1}}{\prod_{n=0}^{k}(p^\alpha q^{-n}\mp\lambda)}. \tag{6}$$

The Eq. (6) has been used to obtain a fractional generalization of the $1/q-$Poisson distribution.

3. $1/q-$Poisson and $1/q-$gamma distributions

Consider a non-negative integer-valued random variable X with the probability mass function $f_X(x)=P(X=x),\ x=0,1,\dots$. We refer to the $1/q-$number transformation $Y=[X]_{1/q}=\frac{1-q^{-X}}{1-q^{-1}}$ as a $1/q-$deformation similar to the $q-$deformation in the language of quantum physics [22]. The distribution of the random variable Y, with the probability function

$$f_Y([x]_{1/q})=P(Y=[x]_{1/q})=P(X=x)=f_X(x),\ \ x=0,1,\dots,$$

which is referred to as a $1/q-$deformed distribution (similar to the $q-$deformed distribution in quantum physics). The mean and the variance of the $1/q-$deformed distribution of Y are the $1/q-$mean and the $1/q-$variance of the distribution of X.

Definition 3.1. *The random variable $[X]_{1/q}$ has a $1/q-$ Poisson distribution with (λ,q) parameters if its probability mass function (pmf) is given by*

$$P([X]_{1/q}=[x]_{1/q})=e_q(-\lambda)\frac{q^{\binom{x}{2}}\lambda^x}{[x]_q!},\ x=0,1,\dots,$$

where $0<q<1,\ 0<\lambda<\infty$. The distribution is denoted by $[X]_{1/q}\sim Po_{1/q}(\lambda)$.

Since $[X]_{r,1/q}=q^{-xr+\binom{r+1}{2}}[X]_{r,q}$, for this distribution we have

$$\begin{aligned}E([X]_{r,1/q})&=e_q(-\lambda)\sum_{x=r}^{\infty}\frac{[X]_{r,1/q}q^{-rx+\binom{r+1}{2}}q^{x(x-1)/2}\lambda^x}{[x]_q!}\\&=\lambda^r e_q(-\lambda)\sum_{x=r}^{\infty}\frac{q^{-rx+\binom{r+1}{2}}q^{r(x-r)}q^{r(r-1)/2}q^{(x-r)(x-r-1)}\lambda^{x-r}}{[x-r]_q!}\\&=\lambda^r e_q(-\lambda)E_q(\lambda)=\lambda^r,\end{aligned}$$

then $E([X]_{1/q})=\lambda$ and $Var([X]_{1/q})=\lambda(1-\lambda(1-1/q))$, where $Var([X]_{1/q})=1/qE[X]_{2,1/q}+E[X]_{1/q}-E^2[X]_{1/q}$. For convenience, in the rest of the paper, we denote $[X]_{1/q}$ as $X_{1/q}$.

Definition 3.2. *It is said that the random variable $X_{1/q}$ has a $1/q-$gamma distribution with (α,β,q) parameters if its pdf is given by*

$$f_{X_{1/q}}(x_{1/q})=\frac{e_q(-\beta x)q^{\binom{\alpha}{2}}x^{\alpha-1}\beta^\alpha}{\Gamma_q(\alpha)},\qquad x\in\mathbb{R}_q^+,$$

where $\alpha > 0$, $\beta > 0$, $0 < q < 1$ and $\binom{\alpha}{2} = \frac{\Gamma(\alpha+1)}{2!\Gamma(\alpha-2+1)}$. The distribution is denoted as $\Gamma_{1/q}(\alpha, \beta)$. A special case of the $1/q-$gamma distribution is important for us: when the parameter α is an integer value, we call it $1/q-$Erlang distribution. The special case $\alpha = 1$, $f_{X_{1/q}}(x_{1/q}) = \beta e_q(-\beta x)$ is $1/q-$exponential distribution. To continue, we will investigate whether there is a relationship between the $1/q-$Poisson and $1/q-$Erlang distributions.

