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Gadient Ricci soliton on Schwarzschild black hole and Ricci-Hessian type space-time warped product

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Abstract. In this article, we study gradient Ricci soliton on generalized Schwarzschild black hole. Then we discuss Ricci-Hessian type space-time warped product and obtain Bochner-Weitzen*ö*ck formula for space-time. We also provide the existence results of gradient Ricci soliton space-time warped product.

1. Introduction and preliminaries

The Ricci flow on a Riemannian manifold (M, g) is an one parameter family of the metric $g(t)$ which satisfies the following equation,

$$\frac{\partial g}{\partial t} = -2Rc, \quad g(0) = g_0,$$

where Rc is the Ricci curvature tensor and g_0 is the initial metric. R. Hamilton [19], introduced the concept of Ricci flow in 1982. A self similar solution of Ricci flow is called Ricci soliton which can be described by the following equation,

$$Rc + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

where X is a vector field on M and λ is some real number. If we consider a smooth function ϕ on M such that $X = \nabla\phi$, then the manifold is said to be gradient Ricci soliton and then the corresponding equation becomes,

$$Rc + \nabla^2\phi = \lambda g.$$

A gradient Ricci soliton is shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. Gradient Ricci soliton is natural generalization of Einstein manifold. In the above expression if we take function ϕ, as a constant function, then it becomes Einstein manifold. In 2020, H. Alodan proved the necessary and sufficient condition for a submanifold of Euclidean space to become Ricci soliton. Nicolas Ginoux et. al. [16], studied

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and partially classified some manifold which satisfy $\nabla^2 f = -fRc$, where f is non-zero smooth function and Rc is Ricci curvature tensor. V. Borges and K. Tenenblat [23], discussed the behavior of Ricci soliton on a warped product whenever the warping function depends on the fiber manifold. In 2011, G. Catino[9], introduced the notion of generalized quasi-Einstein manifold and this notion generalizes the concepts of Ricci soliton. S. Guler and S. A. Demirbag [22], investigated the relationshps between generalized quasi Einstein warped product and Ricci-Hessian type manifold and obtained some rigidity condition. S. Deshmukh and H. Alsodais [21], characterized the trivial Ricci soliton and studied the role of energy function in this characterization. In 2017, F.E.S. Feitosa et. al. [7, 8], proved that if a warped product is a gradient Ricci soliton, then its base manifold is a Ricci-Hessian type manifold and the fiber manifold is an Einstein manifold.

Mathematically, space-time is the union of space and time interpreted by product manifold of space with an closed interval of $\mathbb{R}$. It was first introduced by Hermann Minkowski in 1908. In the currect scenerio, space-time is torsionless, time-oriented Lorentzian manifold. Grisha Perelman [10], discussed the concept of potentially infinite metric. Also, he discussed "Ricci flow as a gradient flow." In 1920, Friedmann-Lemaitre and Robertson-Walker introduced a space time assuming the space homogeneous and isotropic. Sun-Chin Chu [1], gave the concept of space-time connection and modified Ricci flow for degenerate metrics. In 2002, B. Chow and Sun-Chin Chu [2], gave the space-time formulation for Ricci flow and linearized it. Further they showed the variation of 2-parameter family of metrics. In 1916, Einstein presented his theory of general relativity. In this theory, first time the existence of black hole was predicted. Then, German physicist and astronomer Karl Schwarzschild obtained the first modern solution for general relativity theory.

In the coordinate (t, r, θ, ϕ), the Schwarzschild metric has form

$$ds^2 = -\left(1 - \frac{s_r}{r}\right)c^2dt^2 + \left(1 - \frac{s_r}{r}\right)^{-1} dr^2 + r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right],$$

where $s_r = \dfrac{2G\mathcal{M}}{c^2}$ represent Schwarzschild radius.
The generalisation of above leads us to Schwarzschild metric for n-dimension as follow:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\omega^2_{(n-2)},$$

where $f(r) = 1 - \dfrac{m}{r^{(n-3)}}$, m represent geometric mass.

In 2003, R. A. Konoplya [18], studied the characteristic (quasinormal) modes of a D-dimensional Schwarzschild black hole. F. Darabi et. al. [5], showed that the entropy of generalized BTZ black hole can be described by Cardy-Verlinde formula. They also discussed the thermodynamics of generalized BTZ black hole. In 2019, J. P. d. Santo and B. Leandro [14], obtained all the solutions of reduced system of differential equations, where classical Schwarzschild solution behaves like a particular solution.

Warped product is a product manifold of two Riemannian manifolds (B, g_B) and (F, g_F) along with a warping function f which is a positive smooth function on the manifold B with the metric $g_B + f^2 g_F$. The manifold B is called base and F is called fiber manifold. It is a generalization of Riemannian product. If warping function f is constant, then warped product becomes Riemannian product. In 1908 Bishop and O'Neil [17], first introduced warped product during constrcution of some examples of manifold with negative curvature. Study of warped product is not limited to Riemannian manifold. Khalid Ali Khan et. al. [15], proved results on the non-existence of warped product submanifolds of certain types in cosymplectic manifolds. Chenxu He et. al. [4], discussed behaviour of the warped product when it is an Einstein manifold. Recently, J. Melẽndez and M. Hernãndez [12] proved that the warping function of a warped product with non-negative sectional curvature and parabolic base manifold is constant. In [4, 6, 24], author studied geometric flows and some classifications on warped product.

Let (M, g) be a complete Riemannian manifold of dimension greater than two and let $h : M \to \mathbb{R}$ be a smooth function on M. Then k-Bakry-Emery Ricci tensor is given by

$$Rc_h^k = Rc + H^h - \frac{1}{k} dh \otimes dh, \tag{1}$$

is an extenstion of Ricci curvature tensor, where k is a positive integer and λ is some real number. An ∞-Bakry-Emery Ricci tensor is known as Bakry-Emery Ricci tensor and is given by

$$Rc_h = Rc + H^h. \tag{2}$$

The function h is called potential function. Yasemin Soylu [25], used k-Bakry-Emery Ricci tensor to prove Myers-type compactness theorem.

Definition 1.1 (Ricci-Hessian type equation:). *Let (M, g) be a Riemannian manifold, with Ricci curvature tensor Rc. Then, the following equation*

$$Rc + \alpha \nabla^2 \phi = \gamma g, \tag{3}$$

is called Ricci-Hessian type equation, where α, γ and ϕ are the smooth functions on M. The manifold on which this equation is satisfied is called Ricci-Hessian type manifold.

If warped product $B \times_f F$ is gradient Ricci soliton then the base manifold B satisfy the following Ricci-Hessian type equation [7]:

$$Rc + \nabla^2 \phi = \lambda g + \frac{k}{f} \nabla^2 f. \tag{4}$$

Let $(B \times I) \times_f F$ with metric $g_B + (R + \frac{N}{2t})dt^2 + f^2 g_F$ be space-time warped product with potentially infinite metric, where R is the scalar curvature of the manifold (B, g_B) and N is a large number such that $(R + \frac{N}{2t}) > 0$. Let ${}^{BI}Rc$ and Rc are the Ricci curvature tensors on $B \times I$ and B, respectively. Now, we consider the space-time $B \times I$ satisfy the above Ricci-Hessian type equation (4). Let $\tilde{\nabla}$ and ∇ be the connection on $B \times I$ and B respectively. Then, we have

$${}^{BI}Rc + \tilde{\nabla}^2 \phi = \lambda g + \frac{k}{f} \tilde{\nabla}^2 f. \tag{5}$$

Let $(B \times I, g_B + (R + \frac{N}{2t})dt^2)$ be n-dimensional space-time manifold. The Ricci-curvature tensors on $B \times I$ are given by the lemma (1.2).

Lemma 1.2 ([3]). *Let $B \times I$ be a space-time with metric $g = g_B + (R + \frac{N}{2t})dt^2$ and $X, Y, Z \in \Gamma(B)$, $\partial_t (= \frac{\partial}{\partial t}) \in \Gamma(I)$. Then*

(i) ${}^{BI}Rc(X, Y) = {}^{B} Rc(X, Y) - \left(\frac{H^R(X, Y)}{2(R + \frac{N}{2t})} - \frac{XRYR}{4(R + \frac{N}{2t})^2} \right),$

(ii) ${}^{BI}Rc(X, \partial_t) = 0,$

(iii) ${}^{BI}Rc(\partial_t, \partial_t) = -\frac{1}{2} \Delta R - \frac{1}{4(R + \frac{N}{2t})} |\nabla R|^2,$

where ${}^{B}Rc$ denote Ricci curvature of B.

So, the equation (5) becomes:

$$\begin{aligned} &{}^{B}Rc - \left(\frac{H^R}{2(R + \frac{N}{2t})} - \frac{dR \otimes dR}{4(R + \frac{N}{2t})^2} \right) - \left(\frac{1}{2} \Delta R + \frac{1}{4(R + \frac{N}{2t})} |\nabla R|^2 \right) dt^2 + \tilde{\nabla}^2 \phi \\ &= \lambda g + \frac{k}{f} \tilde{\nabla}^2 f. \end{aligned} \tag{6}$$

Equations (5) and (6) are equivalent.

