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Semi-weak k-hyponormality of recursively generated weighted shifts with first two equal weights

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Abstract. Semi-weak k-hyponormality has been considered to study the weak subnormality of Hilbert space operators. In this paper, we consider a recursive weight sequence $\alpha(a, b, \rho)$ induced by two atomic Berger measure with atoms $\{a, b\}$ and density ρ for $0 < a, b, \rho < 1$, and the corresponding weighted shift $W_{\alpha(a,b,\rho)}$. For all $k \geq 2$, we characterize semi-weak k-hyponormalities of recursively generated weighted shifts with first two equal weights. We also show that a semi-weakly k-hyponormal weighted shift needs not satisfy the flatness property, in which equality of first two weights forces all weights to be equal.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A bounded operator T is *subnormal* if it is the restriction of a normal operator to a (closed) invariant subspace. For A and B in $\mathcal{B}(\mathcal{H})$, we let $[A, B] := AB - BA$. A k-tuple $(T_1, ..., T_k)$ of bounded operators in $\mathcal{B}(\mathcal{H})$ is called *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^k$ is positive on the direct sum of $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (k-copies). An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be (*strongly*) *k-hyponormal* if $(T, ..., T^k)$ is hyponormal ([5],[8],[9]). Obviously, 1-hyponormal operator T is hyponormal. It is well known that according to the Bram-Halmos' criterion, an operator T is subnormal if and only if T is k-hyponormal for all $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers ([3]).

An operator T is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for all complex polynomials p. For $k \in \mathbb{N}$, an operator T is *weakly k-hyponormal* if for every polynomial p of degree k or less, $p(T)$ is hyponormal ([8],[15],[16]). For $k = 2$, T is said to be *quadratically hyponormal*. An operator T is called *semi-weakly k-hyponormal* if $T + sT^k$ is hyponormal for all s in the set $\mathbb{C}$ of complex numbers ([17]). An operator T is *completely semi-weakly hyponormal* if T is semi-weakly k-hyponormal for all $k \in \mathbb{N}$ ([21],[23]). Clearly, quadratic hyponormality is equivalent to semi-weak 2-hyponormality. The following implications hold: subnormal $\Rightarrow$ polynomially hyponormal $\Rightarrow$ completely semi-weakly hyponormal, and weakly k-hyponormal $\Rightarrow$ semi-weakly k-hyponormal. However it is known that converse implications are not

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always true ([17],[21]). Sometimes [semi-]weak 3- and 4-hyponormality are referred to as [*semi-*]*cubic* and *quartic hyponormality*.

For a bounded weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ of positive real numbers, the *weighted shift* W_α acting on $\ell^2(\mathbb{N}_0)$, with an orthonormal basis $\{e_i\}_{i=0}^{\infty}$, is defined by $W_\alpha e_j = \alpha_j e_{j+1}$ for all $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Weighted shifts have played important roles in detecting properties of weak subnormality ([12], [13], [14]). In the area of gap theory between subnormality and hyponormality, the flatness is important to detect the structure of such weighted shifts (cf. [4], [5], [6], [20]). The flatness of subnormal weighted shifts was begun by J. Stampfli ([24]); he proved that if W_α is a subnormal weighted shift with the weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ and $\alpha_0 = \alpha_1$, then $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$, i.e. *flat*. In [6] R. Curto improved his result as that if W_α is a 2-hyponormal weighted shift with first two equal weights, then the sequence α is flat. Also he proved that W_α is quadratically hyponormal with the weight $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$. Hence the following problem arose naturally.

Problem 1.1 ([7, Problem 4]). Describe all quadratically hyponormal weighted shifts W_α with the first two equal weights.

Since R. Curto introduced Problem 1.1 in 1991, several operator theorists have studied this problem for more 30 years. In [4], Choi proved this flatness in the case of polynomially hyponormal weighted shift. Li-Cho-Lee in [20] proved that if W_α a cubically hyponormal weighted shift with α satisfying the first two equal weights, then α forces flatness of W_α.

There are another family of subnormal shifts arising from *Stampfli's subnormal completion* ([24]): for positive real numbers u, v, w with $u < v < w$, there exists a recursively subnormal weighted shift $W_{(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge}$ (cf. [2], [10], [18], [22]). In [9] Curto-Fialkow proved that there exists $1 < x < y$ such that $W_{1,(1, \sqrt{x}, \sqrt{y})^\wedge}$ is quadratically hyponormal. For the weighted shift $W_{\alpha(x)}$ with $\alpha(x) : \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ $(x < u < v < w)$, it is well-known [11] that 2-hyponormality of $W_{\alpha(x)}$ is equivalent to subnormality. Moreover, in [23] Li-Lee-Baek proved that subnormality of $W_{\alpha(x)}$ is equivalent to polynomial hyponormality and completely semi-weak hyponormality (cf. Proposition 2.3).

Due to the result in [20], it holds that every weakly k-hyponormal weighed shift with first two equal weights satisfies the flatness property for all $k \geq 3$, *that is*, Problem 1.1 does not extend to the case of weak k-hyponormality. On the other hand, in [17] authors provided an example that a weighted shift W_α with $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$ is semi-cubically hyponormal but not semi-weakly k-hyponormal for any $k \geq 4$. So it is natural question whether a semi-weakly k-hyponormal weighted shift with first two equal weights has the flatness property. Hence it is meaningful studying on the following Problem 1.2.

Problem 1.2. Describe semi-weakly k-hyponormal weighted shifts with first two equal weights for each integer $k \geq 2$.

Authors in [1] described a nonempty range of x, which provides a weighted shift $W_{\alpha(x)}$ with the weight $\alpha(x) : 1, 1, \sqrt{x}, \left(\sqrt{\frac{111}{100}}, \sqrt{\frac{112}{100}}, \sqrt{\frac{113}{100}}\right)^\wedge$ being a semi-cubically hyponormal. Therefore it is worthwhile to find appropriate weighted shifts with first two equal weights which can provide a positive answer to Problem 1.2.

This paper consists of four sections. In Section 2 for an arbitrary given triplet (a, b, ρ) with $0 < a, b, \rho < 1$, we introduce a new notion of a recursively generated weight sequence $\alpha(a, b, \rho)$ induced by two atomic Berger measure. And we show the relationship between $\alpha(a, b, \rho)$ and Stampfli's weight sequence $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ $(0 < u < v < w)$.