Theorem 3.3 ($1/q-$gamma$-$Poisson relationship). *Suppose that $X_{1/q} \sim \Gamma_{1/q}(\alpha, 1)$ and $Y_{1/q} \sim Po_{1/q}(x, q)$, where α is an integer, then $P(X_{1/q} \leqslant x) = P(Y_{1/q} \geqslant \alpha)$.*

Proof. Let us show that

$$\int_0^x \frac{e_q(-qt)t^{\alpha-1}q^{\binom{\alpha}{2}}}{\Gamma_q(\alpha)} \nabla_q t = \sum_{y=\alpha}^{\infty} \frac{e_q(-x)x^y q^{\binom{y}{2}}}{[y]_q!},$$

the right-hand side is equal to $1 - e_q(-x)\sum_{y=0}^{\alpha-1} \frac{x^y q^{\binom{y}{2}}}{[y]_q!}$. Now, we apply the product rule for the $q-$derivative and get

$$\begin{aligned}
=&e_q(-x)\sum_{y=0}^{\alpha-1} \frac{(qx)^y q^{\binom{y}{2}}}{[y]_q!} - e_q(-x)\sum_{y=1}^{\alpha-1} \frac{x^{y-1} q^{\binom{y}{2}}}{[y-1]_q!}\\
=&e_q(-x)\sum_{y=0}^{\alpha-1} \frac{(qx)^y q^{\binom{y}{2}}}{[y]_q!} - e_q(-x)\sum_{y=0}^{\alpha-2} \frac{x^{y} q^{\binom{y+1}{2}}}{[y]_q!}\\
=&e_q(-x)\frac{x^{\alpha-1} q^{\binom{\alpha}{2}}}{\Gamma_q(\alpha)}.
\end{aligned}$$

□

4. Poisson Process on Quantum Time Scale

In this section, the intention is to provide a generalization of the Poisson processes via fractional calculus on quantum time scales, which are known to play a fundamental role in $1/q-$renewal theory. We first provide the basic $1/q-$renewal theory including its fundamental concepts like waiting time between events, the survival probability, and the counting function. If the waiting time is $1/q-$exponentially distributed we have a $1/q-$Poisson process. In this context, we analyze a $1/q-$renewal process with a waiting time distribution described by the $1/q-$Mittag-Leffler function. This distribution contains the $1/q-$exponential as a particular case.

4.1. Renewal Theory on Quantum Time Scale

A renewal process is a stochastic model that describes a class of counting processes where the time intervals between consecutive events are i.i.d. non-negative random variables with a specified probability distribution. These times are referred to as waiting times or inter-arrival times. For waiting times $T_1, T_2, \ldots$ we define variables

$$t_0 = 0, \;\; t_k = \sum_{j=1}^{k} q^{j-k} T_j, \;\; k \geq 1.$$

and named them the $1/q-$sum of the first k waiting times. So, $t_1 = T_1$ is the time of the first $1/q-$renewal, $t_2 = q^{-1}T_1 + T_2$ is the time of the second $1/q-$renewal and so on. In general, t_k denotes the kth $1/q-$renewal.

Now we determine the probability distribution for the waiting times in this process. To this end, we define the pdf $\phi(t)$ and the distribution function $\Phi(t)$ as:

$$\phi(t) := \frac{\nabla_q}{\nabla_q t}\Phi(t), \quad \Phi(t) := P(T \leqslant t) = \int_0^t \varphi(t')\nabla_q t'.$$

We refer to $\Phi(t)$ as the failure probability. Then the survival probability will be

$$\Psi(t) := P(T > t) = \int_t^\infty \varphi(t')\nabla_q t' = 1 - \Phi(t). \tag{7}$$

