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Ergodic type theorems via statistical convergence

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Abstract. In the present paper we obtain some mean ergodic and uniform ergodic type theorems via statistical convergence in a Banach space. We prove, in this case that, the mean ergodic decomposition remains true. We also characterize statistical uniform ergodicity for an operator $T \in B(X)$ under the condition $st - \lim_n \frac{\|T^n\|}{n} = 0$.

1. Introduction

The main idea of the ergodic theory is to investigate the convergence of the sequence given by $M_n(T) := \frac{1}{n}\sum_{k=0}^{n-1} T^k$, where T is a bounded linear operator on a Banach space X and the iterates of the operator T are defined by induction $T^0 = I$ and $T^n = T \circ T^{n-1}$. An operator T is called mean ergodic, respectively uniformly ergodic, if its Cesàro averages $\{M_n(T)\}$ is strongly, respectively uniformly convergent in $B(X)$. We denote by P the strong limit in $B(X)$ of $\{M_n(T)\}$, it is a projection onto the kernel of the operator $I - T$, corresponding to the ergodic decomposition $X = N(I - T) \oplus \overline{(I - T)X}$. Recall that an operator $T \in B(X)$ is called power bounded if $\sup_n \|T^n\| < \infty$.

Initially, the mean ergodic theorem of von Neumann was proved for unitary operators in a complex Banach space X [17]. Afterwards, this theorem was given by Riesz [18] for power bounded operators on $L_p, 1 < p < \infty$, by Kakutani [8] and Yosida [19] (independently) for power bounded operators in a reflexive Banach space.

In the present paper, using statistical convergence we prove some mean and uniform ergodic type theorems. We show, among other things that the mean ergodic decomposition remains true.

Now we give some basic notation concerning the concept of statistical convergence. The notion of statistical convergence was introduced by Fast [5] and developed by many authors (see, e.g, [2], [3], [7], [9], [10], [13]). Let K be a subset of the natural numbers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of the set K_n. The natural (asymptotic) density of K is given by $\delta(K) := \lim_n \frac{1}{n}|K_n|$ whenever the limit exists. K is said to be statistically dense if $\delta(K) = 1$ [2]. A sequence (x_k) of (real or complex) numbers is said to be statistically convergent to some number L, if for every $\varepsilon > 0$, the set $K = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero; in this case, we write $st - \lim x_k = L$.

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The real number sequence x is said to be statistically bounded if there is a number B such that $\delta(\{k : |x_k - L| > B\}) = 0$ [7].

It is well-known that $st - \lim x_k = L$ if and only if there exists a subset $K = \{n_1 < n_2 < ...\}$ with $\delta(K) = 1$ such that $\lim_k x_{n_k} = L$ (i.e., $\lim_{k \in K} x_k = L$), (see, e.g, [3], [6], [10], [13]). Note that a convergent sequence is also statistically convergent to its limit value and the theory of statistical convergence differs from ordinary theory of convergence in at least one important way: a statistically convergent sequence need not be bounded [6].

Throughout the paper we will call an operator $T \in B(X)$ a statistically mean ergodic operator, respectively a statistically uniformly ergodic operator, if the statistical limit of $\{M_n(T)x\}$, respectively the statistical limit of $\{M_n(T)\}$ exists.

2. Mean ergodic theorems via statistical convergence

In this section, using the concept of statistical convergence we give some extensions of the mean ergodic type theorems. Our main result is motivated by that of Theorem 3.6.9 in [1].

Let X be a Banach space and let $U \subset X$, $l \in X$. Then we say that l is in the st-hull of U if there is a sequence (x_n) of elements in U so that $st - \lim x_n = l$. The set U is st-closed if it contains all of the elements in its st-hull. By $\overline{U}^{st}$ we denote the st-hull of U. Observe that $U \subset \overline{U} \subset \overline{U}^{st}$, and U is st-closed if and only if $\overline{U}^{st} = U$.

In the proof of the main theorem we will make use of the following lemma which is essentially proved in [4].

Lemma 2.1. *Let X be a Banach space and let $U \subset X$. Then $\overline{U} = \overline{U}^{st}$.*

We now present the main theorem of this section.