In Section 3 for a recursive sequence $\alpha(a, b, \rho)$ with $0 < a, b, \rho < 1$, we formulate a rather simple formula for quadratic hyponormality of the weighted shift $W_{\alpha(a,b,\rho)}$, and provided some related examples.

In Section 4 we provide a concrete model which gives the affirmative answers to Problem 1.2. Using our model, we characterize semi-weak k-hyponormalities of recursively generated weighted shifts with first two equal weights for all $k \geq 2$ (see Theorem 4.3 or Theorem 4.5).

Some of the calculations in this paper were accomplished by using the software tool *Mathematica* ([25]).

2. Recursively generated weight sequence by two atoms

Let a and b be positive real numbers with $a, b \le 1$ and denote

$$\varphi_0 = -ab \text{ and } \varphi_1 = a + b. \tag{2.1}$$

Define a generating function f as follows:

$$f(t) = t^2 - \varphi_1 t - \varphi_0 \quad (t \in \mathbb{R}).$$

Consider two real numbers $0 < \rho_0, \ \rho_1 < 1$ satisfying $\rho_0 + \rho_1 = 1$ and the Vandermonde equation

$$\begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \rho_0 a + \rho_1 b \end{pmatrix}.$$

Put $\rho = \rho_0$. Then $\rho_1 = 1 - \rho$. Now we consider a (two atomic) probability measure

$$\mu := \mu_{(a,b,\rho)} = \rho\delta_{\{a\}} + (1-\rho)\,\delta_{\{b\}} \text{ with } 0 < \rho < 1.$$

Then the measure μ is the Berger measure and there exists a sequence $\{\gamma_n\}_{n=0}^{\infty} \subset \mathbb{R}_+$ such that

$$\gamma_n = \int_{\mathbb{R}_+} t^n d\mu(t) = \rho a^n + (1-\rho)\,b^n \quad (n \ge 0). \tag{2.2}$$

Define

$$\alpha_n \equiv \alpha_n(a, b, \rho) = \sqrt{\frac{\gamma_{n+1}}{\gamma_n}} \quad (n \ge 0).$$

This produces a sequence $\alpha(a, b, \rho) := \{\alpha_n\}_{n=0}^{\infty}$ such that

$$\alpha_n = \sqrt{\frac{\rho a^{n+1} + (1-\rho)\,b^{n+1}}{\rho a^n + (1-\rho)\,b^n}} \ (n \ge 0). \tag{2.3}$$

Using the notations (2.1) and (2.3), the sequence $\alpha(a, b, \rho)$ turns out to be recursively generated (or simply recursive) by the triplet (a, b, ρ) satisfying

$$\alpha_n^2 = \varphi_1 + \frac{\varphi_0}{\alpha_{n-1}^2}, \quad n \ge 1. \tag{2.4}$$

We note that if two atoms are equal, i.e. $a = b$, then the sequence $\alpha(a, b, \rho)$ forces the flatness regardless of the density value ρ. To avoid this trivial case, we consider two different atoms in this paper.

Proposition 2.1. *For a weight sequence $\alpha(a, b, \rho)$, the following assertions hold.*
(i) *The sequence $\alpha(a, b, \rho)$ is monotone increasing;*
(ii)

$$\lim_{n\to\infty} \alpha_n^2 = \begin{cases} a, & \text{if } a > b, \\ b, & \text{if } a < b. \end{cases}$$

Proof. (i) Using (2.3), it follows from simple computations that

$$\alpha_{n+1}^2 - \alpha_n^2 = \frac{\rho(1-\rho)a^{n-1}(a-b)^2 b^{n-1}}{(\rho a^n + (1-\rho)b^n)\,(\rho a^{n-1} + (1-\rho)b^{n-1})} \quad (n \ge 0).$$

Since $0 < \rho < 1$, $\alpha_{n+1}^2 > \alpha_n^2$ $(n \geq 0)$, which implies the result.
(ii) Suppose that $0 < b < a \leq 1$. Then $\frac{b}{a} < 1$. So

$$\lim_{n\to\infty} \alpha_n^2 = \lim_{n\to\infty} \frac{a\rho + (1-\rho)\left(\frac{b}{a}\right)^{n+1}}{\rho + (1-\rho)\left(\frac{b}{a}\right)^n} = a.$$

For the other case $0 < a < b \leq 1$, the proof is similar and easy. □

It turns out from Proposition 2.1 that $\alpha(a, b, \rho)$ becomes a bounded increasing sequence of positive numbers.

Lemma 2.2. *For a weight sequence $\alpha(a, b, \rho)$, put*

$$K := -\frac{\varphi_1^2}{2\varphi_0}\left(\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_0}\right) \text{ and } L^2 := \frac{1}{2}\left(\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_0}\right).$$

Suppose $a < b$. Then

$$K = \frac{(a+b)^2}{a} \text{ and } L^2 = b.$$

Proof. It is straightforward from simple computations. □

For a given triplet (a, b, ρ), let the sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ be given as in (2.2). Consider a Hankel matrix $M(i)$ for all $i \geq 0$,

$$M(i) = \left[\gamma_{i+j}\right]_{j=0,1,2}.$$

It follows from (2.4) that the sequence $\gamma = \{\gamma_n\}$ satisfies the followings:

$$\gamma_0 = 1,\ \gamma_1 = \rho a + (1-\rho)b,\ \gamma_n = \varphi_0\gamma_{n-2} + \varphi_1\gamma_{n-1}\ \ (n \geq 2).$$

Then the rank of the matrix $M(i)$, $\text{rank}M(i) = 2$ for all $(i \geq 0)$. Denote $W_{\alpha(a,b,\rho)}$ for the corresponding weighted shift with a recursive sequence $\alpha(a, b, \rho)$. It is obvious that $W_{\alpha(a,b,\rho)}$ is a subnormal recursively generated weighted shift (cf. [8, p. 220]).