The non-negative random variable represents the lifetime of technical systems, and $\Phi(t)$ and $\Psi(t)$ are the respective probabilities that the system does or does not fail in the interval $(0, T]$. Also, we need a function that represents the effective number of events before or at instant t. A such quantity is the counting function $N(t)$ defined as:

$$N(t) := max\{k|t_k \leq t \leq t, k = 0, 1, 2, \ldots\}.$$

In a particular case, we have $\Psi(t) = P(N(t) = 0)$. Let us set $F_1(t) = \Phi(t)$, $f_1(t) = \phi(t)$. In general,

$$F_k(t) := P(t_k = q^{1-k}T_1 + q^{2-k}T_2 + ... + T_k \leq t), \ \ f_k(t) = \frac{\nabla_q}{\nabla_q t}F_k(t), \ \ k \geq 1,$$

thus $F_k(t)$ represents the probability that the $1/q-$sum of the first k waiting times is less or equal t and $f_k(t)$ is its density. It can be easily seen that for any fixed $k \geq 1$, the normalization condition for $F_k(t)$ is satisfied, because

$$\lim_{t\to\infty} F_k(t) = P(t_k = q^{1-k}T_1 + q^{2-k}T_2 + ... + T_k < \infty) = 1.$$

In fact, the $1/q-$sum of k random variables each of which is finite with probability 1 is finite with probability 1 itself. Note that for $k \geq 0$ we have

$$P(N(t) = k) := P(t_k \leq t, t_{k+1} > t) = \int_0^t f_k(t')\Psi(t - qt')_q \nabla t'. \tag{8}$$

We now find it convenient to introduce the simplified " $*$ " notation for the $q-$Laplace convolution between two functions $f(t)$ and $g(t)$ as

$$(f * g)(t) = \int_0^t f(t')g(t - qt')_q \nabla t',$$

such that if $g(t) = t^{\beta-1}$, then we have the same convolution as in Lemma 2.7.
The main importance of $q-$Laplace transforms in $1/q-$renewal theory lies in the connection with $1/q-$sums of independent random variables. Using the fact that $f_k(t)$ is the pdf of the $1/q-$sum of the k i.i.d. random variables $T_1, \ldots, T_k$ whose pdf is $\phi(t)$, we can easily obtain that $f_k(t)$ turns out to be the $k-$fold convolution of $\phi(t)$ with itself:

$$f_k(t) = (\phi^{*k})(t).$$

Therefore, Eq. (8) can be simply written as

$$P(N(t) = k) = (\phi^{*k} * \Psi)(t). \tag{9}$$

For example, we can see that if

$$\Psi(t) = e_q(-\lambda t)$$

and

$$f_k(t') = \lambda q^{\binom{k}{2}} \frac{(\lambda t')^{k-1}}{[k-1]_q!} e_q(-\lambda t'),$$

then, we can write

$$\begin{aligned}
&\int_0^t \lambda q^{\binom{k}{2}} \frac{(\lambda t')^{k-1}}{[k-1]_q!} e_q(-\lambda t') e_q(-\lambda(t-qt')_q) \nabla_q t' \\
&= \frac{\lambda^k q^{\binom{k}{2}}}{[k-1]_q!} \int_0^t (t')^{k-1} e_q(-\lambda t') e_q(-\lambda(t-qt')_q) \nabla_q t' \\
&= e_q(-\lambda t) \frac{q^{\binom{k}{2}}}{[k]_q!} (\lambda t)^k,
\end{aligned}$$

Note that the result follows from the definition of the nabla $q-$integral given by

$$\int_0^t (t')^{k-1} \nabla_q t' = (1-q)t \sum_{i=0}^{\infty} q^i (tq^i)^{k-1} = \frac{t^k}{[k]_q}.$$

Further, we have

$$\begin{aligned}
e_q(-\lambda t') e_q(-\lambda(t-qt')_q) &= \sum_{n=0}^{\infty} \frac{(-\lambda t')^n}{[n]_q!} \sum_{n=0}^{\infty} \frac{(-\lambda(t-qt')_q)^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{(t')^k}{[k]_q!} \frac{(t-qt')_q^{n-k}}{[n-k]_q!} \right) \lambda^n \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n}{[n]_q!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (t-qt')_q^{n-k} (t')^k \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n}{[n]_q!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q t^{n-k} (1-q\frac{t'}{t})_q^{n-k} t'^k.
\end{aligned}$$

where we apply the result of [11], i.e.