Theorem 2.2. *Let X be a Banach space and $T \in B(X)$. Assume that there are a constant $c \geq 1$ and a set $E \subset \mathbb{N}$ with $\delta(E) = 1$ and for all $n \in E$, $\|T^n\| \leq c$. For $n \in \mathbb{N}$ define the linear operator $M_n : X \to X$ by*

$$M_n = M_n(T) := \frac{1}{n} \sum_{k=0}^{n-1} T^k.$$

Then the following assertions hold:

(i) Let $x \in X$. The sequence $(M_n x)_{n \in \mathbb{N}}$ statistically converges if and only if it has a statistically dense subsequence which is weakly statistically convergent.

(ii) The set

$$Z := \{x \in X : \text{the sequence } (M_n x)_{n \in \mathbb{N}} \text{ statistically converges}\}$$

is a (statistically) closed T-invariant linear subspace of X and

$$Z = N(I - T) \oplus \overline{R(I - T)}$$

Furthermore, if X is reflexive, then $Z = X$.

(iii) Define the bounded linear operator $P : Z \to Z$ by $P(x + y) := x$, for $x \in N(I - T)$ and $y \in \overline{R(I - T)}$. Then for all $z \in Z$,

$$st - \lim_n M_n z = Pz$$

and $TP = PT = P = P^2$, $\|P\| \leq c$.

Proof. For convenience, we will examine the proof in eight steps.

Step 1. Let $E \subset \mathbb{N}$. Then $\|M_n\| \leq c$ and $\|M_n(I-T)\| \leq \dfrac{1+c}{n}$, for all $n \in E$ such that $\delta(E) = 1$.

Since $\|T^n\| \leq c$ for all $n \in E$ such that $\delta(E) = 1$, we have

$$\|M_n\| = \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k \right\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k\| \leq c, \quad (n \in E)$$

and

$$\|M_n(I-T)\| = \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k(I-T) \right\| \leq \frac{1}{n}(\|I\| + \|T^n\|) \leq \frac{1+c}{n}, \quad (n \in E).$$

Hence the proof of Step 1 is completed.

Step 2. Let $x \in X$ such that $Tx = x$. Then $M_n x = x$ for all $n \in \mathbb{N}$ and

$$\|x\| \leq c\|x + \xi - T\xi\|, \quad \text{for all } \xi \in X.$$

From Step 1, we have $\|M_n(I-T)\| \leq \dfrac{1+c}{n}$. Applying the operator $\lim\limits_{n\in E}$ to both sides of the inequality, we get that

$$st - \lim_n \|M_n(I-T)\xi\| = 0, \quad (\text{for all } \xi \in X).$$

Hence we obtain

$$\begin{aligned} \|x\| &= st - \lim_n \|x + M_n(I-T)\xi\| = st - \lim_n \|M_n(x + \xi - T\xi)\| \\ &\leq (st - \lim_n \|M_n\|)\|x + \xi - T\xi\| \\ &\leq c\|x + \xi - T\xi\|. \end{aligned}$$

This proves Step 2.

Step 3. If $x \in N(I-T)$ and $y \in \overline{R(I-T)}^{st}$ then $\|x\| \leq c\|x+y\|$.

From Lemma 2.1, we know that $\overline{R(I-T)} = \overline{R(I-T)}^{st}$. If we apply the same technique as in [1, page 150], the proof of Step 3 holds.

Step 4. $N(I-T) \cap \overline{R(I-T)} = \{0\}$ and the direct sum $Z = N(I-T) \oplus \overline{R(I-T)}$ is a (statistically) closed subspace of X.

As in the proof of the Step 3, first assumption clearly holds from [1, page 150]. In order to show that Z is closed, take $z_n \in Z$. Hence $z_n = x_n + y_n$ such that $x_n \in N(I-T)$ and $y_n \in \overline{R(I-T)}$ and (z_n) statistically converges to some element $z \in X$. Then (z_n) is a Cauchy sequence and hence (x_n) is a Cauchy sequence from Step 3. This implies that $y_n = z_n - x_n$ is a Cauchy sequence and hence $z = x + y$, where $x := st - \lim x_n \in N(I-T)$ and $y := st - \lim y_n \in \overline{R(I-T)}$. This concludes the proof of Step 4.