For the reader's convenience, we recall Stampfli's subnormal completion (cf. [9],[24]). For given real numbers $\sqrt{u}(\equiv \alpha_0)$, $\sqrt{v}(\equiv \alpha_1)$, $\sqrt{w}(\equiv \alpha_2)$ with $u < v < w$, define

$$\alpha_n^2 = \Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2}, \qquad n \geq 2,$$

where $\Psi_0 = -\frac{uv(w-v)}{v-u}$ and $\Psi_1 = \frac{v(w-u)}{v-u}$. Then we obtain a recursively generated weight sequence and denote it by $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ and the associated weighted shift $W_{(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge}$ is subnormal ([24]). For a weighted shift $W_{\alpha(x)}$ with $\alpha(x) : \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ $(x < u < v < w)$, it is well known [9] that

$$W_{\alpha(x)} \text{ is 2-hyponormal } \Leftrightarrow 0 < x \leq \sqrt{\frac{uv(w-v)}{u^2 - 2uv + vw}}\ (\equiv H_2).$$

Given a triplet (a, b, ρ) with $0 < a, b, \rho < 1$, consider three positive real numbers $u < v < w$ of Stampfli's weight sequence $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ as follows:

$$u = \rho a + (1-\rho)\, b,\ v = \frac{\rho a^2 + (1-\rho)\, b^2}{\rho a + (1-\rho)\, b},\ w = \frac{\rho a^3 + (1-\rho)\, b^3}{\rho a^2 + (1-\rho)\, b^2}. \tag{2.5}$$

From (2.4), we note that the sequence $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ is exactly the same as the weight sequence $\alpha(a, b, \rho)$ induced by a triplet (a, b, ρ).

Let a weight sequence $\alpha(x; a, b, \rho)$ be a backward extension of $\alpha(a, b, \rho)$ as follows:

$$\alpha(x; a, b, \rho): \sqrt{x}, \sqrt{\frac{\rho a^n + (1-\rho) b^n}{\rho a^{n-1} + (1-\rho) b^{n-1}}} \quad (n \geq 1). \tag{2.6}$$

For the recursive weight sequence $\alpha(a, b, \rho)$, using formulas in (2.5), we have

$$H_2(a, b, \rho) = \sqrt{\frac{ab}{a(1-\rho) + b\rho}}. \tag{2.7}$$

Combining (2.7) for the weight sequence $\alpha(a, b, \rho)$ and the results of [11, Theorem 1.3] and [22, Theorem 4.3], we can obtain the following proposition.

Proposition 2.3. *For a recursive weight sequence $\alpha(x; a, b, \rho)$ as in* (2.6)*, let $W_{\alpha(x;a,b,\rho)}$ be the associated weighted shift. Then the following assertions are equivalent:*
(i) $W_{\alpha(x;a,b,\rho)}$ *is* 2*-hyponormal;*
(ii) $W_{\alpha(x;a,b,\rho)}$ *is subnormal;*
(iii) $W_{\alpha(x;a,b,\rho)}$ *is polynomially hyponormal;*
(iv) $W_{\alpha(x;a,b,\rho)}$ *is weakly k-hyponormal, for any positive integer $k \geq 2$;*
(v) $W_{\alpha(x;a,b,\rho)}$ *is semi-weakly k-hyponormal, for any positive integer $k \geq 2$;*
(vi) $W_{\alpha(x;a,b,\rho)}$ *is completely semi-weakly hyponormal;*
(vii) $0 < x \leq \frac{ab}{a(1-\rho)+b\rho}$.

Example 2.4. *For the case of $a = \frac{1}{2}$, $b = \frac{2}{3}$ and $\rho = \frac{1}{3}$ in $\alpha(a, b, \rho)$, we consider a backward extension weight sequence $\alpha(x; \frac{1}{2}, \frac{2}{3}, \frac{1}{3})$ as follows:*

$$\alpha\left(x; \frac{1}{2}, \frac{2}{3}, \frac{1}{3}\right): \sqrt{x}, \sqrt{\frac{11}{18}}, \sqrt{\frac{41}{66}}, \sqrt{\frac{155}{246}}, \sqrt{\frac{593}{930}}, \cdots.$$

Let $W_{\alpha(x;\frac{1}{2},\frac{2}{3},\frac{1}{3})}$ be the corresponding weighted shift. From Proposition 2.3, we can see that

$$W_{\alpha(x;\frac{1}{2},\frac{2}{3},\frac{1}{3})} \text{ is 2-hyponormal} \Longleftrightarrow W_{\alpha(x;\frac{1}{2},\frac{2}{3},\frac{1}{3})} \text{ is subnormal} \Longleftrightarrow 0 < x \leq \frac{3}{5}.$$

3. Quadratic hyponormality of a weighted shift $W_{\alpha(x;a,b,\rho)}$

Recall that T is *semi-weakly k-hyponormal* if $T + sT^k$ is hyponormal for $k \geq 2$, i.e.

$$\left[\left(T + sT^k\right)^*, T + sT^k\right] \geq 0,$$

for all $s \in \mathbb{C}$ ([17]). It is obvious that the semi-weak 2-hyponormality is equivalent to the quadratic hyponormality. Throughout this paper we may consider $k \geq 2$. In this section, we first formulate a rather simple criterion for the quadratic hyponormality of a recursively generated weighted shifts $W_{\alpha(x;a,b,\rho)}$ with the weight sequence $\alpha(x; a, b, \rho)$ as in (2.6).

For a weighted shift $W_{\alpha(x)}$ with $\alpha(x): \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}$ $(x < u < v < w)$, it is well-known the following formula:

$$h_2^+ = \min\left\{u, \frac{u^2v^2w + uv^2(w-u)K + uv(w-v)K^2}{u^3v + uv(w-u)K + (u^2 + vw - 2uv)K^2}\right\}, \tag{3.1}$$

where $h_2^+ = \left(\sup\{x > 0 : W_{\alpha(x)} \text{ is positive quadratically hyponormal}\}\right)^{1/2}$([9, Theorem 4.3]).