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q t^{n-k} (1-q\frac{t'}{t})_q^{n-k} t'^k = 1.$$

By applying the $q-$Laplace convolutions, a $1/q-$renewal process can be suited for the $q-$Laplace transform method. We recognize that Eq. (9) can be written in the $q-$Laplace domain as

$$\mathcal{L}_q\{P(N(t)=k); s\} = (\tilde{\phi}(s))_q^k \tilde{\Psi}(s),$$

where $(.)_q^k$ means the $q-$factorial function. By using Eq. (7), Theorems 2.1 and 2.3 from [9], we get the following identity

$$\tilde{\Psi}(s) = \frac{1-\tilde{\phi}(s)}{s}.$$

Similarly, one can define the $q-$Poisson process as a $q-$Renewal process. The $q-$Poisson process or the Euler stochastic process (see Section 3.4 in [10]), is a $q-$renewal process that can be characterized by a waiting time pdf of $q-$exponential type,

$$\phi(t) = \lambda E_q(-q\lambda t),\ \lambda > 0,\ t \geq 0,\ 0 < q < 1.$$

Its moments turn out to be

$$\mathbb{E}(T) = \frac{1}{\lambda},\ \mathbb{E}(T^2) = \frac{[2]_q!}{\lambda^2}, \dots, \mathbb{E}(T^n) = \frac{[n]_q!}{\lambda^n}, \dots$$

(see corollary 3.1 in [10]). The survival probability is

$$\Psi(t) := P(T > t) = E_q(-q\lambda t),\ \ \lambda > 0,\ t \geq 0,\ 0 < q < 1.$$

In this case, the probability that k events occur in the interval of length t is

$$P(N(t) = k) = \frac{(\lambda t)^k}{[k]_q!} E_q(-\lambda t),\ t \geq 0,\ k = 0, 1, 2, \dots. \tag{10}$$

The probability distribution related to the $q-$sum of k i.i.d. $q-$exponential random variables is known to be the so-called $q-$Erlang distribution (of order k). Then, the corresponding density (the $q-$Erlang pdf) is

$$f_k(t) = \lambda \frac{(\lambda t)^{k-1}}{[k-1]_q!} E_q(-q\lambda t),\ t \geq 0,\ k = 1, 2, \dots.$$

So, the $q-$Erlang distribution function of order k turns out to be

$$F_k(t) = \int_0^t f_k(t)\nabla_q t = 1 - \sum_{n=0}^{k-1} \frac{(\lambda t)^n}{[n]_q!} E_q(-q\lambda t) = \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{[n]_q!} E_q(-q\lambda t),\quad t \geq 0. \tag{11}$$

The results (10)-(11) can be easily obtained by using the technique of the $q-$Laplace transform sketched in the previous section. Further, for the $q-$Poisson process, we have

$$\tilde{\phi}(s) = \frac{\lambda}{\lambda + s},\ \ \tilde{\Psi}(s) = \frac{1}{\lambda + s},$$

and for the $q-$Erlang distribution:

$$\tilde{f}_k(s) = [\tilde{\phi}(s)]^k = \frac{\lambda^k}{(\lambda + s)_q^k},\ \ \tilde{F}_k(s) = \frac{[\tilde{\phi}(s)]^k}{s} = \frac{\lambda^k}{s(\lambda + s)_q^k}.$$

Also, the survival probability for the $q-$Poisson renewal process obeys the $q-$differential equation

$$\frac{\nabla_q \Psi(t)}{\nabla_q} = -\lambda \Psi(qt),\ \ t \geq 0;\ \ \Psi(0^+) = 1.$$

4.2. The $1/q-$Renewal Process of $1/q-$Mittag-Leffler Type

In this section, we introduce a fractional generalized $1/q-$Poisson distribution by replacing the first $q-$derivative operator with the Caputo $q-$fractional difference of order ν in its generating difference equation. Hence, we have now the new ordinary $q-$fractional difference equation,

$${}_qC_0^{\nu}\Psi(t) = -\lambda \Psi(t),\quad t \geq 0, 0 < \nu \leq 1,\ \ \Psi(0^+) = 1. \tag{12}$$