The organization of this paper is as follow. In section 2, we write generalized Schwarzschild black hole in the form of space-time warped product. Then, we find conditions under which generalized Schwarzschild black hole become gradient Ricci soliton. We also show that if warping function reaches to maximum and minimum value then generalized Schwarzschild black hole becomes Riemannian product space. In section 3, we discuss space-time manifold when it satisfy Ricci-Hessian type equation. We also obtain the Bochner-Weitzenböck formula for space-time manifold. Next, we discuss some results which insures the existence or non existence of gradient Ricci-soliton space-time warped product. In section 4, we obtain the components of Ricci curvature tensor of space time manifold $B \times I$, when it satisfy a Ricci-Hessian type equation and then approximate them upto $O(N^{-1})$. Further, we compute Potentially gradient Ricci soliton identities.

2. Generalized Schwarzschild Black hole

In [18], R. Konoplya defined the metric for Schwarzschild black hole as follows:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\omega_{n-2}^2, \tag{7}$$

where $f(r) = (1 - \frac{m}{r^{n-3}})$ and m is geometric mass. Now, assuming that $f(r) > 0$ let us consider a transformation

$$dv = \sqrt{f(r)^{-1}}dr,$$

this gives,

$$v = \int_0^r \sqrt{f(r)^{-1}}dr = F(r).$$

From this we have, $r = F^{-1}(v)$, then we get

$$f(r) = 1 - \frac{m}{(F^{-1}(v))^{n-3}}.$$

Then, equation (7) becomes

$$ds^2 = dv^2 + \left(\frac{m}{(F^{-1}(v))^{n-3}} - 1\right)dt^2 + \left(F^{-1}(v)^2\right)d\omega_{n-2}^2.$$

The above expression looks like the metric of space-time warped product with potentialy infinte metric, i.e. of the form

$$g = g_B + \left(R + \frac{N}{2t}\right)dt^2 + f^2 g_F.$$

After comparision, we get to know that dimension of the base manifold B is 1, so the scalar curvature $R = 0$. Thus, metric of some space-time warped product with potentialy infinte metric with 1 dimensional base becomes,

$$g = dx^2 + \left(\frac{N}{2t}\right)dt^2 + f^2 g_F.$$

Here, we consider that the base manifold possesses Ricci flow, which means the geometric mass of the black hole depends on t. Therefore, the comparison of the metrics gives us

$$\left(\frac{m}{(F^{-1}(v))^{n-3}} - 1\right) = \frac{N}{2t}$$

i.e.

$$m(t) = \frac{(\frac{N}{2t} + 1)}{(F(v))^{n-3}}$$

Hence, whenever geometric mass of black hole m satisfies

$$m(t) = \frac{(\frac{N}{2t} + 1)}{(F(v))^{n-3}},$$

n-dimensional Schwarzschild black hole becomes a space-time warped product with potentially infinite metric.

In this section, we discuss the property of generalized Schwarzschild black hole gradient Ricci soliton. We represent generalized Schwarzschild black hole as a space-time warped product $\tilde{M} = (\mathbb{R} \times I) \times_f F$ with the metric

$$ds^2 = dx^2 + \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)dt^2 + (F^{-1}(x))^2 d\omega_{n-2}^2. \tag{8}$$

The components of the connections on generalized Schwarzschild black hole are given in the following lemma.

Lemma 2.1. *Let $\tilde{M} = (\mathbb{R} \times I) \times_f F$ be the Schwarzschild black hole with metric*

$$ds^2 = dx^2 + \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)dt^2 + (F^{-1}(x))^2 d\omega_{n-2}^2. \tag{9}$$

Then, for the vector fields $X, Y \in \Gamma(\mathbb{R})$, $T, T_1, T_2 \in \Gamma(I)$ and $V, W \in \Gamma(F)$, we have

(i) $D_X Y$ is lift of $\nabla_X Y$ on B,

(ii) $D_T X = D_X T = \frac{X\rho}{2\rho}T$,

(iii) $D_{T_1} T_2 = -\frac{\nabla\rho}{2\rho} g(T_1, T_2) +^I \nabla_{T_1} T_2$,

(iv) $D_X V = D_V X = \frac{Xf}{f}V$,

(v) $D_V W = -\frac{g(V, W)}{f}\nabla f +^F \nabla_V W$,

where $\rho = \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)$ and $f(x) = F^{-1}(x)$.

Let (M, g) be a Riemannian manifold and D is the Levi-Civita connection, then the Riemannian curvature tensor is given by

$$R_{XY}Z = -D_X D_Y Z + D_Y D_X Z + D_{[X,Y]}Z. \tag{10}$$

Let e_i be the orthogonal basis for tangent space of the manifold (M, g), then the Ricci curvature tensor

$$Rc(X, Y) = \sum_{i=1}^{n} g(R_{Xe_i} Y, e_i). \tag{11}$$

Now, the components of Ricci curvature tensor on generalized Schwarzschild black hole is given by

Lemma 2.2. *Let $\tilde{M} = (\mathbb{R} \times I) \times_f F$ be the Schwarzschild black hole with metric*

$$ds^2 = dx^2 + \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)dt^2 + (F^{-1}(x))^2 d\omega_{n-2}^2. \tag{12}$$

Then, the components of Ricci curvature tensor on Schwarzschild black hole are:

(i) $Rc(X, Y) = -\frac{1}{2\rho}H^\rho(X, Y) + \frac{1}{4\rho^2}d\rho \otimes d\rho(X, Y) - \frac{(n-2)}{f}H^f(X, Y),$

(ii) $Rc(T_1, T_2) = \left(\frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2 - \frac{1}{2\rho}\frac{\partial^2\rho}{\partial x^2} - \frac{(n-2)}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right)g(T_1, T_2),$

(iii) $Rc(X, V) = 0,$

(iv) $Rc(V, W) = {}^F Rc(V, W) - \left(\frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x} + \frac{(n-3)}{f^2}\left(\frac{\partial f}{\partial x}\right)^2\right)g(V, W)$, *where* $\rho = \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)$ *and* $f(x) = F^{-1}(x)$.

Note: To prove these Lemma 2.1 and Lemma 2.2, the readers are refered to see [3].

In [20], Richard Hamilton proved that if $(M, g, \nabla\phi, \lambda)$ is gradient Ricci soliton. Then, one has

$$2\lambda\phi - |\nabla\phi|^2 + \Delta\phi = c. \tag{13}$$

In the discussion of Schwarzschild black hole gradient Ricci soliton, first we prove the above result for Schwarzschild black hole.

Proposition 2.3. *Let $\tilde{M} = (\mathbb{R} \times I) \times_f F$ be generalized Schwarzschild black hole with the metric*

$$ds^2 = dx^2 + \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)dt^2 + (F^{-1}(x))^2 d\omega_{n-2}^2. \tag{14}$$

If ϕ be smooth function on $\mathbb{R} \times I$ so that the Schwarzschild black hole is gradient Ricci soliton, then we have

$$2\lambda\phi - \left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{\rho}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2} + \frac{(n-2)}{f}\frac{\partial f}{\partial x}\frac{\partial\phi}{\partial x} = c. \tag{15}$$

Proof. For a smooth function ϕ, we have

$$|\nabla\phi|^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{\rho}\left(\frac{\partial\phi}{\partial t}\right)^2 \tag{16}$$

and

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2} + \frac{(n-2)}{f}\frac{\partial f}{\partial x}\frac{\partial\phi}{\partial x}. \tag{17}$$

Using equations (16) and (17) into equation (13), we get the required result.
□

Proposition 2.4. *Let the generalized Schwarzschild black hole with the metric*

$$ds^2 = dx^2 + \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)dt^2 + (F^{-1}(x))^2 d\omega_{n-2}^2 \tag{18}$$

be a gradient Ricci soliton with potential function ϕ. Then

(*i*) $-\frac{1}{2}\frac{\partial^2\rho}{\partial x^2} + \frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2 + \frac{\partial^2\phi}{\partial x^2} = \lambda + \frac{(n-2)}{f}\frac{\partial^2 f}{\partial x^2}.$

(*ii*) $\frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2 - \frac{1}{2\rho}\frac{\partial^2\rho}{\partial x^2} - \frac{(n-2)}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} = \lambda.$

(*iii*) ${}^F Rc = \mu d\omega_{n-2}^2$, *where μ is given by following equation:*

$$\mu = \left(\lambda f^2 + f\frac{\partial^2 f}{\partial x^2} + \frac{f}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x} + (n-3)\left(\frac{\partial f}{\partial x}\right)^2 - f\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x}\right), \tag{19}$$

along with $\rho = \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)$ and $f = F^{-1}$.

