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Extended Bernoulli wavelet approximation method and its applications in solving the Lane-Emden differential equation and linear integral equation

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Abstract. In this paper, we generalized the Bernoulli wavelet called extended Bernoulli wavelets (EBWs). The EBWs are derived by dilation and translation of the Bernoulli polynomials. We have solved the Lane-Emden differential equation with the help of extended Bernoulli wavelet method and compared proposed method with the Legendre wavelet method, Chebyshev wavelet method (first kind and second kind) and ODE 45 method. Also, we have solved linear integral equation with the help of proposed method and compared with the solution obtained by the Legendre wavelet method. Illustrative examples have been discussed to demonstrate the validity and applicability of the present method.

1. Introduction

Wavelets theory is a newly emerging area in mathematical research fields. It has been applied in engineering disciplines, bioscience, biotechnology, viscoelastic materials, statical mechanics, the detection of submarines and aircraft, and other models of real- life problems. Wavelets allow a wide range of functions and operators to be represented accurately. In addition, wavelets establish a connection with fast numerical algorithm [2]. Wavelets are strong tools that can be used to explore new directions in the solution of differential equations and integral equations. Many researchers have started to use various wavelets for the analysis of problems of high computing complexity.

The application of Legendre wavelets for solving differential and integral equations is thoroughly considered by many authors [16, 17, 19–21]. Also, Chebyshev wavelets are used for solving some differential and integral equations [1, 10]. In recent years many mathematicians and physicists have been interested in studying singular initial and boundary value problems for second order ordinary differential equations (ODEs). One of the equations describing this type is the Lane–Emden type equation. Various types of Lane-Emden type equations have been studied in numerous research works because of their significant applications in various scientific domains [13, 23, 26, 28]. Numerous phenomena in mathematical physics and astrophysics, including the theory of star structure, the thermal behavior of a spherical cloud of

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gas, isothermal gas spheres, and the theory of thermionic currents, were modeled using the Lane-Emden differential equation.

Solution of linear integral equation and Lane-Emden IVPs & BVPs have been discussed by many authors. Authors like Kumar and Pandey [9], Wazwaz [26], Yousefi [28], Liao [14], and Yigider [27] have discussed various wavelet methods and other methods for solving the Lane - Emden equation. Parand and Pirkhedri [18] have discussed some other methods for solving the astrophysics equations. Lal and Yadav [13] have presented solution of Lane - Emden equation by using Gegenbauer wavelets. Doha et.al [3] have investigated the solution of the Lane - Emden equation using second kind Chebyshev wavelet method. Recently, some other approximate solutions using the perturbation method [25] for the Lane–Emden equation are obtained. Lal and Kumar [11] discussed linear integral equation and its numerical solution, Lal and Yadav [12] investigate solution of Fredholm integral equation. According to the best of my knowledge, no one has been used the extended Bernoulli wavelets method to solve linear integral equation and Lane-Emden differential equation.

In this paper, the attempt is made to solve Lane-Emden differential equation using extended Bernoulli wavelet collocation method (EBWCM). This method consists of reducing the differential equation into a set of algebraic equations by first expanding the extended Bernoulli wavelets with unknown coefficients. by solving these coefficients, we get the required solution. Also, we solved linear integral equation by using extended Bernoulli wavelet method. Here we demonstrate the method by considering the some of illustrative examples.

The paper is organized as follows: Section 1 is introductory. Definitions and preliminaries of wavelets are given in section 2. Convergence analysis of wavelets are given in section 3. Section 4, contains the explanation of numerical approximation of function. Method of solution and numerical examples are presented in section 5. The conclusion of the work is drawn in section 6.

2. Definitions and Preliminaries

2.1. Bernoulli wavelet

Wavelets are a family of functions constructed from dilation and translation of a single function $\psi(t)$ called the mother wavelet. When the dilation parameter a and translation parameter b vary continuously, we have the following family of continuous wavelets as

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a \neq 0, \ a, b \in \mathbb{R}.$$

If we restrict the parameters a and b to discrete values as: $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, where $a_0 > 1, b_0 > 0$, and n and k are positive integers, we have following family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi\left(a_0^k t - nb_0\right),$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(t)$ form an orthonormal basis.

Bernoulli wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ were first defined in [7] and have four arguments; $\hat{n} = n-1$, $n = 1, 2, \ldots, 2^{k-1}$, k is assumed to be any positive integer numbers, m is the degree of the Bernoulli polynomials and t is the normalized time. On the interval [0, 1), these wavelets are defined as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{\beta}_m(2^{k-1}t - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \leq t < \frac{\hat{n}+1}{2^{k-1}}; \\ 0, & otherwise, \end{cases}$$

with

$$\tilde{\beta}_m(t) = \begin{cases} 1, & m = 0; \\ \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}} \beta_m(t), & m > 0, \end{cases}$$

where $m = 0, 1, 2, \dots, M-1$, $n = 1, 2, \dots, 2^{k-1}$.
The coefficient $\frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is for normality, the dilation parameter is $a = 2^{-(k-1)}$ and the translation parameter $b = \hat{n}2^{-(k-1)}$. Here, $\beta_m(t)$ are the well-known Bernoulli polynomials of order m which can be defined by

$$\beta_m(t) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-i} t^i,$$

where α_i, $i = 0, 1, \dots, m$ are Bernoulli numbers and can be defined by the identity

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.$$

The first few Bernoulli numbers are

$$\alpha_0 = 1,\ \alpha_1 = \frac{-1}{2},\ \alpha_2 = \frac{1}{6}, \alpha_4 = \frac{-1}{30},\ \alpha_6 = \frac{1}{42},\ \alpha_8 = \frac{-1}{30}, \dots \ \textit{and} \quad \alpha_{2i+1} = 0,\ i = 1, 2, 3, \dots$$