We also allow the limiting case $\nu = 1$ where all the results of the previous section are expected to be recovered. For our purpose, we need to recall the $1/q-$Mittag-Leffler function as the natural fractional generalization of the $1/q-$exponential function, that characterizes the $1/q-$Poisson process. By taking a $q-$Laplace transform from both sides of Eq. (12) and applying Lemma 2.7, it can be written as:

$$\begin{aligned} &\mathcal{L}_q\{{}_qC_0^{\nu}\Psi(t)\} = -\lambda \mathcal{L}_q\{\Psi(t)\}, \\ &s^{\nu}\tilde{\Psi}(s) - s^{\nu-1}\Psi(0^+) = -\lambda \tilde{\Psi}(s), \\ &\tilde{\Psi}(s) = s^{\nu-1}(s^{\nu} + \lambda)^{-1}. \end{aligned} \tag{13}$$

On the other hand, we know that

$$\mathcal{L}_q\{{}_{1/q}E_\nu(-\lambda,t)\} = s^{\nu-1}(s^\nu+\lambda)^{-1}.$$

By simple inspection, we can see that Eq. (13) automatically yields the solution $\Psi(t)$, which is the $1/q-$Mittag-Leffler function

$${}_{1/q}E_\nu(-\lambda,t) = \sum_{n=0}^{\infty} \frac{(-\lambda t^\nu)^n}{\Gamma_q(1+\nu n)},$$

mentioned earlier.

Furthermore, the $1/q-$Poisson and $1/q-$Erlang distributions (corresponding to the n^{th} arrival or event time, $n \in \mathbb{N}$) are generalized in the following way. By considering corollary 2.5 in [9], it can be shown that

$$\mathcal{L}_q\{\frac{q^{\binom{n}{2}}(\lambda t)^n}{[n]_q!}e_q(-\lambda t)\} = \frac{\lambda^n}{(s+\lambda)^n_{1/q}(s+q^n\lambda)},$$

$$\mathcal{L}_q\{\lambda\frac{q^{\binom{n}{2}}(\lambda t)^{n-1}}{[n-1]_q!}e_q(-\lambda t)\} = \frac{\lambda^n}{(s+\lambda)^n_{1/q}}.$$

We see from Eq. (6),

$$\mathcal{L}_q\{t^{\nu n}\,{}_{1/q}E_\nu^{(n)}(-\lambda,t)\} = \frac{q^{-\binom{n+1}{2}}[n]_q!s^{\nu-1}}{(s^\nu+\lambda)^n_{1/q}(s^\nu+q^n\lambda)},$$

where

$${}_{1/q}E_\nu^{(n)}(y) = \frac{\nabla_q^n\,{}_{1/q}E_\nu(y)}{\nabla_q y^n}.$$

This implies that a generalization of the $1/q-$Poisson distribution is given by

$$P_n^\nu(t) = P(N_\nu(t)=n) = \frac{q^{\binom{n}{2}}t^{\nu n}\lambda^n}{[n]_q!}\,{}_{1/q}E_\nu^{(n)}(-\lambda,t), \tag{14}$$

where the $q-$Laplace transform of the pmf is

$$\mathcal{L}_q\{P_n^\nu(t)\} = \frac{\lambda^n s^{\nu-1}}{(s^\nu+\lambda)^n_{1/q}(s^\nu+q^n\lambda)}.$$

Accordingly, a generalization of the $1/q-$Erlang distribution is shown to be

$$f(T = q^{1-n}T_1 + q^{2-n}T_2 + ... + T_n) = f_n^\nu(t) = \lambda^n\nu\frac{q^{\binom{n}{2}}t^{\nu n-1}}{[n-1]_q!}\,{}_{1/q}E_\nu^{(n)}(-\lambda,t), \tag{15}$$

where the $q-$Laplace transform of the pdf is

$$\mathcal{L}_q\{f_n^\nu(t)\} = \frac{\lambda^n}{(s^\nu+\lambda)^n_{1/q}}.$$

It is interesting to know that, when $\nu \to 1$ the distributions (14) and (15) converge to $1/q-$Poisson and $1/q-$Erlang distributions, respectively. Also, when $\nu \to 1$ and $q \to 1$, the distributions (14) and (15) converge to Poisson and Erlang distributions, respectively.