We now consider the weight sequence $\alpha(a, b, \rho)$ induced by a triplet (a, b, ρ) with $a < b \le 1$ and $0 < \rho < 1$, it holds from Lemma 2.2 that $K = \frac{1}{a}(a+b)^2$. Hence we can obtain the formula h_2^+ in (3.1) for $\alpha(a, b, \rho)$ as follows:

$$h_2^+ = \min\left\{\rho a + (1-\rho)b, \frac{a\sum_{l=0}^{7}\varphi_l(\rho)\left(\frac{b}{a}\right)^l}{\sum_{l=0}^{7}\psi_l(\rho)\left(\frac{b}{a}\right)^l}\right\}, \tag{3.2}$$

where

$$\begin{cases} \varphi_0(\rho) = \rho^2, \\ \varphi_1(\rho) = \rho(2\rho+1)(1-\rho), \\ \varphi_2(\rho) = \rho(1-\rho)(3-2\rho), \\ \varphi_3(\rho) = b^3\rho(1-3\rho)(1-\rho), \\ \varphi_4(\rho) = -\rho(1-\rho)(5-2\rho), \\ \varphi_5(\rho) = -2\rho(1-\rho)^2, \\ \varphi_6(\rho) = (1-\rho)(\rho+1), \\ \varphi_7(\rho) = \rho(1-\rho)(2-\rho), \end{cases} \quad \text{and} \quad \begin{cases} \psi_0(\rho) = \rho\left(1-\rho+\rho^3\right), \\ \psi_1(\rho) = \rho(1-\rho)\left(3\rho^2+3-\rho\right), \\ \psi_2(\rho) = \rho^2(1-\rho)(3-2\rho), \\ \psi_3(\rho) = -\rho(1-\rho)\left(5-4\rho+2\rho^2\right), \\ \psi_4(\rho) = -3\rho^2(1-\rho)(2-\rho), \\ \psi_5(\rho) = \left(\rho+\rho^2+1\right)(1-\rho)^2, \\ \psi_6(\rho) = 2\rho(1-\rho), \\ \psi_7(\rho) = \rho^2(1-\rho). \end{cases}$$

We also recall that for a weighted shift $W_{\alpha(x)}$ with $\alpha(x) : \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}$, Jung-Park proved the following([19, Theorem 4.6]):

the quadratic hyponormality and positive quadratic hyponormality of $W_{\alpha(x)}$ are equivalent to each other.

Then we can obtain the following result for the recursively generated weighted shifts $W_{\alpha(x;a,b,\rho)}$.

Proposition 3.1. *Let a weight sequence $\alpha(x; a, b, \rho)$ be given as in* (2.6) *and let $W_{\alpha(x;a,b,\rho)}$ be the corresponding weighted shift. Then the followings are equivalent:*
(i) *$W_{\alpha(x;a,b,\rho)}$ is quadratically hyponormal;*
(ii) $0 < x \le h_2^+$.

Example 3.2. *(Continued Example 2.4) Let $W_{\alpha(x;\frac{1}{2},\frac{2}{3},\frac{1}{3})}$ be a weighted shift with a sequence $\alpha(x; \frac{1}{2}, \frac{2}{3}, \frac{1}{3})$. Applying to* (3.2)*, we get $h_2^+ = \min\left\{\frac{11}{18}, \frac{295465}{468742}\right\}$, so $h_2^+ = \frac{11}{18} \approx 0.61111$. Hence from Proposition 3.1, $W_{\alpha(x;\frac{1}{2},\frac{2}{3},\frac{1}{3})}$ is quadratically hyponormal if and only if* $0 < x \le \frac{11}{18}$.

For a triplet (a, b, ρ) with $0 < a < b \le 1$ and $0 < \rho < 1$, without loss of generality, we may assume that $b = 1$. In fact we can take $a = \frac{a'}{b'}$ and $b = \frac{b'}{b'}$ for the triplet (a', b', ρ) with $a' < b' \le 1$. Hence we consider $(a, 1, \rho)$ instead of (a, b, ρ) for our convenience. Now we define a backward extension weight sequence $\alpha(x; a, 1, \rho)$ of $\alpha(a, 1, \rho)$ with $0 < a, \rho < 1$ as follows:

$$\alpha(x; a, 1, \rho): \sqrt{x}, \sqrt{\frac{\rho a^n + 1 - \rho}{\rho a^{n-1} + 1 - \rho}} \quad (n \ge 1). \tag{3.3}$$

Proposition 3.3. *Let $W_{\alpha(x;a,1,\rho)}$ be the weighted shift with a weight $\alpha(x;a,1,\rho)$ in (3.3). Then the following assertions are equivalent:*
(i) *$W_{\alpha(x;a,1,\rho)}$ is quadratically hyponormal;*
(ii) *$0 < x \le h_2^+ = \min\left\{1-\omega, \frac{a\left(\eta_3\omega^3+\eta_2\omega^2-\eta_1\omega+a\right)}{\zeta_4\omega^4-\zeta_3\omega^3-\zeta_2\omega^2-\zeta_1\omega+a^2}\right\}$, where $\omega = (1-a)\,\rho$,*

$$\begin{cases} \eta_3 = (a+1)\left(2a^2+2a+1\right), \\ \eta_2 = a^5+3a^4-7a^2-7a-3, \\ \eta_1 = 4a^3+7a^2+3a+1, \end{cases} \quad \textit{and} \quad \begin{cases} \zeta_4 = a^2\,(a+1)\,, \\ \zeta_3 = 4a^3+7a^2+3a+1, \\ \zeta_2 = a^5+6a^4+8a^3+a^2-1, \\ \zeta_1 = a\left(a^5+4a^4+4a^3-a^2-a-2\right). \end{cases}$$

Proof. To prove this result, we apply the weight sequence $\alpha(x;a,1,\rho)$ to Proposition 3.1. By substituting $b = 1$ into the formula in (3.2), we can obtain the same formula h_2^+ in the result. □

In particular, if we consider the case $\rho = \frac{1}{2}$ in the weight sequence $\alpha(a,1,\rho)$, then from Proposition 3.3 the following result holds.

Corollary 3.4. *Let $W_{\alpha(x;a,1,1/2)}$ be the weighted shift with $\alpha(x;a,1,1/2)$ as follows:*

$$\alpha\left(x;a,1,\frac{1}{2}\right):\ \sqrt{x},\ \sqrt{\frac{a^n+1}{a^{n-1}+1}} \quad (n \ge 1).$$

Then the following assertions are equivalent:
(i) *$W_{\alpha(x;a,1,1/2)}$ is quadratically hyponormal;*
(ii) *$0 < x \le h_2^+$, where*

$$h_2^+ = \begin{cases} \frac{2a\left(2a^5+2a^3-5a^2+3\right)}{(a+1)\left(5a^4-2a^3-5a^2+2a+2\right)} & \textit{if } a \le c_0, \\ \frac{a+1}{2} & \textit{if } c_0 < a, \end{cases}$$

for some $c_0 \in (0,1)$.