The first few Bernoulli polynomials are

$$\beta_0(t) = 1,\ \beta_1(t) = t - \frac{1}{2},\ \beta_2(t) = t^2 - t + \frac{1}{6},\ \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,\ \beta_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}, \dots$$

2.2. *Properties of Bernoulli polynomial*

1. $\beta_m^{'}(t) = m\beta_{m-1}(t), \quad m \in \mathbb{Z}^+$
2. $\int_0^1 |\beta_m(t)|\, dt < 16 \frac{m!}{(2\pi)^{m+1}} \alpha_{m+n}(t), \quad m \geq 0$
3. $\beta_m(1-t) = (-1)^m \beta_m(t)$,
4. $\int_0^1 \beta_m(t)\, dt = 0$,
5. $\int_a^t \beta_m(x)\, dx = \frac{\beta_{m+1}(t) - \beta_{m+1}(a)}{m+1}$,
6. $\int_0^1 \beta_n(t)\beta_m(t)\, dt = (-1)^{n-1} \frac{m!n!}{(m+n)!} \alpha_{n+m}, \quad m, n \geq 1$ [7, 15, 22].

According to Kreyszig [8], Bernoulli polynomials, form a complete basis over the interval [0, 1].

2.3. *Extended Bernoulli wavelets*

The EBWs on the interval $[0, 1)$ are defined by

$$\psi_{n,m}^{(\mu)}(t) = \begin{cases} \mu^{\frac{k-1}{2}} \tilde{\beta}_m(\mu^{k-1}t - \hat{n}), & \frac{\hat{n}}{\mu^{k-1}} \leq t < \frac{\hat{n}+1}{\mu^{k-1}}; \\ 0, & \textit{otherwise}, \end{cases}$$

with

$$\tilde{\beta}_m(t) = \begin{cases} 1, & m = 0; \\ \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}} \beta_m(t), & m > 0, \end{cases}$$

where $\hat{n} = n - 1$, $n = 1, 2, \dots, \mu^{k-1}$, $\mu = 2, 3, 4, \dots$ $k = 1, 2, 3, \dots$ m is order of the Bernoulli polynomial ($m = 0, 1, 2, \dots, M-1$) and t is normalized time. In the above definition, the polynomials $\beta_m(t)$ are the Bernoulli polynomials of degree m which are defined in [7].

2.4. Function approximation by extended Bernoulli wavelet

A function $f(t) \in L^2(\mathbb{R})$ defined over $[0, 1)$ is expanded in terms of EBWs series as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}^{(\mu)} \psi_{n,m}^{(\mu)}(t), \tag{1}$$

where $c_{n,m}^{(\mu)} = \langle f, \psi_{n,m}^{(\mu)} \rangle$ on $L^2[0, 1]$ [19].
If the above infinite series is truncated then Eq. (1) is written as

$$S_{\mu^{k-1},M}(t) = \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{(\mu)} \psi_{n,m}^{(\mu)}(t) = C^T \psi^{(\mu)}(t), \tag{2}$$

where

$$C = \left[c_{1,0}^{(\mu)}, c_{1,1}^{(\mu)}, \ldots, c_{1,M-1}^{(\mu)}, c_{2,0}^{(\mu)}, \ldots, c_{2,M-1}^{(\mu)}, \ldots, c_{\mu^k,0}^{(\mu)}, \ldots, c_{\mu^{k-1},M-1}^{(\mu)} \right]^T,$$

$$\psi^{(\mu)}(t) = \left[\psi_{1,0}^{(\mu)}, \psi_{1,1}^{(\mu)}, \ldots, \psi_{1,M-1}^{(\mu)}, \psi_{2,0}^{(\mu)}, \ldots, \psi_{2,M-1}^{(\mu)}, \ldots, \psi_{\mu^{k-1},0}^{(\mu)}, \ldots, \psi_{\mu^{k-1},M-1}^{(\mu)} \right]^T. \tag{3}$$

3. Convergence Analysis

Theorem 3.1. *Suppose that $f(t) \in C^m[0, 1]$ and $C^T \psi^{(\mu)}(t)$ is the approximate solution using extended Bernoulli wavelet then the error bound would be given by*

$$||E|| \le \left\| \frac{2}{m! 4^m \mu^{m(k-1)}} \max_{t \in [0,1]} |f^m(t)| \right\|,$$

where $\mu = 2, 3, 4, \ldots$.

Proof.

Applying the definition of norm in the normed space, we have

$$||E||^2 = \int_0^1 \left[f(t) - C^T \psi(t) \right]^2 dt.$$

Dividing interval $[0, 1]$ into μ^{k-1} subintervals $I_n = \left[\frac{n-1}{\mu^{k-1}}, \frac{n}{\mu^{k-1}} \right]$, $n = 1, 2, 3, \ldots, \mu^{k-1}$.
Therefore,

$$\begin{aligned} ||E||^2 &= \sum_{n=1}^{\mu^{k-1}} \int_{\frac{n-1}{\mu^{k-1}}}^{\frac{n}{\mu^{k-1}}} \left[f(t) - C^T \psi(t) \right]^2 dt \\ &\le \sum_{n=1}^{\mu^{k-1}} \int_{\frac{n-1}{\mu^{k-1}}}^{\frac{n}{\mu^{k-1}}} [f(t) - p_m(t)]^2 dt. \end{aligned}$$