5. Simulation study

In this section, we discuss the maximum likelihood (ML) method for the estimation of the distributions parameters. The two distributions that we consider in this context are the $1/q-$Poisson and $1/q-$Erlang distributions. The function that we maximize is of the form

$$L = \sum_{i=1}^{N} \log f(X_i; \boldsymbol{\theta}),$$

where $f(.)$ is the corresponding probability function, and $\boldsymbol{\theta}$ is the vector of parameters. The maximization of the above function, for a given data set $\{X_i\}_{i=\overline{1,N}}$, is obtained numerically.

The ML method is tested on the simulated data sets. Using the $1/q-$Poisson distribution we generate 100 samples of lengths 100, 250, and 500. The samples are generated with three sets of parameters: a) $\lambda = 0.7$, $\nu = 0.7$, $q = 0.8$; b) $\lambda = 1$, $\nu = 0.5$, $q = 0.3$; and c) $\lambda = 0.5$, $\nu = 0.1$, $q = 0.8$. On this simulated data sets the efficiency of the ML method for parameter estimation is tested. Beside the estimated values, the standard error of the estimates is observed and discussed. The parameters values were chosen to be similar to the values that will be met in the next section when we discuss the real-world data examples. The same procedure is conducted for the $1/q-$Erlang distribution where the three sets of parameters used to generated data sets are: a) $\lambda = 1.5$, $\nu = 0.8$, $q = 0.5$; b) $\lambda = 1$, $\nu = 0.3$, $q = 0.7$; and c). For all three sets, the value of k is set to $k = 3$. The results are presented in Table 1. Under the estimated values, the standard errors of the estimates are given.

Table 1: Estimated parameters with their standard errors obtained with the ML method:

(a) $1/q$-Poisson distributions.

size	λ	ν	q
a) $\lambda = 0.7, \nu = 0.7, q = 0.8$			
100	0.8464	0.7406	0.8142
	(0.1788)	(0.0455)	(0.1414)
250	0.7953	0.6504	0.8201
	(0.0118)	(0.0404)	(0.0928)
500	0.7406	0.7199	0.8117
	(0.0105)	(0.0280)	(0.0752)
b) $\lambda = 1.0, \nu = 0.5, q = 0.3$			
100	0.9528	0.6141	0.2834
	(0.1531)	(0.1738)	(0.1025)
250	0.9828	0.5711	0.3434
	(0.1405)	(0.1029)	(0.0799)
500	1.0273	0.4402	0.3360
	(0.0891)	(0.0758)	(0.0535)
c) $\lambda = 0.5, \nu = 0.1, q = 0.8$			
100	0.4654	0.2576	0.9151
	(0.0789)	(0.0849)	(0.1066)
250	0.4426	0.2045	0.7541
	(0.0505)	(0.0234)	(0.0845)
500	0.4755	0.1583	0.8190
	(0.0383)	(0.0176)	(0.0581)

(b) $1/q$-Erlang distributions.

size	λ	ν	q
a) $\lambda = 1.5, \nu = 0.8, q = 0.5$			
100	1.408	0.6685	0.5975
	(0.0329)	(0.0521)	(0.034)
250	1.4086	0.6865	0.5981
	(0.0218)	(0.0211)	(0.0237)
500	1.4839	0.7676	0.4495
	(0.0035)	(0.009)	(0.0032)
b) $\lambda = 1.0, \nu = 0.3, q = 0.7$			
100	0.9196	0.2152	0.6627
	(0.0642)	(0.0587)	(0.1250)
250	0.9736	0.3193	0.6688
	(0.0534)	(0.0317)	(0.1034)
500	1.039	0.2997	0.6781
	(0.038)	(0.0085)	(0.0038)
c) $\lambda = 2.0, \nu = 0.5, q = 0.5$			
100	1.846	0.4517	0.5718
	(0.115)	(0.0348)	(0.0884)
250	2.071	0.4553	0.5384
	(0.0701)	(0.0102)	(0.0352)
500	2.064	0.4727	0.5178
	(0.0331)	(0.0012)	(0.0109)