Step 5. If $z \in Z$ then $Tz \in Z$.

Let $z \in Z$. Then $z = x + y$ such that $x \in N(I-T)$ and $y \in \overline{R(I-T)}$. Similar to the proof of Step 4, the proof of Step 5 is observed from [1, page 151].

Step 6. Let $x \in N(I-T)$ and $y \in \overline{R(I-T)}$. Then $x = st - \lim\limits_n M_n(x+y)$.

Since $x \in N(I-T)$, we get $M_n x = x$. So we have

$$st - \lim_n M_n x = x. \tag{1}$$

On the other hand, since $y \in \overline{R(I-T)}$, there exists $y_j \in R(I-T)$ such that $y_j \to y$. For $\xi_j \in X$, we can write $y_j = \xi_j - T\xi_j$. From Step 1, we have for $n \in E$ with $\delta(E) = 1$ that

$$\|M_n y_j\| = \|M_n(\xi_j - T\xi_j)\| \leq \frac{1+c}{n}\|\xi_j\|.$$

This implies that

$$st-\lim_n \|M_n y_j\| = 0. \tag{2}$$

Furthermore, we can write, for $y \in \overline{R(I-T)}$ that,

$$\|M_n y\| \leq \|M_n(y_j - y)\| + \|M_n y_j\|.$$

Given $\epsilon > 0$, there exist $j_0(\epsilon)$ so that for all $j \geq j_0(\epsilon)$ we have $\|y_j - y\| \leq \dfrac{\epsilon}{c}$. Hence we necessarily have, for all $j \in E$, $j \geq j_0(\epsilon)$, that

$$\|M_n(y_j - y)\| = \left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k(y_j - y) \right\| \leq \frac{1}{n}\sum_{k=0}^{n-1} \|T^k\| \|(y_j - y)\| \leq c\|(y_j - y)\| < \varepsilon$$

from which we immediately get

$$st-\lim_n \|M_n(y_j - y)\| = 0. \tag{3}$$

Hence, Step 6 follows from (1), (2) and (3).

Step 7. Let $x, z \in X$. The following are equivalent:

(a) $Tx = x$ and $z - x \in \overline{R(I-T)}$.

(b) $st-\lim_n \|M_n z - x\| = 0$.

(c) There is a sequence of integers $1 \leq n_1 \leq n_2 \leq n_3 \leq ...$ such that for all $f \in X'$ we have

$$st-\lim_i \|f(M_{n_i} z)\| = f(x).$$

The implication $(a) \Rightarrow (b)$ follows from Step 6.

We now prove $(b) \Rightarrow (c)$ Since $(M_n z)$ is statistically convergent, a statistically dense subsequence of it also statistically converges. Hence the subsequence weak statistically converges to same limit.

$(c) \Rightarrow (a)$ Let $f \in X'$. Then

$$T^* f = f \circ T : X \to \mathbb{R}$$

is a bounded linear functional and

$$\begin{aligned} f(x - Tx) &= ((I-T)^* \circ f)(x) \\ &= st-\lim_i ((I-T)^* \circ f)(M_{n_i} z) \\ &= st-\lim_i (f \circ (I-T))(M_{n_i} z). \end{aligned}$$

Let $B := \{n_i : i = 1, 2, ...\}$, then we have $\delta(B) = 1$. Moreover, from Step 1 we know that the sequence $M_n(I-T)z$ statistically converges to zero. Recall that $\delta(E) = 1$. Now let $E := \{n_j : j = 1, 2, ...\}$. Then by Step 1, $st-\lim_j M_{n_j}(I-T)z = 0$. Hence we have

$$\lim_{k \in B \cap E} (f(I-T))(M_k z) = 0. \tag{4}$$

So, we obtain for every $f \in X'$ that $f(x - Tx) = 0$ by (4) which yields $Tx = x$. Following the similar method as in [1, page 151] one can show that $z - x \in \overline{R(I-T)}$.

Step 8. Now we prove Theorem 2.2.