Where $p_m(t)$ is the interpolating polynomial of degree m which approximates $f(t)$ on I_n. By using the maximum error estimate for polynomial on I_n, then

$$\begin{aligned} ||E||^2 &\leq \sum_{n=1}^{\mu^{k-1}} \int_{\frac{n-1}{\mu^{k-1}}}^{\frac{n}{\mu^{k-1}}} \left[\frac{2}{m!4^m \mu^{m(k-1)}} \max_{t\in I_n} |f^m(t)|\right]^2 dt \\ &\leq \sum_{n=1}^{\mu^{k-1}} \int_{\frac{n-1}{\mu^{k-1}}}^{\frac{n}{\mu^{k-1}}} \left[\frac{2}{m!4^m \mu^{m(k-1)}} \max_{t\in [0,1]} |f^m(t)|\right]^2 dt \\ &= \int_0^1 \left[\frac{2}{m!4^m \mu^{m(k-1)}} \max_{t\in [0,1]} |f^m(t)|\right]^2 dt. \end{aligned}$$

Hence,

$$||E|| \leq \left\|\frac{2}{m!4^m \mu^{m(k-1)}} \max_{t\in [0,1]} |f^m(t)|\right\|.$$

Here we have used the well-known maximum error bound for the interpolation.

4. Numerical verification of wavelet approximation

This section is designed to see the numerical accuracy of the calculated approximation for the function

$$f(t) = \begin{cases} e^{t^2}, & t \in [0,1]; \\ 0, & otherwise. \end{cases}$$

For this, let us derive the basis functions of the extended Bernoulli wavelets for $\mu = 2, 3, 5;\ k = 2$ & $M = 3$.

For $\mu = 2$; k=2 & M=3

$$\psi_{1,0}(t) = \begin{cases} \sqrt{2}, & 0 \leq t < \frac{1}{2}; \\ 0, & otherwise, \end{cases} \qquad \psi_{2,0}(t) = \begin{cases} \sqrt{2}, & \frac{1}{2} \leq t < 1; \\ 0, & otherwise, \end{cases}$$

$$\psi_{1,1}(t) = \begin{cases} 2\sqrt{6}\left\{(2t) - \frac{1}{2}\right\}, & 0 \leq t < \frac{1}{2}; \\ 0, & otherwise, \end{cases} \qquad \psi_{2,1}(t) = \begin{cases} 2\sqrt{6}\left\{(2t-1) - \frac{1}{2}\right\}, & \frac{1}{2} \leq t < 1; \\ 0, & otherwise, \end{cases}$$

$$\psi_{1,2}(t) = \begin{cases} 6\sqrt{10}\times \\ \left\{(2t)^2 - (2t) + \frac{1}{6}\right\}, & 0 \leq t < \frac{1}{2}; \\ 0, & otherwise, \end{cases} \qquad \psi_{2,2}(t) = \begin{cases} 6\sqrt{10}\times \\ \left\{(2t-1)^2 - (2t-1) + \frac{1}{6}\right\}, & \frac{1}{2} \leq t < 1; \\ 0, & otherwise. \end{cases}$$

For $\mu = 3$; k=2 & M=3

$$\psi_{1,0}(t) = \begin{cases} \sqrt{3}, & 0 \le t < \frac{1}{3}; \\ 0, & otherwise, \end{cases}$$

$$\psi_{1,1}(t) = \begin{cases} 6\left\{(3t) - \frac{1}{2}\right\}, & 0 \le t < \frac{1}{3}; \\ 0, & otherwise, \end{cases}$$

$$\psi_{1,2}(t) = \begin{cases} 6\sqrt{15}\times & \\ \left\{(3t)^2 - (3t) + \frac{1}{6}\right\}, & 0 \le t < \frac{1}{3}; \\ 0, & otherwise, \end{cases}$$

$$\psi_{2,0}(t) = \begin{cases} \sqrt{3}, & \frac{1}{3} \le t < \frac{2}{3}; \\ 0, & otherwise, \end{cases}$$

$$\psi_{2,1}(t) = \begin{cases} 6\left\{(3t-1) - \frac{1}{2}\right\}, & \frac{1}{3} \le t < \frac{2}{3}; \\ 0, & otherwise, \end{cases}$$

$$\psi_{2,2}(t) = \begin{cases} 6\sqrt{15}\times & \\ \left\{(3t-1)^2 - (3t-1) + \frac{1}{6}\right\}, & \frac{1}{3} \le t < \frac{2}{3}; \\ 0, & otherwise, \end{cases}$$

$$\psi_{3,0}(t) = \begin{cases} \sqrt{3}, & \frac{2}{3} \le t < 1; \\ 0, & otherwise, \end{cases}$$

$$\psi_{3,1}(t) = \begin{cases} 6\left\{(3t-2) - \frac{1}{2}\right\}, & \frac{2}{3} \le t < 1; \\ 0, & otherwise, \end{cases}$$

$$\psi_{3,2}(t) = \begin{cases} 6\sqrt{15}\times & \\ \left\{(3t-2)^2 - (3t-2) + \frac{1}{6}\right\}, & \frac{2}{3} \le t < 1; \\ 0, & otherwise. \end{cases}$$