As we can conclude from Table 1, the estimated values converge to their true values as we increase the sample size. Also, the standard errors of the estimates become smaller with the increase of the sample size. For the $1/q-$Poisson distribution, the convergence of estimates for parameter ν are a bit slower, especially when the true value of ν is a small number. While some estimates are off their true values for samples of the length 100, the values estimated from samples 500 are quite close to the true values. The similar conclusions can be made for the $1/q-$Erlang distribution.

6. Applicability to practical data

In this section, we discuss the application of the $1/q-$Poisson distribution where we observe a data set from the real world and compare the results with the standard Poisson distribution and the negative binomial distribution (NB). The data set that we observe contains the records of serious earthquakes in Turkey (the region of the North Anatolian Fault Zone) in the 20th century. The data was gathered at the Kandilli Observatory (Turkish observatory) and includes the date and time, location, magnitude, number of fatalities, and the count of damaged structures. The interested reader can also find these data in the research paper [5].

We fit the distributions to the data where we observe the number of earthquakes per year. The mean value and the variance are 0.81 and 0.97, respectively, which shows the small over-dispersed behavior of the data. The bar plot of the data is presented with Figure 1, and the main statistical values are presented in Table 2.

Table 2: Properties of the observed data set.

Mean	Variance	Index of dispersion
0.81	0.97	1.19

Before we start the modeling of the data, let us perform some tests to confirm that the series is independent, stationary and that the data set follows the Poisson distribution. The results of these tests are summarized in Table 3 and analyzed in the forthcoming discussion.

Table 3: Statistical tests of the observed data set.

Test statistics	value	p-value
Von-Neumann's	-1.86	0.062
Wald–Wolfowitz	1.91	0.055
χ^2	6.58	0.086

To test the independency, we used Von-Neumann's test (e.g. [24], [6], [19]). The null hypothesis is that the data are i.i.d. random quantities, and the alternative is that the data are not randomly distributed. The value of the test statistics is -1.86, while the p-value is 0.062. Thus we accept the null hypothesis.

For testing the stationarity the Wald–Wolfowitz test was applied (e.g. [25]). The null hypothesis is that the data are stationary. The test statistics returns a value of 1.91, while the p-value is 0.055, so we accept the null hypothesis.

Finally, we used the χ^2-test to test the hypothesis that the data follows the Poisson distribution. The obtained value of the test statistics is 6.58, while the p-value is 0.086. Thus we can conclude that the data can be modeled with the Poisson distribution.

In all these tests, we observe that the p-value is near the critical value, yet none of the tests indicate rejecting the null hypothesis for the confidence level $\alpha = 0.05$.

The parameters of the observed distributions are estimated by the maximum likelihood method. The results are summarized in Table 4, where the estimated parameters and the values of the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) are given. According to the values shown in Table 4, we can conclude that $1/q-$Poisson slightly better fits the observed data than the other two considered distributions. The maximum likelihood for the NB(n, p) distribution is obtained for $n = 3$. We included in the discussion the NB distribution since the index of dispersion of the observed data is slightly over one, i.e. it is 1.19. The theoretical index of dispersion for $1/q-$Poisson distribution (Eq. (16)) is 1.11, while for the NB it is 1.27. So, they are both really close to the empirical value. Additionally, to justify the adequacy of the $1/q-$Poisson distribution for the observed data, we perform the χ^2 test. The value of the test is 5.982, while the p-value is 0.112. Also, it should be noticed that the value of the χ^2 test for the NB distribution is

1.356 with the p-value 0.715. Thus, all three discussed distributions can be applied to the observed data set.