(*i*) follows from Step 7. Combining Step 4 and Step 5, (*ii*) is obtained. We just prove (*iii*). Since the operator $P : Z \to Z$ is defined by $Pz = P(x + y) := x$, we obtain from Step 6 that $st - \lim_n M_n z = Pz$. On the other hand, Step 1 implies that

$$\|P\| = \sup_{\|z\|\leq 1} \|Pz\| = \sup_{\|z\|\leq 1} \|st - \lim_n M_n z\| = \sup_{\|z\|\leq 1} st - \lim_n \|M_n z\| \leq \sup_{\|z\|\leq 1} c\|z\| \leq c.$$

Finally we need to show that $TP = PT = P = P^2$. We already know that $Pz := st - \lim_n M_n z = x$, for $x \in N(I - T)$. Therefore for all $z \in Z$, we get

$$Tx = TPz = x = Pz.$$

This directly gives us $TP = P$ and $T^k P = P$ for all $k \in \mathbb{N}$. From this, we obtain

$$st - \lim_n M_n Pz = st - \lim \frac{1}{n} \sum_{k=0}^{n-1} T^k Pz = Pz.$$

Hence for all $z \in Z$ we immediately have $P^2 z = Pz$ which yields $P^2 = P$. By Step 1, we know for all $z \in Z$ that $st - \lim_n \|M_n(I - T)z\| = 0$. Hence we obtain

$$st - \lim_n M_n z = st - \lim_n M_n Tz$$

which yields $P = PT$. This concludes the proof. □

Fridy and Orhan [7] proved, for a real sequence (x_n), that

$$\liminf x_n \leq st - \liminf x_n \leq st - \limsup x_n \leq \limsup x_n.$$

Hence our next result is sharper than that of Proposition 2.1 of [15].

Theorem 2.3. *Let X be a Banach space and $T \in B(X)$. Assume further that $st - \lim_n \frac{\|T^n x\|}{n} = 0$ (for all $x \in X$) and there exists a set $E \subset \mathbb{N}$ with $\delta(E) = 1$ such that $S_T := \sup_{n\in E} \|M_n\| < \infty$. Let $Y = \overline{R(I - T)}$. Then for all $x \in X$, the following inequality holds:*

$$\operatorname{dist}(x, Y) \leq st - \liminf_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k x \right\| \leq st - \limsup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k x \right\| \leq S_T \operatorname{dist}(x, Y),$$

where dist(x, Y) *stands for the distance of x to the set Y.*

Proof. Let $x \in X$. Since $st - \lim_n \frac{\|T^n x\|}{n} = 0$, we get a set $F \subseteq \mathbb{N}$ with $\delta(F) = 1$ so that $\lim_{n\in F} \frac{\|T^n x\|}{n} = 0$. On the other hand by Step 1, we already know that

$$\lim_{n\in E} \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k y \right\| = 0 \tag{5}$$

with $\delta(E) = 1$ and for all $y \in Y$. Observe now that the set $G := E \cap F$ has denstiy 1. Hence we can write for all $n \in G$ that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k x - \frac{1}{n} \sum_{k=0}^{n-1} T^k y \right\| \leq S_T \|x - y\|. \tag{6}$$

Combining this with (5) we obtain

$$st-\limsup_n \left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k x \right\| \le S_T \operatorname{dist}(x,Y)$$

by (6). On the other hand let us take $\varphi \in X'$ such that $T^*\varphi = \varphi$ and $\|\varphi\| \le 1$ where T^* is the adjoint operator of T. This clearly gives $\varphi(T^k x) = \varphi(x)$. Then for all $x \in X$, we obtain

$$|\varphi(x)| \le st-\liminf_n \left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k x \right\|,$$

which implies that

$$\operatorname{dist}(x,Y) \le st-\liminf_n \left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k x \right\|$$

because of the fact that $\operatorname{dist}(x,Y) = \sup\{|\varphi(x)| \,:\, T^*\varphi = \varphi,\ \|\varphi\| \le 1\}$ (see, e.g, [14]). Hence the proof is completed.
□

As a result of this theorem we can easily obtain the following:

Corollary 2.4. *The set $Y = \overline{R(I-T)}$ is characterized as follows:*

$$Y := \{x \in X \,:\, st-\lim_n \left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k x \right\| = 0\}.$$

Remark 2.5. *The proofs of Step 6 and Step 7 of Theorem 2.2 can also be achieved with the help of Corollary 2.4.*

The next theorem is motivated by Lemma 4.2 in [16].