Proof. If Schwarzschild black hole is a gradient Ricci soliton with potential funtion ϕ, then we have

$$Rc + \nabla^2\phi = \lambda g, \tag{20}$$

where λ is some constant and Rc is Ricci curvature tensor on Schwarzschild black hole. For the vector fields X, Y corresponding to metric dx^2, we have

$$-\frac{1}{2\rho}H^\rho(X,Y) + \frac{1}{4\rho^2}d\rho\otimes d\rho(X,Y) - \frac{(n-2)}{f}H^f(X,Y) + \nabla^2\phi(X,Y) = \lambda g(X,Y) \tag{21}$$

or equivalently,

$$-\frac{1}{2}\frac{\partial^2\rho}{\partial x^2} + \frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2 - \frac{(n-2)}{f}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2} = \lambda. \qquad \because g(\partial_x, \partial_x) = 1. \tag{22}$$

Now, for the vector fields T_1, T_2 corresponding to the time space with metric dt^2, we have

$$\nabla^2\phi(T_1,T_2) = \left(\frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t}\right)g(T_1,T_2). \tag{23}$$

Using this value of Hessian of ϕ and part (*ii*) of the lemma (2.2) , the equation (20) becomes:

$$\left(\frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2 - \frac{1}{2\rho}\frac{\partial^2\rho}{\partial x^2} + \frac{n-2}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right)g(T_1,T_2) + \left(\frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t}\right)g(T_1,T_2) = \lambda g(T_1,T_2). \tag{24}$$

Simplifying and rearranging the above equation, we achieve

$$\frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2-\frac{1}{2\rho}\frac{\partial^2\rho}{\partial x^2}+\frac{n-2}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}+\frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x}+\frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2}-\frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t}=\lambda. \tag{25}$$

Further, for the vector fields V, W corresponding to the metric $d\omega^2_{n-2}$, the Hessian of potential function ϕ is as follows

$$\nabla^2\phi(V,W)=g(D_V\nabla\phi,w)=f\frac{\partial\phi}{\partial x}\frac{\partial f}{\partial x}d\omega^2_{n-2}. \tag{26}$$

Now, using part (iv) of lemma (2.2) and hessian of ϕ, $\nabla^2\phi(V,W)$, also using the fact $g(V,W)=f^2d\omega^2_{n-2}$ into the equation (20), we have

$$^{\omega}Rc(V,W)-\left(\frac{1}{f}\frac{\partial^2 f}{\partial x^2}+\frac{1}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}+\frac{(n-3)}{f^2}\left(\frac{\partial f}{\partial x}\right)^2\right)f^2d\omega^2_{n-2}+f\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x}d\omega^2_{n-2}=\lambda f^2d\omega^2_{n-2}. \tag{27}$$

Rearranging the terms in above equation, we obtain

$$^{\omega}Rc(V,W)=\left(\lambda f^2+f\frac{\partial^2 f}{\partial x^2}+\frac{f}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}+(n-3)\left(\frac{\partial f}{\partial x}\right)^2-f\frac{\partial\phi}{\partial x}\frac{\partial f}{\partial x}\right)d\omega^2_{n-2}. \tag{28}$$

Hence the proof. □

In next proposition, we prove that μ is constant. For this we will use following some well known results.

$$\mathrm{div}\nabla^2\phi=Rc(\nabla\phi,\cdot)+d(\Delta\phi)$$

and

$$\frac{1}{2}d|\nabla\phi|^2=\nabla^2\phi(\nabla\phi,\cdot).$$

Proposition 2.5. *Let the* $(\mathbb{R}\times I, dx^2+(\frac{m}{(F^{-1}(x))^{n-3}}-1)dt^2)$ *be space-time manifold with two smooth functions* f *and* ϕ*, where* $f>0$ *be function of 'x' only and* ϕ *be function of 'x' and 't' such that* f *and* ϕ *satisfies*

$$2\lambda\phi-\left(\frac{\partial\phi}{\partial x}\right)^2-\frac{1}{\rho}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{\partial^2\phi}{\partial x^2}+\frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x}-\frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t}+\frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2}+\frac{(n-2)}{f}\frac{\partial f}{\partial x}\frac{\partial\phi}{\partial x}=c, \tag{29}$$

$$-\frac{1}{2}\frac{\partial^2\rho}{\partial x^2}+\frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2-\frac{(n-2)}{f}\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2\phi}{\partial x^2}=\lambda \tag{30}$$

and

$$\frac{1}{4\rho^2}\left(\frac{\partial\rho}{\partial x}\right)^2-\frac{1}{2\rho}\frac{\partial^2\rho}{\partial x^2}-\frac{(n-2)}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}+\frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x}+\frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2}-\frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t}=\lambda. \tag{31}$$

Then, μ given by the equation (47), is constant.

Proof. On the space-time $\left(\mathbb{R}\times I, dx^2 + (\frac{m}{(F^{-1}(x))^{n-3}} - 1)dt^2\right)$, we have

$$\Delta\phi = \text{trace}(\nabla^2\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2} \tag{32}$$

and

$$\Delta f = \text{trace}(\nabla^2 f) = \frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}. \tag{33}$$

We can write equations (30) and (31) in combined form as follows:

$$Rc + \nabla^2\phi = \lambda g + \frac{(n-2)}{f}\nabla^2 f. \tag{34}$$

Taking trace of above equation, we get

$$\begin{aligned} S =& 2\lambda + \frac{(n-2)}{f}\Delta f - \Delta\phi \\ =& 2\lambda + \frac{(n-2)}{f}\left(\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right) - \left(\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2}\right). \end{aligned} \tag{35}$$

Thus,

$$\begin{aligned} dS =& -\frac{(n-2)}{f^2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right)df + \frac{(n-2)}{f}d\left(\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right) \\ &- d\left(\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2}\right). \end{aligned} \tag{36}$$

Now, taking the divergence of both side of the equation (34), we get

$$\begin{aligned} \text{div}Rc &= 0 + (n-2)\left\{\frac{1}{f}\text{div}(\nabla^2 f) - \frac{1}{f^2}\nabla^2 f(\nabla f, \cdot)\right\} - \text{div}(\nabla^2\phi) \\ &= \frac{(n-2)}{f}\Big(Rc(\nabla f, \cdot) + d(\Delta f)\Big) - \frac{k}{2f^2}d(|\nabla f|^2) - Rc(\nabla\phi, \cdot) - d(\Delta\phi). \end{aligned} \tag{37}$$

From equation (34), we have

$$Rc(\nabla f, \cdot) = \lambda df + \frac{(n-2)}{2f}d\left(\frac{\partial f}{\partial x}\right)^2 - \nabla^2\phi(\nabla f, \cdot) \tag{38}$$

and

$$Rc(\nabla\phi, \cdot) = \lambda d\phi + \frac{(n-2)}{f}\nabla^2 f(\nabla\phi, \cdot) - \frac{1}{2}d\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{\rho}\left(\frac{\partial\phi}{\partial t}\right)^2\right\}. \tag{39}$$

Using equations (38) and (39) into the equation (37), we achieve

$$\begin{aligned} \text{div}Rc =& \frac{(n-2)}{f}\left(\lambda df + \frac{k}{2f}d\left(\frac{\partial f}{\partial x}\right)^2 - \nabla^2\phi(\nabla f, \cdot) + d\left(\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right)\right) - \frac{(n-2)}{2f^2}d(|\nabla f|^2) - \lambda d\phi \\ &+ \frac{(n-2)}{f}\nabla^2 f(\nabla\phi, \cdot) - \frac{1}{2}d\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{\rho}\left(\frac{\partial\phi}{\partial t}\right)^2\right\} - d\left(\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2}\right). \end{aligned} \tag{40}$$

Since,

$$d(\nabla\phi(f)) = d(\frac{\partial\phi}{\partial x}\frac{\partial f}{\partial x}) = \nabla^2\phi(\nabla f,\cdot) + \nabla^2 f(\nabla\phi,\cdot).$$

Thus, we get

$$\begin{aligned}\mathrm{div}Rc =& \frac{(n-2)}{f}\left\{\lambda df + \frac{(n-2)}{2f}d\left(\frac{\partial f}{\partial x}\right)^2 + d\left(\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right)\right\} - \frac{(n-2)}{2f^2}d(|\nabla f|^2) - \lambda d\phi \\ & - \frac{1}{2}d\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{\rho}\left(\frac{\partial\phi}{\partial t}\right)^2\right\} - d\left(\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2}\right) - \frac{(n-2)}{f}d(\frac{\partial\phi}{\partial x}\frac{\partial f}{\partial x}).\end{aligned} \tag{41}$$

Then, using second contracted Bianchi identity

$$-\frac{1}{2}ds + \mathrm{div}Rc = 0,$$

we obtain

$$\begin{aligned}&\frac{(n-2)}{2f^2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right)df + \frac{(n-2)}{2f}d\left(\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right) + \frac{1}{2}d\left(\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t}\right. \\ &\quad \left. + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2}\right) + \frac{(n-2)}{f}\left\{\lambda df + \frac{(n-2)}{2f}d\left(\frac{\partial f}{\partial x}\right)^2 + d\left(\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right)\right\} - \frac{(n-2)}{2f^2}d(|\nabla f|^2) - \lambda d\phi \\ &\quad - \frac{1}{2}d\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{\rho}\left(\frac{\partial\phi}{\partial t}\right)^2\right\} - \frac{(n-2)}{f}d(\frac{\partial\phi}{\partial x}\frac{\partial f}{\partial x}) - d\left(\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2}\right) = 0.\end{aligned} \tag{42}$$