Proof. For the sequence $\alpha(x;a,1,\frac{1}{2})$ with $0 < a, \rho < 1$, it follows from (3.2) that $1-\omega = \frac{a+1}{2}$ and

$$h_2^+ = \min\left\{\frac{a+1}{2}, \frac{2a\left(2a^5+2a^3-5a^2+3\right)}{(a+1)\left(5a^4-2a^3-5a^2+2a+2\right)}\right\}.$$

Since

$$\frac{2a\left(2a^5+2a^3-5a^2+3\right)}{(a+1)\left(5a^4-2a^3-5a^2+2a+2\right)} - \frac{a+1}{2} = \frac{(a-1)^2\left(3a^4-2a^3+5a^2+2a-2\right)}{2(a+1)\left(5a^4-2a^3-5a^2+2a+2\right)},$$

we consider a function $f(a)$ on $(0,1)$ as

$$f(a) = \frac{3a^4-2a^3+5a^2+2a-2}{5a^4-2a^3-5a^2+2a+2}.$$

From simple computations,

$$f'(a) = \frac{4(a-1)(a^5-19a^4-17a^3-a^2-2a-2)}{(5a^4-2a^3-5a^2+2a+2)^2}.$$

For $0 < a < 1$, it is clear that $a^5-19a^4-17a^3-a^2-2a-2 < 0$, which implies $f'(a) > 0$, i.e. $f(a)$ is increasing function on $(0,1)$. Since $f(0) < 0$, the equation $f(a) = 0$ has a unique positive solution $c_0 \approx 0.472\,5249$ on $(0,1)$. Hence we have completed the proof. □

Example 3.5. *For the case $a = 1/3$ and $\rho = 1/2$ in $\alpha(a, 1, \rho)$, we consider a weighted shift $W_{\alpha(x;1/3,1,1/2)}$ with the weight sequence $\alpha(x; 1/3, 1, 1/2)$ as follows:*

$$\alpha\left(x; \frac{1}{3}, 1, \frac{1}{2}\right): \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{14}{15}}, \sqrt{\frac{41}{42}}, \cdots.$$

Since $a = 1/3 < c_0$, from Corollary 3.4, it holds $h_2^+ = \frac{307}{510} \approx 0.601\,96$. Hence it holds that $W_{\alpha(x;1/3,1,1/2)}$ is quadratically hyponormal $\Leftrightarrow 0 < x \leq \frac{307}{510}$.

Now to show more simple formula for h_2^+, we consider a backward extension weight sequence $\alpha(x; a, 1, a)$, i.e. $\rho = a$. From (3.3), the sequence $\alpha(x; a, 1, a)$ with $0 < a < 1$ is defined by

$$\alpha(x; a, 1, a): \sqrt{x}, \sqrt{\frac{a^{n+1} - a + 1}{a^n - a + 1}} \quad (n \geq 1). \tag{3.4}$$

Then we can have the following result.

Corollary 3.6. *Let $\alpha(x; a, 1, a)$ be a weight sequence as in (3.4) and let $W_{\alpha(x;a,1,a)}$ be the corresponding weighted shift. Then the following assertions are equivalent:*
(i) *$W_{\alpha(x;a,1,a)}$ is quadratically hyponormal;*
(ii) *$0 < x \leq h_2^+$, where*

$$h_2^+ = \begin{cases} \frac{a^8-3a^7+a^6+3a^5-6a^4+a^3+2a^2+3a-3}{-a^9+3a^8-5a^7+6a^6+3a^5-19a^4+12a^3+4a-4}, & \text{if } a \leq c_1, \\ a^2 - a + 1, & \text{if } c_1 < a, \end{cases}$$

for some $c_1 \in (0, 1)$.

Proof. Using the similar computations in the proof of Corollary 3.4 for $\alpha(x; a, 1, a)$, we can obtain

$$h_2^+ = \min\left\{a^2 - a + 1, \frac{a^8 - 3a^7 + a^6 + 3a^5 - 6a^4 + a^3 + 2a^2 + 3a - 3}{-a^9 + 3a^8 - 5a^7 + 6a^6 + 3a^5 - 19a^4 + 12a^3 + 4a - 4}\right\}.$$

Also we have

$$\begin{aligned} &\frac{a^8 - 3a^7 + a^6 + 3a^5 - 6a^4 + a^3 + 2a^2 + 3a - 3}{-a^9 + 3a^8 - 5a^7 + 6a^6 + 3a^5 - 19a^4 + 12a^3 + 4a - 4} - \left(a^2 - a + 1\right) \\ &= \frac{(a-1)^3\left(-a^8 + a^7 - 3a^6 + 5a^4 - 5a^3 + a^2 - 2a + 1\right)}{a^9 - 3a^8 + 5a^7 - 6a^6 - 3a^5 + 19a^4 - 12a^3 - 4a + 4} \equiv (a-1)^3 g(a). \end{aligned}$$

For $0 < a < 1$, it is obvious from some computations that $g'(a)$ is always negative, which implies that the function $g(a)$ is decreasing on $(0, 1)$. Since $g(0) > 0$ and $g(1) < 0$, the equation $g(a) = 0$ has a unique positive solution $c_1 \approx 0.461\,028$ on $(0, 1)$. Hence we have completed the proof. □

Example 3.7. *For the case $a = 1/2$ in $\alpha(a, 1, a)$, we consider a weighted shift $W_{\alpha(x;1/2,1,1/2)}$ with the sequence $\alpha(x; 1/2, 1, 1/2)$ as follows:*

$$\alpha\left(x; \frac{1}{2}, 1, \frac{1}{2}\right): \sqrt{x}, \sqrt{\frac{3}{4}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{9}{10}}, \sqrt{\frac{17}{18}}, \cdots.$$

Since $c_1 < a$, by Corollary 3.6 we have $h_2^+ = a^2 - a + 1 = \frac{3}{4}$. Hence $W_{\alpha(x;1/2,1,1/2)}$ is quadratically hyponormal $\Leftrightarrow 0 < x \leq \frac{3}{4}$.