Theorem 2.6. *Let X be a Banach space and $T \in B(X)$ be a power bounded operator. Assume that $x \in X$. Then the sequence $\{T^n x\}$ is statistically convergent if and only if $st-\lim_n \|T^{n+1}x - T^n x\| = 0$ and the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}$ is statistically convergent.*

Proof. First assume that $\{T^n x\}$ is statistically convergent. Then $\{T^n x\}$ is strongly C_1-convergent since $\{T^n x\}$ is bounded [3]. Hence $\{T^n x\}$ is C_1-convergent. For the sufficiency, suppose that $\left\{\frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}$ is statistically convergent sequence. Now let

$$V := \{y \in X \,:\, st-\lim_n \|T^{n+1}y - T^n y\| = 0\}.$$

The subspace V is T-invariant and closed. Since $st-\lim_n \|T^{n+1}y - T^n y\| = 0$ for all $y \in V$ and T is a power bounded operator, we obtain

$$st-\lim_n \|T^n y\| = 0, \qquad \text{for all } y \in \overline{(I-T)V}. \tag{7}$$

Now consider the set $W := \{x \in V \,:\, st-\lim_n M_n x \ \text{ exists}\}$. By Theorem 2.2, $x \in W$ may be written as $x = x_0 + y_0$ such that $x_0 \in N(I-T)$ and $y_0 \in \overline{(I-T)V}$. Hence we have $T^n x = T^n x_0 + T^n y_0$ which yields $T^n x = x_0 + T^n y_0$. This implies, by (7), that

$$st-\lim_n \|T^n x - x_0\| = 0.$$

Hence the proof is completed. □

3. Uniform ergodic theorems via statistical convergence

In this section, we give some extensions of the uniform ergodic type theorems via the concept of the statistical convergence. First, we give a proposition which will be used in the proof of the main theorem of this section.

Proposition 3.1. *Let X be a Banach space and $T \in B(X)$ and $N(I-T) = \{0\}$. Assume that $st - \lim_n \frac{\|T^n\|}{n} = 0$. Then the following assertions are equivalent:*

(i) $I - M_n$ *is surjective,* (*for all* $n \in \mathbb{N}$).
(ii) $I - T$ *is surjective.*
(iii) $st - \lim_n \|M_n\| = 0$.

Proof. $(i) \Rightarrow (ii)$ From the assumption there exists an $x \in X$ such that $(I - M_n)x = y$. Thus we have

$$y = (I - M_n)x = (I-T)\frac{1}{n}\sum_{p=0}^{n-1}\sum_{k=0}^{p} T^k x,$$

which yields $I - T$ is surjective.
$(ii) \Rightarrow (iii)$ Because of the assumption that $\ker(I-T) = \{0\}$, $I - T$ is injective and onto by (ii). Furthermore it is obvious that $I - T$ is continuous. By the Open Mapping Theorem, the inverse operator $(I-T)^{-1}$ is continuous as well. Let us denote by B the closed unit ball in X. Then $C := (I-T)^{-1}B$ is bounded. Let $K = \sup_{x\in C} \|x\|$. Then we get

$$\begin{aligned}\|M_n\| &= \sup_{z\in B}\|M_n z\| = \sup_{x\in C}\|M_n(I-T)x\| = \sup_{x\in C}\|\frac{1}{n}(T - T^{n+1}x)\| \\ &\le \frac{1}{n}\sup_{x\in C}\|Tx\| + \frac{n+1}{n}\sup_{x\in C}\left\|\frac{1}{n+1}T^{n+1}x\right\|.\end{aligned}$$

Applying the operator $st - \lim_n$ to both sides of the inequality given above, we get that

$$st - \lim_n \|M_n\| = 0.$$

$(iii) \Rightarrow (i)$ Since $st - \lim_n \|M_n\| = 0$ we have $\delta(\{n : \|M_n\| < \epsilon\}) = 1$ for every $\epsilon > 0$. Taking $\epsilon = 1$ we conclude that $\|M_{n_0}\| < 1$ for an $n_0 \in \mathbb{N}$. Thus, we have that $I - M_n$ is invertible which yields $I - M_n$ is surjective. □

The next theorem is an extension of the Uniform Ergodic Theorem given by Lin [11].