$$ID = 1 - \lambda\left(1 - \frac{1}{q}\right) \tag{16}$$

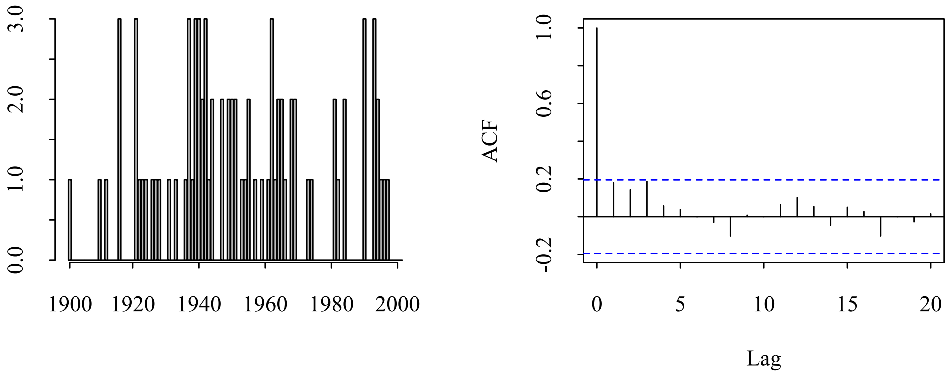

Figure 1: Number of earthquakes per year in the North Anatolian Fault Zone, Turkey, and the corresponding autocorrelation values of the series.

Table 4: Estimated parameters and AIC, BIC, and χ^2-test values obtained from $1/q$-Poisson, Poisson, and NB distributions.

distribution	parameters	AIC	BIC	χ^2-test	p-value
$1/q-$Poisson	$\lambda = 0.7016$, $\nu = 0.714$, $q = 0.858$	127.06	134.91	5.982	0.112
Poisson	$\lambda = 0.811$	251.82	254.43	6.58	0.086
NB	$n = 3$, $p = 0.787$	251.82	257.05	1.356	0.715

Considering the same data set, we apply the $1/q-$Erlang and Erlang distributions to model the probability of striking three earthquakes in the period of one decade. For this purpose, the data set is restructured to represent the time between earthquakes. The time between two earthquakes is set as a fraction of decade. For this test, we set $k = 3$, but we can utilize a different value if needed. The results are presented in Table 5. We can notice that AIC and BIC values are lower for the $1/q-$Erlang distribution than for the Erlang distribution, which makes it a little bit more favourable for the particular data set, and the pre-assumed test specifications.

Table 5: Estimated parameters and AIC and BIC values obtained from $1/q$-Erlang, and Erlang distributions for $k = 3$.

distribution	parameters	AIC	BIC
$1/q-$Erlang	$\lambda = 1.877$, $\nu = 0.208$, $q = 0.893$	373.92	381.76
Erlang	$\lambda = 3.71$	447.6	450.21

Conclusion

Firstly, we provided the basics of the $1/q-$renewal theory, including its fundamental concepts like waiting time between events, the survival probability, and the counting function. If the waiting time is $1/q-$exponentially distributed, we have a $1/q-$Poisson process. Also, we analyzed a $1/q-$renewal process with a waiting time distribution described by the $1/q-$Mittag-Leffler function. The probability distribution related to the $1/q-$sum of i.i.d. $1/q-$ random variables is the $1/q-$distribution.

Choosing the appropriate distribution for a stochastic process results in more accurate predictions and inferences about the process. Using the real-world data, we found that the fitting with the $1/q-$Poisson distribution yields a higher log-likelihood, a smaller AIC value, and a smaller BIC value than the fitting with the Poisson and the NB distributions. We used the $1/q-$Poisson distribution to fit the number of earthquakes per year, where the distribution function results from the fractional calculation on quantum time scale.

As another application, we applied the $1/q-$Erlang and the Erlang distributions to model the waiting time for three earthquakes to occur within a ten-year span. The AIC and the BIC values are lower for the $1/q-$Erlang distribution than for the Erlang distribution. Both fitting schemes confirm the superiority of quantum distributions over ordinary distributions.

The parameters of the $1/q-$Poisson and the $1/q-$Erlang distributions were estimated by the maximum likelihood method and performance of the method was discussed on simulated data sets. Future topics for an interested researcher might be exploring in details the estimation methods for parameters of the presented distributions.

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