4. Main results

In this section we discuss with Problem 1.2. To obtain an affirmative answer about Problem 1.2, it is worthwhile to consider a weighted shift W_β with a weight sequence $\beta : 1, 1, \beta_2, \beta_3, \cdots$ satisfying $1 < \beta_2 < \beta_3 < \cdots$. For this purpose, we deal with a recursive weight sequence $\alpha(a, 1, \rho)$ with two atoms $\{a, 1\}$ and density ρ $(0 < a, \rho < 1)$ introduced in Section 2. Now for our convenience and without loss of generality, we may consider new two atoms as follows:

$$\frac{a}{\rho a + 1 - \rho} \text{ and } \frac{1}{\rho a + 1 - \rho} \quad (0 < a, \rho < 1). \tag{4.1}$$

Then the associated two coefficients ψ_0 and ψ_1 of the quadratic generating function for two atoms as in (4.1) become

$$\psi_0 = -\frac{a}{(\rho a + 1 - \rho)^2} \text{ and } \psi_1 = \frac{a+1}{\rho a + 1 - \rho}. \tag{4.2}$$

Hence according to the usual methods in Section 2, we can have a recursively generated weight sequence $\beta(a, \rho)$, *that is*

$$\beta(a, \rho) : 1, \sqrt{\frac{\rho a^2 + 1 - \rho}{(\rho a + 1 - \rho)^2}}, \sqrt{\frac{\rho a^{n+1} + 1 - \rho}{(\rho a + 1 - \rho)(\rho a^n + 1 - \rho)}} \quad (n \geq 2). \tag{4.3}$$

We now consider the main weight sequence $\beta(1; a, \rho) = \{\beta_n\}_{n=0}^{\infty}$, a backward extension of $\beta(a, \rho)$ in (4.3) with first two equal weights as follows:

$$\beta(1; a, \rho) : 1, 1, \sqrt{\frac{\rho a^n + 1 - \rho}{(\rho a + 1 - \rho)(\rho a^{n-1} + 1 - \rho)}} \ (n \geq 2) \tag{4.4}$$

and denote $W_{\beta(1;a,\rho)}$ for the weighted shift with $\beta(1; a, \rho)$. Then we can obtain the fundamental lemma.

Lemma 4.1. *Let a weight sequence $\beta(1; a, \rho)$ be given as in* (4.4). *Set*

$$Q_k := \frac{M^k - N^k}{\sqrt{\psi_1^2 + 4\psi_0}} \left(\frac{1 - \left(-\frac{M^2}{\psi_0}\right)^k}{1 + \frac{M^2}{\psi_0}} \right),$$

where

$$M := \frac{\psi_1 + \sqrt{\psi_1^2 + 4\psi_0}}{2}, \ N := \frac{\psi_1 - \sqrt{\psi_1^2 + 4\psi_0}}{2}.$$

Then

$$Q_k = \frac{\left(\sum_{i=1}^{k} a^{i-1}\right)^2}{(a(\rho a + 1 - \rho))^{k-1}}. \tag{4.5}$$

Proof. It follows from (4.2) that

$$\sqrt{\psi_1^2 + 4\psi_0} = \frac{1-a}{\rho a + 1 - \rho}, \ M = \frac{1}{\rho a + 1 - \rho}, \ N = \frac{a}{\rho a + 1 - \rho}.$$

Since $0 < a < 1$, $\frac{a}{\rho a+1-\rho} < \frac{1}{\rho a+1-\rho}$. It follows from Lemma 2.2 that $L^2 = \frac{1}{\rho a+1-\rho} = M$, which implies $\frac{M^2}{\psi_0} = -\frac{1}{a}$. Also for all $k \geq 2$,

$$\frac{M^k - N^k}{\sqrt{\psi_1^2 + 4\psi_0}} = \frac{\rho a + 1 - \rho}{1-a}\left(\frac{1}{(\rho a + 1 - \rho)^k} - \frac{a^k}{(\rho a + 1 - \rho)^k}\right) = \frac{(\rho a + 1 - \rho)(1-a^k)}{(1-a)(\rho a + 1 - \rho)^k}. \tag{4.6}$$

To obtain the formula of Q_k, using the third formula in (4.6), it holds that for all $k \geq 2$

$$\begin{aligned} Q_k &= \frac{(\rho a + 1 - \rho)(1-a^k)}{(1-a)(\rho a + 1 - \rho)^k} \cdot \frac{1 - \left(\frac{1}{a}\right)^k}{1 - \frac{1}{a}} \\ &= \frac{(\rho a + 1 - \rho)(1-a^k)}{(1-a)(\rho a + 1 - \rho)^k} \cdot \frac{a(1-a^k)}{(1-a)a^k} \\ &= \frac{(1-a^k)^2}{a^{k-1}(1-a)^2(\rho a + 1 - \rho)^{k-1}} \\ &= \frac{\left(1 + a + a^2 + \cdots + a^{k-1}\right)^2}{\left(a\left(a\rho + 1 - \rho\right)\right)^{k-1}}. \end{aligned}$$

Hence the proof is completed. □

For the recursive sequence $\beta(1; a, \rho)$ with first two equal weights as in (4.4), applying to the formula [23, Proposition 3.3] via some computations for formulas in (2.5), we can obtain the following result.

Proposition 4.2. *For $0 < a, \rho < 1$, let $\beta(1; a, \rho)$ be given as in* (4.4) *and let $W_{\beta(1;a,\rho)}$ be the associated weighted shift. Suppose $k \geq 2$. Then $W_{\beta(1;a,\rho)}$ is semi-weakly k-hyponormal if and only if $\Theta_k(a, \rho) \geq 1$, where*

$$\Theta_k := \frac{\beta_k^2\beta_{k+1}^2 \cdots \beta_{2k-1}^2(\beta_1^2\beta_2^2 \cdots \beta_k^2 + Q_k) - Q_k\beta_1^2\beta_2^2 \cdots \beta_{k-1}^2\beta_k^4}{\beta_1^2\beta_2^2 \cdots \beta_{k-1}^2(\beta_1^2\beta_2^2 \cdots \beta_k^2 + Q_k - 2\beta_k^2 Q_k) + Q_k\beta_k^2\beta_{k+1}^2 \cdots \beta_{2k-1}^2}.$$

From Lemma 4.1 and Proposition 4.2, we obtain the main theorem of the paper.