Multiplying whole equation by $\frac{2f^2}{(n-2)}$, we get

$$\begin{aligned}&d\left\{\left(\frac{\partial^2 f}{\partial x^2} - \frac{1}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x}\right) + \lambda f^2 + (n-3)\left(\frac{\partial f}{\partial x}\right)^2\right\} - \frac{f^2}{(n-2)}d\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t}\right. \\ &\quad \left. + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2} + 2\lambda\phi - \left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{\rho}\left(\frac{\partial\phi}{\partial t}\right)^2\right\} - 2fd\left(\frac{\partial\phi}{\partial x}\frac{\partial f}{\partial x}\right) = 0.\end{aligned} \tag{43}$$

Now, using the hypothesis (29), we have

$$\frac{f^2}{(n-2)}d\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2} + 2\lambda\phi - \left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{\rho}\left(\frac{\partial\phi}{\partial t}\right)^2\right\} - fd\left(\frac{\partial\phi}{\partial x}\frac{\partial f}{\partial x}\right) = -\frac{\partial\phi}{\partial x}\frac{\partial f}{\partial x}df. \tag{44}$$

Thus, we obtain

$$d\left(\lambda f^2 + f\frac{\partial^2 f}{\partial x^2} + \frac{f}{2\rho}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x} + (n-3)\left(\frac{\partial f}{\partial x}\right)^2 - f\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x}\right) = 0. \tag{45}$$

Therefore, μ is constant. □

Theorem 2.6. *Let the Schwarzschild black hole* $(\tilde{M}, ds^2)$ *with metric*

$$ds^2 = dx^2 + \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)dt^2 + (F^{-1}(x))^2 d\omega_{n-2}^2 \tag{46}$$

be a expanding or steady gradient Ricci soliton with potential function ϕ*. Also, if the dimension of Schwarzschild black hole is atleast* 4 *and* $f = F^{-1}$ *attains its maximum and minimum values, then Schwarzschild black hole is a Riemannian product space, i.e.* f *is constant.*

Proof. By proposition (2.5), we have

$$\mu = \left(\lambda f^2 + f\frac{\partial^2 f}{\partial x^2} + \frac{f}{2\rho}\frac{\partial f}{\partial x}\frac{\partial \rho}{\partial x} + (n-3)\left(\frac{\partial f}{\partial x}\right)^2 - f\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial x}\right). \tag{47}$$

is constant.
Let x_1 and x_2 be the points at which f attains its maximum and minimum values, respectively. Then

$$\left.\frac{\partial f}{\partial x}\right|_{x_1} = 0 = \left.\frac{\partial f}{\partial x}\right|_{x_2} \quad \text{and} \quad \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_1} \leq 0 \leq \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_2}.$$

Since $f > 0$ and $\lambda \leq 0$, because space is steady or expanding gradient Ricci soliton. Then we have

$$-\lambda f(x_1)^2 \geq -\lambda f(x_2)^2.$$

Combining these results with equation (47), we obtain

$$\mu - \lambda f(x_1)^2 = f(x_1)\frac{\partial^2 f}{\partial x^2}(x_1)$$

and

$$\mu - \lambda f(x_2)^2 = f(x_2)\frac{\partial^2 f}{\partial x^2}(x_2)$$

Thus, we have

$$0 \geq f(x_1)\frac{\partial^2 f}{\partial x^2}(x_1) = \mu - \lambda f(x_1)^2 = \mu - \lambda f(x_2)^2 = f(x_2)\frac{\partial^2 f}{\partial x^2}(x_2) \geq 0.$$

This leads us to the following

$$\mu - \lambda f(x_1)^2 = 0 = \mu - \lambda f(x_2)^2.$$

Now, we have two cases

Case 1:($\lambda < 0$) In this case, we obtain

$$f(x_1) = f(x_2).$$

Hence f is constant.

Case 2:($\lambda = 0$) In this case, $\mu = 0$, so the equation (47) gives us

$$\left(f\frac{\partial^2 f}{\partial x^2} + \frac{f}{2\rho}\frac{\partial f}{\partial x}\frac{\partial \rho}{\partial x} + (n-3)\left(\frac{\partial f}{\partial x}\right)^2 - f\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial x}\right) = 0. \tag{48}$$

Then

$$f\left(\frac{\partial^2}{\partial x^2} + \frac{1}{2\rho}\frac{\partial \rho}{\partial x}\frac{\partial}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}\right)f = -(n-3)\left(\frac{\partial f}{\partial x}\right)^2.$$

The above expression can be rewrite as follows:

$$\mathcal{L}f = \frac{3-n}{f}\left(\frac{\partial f}{\partial x}\right)^2 \leq 0,$$

where $\mathcal{L} := \frac{\partial^2}{\partial x^2} + \frac{1}{2\rho}\frac{\partial \rho}{\partial x}\frac{\partial}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}$.

Therefore using strong maximum principle, we conclude that f is constant. In both the cases, we obtain that Schwarzschild black hole is a Riemannian product space. □

Next, we show that the compactness criterion of Schwarzschild black hole $\tilde{M} = (\mathbb{R} \times I) \times_f F$ when the base $\mathbb{R} \times I$ is compact.

Theorem 2.7. *Let $\tilde{M} = (\mathbb{R} \times I) \times_f F$ be Schwarzschild black hole with metric*

$$ds^2 = dx^2 + \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)dt^2 + (F^{-1}(x))^2 d\omega_{n-2}^2, \tag{49}$$

and ϕ be smooth function on $\mathbb{R} \times I$ so that $(\tilde{M}, ds^2, \nabla\phi, \lambda)$ be shrinking gradient Ricci soliton. If $\mathbb{R} \times I$ is compact and $n \geq 4$, then Schwarzschild black hole $\tilde{M}$ is compact.

Proof. Let us assume that the Schwarzschild black hole is a gradient Ricci soliton with $\nabla^2\phi = \lambda g + \frac{k}{f}\nabla^2 f$ and ${}^F Rc = \mu g_F$, where μ is constant and given by

$$\mu = \left(\lambda f^2 + f\frac{\partial^2 f}{\partial x^2} + \frac{f}{2\rho}\frac{\partial f}{\partial x}\frac{\partial \rho}{\partial x} + (n-3)\left(\frac{\partial f}{\partial x}\right)^2 - f\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial x}\right). \tag{50}$$

The equation (50) can be written as,

$$\mu = \lambda f^2 + fLf + \frac{n-3}{f^2}\left(\frac{\partial f}{\partial x}\right)^2, \tag{51}$$

where $\mathcal{L} := \frac{\partial^2}{\partial x^2} + \frac{1}{2\rho}\frac{\partial \rho}{\partial x}\frac{\partial}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}$.
On integration of both side of equation (51), we obtain

$$\int_B \mu e^{-\varphi} dB = \int_B \lambda f^2 e^{-\varphi} dB + \int_B fLf e^{-\varphi} dB + \int_B \frac{n-3}{f^2}\left(\frac{\partial f}{\partial x}\right)^2 e^{-\varphi} dB. \tag{52}$$

Therefore, we get

$$\mu \mathrm{vol}_\phi B = \lambda \int_B f^2 e^{-\varphi} dB + (n-3)\int_B \frac{1}{f^2}\left(\frac{\partial f}{\partial x}\right)^2 e^{-\varphi} dB. \tag{53}$$

Since $\lambda > 0$ and $n \geq 4$, we conclude that $\mu > 0$. Then, by Bonnet-Myers compactness theorem, the fiber manifold corresponding to the metric $d\omega_{n-2}^2$ is compact and hence the Schwarzschild black hole is compact. □

The next result is the necessary and sufficient condition on generalized Schwarzschild black hole to become gradient Ricci soliton.

Theorem 2.8. *Let $\left(\mathbb{R} \times I, dx^2 + \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)dt^2\right)$ be complete Riemannian space and f and ϕ be the smooth functions on $\mathbb{R}$ and $\mathbb{R} \times I$ respectively, satisfying the followings*

(i) $-\frac{1}{2}\frac{\partial^2 \rho}{\partial x^2} + \frac{1}{4\rho^2}\left(\frac{\partial \rho}{\partial x}\right)^2 + \frac{\partial^2 \phi}{\partial x^2} = \lambda + \frac{(n-2)}{f}\frac{\partial^2 f}{\partial x^2}.$

(ii) $\frac{1}{4\rho^2}\left(\frac{\partial \rho}{\partial x}\right)^2 - \frac{1}{2\rho}\frac{\partial^2 \rho}{\partial x^2} - \frac{(n-2)}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial \rho}{\partial x} + \frac{1}{2\rho}\frac{\partial \phi}{\partial x}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{2\rho^2}\frac{\partial \phi}{\partial t}\frac{\partial \rho}{\partial t} = \lambda,$

along with equation (15). Take a complete Riemannian manifold $(F, d\omega_{n-2}^2)$ such that ${}^F Rc = \mu d\omega_{n-2}^2$, where μ, given by equation (47), is constant. Then the Schwarzschild black hole $(\mathbb{R} \times I) \times_f F$with metric

$$ds^2 = dx^2 + \left(\frac{m}{(F^{-1}(x))^{n-3}} - 1\right)dt^2 + (F^{-1}(x))^2 d\omega_{n-2}^2, \tag{54}$$

is gradient Ricci soliton.