Similarly, we can find wavelet basis for $\mu = 5, k = 2, M = 3$. Then the value of $S_{\mu^{k-1},M}$ are calculated and are given as

$$S_{2^1,3}(t) = \begin{cases} (0.7707281540549176)\sqrt{2} + (0.05649436808579126) \times 2\sqrt{6}\left((2t) - \frac{1}{2}\right) & \\ +(0.016259969973289524) \times 6\sqrt{10}(((2t)^2 - 2t) + \frac{1}{6}), & 0 \le t < \frac{1}{2}, \\ (1.2977737820357043)\sqrt{2} + (0.2829623566596548) \times 2\sqrt{6}\left((2t-1) - \frac{1}{2}\right) & \\ +(0.05147767834313928) \times 6\sqrt{10}\left((2t-1)^2 - (2t-1) + \frac{1}{6}\right), & \frac{1}{2} \le t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{3^1,3}(t) = \begin{cases} (0.5994656620865749)\sqrt{3} + (0.01936663454990928) \times 6\left((3t) - \frac{1}{2}\right) & \\ +(0.005254880201759972) \times 6\sqrt{15}\left((3t)^2 - (3t) + \frac{1}{6}\right), & 0 \le t < \frac{1}{3}, \\ (0.7517489243185208)\sqrt{3} + (0.07273786541946725) \times 6\left((3t-1) - \frac{1}{2}\right) & \\ +(0.009363113213045962) \times 6\sqrt{15}\left((3t-1)^2 - (3t-1) + \frac{1}{6}\right), & \frac{1}{3} \le t < \frac{2}{3}, \\ (1.1821725512854664)\sqrt{3} + (0.19001036697162021) \times 6\left((3t-2) - \frac{1}{2}\right) & \\ +(0.023384803314553896) \times 6\sqrt{15}\left((3t-2)^2 - (3t-2) + \frac{1}{6}\right), & \frac{2}{3} \le t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{5^1,3}(t) = \begin{cases} (0.45324868441877236)\sqrt{5} + ((0.005247494087688237) \times 2\sqrt{15}\left((5t) - \frac{1}{2}\right) & \\ +(0.0013796886183712331) \times 30\left((5t)^2 - (5t) + \frac{1}{6}\right), & 0 \le t < \frac{1}{5}, \\ (0.491261038102119)\sqrt{5} + (0.017059091405668525) \times 2\sqrt{15}\left((5t-1) - \frac{1}{2}\right) & \\ +(0.0017300990784008263) \times 30\left((5t-1)^2 - (5t-1) + \frac{1}{6}\right), & \frac{1}{5} \le t < \frac{2}{5}, \\ (0.5771167924340004)\sqrt{5} + (0.03338642939956284) \times 2\sqrt{15}\left((5t-2) - \frac{1}{2}\right) & \\ +(0.0025833932219967437) \times 30\left((5t-2)^2 - (5t-2) + \frac{1}{6}\right), & \frac{2}{5} \le t < \frac{3}{5}, \\ (0.7348359974376159)\sqrt{5} + (0.05947656908972121) \times 2\sqrt{15}\left((5t-3) - \frac{1}{2}\right) & \\ +(0.004340039319235545) \times 30\left((5t-3)^2 - (5t-3) + \frac{1}{6}\right), & \frac{3}{5} \le t < \frac{4}{5}, \\ (1.0141262188647)\sqrt{5} + (0.1054437226038818) \times 2\sqrt{15}\left((5t-4) - \frac{1}{2}\right) & \\ +(0.007918706373111206) \times 30\left((5t-4)^2 - (5t-4) + \frac{1}{6}\right), & \frac{4}{5} \le t < 1, \\ 0, & otherwise. \end{cases}$$

The graphs of $S_{\mu^{k-1},M}$ and $f(t)$ has been plotted for $\mu = 2, 3, 5;\ M = 3$ and $k = 2$ in Figures 1, 2, and 3.