Theorem 4.3. *For $0 < a, \rho < 1$, let $\beta(1; a, \rho)$ be given as in* (4.4) *and let $W_{\beta(1;a,\rho)}$ be the associated weighted shift. Suppose $k \geq 2$. Then $W_{\beta(1;a,\rho)}$ is semi-weakly k-hyponormal if and only if $h_k(a, \rho) \geq 0$, where*

$$h_k(a, \rho) = \left(a^k - 1\right)^2 \rho^2 + \left(2a^k - a^{k-1} - 1\right)\rho + a^{k-1}. \tag{4.7}$$

Proof. We first note the formula of Θ_k in Proposition 4.2 for the weight sequence $\beta(1; a, \rho)$ as following:

$$\Theta_k = \frac{\beta_{k+1}^2\beta_{k+2}^2 \cdots \beta_{2k}^2(\beta_2^2 \cdots \beta_{k+1}^2 + Q_k) - Q_k\beta_2^2 \cdots \beta_k^2\beta_{k+1}^4}{\beta_2^2 \cdots \beta_k^2(\beta_2^2 \cdots \beta_{k+1}^2 + Q_k - 2\beta_{k+1}^2 Q_k) + Q_k\beta_{k+1}^2\beta_{k+2}^2 \cdots \beta_{2k}^2}. \tag{4.8}$$

From the definition of the weight sequence $\beta(1; a, \rho)$,

$$\beta_i^2 = \frac{\rho a^i + 1 - \rho}{(\rho a + 1 - \rho)(\rho a^{i-1} + 1 - \rho)} \quad (i \geq 1),$$

which implies that for any $n,\ \ell \geq 1$

$$\beta_n^2\beta_{n+1}^2 \cdots \beta_{n+\ell-1}^2 = \frac{\rho a^{n+\ell-1} + 1 - \rho}{(\rho a + 1 - \rho)^{\ell}\,(\rho a^{n-1} + 1 - \rho)}. \tag{4.9}$$

Applying the formulas Q_k in Lemma 4.1 and (4.9) to (4.8), we can obtain

$$\Theta_k = \frac{\theta_k}{\xi_k},$$

where

$$\theta_k = \frac{\rho a^{2k} + 1 - \rho}{(\rho a + 1 - \rho)^k (\rho a^k + 1 - \rho)} \left(\frac{\rho a^{k+1} + 1 - \rho}{(\rho a + 1 - \rho)^{k+1}} + Q_k \right) - \frac{Q_k(\rho a^{k+1} + 1 - \rho)^2}{(\rho a + 1 - \rho)^{k+2} (\rho a^k + 1 - \rho)}$$

and

$$\xi_k = \frac{\rho a^k + 1 - \rho}{(\rho a + 1 - \rho)^k} \left(\frac{\rho a^{k+1} + 1 - \rho}{(\rho a + 1 - \rho)^{k+1}} + Q_k - \frac{2(\rho a^{k+1} + 1 - \rho)Q_k}{(\rho a + 1 - \rho)(\rho a^k + 1 - \rho)} \right) + \frac{Q_k(\rho a^{2k} + 1 - \rho)}{(\rho a + 1 - \rho)^k (\rho a^k + 1 - \rho)}.$$

It follows from some computations that $\theta_k > 0$ and $\xi_k > 0$ for all $k \geq 2$.
Hence $\Theta_k \geq 1 \Leftrightarrow \theta_k - \xi_k \geq 0$ ($k \geq 2$). Using Q_k in (4.5) and $h_k(a, \rho)$ in (4.7), we have

$$\begin{aligned} \theta_k - \xi_k &= \frac{\rho(1-\rho)(a^k - 1)^2}{(\rho a^k + 1 - \rho)(\rho a + 1 - \rho)^{2k+1}} \left(\frac{-\rho(1-\rho)(a^k - 1)^2}{a^{k-1}} + a^k \rho + 1 - \rho \right) \\ &= \frac{\rho(1-\rho)(a^k - 1)^2 a^{k-1}}{(\rho a^k + 1 - \rho)(\rho a + 1 - \rho)^{2k+1}} h_k(a, \rho). \end{aligned}$$

Since $0 < a, \rho < 1$, it holds that $\Theta_k \geq 1 \Leftrightarrow h_k(a, \rho) \geq 0$, which completes the proof. □

Set for $k \geq 2$,

$$\mathcal{RH}_k := \{(a, \rho) : h_k(a, \rho) \geq 0,\ 0 < a, \rho < 1\}.$$

From Theorem 4.3, we can describe regions of $\mathcal{RH}_k$ for each $k \geq 2$ in Figure 1, which provide distinctions and implications of semi-weak k-hyponormalities of weighted shift $W_{\beta(1;a,\rho)}$ with $0 < a, \rho < 1$.

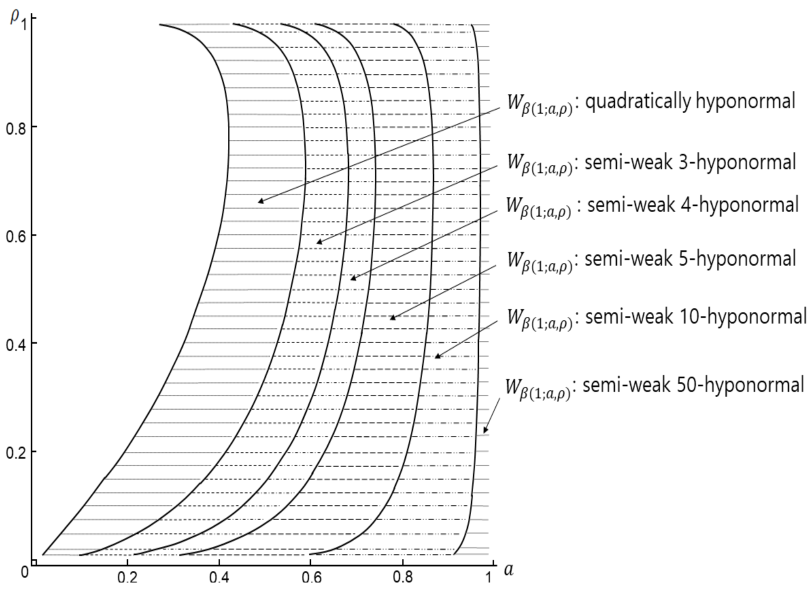

Figure 1: Regions of semi-weak k-hyponormality of $W_{\beta(1;a,\rho)}$.

For more simplicity of characterization of semi-weak k-hyponormality of $W_{\beta(1;a,\rho)}$, we consider the projection of the set $\mathcal{RH}_k$ onto the diagonal set $\{(a,\rho) : 0 < a, \rho < 1,\ \rho = a\}$, that is, we take the recursive weight sequence $\beta(1; a, a)$ $(0 < a < 1)$ as follows:

$$\beta(1;a,a) : 1, 1, \sqrt{\frac{a^3 - a + 1}{(a^2 - a + 1)^2}}, \sqrt{\frac{a^4 - a + 1}{(a^2 - a + 1)(a^3 - a + 1)}}, \cdots . \tag{4.10}$$

Using Theorem 4.3, we can obtain the following assertion.