Proof. We prove this theorem in three cases.

<u>Case 1:</u>(For $X, Y \in \Gamma(\mathbb{R})$)

For the vector fields $X, Y \in \Gamma(\mathbb{R})$, the Hessian of ϕ is

$$\nabla^2\phi(X, Y) = \frac{\partial^2\phi}{\partial x^2} g(X, Y).$$

and the Hessian of f is

$$\nabla^2 f(X, Y) = \frac{\partial^2 f}{\partial x^2} g(X, Y).$$

Using these values and part (i) of lemma (2.2) into the hypothesis (i) of this theorem, we obtain

$$Rc(X, Y) + \nabla^2\phi(X, Y) = \lambda g(X, Y). \tag{55}$$

Hence, in this case, the gradient Ricci soliton equation is satisfied.

<u>Case 2:</u>(For $T_1, T_2 \in \Gamma(I)$)

In this case,

$$\nabla^2\phi(T_1, T_2) = \left(\frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{1}{\rho}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2\rho^2}\frac{\partial\phi}{\partial t}\frac{\partial\rho}{\partial t}\right) g(T_1, T_2) \tag{56}$$

and

$$\nabla^2 f(T_1, T_2) = \left(\frac{1}{2\rho}\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x}\right) g(T_1, T_2). \tag{57}$$

Again, by the hypothesis (ii) of this theorem and part (ii) of lemma (2.2), we have

$$Rc(T_1, T_2) + \nabla^2\phi(T_1, T_2) = \lambda g(T_1, T_2). \tag{58}$$

Thus, the gradient Ricci soliton equation is satisfied.

<u>Case 3:</u>(For $V, W \in \Gamma(F)$)

From part (iii) of the lemma (2.2), we have

$$Rc(V, W) =^F Rc(V, W) - \left(\frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial\rho}{\partial x} + \frac{(n-3)}{f^2}\left(\frac{\partial f}{\partial x}\right)^2\right) g(V, W). \tag{59}$$

For this case, we have μ given by equation (47) satisfying

$${}^F Rc = \mu d\omega_{n-2}^2.$$

Thus, the above equation reduces to

$$\begin{aligned} Rc(V,W) &= \mu d\omega_{n-2}^2(V,W) - \Big(\frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{1}{2\rho f}\frac{\partial f}{\partial x}\frac{\partial \rho}{\partial x} + \frac{(n-3)}{f^2}\Big(\frac{\partial f}{\partial x}\Big)^2\Big)g(V,W) \\ &= \Big(\lambda - \frac{1}{f}\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial x}\Big)g(V,W). \end{aligned} \tag{60}$$

For the vector fields $V.W \in \Gamma(F)$, we have

$$\nabla^2\phi(V,W) = g(D_V\nabla\phi, w) = f\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial x}d\omega_{n-2}^2 = \frac{1}{f}\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial x}g(V,W). \tag{61}$$

Using the above value, we obtain

$$Rc(V,W) + \nabla^2\phi(V,W) = \lambda g(V,W). \tag{62}$$

Therefore, the gradient Ricci soliton equation is again satisfied and hence the proof is complete. □

3. Ricci-Hessian type space-time manifolds

Let $(B\times I, g_B + (R+\frac{N}{2t})dt^2)$ be space-time manifold with Levi-Civita connection $\tilde{\nabla}$ and ∇ on $B\times I$ and B, respectively. Then, for any smooth function h on B, we have

$$\tilde{\nabla}h = \nabla h + \frac{1}{(R+\frac{N}{2t})}\frac{\partial h}{\partial t}\frac{\partial}{\partial t},$$

where $\tilde{\nabla}h$ and ∇h are the gradient of $B\times I$ and B, respetively. Hessian of the smooth function h on B is denoted by $\tilde{\nabla}^2 h$.
On space-time manifold $B\times I$, the Bakry-Emery Ricci tensor is

$$Rc_h =^{BI} Rc + \tilde{\nabla}^2 h. \tag{63}$$

Using Lemma (1.2), the above equation becomes

$$\begin{aligned} Rc_h =^{B} Rc &- \Big(\frac{\nabla^2 R}{2(R+\frac{N}{2t})} - \frac{dR\otimes dR}{4(R+\frac{N}{2t})^2}\Big) + \Big(-\frac{1}{2}\Delta R - \frac{1}{4(R+\frac{N}{2t})}|\nabla R|^2\Big)dt^2 \\ &+ \tilde{\nabla}^2 h. \end{aligned} \tag{64}$$

Then, k-Bakry-Emery Ricci tensor on space-time manifold is given by:

$$\begin{aligned} Rc_h^k =^{B} Rc &- \Big(\frac{\nabla^2 R}{2(R+\frac{N}{2t})} - \frac{dR\otimes dR}{4(R+\frac{N}{2t})^2}\Big) + \Big(-\frac{1}{2}\Delta R - \frac{1}{4(R+\frac{N}{2t})}|\nabla R|^2\Big)dt^2 \\ &+ \tilde{\nabla}^2 h - \frac{1}{k}dh\otimes dh, \end{aligned} \tag{65}$$

for some positve integer k. If we take $f = e^{-h/k}$ with $0 < k < \infty$, the above equation reduces to:

$$\begin{aligned} Rc_h^k =^{B} Rc &- \Big(\frac{\nabla^2 R}{2(R+\frac{N}{2t})} - \frac{dR\otimes dR}{4(R+\frac{N}{2t})^2}\Big) + \Big(-\frac{1}{2}\Delta R - \frac{1}{4(R+\frac{N}{2t})}|\nabla R|^2\Big)dt^2 \\ &- \frac{k}{f}\tilde{\nabla}^2 f, \end{aligned} \tag{66}$$

$k-$ quasi Einstein metric on a smooth manifold satisfies $Rc_h^k = \lambda g$. Therefore, for space-time k-quasi Einstein metric satisfies:

$$ {}^{B}Rc - \left(\frac{\nabla^2 R}{2(R+\frac{N}{2t})} - \frac{dR\otimes dR}{4(R+\frac{N}{2t})^2}\right) + \left(-\frac{1}{2}\Delta R - \frac{1}{4(R+\frac{N}{2t})}|\nabla R|^2\right)dt^2 + \tilde{\nabla}^2 h - \frac{1}{k}dh\otimes dh = \lambda g. \tag{67} $$

Now, we consider a complete weighted space-time manifold $(B\times I, g_B + (R+\frac{N}{2t})dt^2, e^{-\psi}dvol)$, where ψ is a smooth function so that $\psi = \phi - k\ln f$, with $0 < k < \infty$. Hence, we have

$$ Rc_\psi =^{B} Rc - \left(\frac{\nabla^2 R}{2(R+\frac{N}{2t})} - \frac{dR\otimes dR}{4(R+\frac{N}{2t})^2}\right) + \left(-\frac{1}{2}\Delta R - \frac{1}{4(R+\frac{N}{2t})}|\nabla R|^2\right)dt^2 + \tilde{\nabla}^2\phi - \frac{k}{f}\tilde{\nabla}^2 f + \frac{k}{f^2}df\otimes df. \tag{68} $$

We define a modified k-Bakry-Emery Ricci tensor for space-time by:

$$ Rc_{\phi,h}^k = Rc_h^k + \nabla^2\phi. \tag{69} $$

If Riemannian metric for space-time $g_B + (R+\frac{N}{2t})dt^2$ satisfies

$$ Rc_{\phi,h}^k = \lambda g, \tag{70} $$

then, this leads us to the following:

$$ {}^{B}Rc - \left(\frac{\nabla^2 R}{2(R+\frac{N}{2t})} - \frac{dR\otimes dR}{4(R+\frac{N}{2t})^2}\right) + \left(-\frac{1}{2}\Delta R - \frac{1}{4(R+\frac{N}{2t})}|\nabla R|^2\right)dt^2 + \tilde{\nabla}^2\phi - \frac{k}{f}\tilde{\nabla}^2 f = \lambda g, \tag{71} $$

or equivalently

$$ {}^{BI}Rc + \tilde{\nabla}^2\phi - \frac{k}{f}\tilde{\nabla}^2 f = \lambda g. \tag{72} $$

In [8], Feitosa et. al. showed that (72) can be reduced to a Ricci-Hessian type equation.