Theorem 3.2. *Let X be a Banach space and $T \in B(X)$. Assume further that $st - \lim_n \frac{\|T^n\|}{n} = 0$. Then the following assertions are equivalent:*

(i) T *is statistically uniformly ergodic operator.*
(ii) $(I-T)X$ *is closed and* $X = \ker(I-T) \oplus R(I-T)$.
(iii) $(I-T)^2 X$ *is closed.*
(iv) $(I-T)X$ *is closed.*

Proof. Throughout the proof we assume that $Y := \overline{R(I-T)} = \overline{(I-T)X}$.

$(i) \Rightarrow (ii)$ Since T is statistically uniformly ergodic operator, there exists a $P \in B(X)$ such that $st - \lim_n \|M_n - P\| = 0$. This gives that $st - \lim_n \|M_n x - Px\| = 0$. Thus we have $X = \ker(I-T) \oplus \overline{(I-T)X}$ by

Theorem 2.2. Now we show that the subspace $(I-T)X$ is closed. In order to do this, take $x \in X$. Hence $T(I-T)x = (I-T)Tx \in (I-T)X$. Thus we have,

$$T(Y) = T(\overline{(I-T)X}) \subseteq \overline{T(I-T)X} = \overline{(I-T)X} = Y.$$

Hence Y is T-invariant subspace and we can write $S := T_{|Y}$ and $S_n := M_n|_Y$. We also know that $\ker P = Y$ from Corollary 2.4 and so we get $st-\lim_n \|S_n\| = 0$. Moreover it is clear that $ker(I-T) \cap \overline{(I-T)X} = \{0\}$ which yields that $Y \cap \overline{(I-T)X} = \{0\}$. So we get $\ker(I-S) = \{0\}$. Therefore, one can get by Proposition 3.1 that $(I-S)$ is onto. Thus,

$$(I-S)Y = Y = (I-T)Y \subseteq (I-T)X \subseteq Y.$$

This implies that $Y = (I-T)X$ which in turn yields that $(I-T)X$ is closed.
We note that replacing "limit operator" by "st-lim operator" one can prove as in [11] that $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$. So we omit the details.
$(iv) \Rightarrow (i)$ Because of (iv), by the Open Mapping Theorem we find that there exists a $H \geq 0$ such that for each $y \in Y$ there exists $z \in X$ with

$$(I-T)z = y \quad \text{and} \quad \|z\| \leq H\|y\|.$$

Let $y \in Y$. Now, we have, for $n \in \mathbb{N}$, that

$$\|M_n y\| = \|M_n(I-T)z\| \leq \frac{1}{n}\|T - T^{n+1}\|\|z\| \leq \frac{H}{n}(\|T\| + \|T^{n+1}\|)\|y\|.$$

Hence taking supremum over $\|y\| = 1$ and using the fact that $st-\lim_n \frac{\|T^n\|}{n} = 0$ one can get $st-\lim_n \|M_n\| = 0$. We conclude that $I - M_n$ is surjective by Proposition 3.1. This implies that

$$(I-T)X = Y = (I-S)Y = (I-T)^2X.$$

Hence for all $x \in X$ there exists a $y \in Y$ such that $(I-T)x = (I-T)y$. Note that $\ker(I-S) = \{0\}$. Hence $(I-S)$ is invertible. So we can write

$$y = (I-S)^{-1}((I-T)x). \tag{8}$$

Since $(I-S)^{-1}$ is also continuous, one can obtain

$$\|y\| \leq \|(I-S)^{-1}\|\|(I-T)x\|.$$

Since $(I-T)(x-y) = 0$, we observe that $T(x-y) = (x-y)$ and for all $m \in \mathbb{N}$, $T^m(x-y) = (x-y)$ which yields, for all $n \in \mathbb{N}$, that $M_n(x-y) = (x-y)$.