Proposition 4.4. *For $0 < a < 1$, let $W_{\beta(1;a,a)}$ be the weighted shift with $\beta(1; a, a)$ as in* (4.10). *Suppose $k \geq 2$. Then $W_{\beta(1;a,a)}$ is semi-weakly k-hyponormal if and only if $s_k(a) \geq 0$, where*

$$s_k(a) = \begin{cases} a^{2k+1} - 2a^{k+1} + 2a^k - a^{k-1} + a^{k-2} + a - 1 & \text{for } k \geq 3, \\ a^2(a^3 - 2a + 2) & \text{for } k = 2. \end{cases} \tag{4.11}$$

In particular, $W_{\beta(1;a,a)}$ is quadratically hyponormal if and only if $0 < a < 1$.

Proof. Substituting $\rho = a$ to $h_k(a, \rho)$ in (4.7), we have $h_k(a, a) = as_k(a)$. From Theorem 4.3, we obtain the first result.

Next to show the range of the quadratic hyponormality for $W_{\beta(1;a,a)}$, we consider the function $s_2(a)$ as in (4.11). For our convenience, put

$$t(a) = a^3 - 2a + 2.$$

It follows from simple computations that $t(a)$ has the positive local minimum at $a = \sqrt{\frac{2}{3}}$, which implies $s_2(a) > 0$ for all $0 < a < 1$. Hence we have completed the proof. □

Let a weight sequence $\beta(1; a, a)$ be as in (4.10) and let $W_{\beta(1;a,a)}$ be the corresponding weighted shift. We now consider the following set

$$s\mathcal{WH}_k := \{a \in (0, 1) : W_{\beta(1;a,a)} \text{ is semi-weakly } k\text{-hyponormal}\} \text{ for } k \geq 2.$$

From Proposition 4.4 it is obvious that $s\mathcal{WH}_2 = (0, 1)$.

Theorem 4.5. *Suppose that $k \geq 3$ and $0 < a < 1$. Let $W_{\beta(1;a,a)}$ be the weighted shift with $\beta(1; a, a)$ as in* (4.10). *Denote r_k for a positive zero of the function $s_k(a)$ as in* (4.11). *Then the following assertions hold:*
(i) *$\{r_k\}_{k=3}^{\infty}$ is a strictly increasing sequence in* (0, 1);
(ii) *$s\mathcal{WH}_k = [r_k, 1)$ for $3 \leq k < \infty$;*
(iii) *$s\mathcal{WH}_k \setminus s\mathcal{WH}_{k+1} = [r_k, r_{k+1})$ and $(0, 1) \supsetneq s\mathcal{WH}_3 \supsetneq \cdots \supsetneq s\mathcal{WH}_k \supsetneq s\mathcal{WH}_{k+1} \supsetneq \cdots$.*

Proof. Consider the function $s_n(a)$ for $n \geq 3$ in Proposition 4.4,

$$s_n(a) = a^{2n+1} - 2a^{n+1} + 2a^n - a^{n-1} + a^{n-2} + a - 1.$$

Then for all $n \geq 3$, the continuous function $s_n(a)$ passes through two fixed points $(0, -1)$ and $(1, 1)$, which induces that there exists a zero $r_n \in (0, 1)$ of the equation $s_n(a) = 0$. Also from simple computations, we have

$$s_n'(a) = (2n + 1)a^{2n} - 2(n + 1)a^n + 2na^{n-1} - (n - 1)a^{n-2} + (n - 2)a^{n-3} + 1,$$
$$s_n''(a) = a^{n-4}\left(2(2n + 1)na^{n+3} - 2(n + 1)na^3 + 2(n - 1)na^2 - (n - 1)(n - 2)a + (n - 2)(n - 3)\right).$$

For $n \geq 3$, it holds that $s''_n(a) > 0$ for $0 < a < 1$, which induces that the function $s'_n(a)$ is increasing on (0, 1). By $s'_n(0) > 0$, $s'_n(a) > 0$ for $0 < a < 1$. So the function $s_n(a)$ is strictly increasing on $(0, 1)$, which guarantees that there exists a unique positive root $r_n \in (0, 1)$ satisfying $s_n(r_n) = 0$ for each $n \geq 3$. For the positive zero r_n $(3 \leq n < \infty)$, from some computations we can obtain that

$$s_{n+1}(r_n) = r_n\, s_n(r_n) + (r_n - 1)\left(r_n^{2(n+1)} - r_n + 1\right),\ n \geq 3. \tag{4.12}$$

Since the function $s_n(a)$ is increasing on (0, 1), the value of $s_{n+1}(r_n)$ in (4.12) turns out to be negative, *i.e.* $s_{n+1}(r_n) < 0$, which implies $r_n \lneq r_{n+1}$ $(n \geq 3)$. Then by mathematical induction, the following inequalities hold:

$$0 < r_3 \lneq r_4 \lneq \cdots \lneq r_n \lneq r_{n+1} \lneq \cdots < 1. \tag{4.13}$$

Hence the sequence $\{r_n\}_{n=3}^{\infty}$ is a strictly increasing sequence in the interval (0, 1). Also by conditions of the $s_n(a)$ and uniqueness of the positive root r_n for all $n \geq 3$ and $0 < a < 1$, we can obtain that $r_n = \inf s\mathcal{WH}_n$ and the set $s\mathcal{WH}_n$ is a connected interval, *that is*,

$$s\mathcal{WH}_n = [r_n,\ 1)\ \text{ for } 3 \leq n < \infty.$$

Therefore by (4.13), we have completed the proofs. □

Finally, we close this note providing mutually disjoint approximate values for r_k for useful finite numbers of $k = 3, 4, 5$.

Corollary 4.6. *Let $W_{\beta(1;a,a)}$ be the weighted shift. Then the following assertions hold:*
(i) *$W_{\beta(1;a,a)}$ is quadratically but not semi-cubically hyponormal* $\Leftrightarrow 0 < a < r_3(\approx 0.574)$;
(ii) *$W_{\beta(1;a,a)}$ is semi-cubically but not semi-quartically hyponormal* $\Leftrightarrow r_3 \leq a < r_4(\approx 0.682)$;
(iii) *$W_{\beta(1;a,a)}$ is semi-quartically but not semi-quintically hyponormal* $\Leftrightarrow r_4 \leq a < r_5(\approx 0.741)$.

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