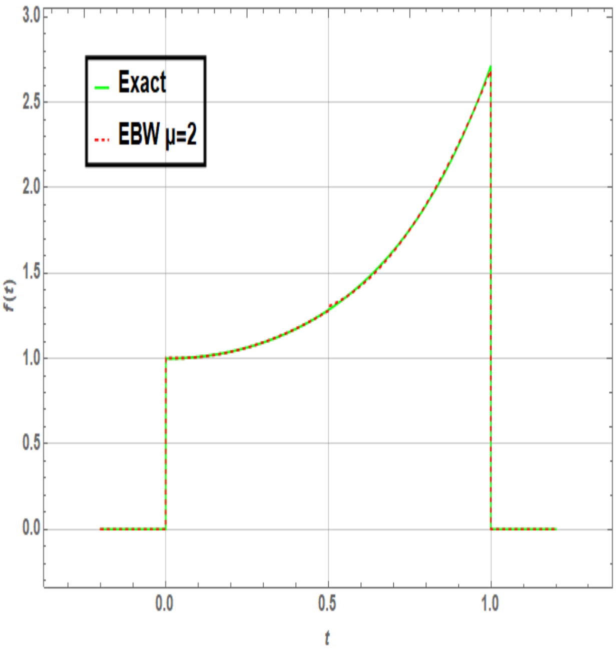

Figure 1: Graphical representation of $S_{2^1,3}$ and $f(t)$

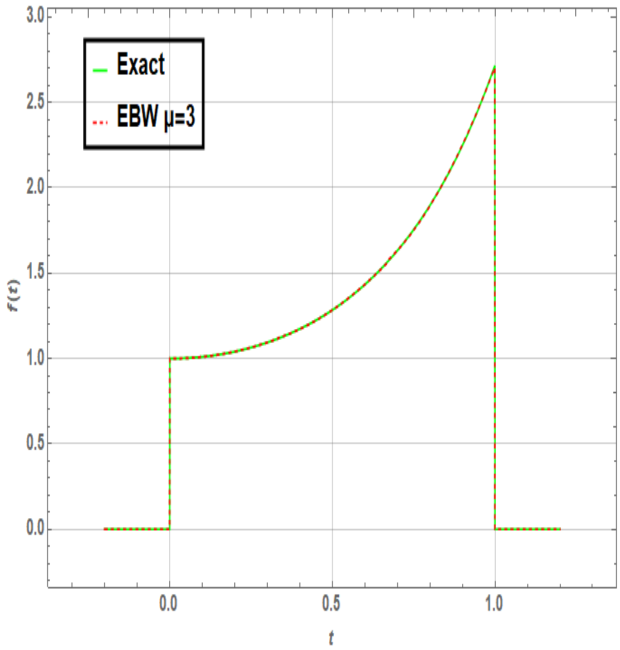

Figure 2: Graphical representation of $S_{3^1,3}$ and $f(t)$

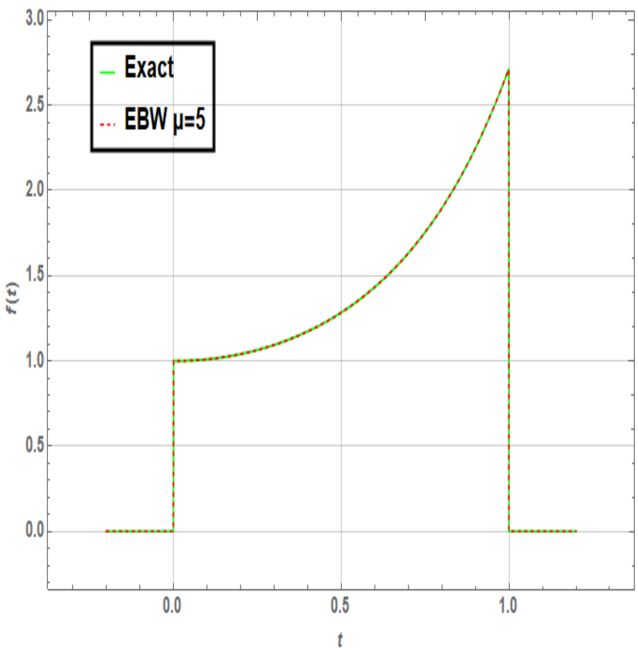

Figure 3: Graphical representation of $S_{5^1,3}$ and $f(t)$

5. Illustrative Examples

5.1. Algorithm for Solving the Lane-Emden differential equation

Let $y(t)$ be the solution of the Lane - Emden differential equation [9]:

$$y'' + \frac{\beta}{t}y' + h(t, y) = g(t), \quad t \in (0, 1], \quad y(0) = a,\ y'(0) = b. \tag{4}$$

Where β, a & b are real constant, $h(t, y)$ is a real valued function and $g(t)$ is nonhomogeneous term. The proposed method follows as
Let us assume that

$$y(t) = \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{(\mu)} \psi_{n,m}^{(\mu)}(t) = C^T \psi^{(\mu)}(t), \tag{5}$$

is the solution of the differential equation. By initial condition of Eq. (4), the Eq. (5) reduces to

$$y(0) = \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{(\mu)} \psi_{n,m}^{(\mu)}(0) = a, \ \ y'(0) = \frac{d}{dt} \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M-1} c_{n,m}^{(\mu)} \psi_{n,m}^{(\mu)}(0) = b.$$

Putting Eq. (5) in Eq. (4), we have

$$(C^T \psi^{(\mu)}(t))'' + \frac{\beta}{t}(C^T \psi^{(\mu)}(t))' + h(t, (C^T \psi^{(\mu)}(t))) = g(t). \tag{6}$$

In Eq. (6), C^T contains $\mu^{k-1}M$ unknown coefficients. Therefore, removing initial conditions, $\mu^{k-1}M - 2$ more conditions are required for the solution of differential equation. For determining the values of $\mu^{k-1}M$ unknown coefficients $c_{n,m}^{(\mu)}$, collocation points $t_i = \frac{i-1}{\mu^{k-1}M}, i = 3, \ldots, \mu^{k-1}M$, are substituted in Eq. (6) to obtain $\mu^{k-1}M - 2$ equations. Hence, these $\mu^{k-1}M$ equations give the values of the unknown coefficients $c_{n,m}^{(\mu)}$. In addition, higher-order differential equations can be solved using this technique.

Example 1.

Consider the Lane - Emden differential equation

$$y'' + \frac{2}{t}y' + y = t^3 + t^2 + 12t + 6, \ t \in (0, 1], \ y(0) = 0, \ y'(0) = 0. \tag{7}$$

the exact solution of Eq. (7) is $y(t) = t^2 + t^3$.
It is now possible to solve the differential equation by applying the EBW technique that was previously discussed. For the approximate solution of the Eq. (7), we take $\mu = 2, 3;\ M = 4$, and $k = 2$.

For various values of t in the interval [0, 1), the exact solution (ES) and approximation solution of the Lane-Emden differential equation determined by the extended Bernoulli wavelet method (EBWM) have been obtained. Also, Tab. (1) of [10] provides comparisons of this solution using the Legendre wavelet method (LWM), first kind Chebyshev wavelet method (FKCWM), second kind Chebyshev wavelet method (SKCWM).
The comparision of the exact solution and approximate solutions of the Lane-Emden differential equation (7) by EBWM, LWM, CWM and ODE 45 method are shown in the Table (1) and Figures (4) & (5).

t	Exact sol. y(t)	LWM [10]	FKCWM [10]	SKCWM [10]	ODE 45 method	EBWM for $\mu = 2$, k=2,M=4	EBWM for $\mu = 3$, k=2,M=4
0.1	0.0110000000000000	0.1105200243	0.0109999998	0.0110000002	0.011000084711643	0.0110002886336427	0.0110000000000000
0.2	0.0480000000000000	0.0483697994	0.0479999999	0.0480000001	0.048000617039672	0.0480002886336427	0.0480000000000000
0.3	0.1169999999999999	0.1181960706	0.1169999999	0.1170000000	0.116999923775231	0.1170002886336428	0.1170000000000000
0.4	0.2240000000000000	0.2267734978	0.2239999999	0.2240000001	0.223999687741197	0.2240002886336427	0.2239999999914254
0.5	0.3750000000000000	0.3803447612	0.3750000001	0.3750000001	0.374999558962347	0.3749999999992738	0.3749999999915540
0.6	0.5760000000000000	0.5851525426	0.5760000001	0.5760000001	0.575999479514219	0.5759999999992873	0.5759999999917186
0.7	0.8329999999999999	0.8474395224	0.8330000003	0.8330000002	0.832999439266877	0.8329999999993025	0.8329999999929758
0.8	1.1520000000000001	1.1734483820	1.1520000000	1.1520000000	1.151999439150442	1.1519999999993207	1.1519999999931549
0.9	1.5390000000000001	1.5694218030	1.5390000010	1.5390000000	1.538999482365030	1.5389999999993419	1.5389999999933526