Now we define $P : X \to X$ by $Px = x - y$ such that y is the unique element defined by (8). It is easily checked that $(I-S)$ is well defined and continuous. Thus, to complete the proof, we show that $st-\lim_n \|M_n - P\| = 0$. To achive this, take $x \in X$ and define y by (8). Then we find, for $n \in \mathbb{N}$, that

$$\begin{aligned}\|(M_n - P)x\| &= \|M_n x - Px\| = \|M_n x - (x-y)\| = \|M_n y\| \\ &= \|M_n(I-S)^{-1}((I-T)x)\| \leq \|(I-S)^{-1}\|\|M_n(I-T)x\| \\ &\leq \|(I-S)^{-1}\|\frac{1}{n}(\|T\| + \|T^{n+1}\|)\|x\|.\end{aligned}$$

Hence taking supremum all over x with $\|x\| = 1$, we get that

$$\|M_n - P\| \leq \frac{1}{n}\|(I-S)^{-1}\|(\|T\| + \|T^{n+1}\|)$$

and applying the operator $st-\lim_n$ to both sides of the inequality, we get that

$$st-\lim_n \|M_n - P\| = 0$$

which concludes the proof. □

The following theorem motivated by Proposition 2.8 in [12] characterizes the statistical convergence of $\{T^n\}$.

Theorem 3.3. *Let $T \in B(X)$ be a power bounded operator. Then the following assertions are equivalent:*

(i) $\{T^n\}$ *uniformly statistically converges.*
(ii) $st-\lim_n \|T^{n+1} - T^n\| = 0$ *and T is a statistically uniformly ergodic operator.*
(iii) $st-\lim_n \|T^{n+1} - T^n\| = 0$ *and $(I-T)X$ is closed.*

Proof. $(i) \Rightarrow (ii)$ Since $\{T^n\}$ uniformly statistically converges, as in the proof of Theorem 2.6 we immediately get (ii). Clearly we have $(ii) \Rightarrow (iii)$.

We now prove $(iii) \Rightarrow (i)$ Since $(I-T)X$ is closed, following the technique used in the proof of Theorem 3.2 we conclude by the Open Mapping Theorem that $(I-S)$ is invertible on Y. Hence we observe that

$$(I-T)X = Y = (I-S)Y = (I-T)Y.$$

Thus, for all $x \in X$, there exists a $y \in Y$ such that $(I-T)x = (I-T)y$ then we may write $y = (I-S)^{-1}((I-T)x)$. Since $(I-S)$ is also continuous, we get

$$\|y\| \le \|(I-S)^{-1}\|\|(I-T)x\|.$$

Since $(I-T)(x-y) = 0$, we observe that $T(x-y) = (x-y)$ and for all $m \in \mathbb{N}$, $T^m(x-y) = (x-y)$. Let us define $P : X \to X$ with $Px = x - y$, where y is the unique element defined by (8). It is easily checked that $(I-S)$ is well defined and continuous. In order to complete the proof we show that $st-\lim_n \|T^n - P\| = 0$. Now take $x \in X$ and define y by (8). Then one can get

$$\begin{aligned}\|(T^n - P)x\| &= \|T^n x - Px\| = \|T^n x - (x-y)\| = \|T^n x - T^n(x-y)\| = \|T^n y\| \\ &= \|T^n (I-S)^{-1}((I-T)x)\| \le \|(I-S)^{-1}\|\|T^n(I-T)x\| \\ &\le \|(I-S)^{-1}\|\|T^{n+1} - T^n\|\|x\|.\end{aligned}$$

Hence taking supremum all over x with $\|x\| = 1$, we observe that

$$\|T^n - P\| \le \|(I-S)^{-1}\|\|T^{n+1} - T^n\|$$

and applying the operator $st-\lim_n$ to both sides of the inequality and using (iii) we find that

$$st-\lim_n \|T^n - P\| = 0,$$

which concludes the proof. □

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