R. Hamilton [20], proved that if (M, g, ϕ, λ) is gradient Ricci soliton then,

$$ 2\lambda\phi - |\nabla\phi|^2 + \Delta\phi = c, \tag{73} $$

for some constant c. For space-time manifold $(B\times I)$, we compute this equation and obtain in following form:

$$ \begin{aligned} 2\lambda\phi - |\nabla\phi|^2 &- \frac{1}{(R+\frac{N}{2t})}\left(\frac{\partial\phi}{\partial t}\right)^2 + \Delta\phi + \frac{1}{(R+\frac{N}{2t})}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{2(R+\frac{N}{2t})^2}\frac{\partial\phi}{\partial t}\frac{N}{2t^2} \\ &+ \frac{\nabla\phi(R)}{2(R+\frac{N}{2t})} + \frac{k}{f}\nabla\phi(f) + \frac{k}{f(R+\frac{N}{2t})}\frac{\partial\phi}{\partial t}\frac{\partial f}{\partial t} = c, \end{aligned} \tag{74} $$

for some constant c. Also, for smooth functions ϕ and $f > 0$, we have

$$\mu = \lambda f^2 + f\Delta f + \frac{f\nabla f(R)}{2(R+\frac{N}{2t})} - f\nabla\phi(f) + \frac{f}{2(R+\frac{N}{2t})^2}\frac{N}{2t^2}\frac{\partial f}{\partial t} + \frac{f}{(R+\frac{N}{2t})}\frac{\partial^2 f}{\partial t^2}$$
$$- \frac{f}{(R+\frac{N}{2t})}\frac{\partial f}{\partial t}\frac{\partial\phi}{\partial t} + (k-1)\left(|\nabla f|^2 + \frac{1}{(R+\frac{N}{2t})}(\frac{\partial f}{\partial t})^2\right). \tag{75}$$

Therefore,

Proposition 3.1. *Let $M = (B\times I)\times_f F$ with metric $g_B + (R+\frac{N}{2t})dt^2 + f^2 g_F$ be gradient Ricci soliton space-time warped product and ϕ be the potential function. Then, space-time manifold $B\times I$ holds equation (71) or equivalently (72) and (74) and fiber manifold F is Einstein manifold with ${}^F Rc = \mu g_F$, where μ is given by equation (75). Conversely, let $B\times I$ be complete Riemannian space-time manifold with two smooth function $f > 0$ and ϕ, which satisfies equations (72) and (74) and constant μ given by (75). Let F be a complete Riemannian manifold such that ${}^F Rc = \mu g_F$. Then, we can construct gradient Ricci solitn space-time warped product.*

Let (M, g) be a complete Riemannian manifold. Then, for any smooth function $w \in C^\infty(M)$, the Bochner-Weitzenböck formula is given by:

$$\frac{1}{2}\Delta|\nabla w|^2 = |\nabla^2 w|^2 + g(\nabla w, \nabla\Delta w) + Rc(\nabla w, \nabla w). \tag{76}$$

For any smooth fuction $\psi \in C^\infty(M)$, we say $\Delta_\psi w = \Delta w - g(\nabla\psi, \nabla w)$ be $\psi-$ Laplacian. In [13], J. N. V. Gomes et.al. mention the Bochner-Weitzenböck formula with respect to the $\psi-$ Laplacian given by:

$$\frac{1}{2}\Delta_\psi|\nabla w|^2 = |\nabla^2 w|^2 + g(\nabla w, \nabla\Delta w) + Rc_\psi(\nabla w, \nabla w). \tag{77}$$

Here, if we combine equations (68) and (69), we obtain

$$Rc_\psi = Rc^m_{\phi h} + \frac{k}{f^2} df\otimes df. \tag{78}$$

Then, we have

Lemma 3.2. *For space-time manifold $B\times I$, the Bochner-Weitzenböck formula becomes*

$$\frac{1}{2}\tilde{\Delta}_\psi|\tilde{\nabla} w|^2 = |\tilde{\nabla}^2 w|^2 + g(\tilde{\nabla} w, \tilde{\nabla}\tilde{\Delta} w) + Rc^k_{\phi h}(\tilde{\nabla} w, \tilde{\nabla} w) + \frac{k}{f^2}\left[g(\nabla w, \nabla\psi)^2 + \frac{1}{(R+\frac{N}{2t})^2}\left(\frac{\partial w}{\partial t}\right)^2\left(\frac{\partial\psi}{\partial t}\right)^2\right]. \tag{79}$$

Now, we consider that the space-time manifold $B\times I$ satisfy Ricci-Hessian type equation (5), which is a necessary condition to construct a gradient Ricci soliton warped product and this implies that the equation (74) is satisfied on $B\times I$. Using equations (72), (74) and (75), we have the following lemma.

Lemma 3.3. *Let $B\times I$ be a Ricci-Hessian type space time manifold which satisfies equation (74), then*

$$\tilde{\Delta}\psi = n\lambda - S + k|\nabla\ln f|^2 + \frac{k}{f^2(R+\frac{N}{2t})}\left(\frac{\partial f}{\partial t}\right)^2, \tag{80}$$

$$\tilde{\Delta}_{\psi}\phi = c - 2\lambda\phi, \tag{81}$$

$$\tilde{\Delta}_{\psi}\ln f = \frac{1}{f^2}(\mu - \lambda f^2), \tag{82}$$

where, S is scalar curvature of $B \times I$.

Now, lemma (3.2) and lemma (3.3) leads us to the following result.

Lemma 3.4. *Let the space-time manifold $B \times I$ be Ricci-Hessian type manifold which satisfies equation (74). Then, following holds*

1. $$\frac{1}{2}\tilde{\Delta}_{\psi}|\tilde{\nabla}\phi|^2 = |\tilde{\nabla}^2\phi|^2 - \lambda|\nabla\phi|^2 - \frac{\lambda}{(R+\frac{N}{2t})}\Big(\frac{\partial\phi}{\partial t}\Big)^2 + \frac{k}{f^2}\Big[g(\nabla f,\nabla\phi)^2 + \frac{1}{(R+\frac{N}{2t})^2}\Big(\frac{\partial f}{\partial t}\Big)^2\Big(\frac{\partial\phi}{\partial t}\Big)^2\Big]. \tag{83}$$

2. $$\begin{aligned}\frac{1}{2}\tilde{\Delta}_{\psi}|\tilde{\nabla}\ln f|^2 =& |\tilde{\nabla}^2\ln f|^2 + \Big(\lambda - \frac{2\mu}{f^2}\Big)\Big(|\nabla\ln f|^2 + \frac{1}{f^2(R+\frac{N}{2t})^2}\Big(\frac{\partial f}{\partial t}\Big)^2\Big)\\ &+ \frac{k}{f^2}\Big(g(\nabla\ln f,\nabla f) + \frac{1}{f(R+\frac{N}{2t})}\Big(\frac{\partial f}{\partial t}\Big)^2\Big)^2.\end{aligned} \tag{84}$$

The following Lemma is an immediate consequence of the lemma (3.4) and well-known Kato's inequality.

Lemma 3.5. *Let $B \times I$ be space-time Ricci-Hessian type manifold satisfying (74). Then, following identities hold*

$$|\tilde{\nabla}\phi|\tilde{\Delta}_{\psi}|\tilde{\nabla}\phi| \geq -\lambda\Big(|\nabla\phi|^2 + \frac{1}{f(R+\frac{N}{2t})}\Big(\frac{\partial\phi}{\partial t}\Big)^2\Big) + \frac{k}{f^2}\Big[g(\nabla f,\nabla\phi)^2 + \frac{1}{(R+\frac{N}{2t})^2}\Big(\frac{\partial f}{\partial t}\Big)^2\Big(\frac{\partial\phi}{\partial t}\Big)^2\Big] \tag{85}$$

and

$$|\tilde{\nabla}\ln f|\tilde{\Delta}_{\psi}|\tilde{\nabla}\ln f| \geq \Big(\lambda - \frac{2\mu}{f^2}\Big)\Big(|\nabla\ln f|^2 + \frac{1}{f^2(R+\frac{N}{2t})}\Big(\frac{\partial f}{\partial t}\Big)^2\Big) + \frac{k}{f^2}\Big(g(\nabla\ln f,\nabla f) + \frac{1}{f(R+\frac{N}{2t})}\Big(\frac{\partial f}{\partial t}\Big)^2\Big)^2. \tag{86}$$

Proof. For a smooth function ϕ on $B \times I$, we have

$$\tilde{\nabla}\phi = \nabla\phi + \frac{1}{(R+\frac{N}{2t})}\frac{\partial\phi}{\partial t}\frac{\partial}{\partial t}. \tag{87}$$

Also, we have the following result

$$\frac{1}{2}\tilde{\Delta}_{\psi}|\tilde{\nabla}\phi|^2 = |\tilde{\nabla}\phi|\tilde{\Delta}_{\psi}|\tilde{\nabla}\phi| + |\tilde{\nabla}|\tilde{\nabla}\phi||^2. \tag{88}$$

The Kato's inequality for the smooth function ϕ is given by:

$$|\tilde{\nabla}^2\phi|^2 \geq |\tilde{\nabla}|\tilde{\nabla}|\phi|^2. \tag{89}$$

Combining equations (88) and (89), we obtain

$$|\tilde{\nabla}\phi|\tilde{\Delta}_{\psi}|\tilde{\nabla}\phi| \geq \frac{1}{2}\tilde{\Delta}_{\psi}|\tilde{\nabla}\phi|^2 - |\tilde{\nabla}^2\phi|. \tag{90}$$