Table 1: Comparison table between the exact solution, Legendre wavelet solution, Chebyshev wavelet solution, ODE 45 method and extended Bernoulli wavelet solution for various values of variable t for $M = 4$ of example 1.

Comparision of absolute errors of the Lane-Emden differential equation (7) by EBWM, LWM, CWM and ODE 45 method are shown in the Table (2) and Figure (6).

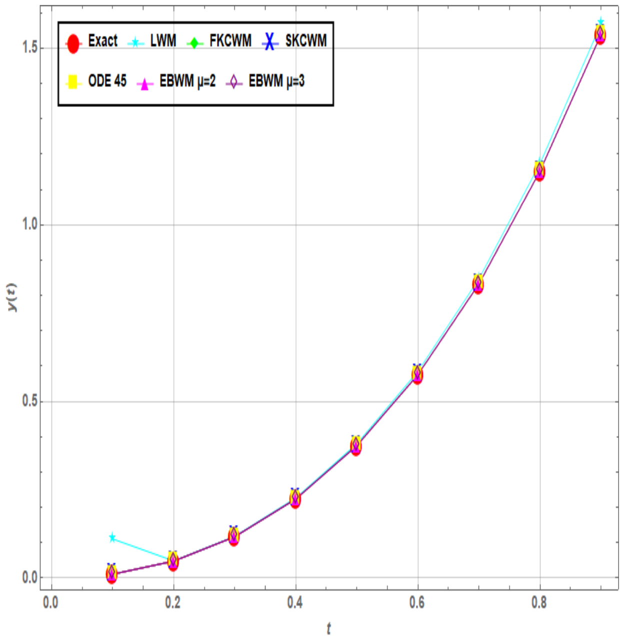

Figure 4: Comparison between the exact solution, Legendre wavelet solution, Chebyshev wavelet solution, ODE 45 method and extended Bernoulli wavelet solution of example 1.

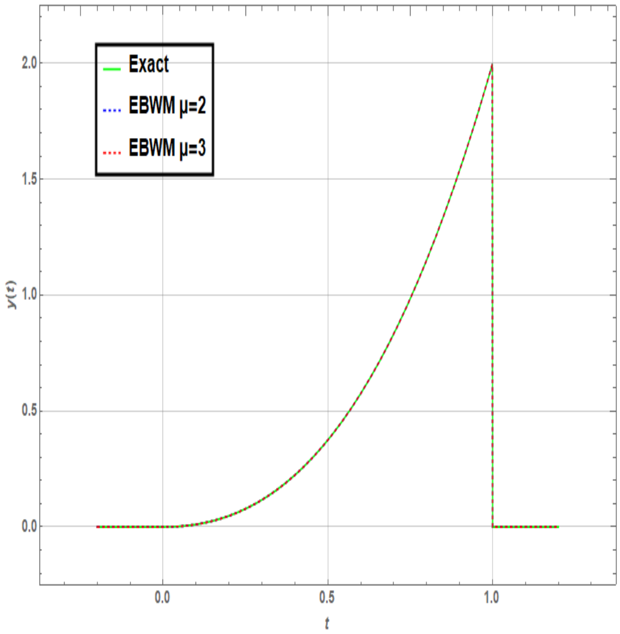

Figure 5: Comparison between the exact solution and extended Bernoulli wavelet solution for various values of variable t for $M = 4$ of example 1.

t	LWM	FKCWM	SKCWM	ODE 45 method	EBWM for μ = 2, k=2, M=4	EBWM for μ = 3, k=2, M=4
0.1	0.000052002	2×10^{-10}	2×10^{-10}	8×10^{-08}	3×10^{-7}	0
0.2	0.000369799	1×10^{-10}	1×10^{-10}	6×10^{-7}	3×10^{-7}	0
0.3	0.001196071	1×10^{-11}	1×10^{-15}	7×10^{-08}	3×10^{-7}	1×10^{-15}
0.4	0.002773498	1×10^{-10}	1×10^{-10}	3×10^{-07}	3×10^{-7}	8×10^{-15}
0.5	0.005344761	1×10^{-10}	1×10^{-10}	4×10^{-07}	7×10^{-13}	8×10^{-12}
0.6	0.009152543	1×10^{-10}	1×10^{-10}	5×10^{-07}	7×10^{-13}	8×10^{-12}
0.7	0.014439522	3×10^{-10}	2×10^{-10}	5×10^{-07}	7×10^{-13}	7×10^{-12}
0.8	0.021448382	0	0	5×10^{-07}	7×10^{-13}	7×10^{-12}
0.9	0.030421803	1×10^{-9}	0	5×10^{-07}	7×10^{-13}	7×10^{-12}

Table 2: Comparison table between the absolute errors of the Legendre wavelet solution, Chebyshev wavelet solution, ODE 45 method and extended Bernoulli wavelet solution of example 1.