Using part (1) of lemma (3.4), we get

$$\begin{aligned}|\tilde{\nabla}\phi|\tilde{\Delta}_{\psi}|\tilde{\nabla}\phi| &\geq \frac{1}{2}\tilde{\Delta}_{\psi}|\tilde{\nabla}\phi|^2 - |\tilde{\nabla}^2\phi| \\ &= -\lambda\Big(|\nabla\phi|^2 + \frac{1}{f(R+\frac{N}{2t})}\Big(\frac{\partial\phi}{\partial t}\Big)^2\Big) + \frac{k}{f^2}\Big[g(\nabla f, \nabla\phi)^2 \\ &+ \frac{1}{(R+\frac{N}{2t})^2}\Big(\frac{\partial f}{\partial t}\Big)^2\Big(\frac{\partial\phi}{\partial t}\Big)^2\Big].\end{aligned} \tag{91}$$

For second part, if we replace ϕ by $\ln f$ in equation (90), then we get

$$|\tilde{\nabla}\ln f|\tilde{\Delta}_{\psi}|\tilde{\nabla}\ln f| \geq \frac{1}{2}\tilde{\Delta}_{\psi}|\tilde{\nabla}\ln f|^2 - |\tilde{\nabla}^2\ln f|. \tag{92}$$

Therefore, part (2) of the lemma (3.4) leads us to the following:

$$|\tilde{\nabla}\ln f|\tilde{\Delta}_{\psi}|\tilde{\nabla}\ln f| \geq \Big(\lambda - \frac{2\mu}{f^2}\Big)\Big(|\nabla\ln f|^2 + \frac{1}{f^2(R+\frac{N}{2t})}\Big(\frac{\partial f}{\partial t}\Big)^2\Big) + \frac{k}{f^2}\Big(g(\nabla\ln f, \nabla f) + \frac{1}{f(R+\frac{N}{2t})}\Big(\frac{\partial f}{\partial t}\Big)^2\Big)^2. \tag{93}$$

Hence the proof. □

4. Some results on space time warped product

Let $\tilde{M} = (B\times I)\times_f F$ be space-time warped product with metric $\tilde{g} = (g_B + (R+\frac{N}{2t})dt^2) + f^2 g_F$. In the next lemmas, we assume that the base manifold $B\times I$ satisfy the Ricci Hessian type equation (5). For fiber manifold, we consider two different cases.

Case :1(Fiber manifold F possesses Ricci flow)

In this case we have

$$^{F}Rc = \frac{1}{f}\frac{\partial f}{\partial t}g.$$

Case :2

In the second case, we consider that the fiber manifold F satisfy

$$^{F}Rc = \mu g_F,$$

where μ is given by the equation (75).

In both the cases, first we investigate whether the space-time warped product become potentially Ricci flat i.e. all the components of the Ricci tensor are equal to zero $O(N^{-1})$. Here, in the lemmas (4.1) and (4.3), we establish that space-time warped product is not potentially Ricci flat. In the lemmas (4.2) and (4.4), we discuss potentially gradient soliton for space-time warped product. Here we also show that space-time warped product is not potentially gradient soliton in both the cases.

Lemma 4.1 (Space-time warped product Ricci curvature upto $O(N^{-1})$). *Let $B \times I$ satisfies Ricci-Hessian type equation, and $(B \times I) \times_f F$ is space-time warped product so that F possesses Ricci flow. Then the components of space-time warped product Ricci curvature tensor are*

(i) $R_{ij} = \lambda g_{ij} - (\nabla^2\phi)_{ij}$,

(ii) $R_{i0} = -\dfrac{\partial^2\phi}{\partial x^i \partial t} + O(N^{-1})$,

(iii) $R_{00} = \lambda(R + \dfrac{N}{2t}) - \dfrac{\partial^2\phi}{\partial t^2} + \dfrac{1}{2}\nabla R(\phi) - \dfrac{1}{2t} + O(N^{-1})$,

(iv) $R_{\alpha\beta} = -\left(\dfrac{\Delta f}{f} + \dfrac{1}{f}\dfrac{\partial f}{\partial t} - (k-1)\dfrac{|\nabla f|^2}{f^2}\right) g_{\alpha\beta} + O(N^{-1})$.

Proof. Here, $B \times I$ satisfies Ricci-Hessian type equation,

$$^{BI}Ric + \nabla^2\phi = \lambda g + \frac{k}{f}\nabla^2 f.$$

Also for vector fields $X, Y \in \Gamma(B \times I)$, we have

$$Ric(X, Y) =^{BI} Ric(X, Y) - \frac{k}{f}\nabla^2 f(X, Y).$$

Therefore, for $X, Y \in \Gamma(B \times I)$, we obtain

$$Ric(X, Y) = \lambda g(X, Y) - \nabla^2\phi(X, Y). \tag{94}$$

(i) For $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$ equation (94) becomes

$$R_{ij} = \lambda g_{ij} - (\nabla^2\phi)_{ij}.$$

(ii) Now, for $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial t}$, we get

$$R_{i0} = Rc(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}) = \lambda g_{io} - \nabla^2\phi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}). \tag{95}$$

Since, we know that

$$\nabla^2\phi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}) = \frac{\partial^2\phi}{\partial x^i \partial t} - \Gamma^k_{i0}\frac{\partial\phi}{\partial x^k} - \Gamma^0_{i0}\frac{\partial\phi}{\partial t}. \tag{96}$$

Using the values of Γ^k_{i0} and Γ^0_{i0} and from equation (96), the equation (95) becomes

$$R_{i0} = -\frac{\partial^2\phi}{\partial x^i \partial t} + \frac{1}{2(R + \frac{N}{2t})}\frac{\partial R}{\partial x^i}\frac{\partial\phi}{\partial t}.$$

Approximating upto order $O(N^{-1})$, we obtain

$$R_{i0} = -\frac{\partial^2\phi}{\partial x^i \partial t} + O(N^{-1}).$$

(iii) In the same manner, for $X = Y = \frac{\partial}{\partial t}$, we have

$$R_{00} = Rc(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \lambda g_{00} - \nabla^2\phi(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}). \tag{97}$$

Since,

$$\nabla^2\phi(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \frac{\partial^2\phi}{\partial t^2} - \Gamma^k_{00}\frac{\partial\phi}{\partial x^k} - \Gamma^0_{00}\frac{\partial\phi}{\partial t}, \tag{98}$$

then we have $\Gamma^k_{00} = -\frac{g^{kj}}{2}\frac{\partial R}{\partial x^j}$ and $\Gamma^0_{00} = -\frac{1}{2(R+\frac{N}{2t})}\left(\frac{N}{2t^2}\right)$, also we have $g_{00} = R + \frac{N}{2t}$. Using these values in equation (97), we have

$$R_{00} = \lambda(R + \frac{N}{2t}) - \frac{\partial^2\phi}{\partial t^2} + \frac{g^{kj}}{2}\frac{\partial R}{\partial x^j}\frac{\partial\phi}{\partial x^k} + \frac{1}{2(R+\frac{N}{2t})}\left(\frac{N}{2t^2}\right). \tag{99}$$

Approximating upto order $O(N^{-1})$, above equation reduces to

$$R_{00} = \lambda(R + \frac{N}{2t}) - \frac{\partial^2\phi}{\partial t^2} + \frac{1}{2}\nabla R(\phi) - \frac{1}{2t} + O(N^{-1}). \tag{100}$$

(iv) In space-time warped product $\tilde{M} = (B \times I)_f F$, the Fiber space F possess Ricci flow. So, we have

$${}^F Rc = \frac{1}{f}\frac{\partial f}{\partial t}g.$$

Also, for $V, W \in \Gamma(F)$, we have

$$Rc(V, W) ={}^F Rc(V, W) - \left(\frac{\tilde{\Delta} f}{f} - (k-1)\frac{|\tilde{\nabla} f|^2}{f^2}\right)g(V, W). \tag{101}$$

Putting $V = \frac{\partial}{\partial x^\alpha}$ and $W = \frac{\partial}{\partial x^\beta}$ and using ${}^F Rc = \frac{1}{f}\frac{\partial f}{\partial t}g$, the equation (101) becomes

$$R_{\alpha\beta} = -\left(\frac{\tilde{\Delta} f}{f} + \frac{1}{f}\frac{\partial f}{\partial t} - (k-1)\frac{|\tilde{\nabla} f|^2}{f^2}\right)g_{\alpha\beta}. \tag{102}$$

Since, the gradient and Laplacian of a function on $B \times I$ has following property

$$|\tilde{\nabla} f|^2 = |\nabla f|^2 + O(N^{-1}), \tag{103}$$

and

$$\tilde{\Delta} f = \Delta f + O(N^{-1}). \tag{104}$$

Thus, we obtain

$$R_{\alpha\beta} = -\left(\frac{\Delta f}{f} + \frac{1}{f}\frac{\partial f}{\partial t} - (k-1)\frac{|\nabla f|^2}{f^2}\right)g_{\alpha\beta} + O(N^{-1}). \tag{105}$$

Again, approximating up to $O(N^{-1})$, we obtain

$$R_{\alpha\beta} = -\left(\frac{\Delta f}{f} + \frac{1}{f}\frac{\partial f}{\partial t} - (k-1)\frac{|\nabla f|^2}{f^2}\right)g_{\alpha\beta} + O(N^{-1}).$$

□

Lemma 4.2 (Potentially gradient Soliton). *Let $B \times I$ satisfies Ricci-Hessian type equation, and $(B \times I) \times_f F$ is space-time warped product so that F possess Ricci flow. Define $h(t)$ so that*

$$\frac{\partial h}{\partial t} = \frac{N}{2t}. \tag{106}$$

Then for any $c, b \in \mathbb{R}$, $b \neq c$, we have

(i) $R_{ij} + c\nabla_i\nabla_j h = \lambda g_{ij} - \nabla^2\phi_{ij}$,

(ii) $R_{i0} + c\nabla_i\nabla_0 h = -\dfrac{\partial^2\phi}{\partial x^i \partial t} - \dfrac{c}{2b}\dfrac{\partial R}{\partial x^i} + O(N^{-1})$,

(iii) $R_{00} + c\nabla_0\nabla_0 h = \lambda(R + \dfrac{N}{2t}) - \dfrac{\partial^2\phi}{\partial t^2} + \dfrac{1}{2}\nabla R(\phi) - \dfrac{1}{2t} - \dfrac{cR}{2bt} - c\dfrac{N}{2t} + O(N^{-1})$,

(iv) $R_{\alpha\beta} + c\nabla\nabla_\beta h = -\left(\dfrac{\Delta f}{f} + \dfrac{1}{f}\dfrac{\partial f}{\partial t} - (k-1)\dfrac{|\nabla f|^2}{f^2}\right) g_{\alpha\beta} + O(N^{-1})$.

Proof. To prove this lemma, we use lemma (4.1).

$$\begin{aligned}
\text{(i)} R_{ij} + c\nabla_i\nabla_j\phi &= \lambda g_{ij} - (\nabla^2\phi)_{ij} + c\nabla_i(\frac{\partial}{\partial x^j}h) \\
&= \lambda g_{ij} - (\nabla^2\phi)_{ij} + 0 \\
&= \lambda g_{ij} - (\nabla^2\phi)_{ij}.
\end{aligned}$$

Thus, we have

$$R_{ij} + c\nabla_i\nabla_j\phi = R_{ij} = \lambda g_{ij} - (\nabla^2\phi)_{ij}.$$

$$\begin{aligned}
\text{(ii)} R_{i0} + c\nabla_i\nabla_0\phi &= -\frac{\partial^2\phi}{\partial x^i\partial t} + O(N^{-1}) - c\Gamma^0_{i0}\frac{\partial}{\partial t}\phi \\
&= -\frac{\partial^2\phi}{\partial x^i\partial t} + O(N^{-1}) - c\frac{1}{2(R + \frac{bN}{2t})}\frac{\partial R}{\partial x^i}\frac{N}{2t} \\
&= -\frac{\partial^2\phi}{\partial x^i\partial t} + O(N^{-1}) - c\frac{1}{2}\frac{\partial R}{\partial x^i}(R + \frac{bN}{2t})^{-1}\frac{N}{2t} \\
&= -\frac{\partial^2\phi}{\partial x^i\partial t} + O(N^{-1}) - \frac{c}{2b}\frac{\partial R}{\partial x^i} + O(N^{-1}) \\
&= -\frac{\partial^2\phi}{\partial x^i\partial t} - \frac{c}{2b}\frac{\partial R}{\partial x^i} + O(N^{-1}).
\end{aligned}$$

$$
\begin{aligned}
\text{(iii)}R_{00} + c\nabla_0\nabla_0\phi &= \lambda(R + \frac{N}{2t}) - \frac{\partial^2\phi}{\partial t^2} + \frac{1}{2}\nabla R(\phi) - \frac{1}{2t} + O(N^{-1}) - c\frac{\partial^2}{\partial t^2}\phi \\
&- c\Gamma^0_{00}\frac{\partial}{\partial t}\phi \\
&= \lambda(R + \frac{N}{2t}) - \frac{\partial^2\phi}{\partial t^2} + \frac{1}{2}\nabla R(\phi) - \frac{1}{2t} + O(N^{-1}) - c(\frac{-N}{2t^2}) \\
&\quad - c\frac{1}{2(R + \frac{bN}{2t})}\frac{\partial(R + \frac{bN}{2t})}{\partial t}\frac{N}{2t} \\
&= \lambda(R + \frac{N}{2t}) - \frac{\partial^2\phi}{\partial t^2} + \frac{1}{2}\nabla R(\phi) - \frac{1}{2t} + O(N^{-1}) - \frac{cR}{2bt} \\
&\quad + O(N^{-1}) - c(\frac{-N}{2t^2}) \\
&= \lambda(R + \frac{N}{2t}) - \frac{\partial^2\phi}{\partial t^2} + \frac{1}{2}\nabla R(\phi) - \frac{1}{2t} - \frac{cR}{2bt} \\
&\quad + O(N^{-1}) - c(\frac{-N}{2t^2}).
\end{aligned}
$$

$$
\begin{aligned}
\text{(iv)}R_{\alpha\beta} + c\nabla_\alpha\nabla_\beta\phi &= -\left(\frac{\Delta f}{f} + \frac{1}{f}\frac{\partial f}{\partial t} - (k-1)\frac{|\nabla f|^2}{f^2}\right)g_{\alpha\beta} + O(N^{-1}) \\
&\quad - c\Gamma^0_{\alpha\beta}\frac{\partial}{\partial t}\phi \\
&= -\left(\frac{\Delta f}{f} + \frac{1}{f}\frac{\partial f}{\partial t} - (k-1)\frac{|\nabla f|^2}{f^2}\right)g_{\alpha\beta} + O(N^{-1}) \\
&\quad - c\frac{1}{f(R + \frac{N}{2t})}\frac{\partial f}{\partial t}g_{\alpha\beta}\frac{N}{2t}.
\end{aligned}
$$

□

In next two lemmas, we consider that fiber manifold is an Einstein manifold satisfying ${}^F Rc = \mu g_F$.

Lemma 4.3. *Let $B \times I$ satisfies Ricci-Hessian type equation and $(B \times I) \times_f F$ is space-time warped product so that ${}^F Rc = \mu g_F$, where μ is given by equation (75). Then the components of space-time warped product Ricci curvature tensor are*

(i) $R_{ij} = \lambda g_{ij} - (\nabla^2\phi)_{ij}$,

(ii) $R_{i0} = -\dfrac{\partial^2\phi}{\partial x^i \partial t} + O(N^{-1})$,

(iii) $R_{00} = \lambda(R + \dfrac{N}{2t}) - \dfrac{\partial^2\phi}{\partial t^2} + \dfrac{1}{2}\nabla R(\phi) - \dfrac{1}{2t} + O(N^{-1})$,

(iv) $R_{\alpha\beta} = \lambda g_{\alpha\beta}$.

Proof. Since, $B \times I$ satisfies Ricci-Hessian type equation

$$
{}^{BI}Ric + \nabla^2\phi = \lambda g + \frac{k}{f}\nabla^2 f
$$

and

$$^{F}Rc = \mu g_F.$$

Then, by proposition (3.1), space-time warped product $(B \times I) \times_f F$ becomes gradient Ricci soliton. Therefore,

$$Rc + \nabla^2\phi = \lambda g.$$

Thus, proof of first three part of this lemma is same as that of lemma (4.1).
(iv) For the last part, we have

$$\nabla^2\phi_{\alpha\beta} = 0.$$

Thus, we obtain

$$R_{\alpha\beta} = \lambda g_{\alpha\beta}.$$

□

Lemma 4.4. *Let $B \times I$ satisfies Ricci-Hessian type equation, and $(B \times I) \times_f F$ is space-time warped product so that $^{F}Rc = \mu g_F$, where μ is given by (75). Define $h(t)$ so that*

$$\frac{\partial h}{\partial t} = \frac{N}{2t}. \tag{107}$$

Then for any $c, b \in \mathbb{R}$, $b \neq c$, we have

(i) $R_{ij} + c\nabla_i\nabla_j h = \lambda g_{ij} - \nabla^2\phi_{ij}$,

(ii) $R_{i0} + c\nabla_i\nabla_0 h = -\dfrac{\partial^2\phi}{\partial x^i \partial t} - \dfrac{c}{2b}\dfrac{\partial R}{\partial x^i} + O(N^{-1})$,

(iii) $R_{00} + c\nabla_0\nabla_0 h = \lambda\left(R + \dfrac{N}{2t}\right) - \dfrac{\partial^2\phi}{\partial t^2} + \dfrac{1}{2}\nabla R(\phi) - \dfrac{1}{2t} - \dfrac{cR}{2bt} - c\dfrac{N}{2t} + O(N^{-1})$,

(iv) $R_{\alpha\beta} + c\nabla_\alpha\nabla_\beta h = \lambda g_{\alpha\beta}$.

Proof. First three part of this lemma is same that of lemma (4.3), so we directly move to last part.
(iv) Since,

$$R_{\alpha\beta} = \lambda g_{\alpha\beta},$$

and

$$\nabla_\alpha\nabla_\beta h = 0.$$

Therefore, we obtain

$$R_{\alpha\beta} + c\nabla_\alpha\nabla_\beta h = \lambda g_{\alpha\beta}.$$

□

Data availability statement

No new data were created or analysed in this study.

Conflict of interest statement

There is no conflict of interest.

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