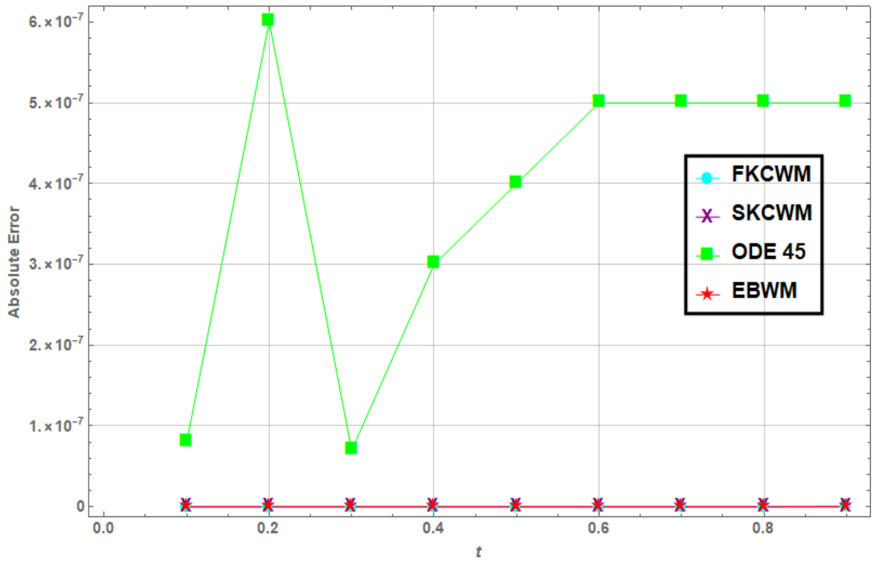

Figure 6: Graphical representation of absolute errors of Legendre wavelet solution, Chebyshev wavelet solution, ODE 45 method and extended Bernoulli wavelet solution of example 1.

5.2. *Algorithm for Solving Linear Integral Equation*

The linear integral equation solving algorithm is going to be covered in this section. Let us consider the linear integral equation

$$y(x) = f(x) + \int_0^1 K(x,t)y(t)\,dt \tag{8}$$

where $K(x,t)$ is continuous in the region $[0,1]\times[0,1]$, $f(x)$ is continuous on $[0,1]$, and $y(x)$ is an unknown function that needs to be evaluated. The functions $y(x)$, $f(x)$, and $K(x,t)$ are now approximated as

$$y(x) = C^T\psi(x),\ f(x) = S^T\psi(x),\ and\ K(x,t) = (\psi(x))^T R\psi(t) \tag{9}$$

where C, S are $\mu^{k-1}M \times 1$ vectors and R is $\mu^{k-1}M \times \mu^{k-1}M$ matrix. Substituting these values in Eq. (8), we have

$$C = (I-R)^{-1}S, \tag{10}$$

where C is a $\mu^{k-1}M \times 1$ column vector. This method is illustrated with help of following example.

Example 2.

Consider the Fredholm integral equation

$$y(x) = \frac{x}{2} + \frac{1}{4}\int_0^1 e^{x+t}y(t)\,dt \tag{11}$$

The exact solution of above Eq. (11) is $y(x) = \frac{x}{2} + \frac{e^x}{9-e^2}$.
For μ=2,3,5 in the interval [0, 1), the exact solution (ES) and approximate solution of Eq. (11) developed using extended Bernoulli wavelet method (EBWM). Also, we have compared this solution using the Legendre wavelet method (LWM), which is given in Table (1) of [11].
The comparision of the exact solution and approximate solutions of the linear integral equation (11) by EBWM, LWM are shown in the Table (3) and Figures (7) & (8).
By Table (3) and Figures (7) & (8), it is evident that the exact solution and EBW solution of the linear integral equation (11) coincide almost everywhere.

The absolute errors between exact solution and approximate solution by extended Bernoulli wavelet method for $\mu = 2,3,5$, and Legendre wavelet method given in Table (4) and Figure (9).

t	Exact sol. $y(t)$	LWM [11] for $k = 3, M = 3$	EBWM for $\mu = 2$, $k = 2, M = 3$	EBWM for $\mu = 3$, $k = 2, M = 3$	EBWM for $\mu = 5$, $k = 2, M = 3$
0.0	0.62075	0.62084	0.62155	0.62097	0.62079
0.1	0.73603	0.73601	0.73575	0.73594	0.73603
0.2	0.85819	0.85822	0.8579	0.85825	0.85824
0.3	0.98793	0.98789	0.98815	0.98791	0.98793
0.4	1.12606	1.12609	1.12637	1.12595	1.12612
0.5	1.27345	1.27360	1.27478	1.27345	1.27345
0.6	1.43109	1.43105	1.43061	1.43120	1.43117
0.7	1.60005	1.60010	1.59965	1.60004	1.60008
0.8	1.78151	1.78145	1.78189	1.78139	1.78161
0.9	1.97681	1.97680	1.97732	1.97700	1.97681

Table 3: Comparison table between the exact solution, Legendre wavelet solution and extended Bernoulli wavelet solution for various values of variable t for $M = 3$ of example 2.

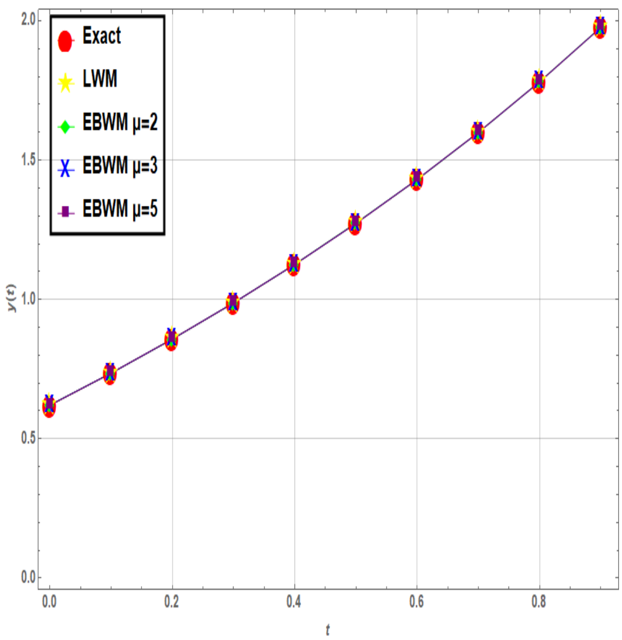

Figure 7: Comparison between the exact solution, Legendre wavelet solution and extended Bernoulli wavelet solution for various values of variable t for $M = 3$ of example 2.

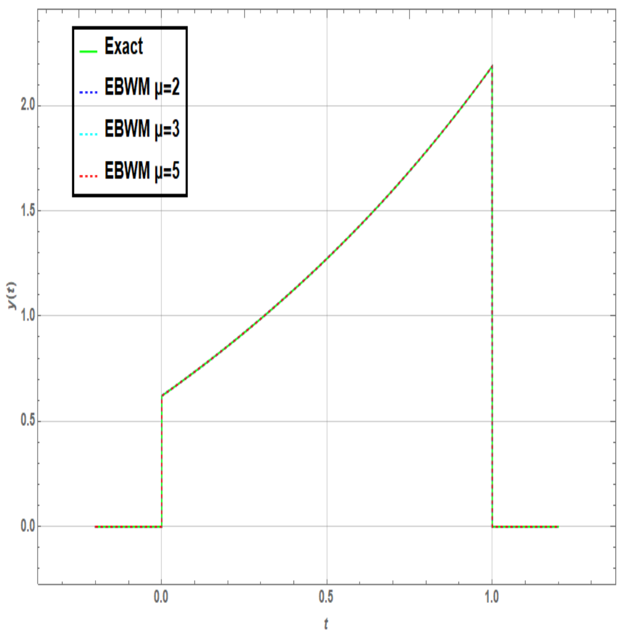

Figure 8: Comparison between the exact solution and extended Bernoulli wavelet solution for various values of variable t for $M = 3$ of example 2.

t	Abs. Error [11] for k=3, M=3	Abs. Error for μ = 2, k=2, M=3	Abs. Error for μ = 3, k=2, M=3	Abs. Error for μ = 5, k=2, M=3
0.0	9×10^{-5}	8.0×10^{-4}	2.2×10^{-4}	4×10^{-5}
0.1	2×10^{-5}	2.8×10^{-4}	9.0×10^{-5}	0
0.2	3×10^{-5}	2.9×10^{-4}	6.0×10^{-5}	5×10^{-5}
0.3	4×10^{-5}	2.2×10^{-4}	2.0×10^{-5}	0
0.4	3×10^{-5}	3.1×10^{-4}	1.1×10^{-4}	6×10^{-5}
0.5	2×10^{-4}	1.3×10^{-3}	0	0
0.6	4×10^{-5}	4.8×10^{-4}	1.1×10^{-4}	8×10^{-5}
0.7	5×10^{-5}	4.0×10^{-4}	1.0×10^{-5}	3×10^{-5}
0.8	6×10^{-5}	3.8×10^{-4}	1.2×10^{-4}	1×10^{-4}
0.9	1×10^{-5}	5.1×10^{-4}	1.9×10^{-4}	0

Table 4: Comparison table between the absolute errors of the Legendre wavelet solution, and extended Bernoulli wavelet solution of example 2.

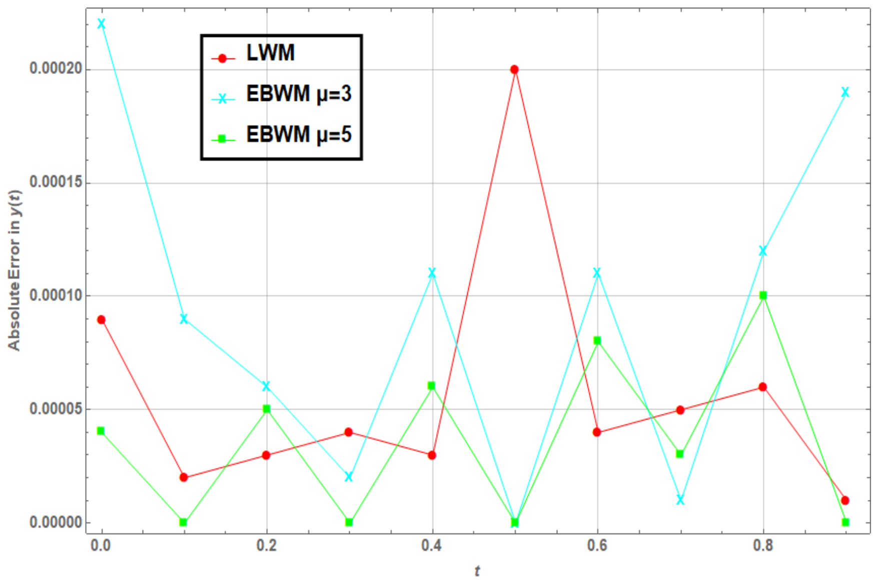

Figure 9: Comparison between the absolute errors of the Legendre wavelet solution, and extended Bernoulli wavelet solution of example 2.

6. Conclusions

An extended Bernoulli wavelet method has been proposed for the numerical solution of Lane-Emden differential equation. The performance of EBWCM superior to the CWM and LWM which is justified through the illustrative examples. Superior accuracy is attained in the case of EBWM over the other methods. Also, we have solved linear integral equation wih the help of proposed method and compared with the solution obtained by the Legendre wavelet method. The main advantage of this method is its simplicity and small computation costs.

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