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On the generalized Fibonacci and Lucas matrix hybrinomials

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Abstract. Hybrid numbers, which are a generalization of complex, dual and hyperbolic numbers, are widely used in a variety of fields. Through a novel approach, this study aims to obtain the generalized Fibonacci matrix hybrinomials by virtue of the bi–periodic Fibonacci matrix polynomials. Moreover, we give the definition of the bi–periodic Lucas matrix polynomials unlocking their specific properties. Then, we obtain the generalized Lucas matrix hybrinomials via bi–periodic Lucas polynomials and leveraging these findings. Ultimately, we give the generating function, Binet's formula and a few summation formulas for the generalized Fibonacci and Lucas matrix hybrinomials.

1. Introduction

A hybrid number is a mathematical concept that generalizes complex, hyperbolic, and dual numbers. The topic of hybrid numbers, which arise in various fields such as mathematics, physics, computer science and engineering, has received a great deal of attention in recent years. For example, Özdemir defined a non-commutative number system which is called as hybrid numbers. Moreover, he provided some algebraic and geometric properties of this number set and examined the roots of a hybrid number based on its type and characteristics [11]. Then, Szynal-Liana introduced the concept of the Horadam hybrid numbers by combining the notions of Horadam numbers and hybrid numbers. The author derived the Binet formula and generating function for these numbers and proved certain significant identities [12]. Szynal-Liana and Włoch proposed the Fibonacci hybrid numbers and obtained certain properties using classical Fibonacci identities [13]. Kızılateş presented a new generalization of hybrid numbers which are called as $q-$Fibonacci hybrid and $q-$Lucas hybrid numbers. Furthermore, he obtained certain algebraic properties of these numbers [7]. Şentürk et al. examined the Horadam hybrid numbers and provided exponential generating function, Poisson generating function, Vajda, Catalan, Cassini, and d'Ocagne identities for these numbers. Also, they investigated the Honsberger formula and some summation formulas for these numbers [15]. Szynal-Liana and Włoch discussed the Fibonacci hybrinomials and the Lucas hybrinomials, which are generalizations of the Fibonacci hybrid and Lucas hybrid numbers [14]. Then, Koçer and Aslan presented the generalized hybrid Fibonacci and Lucas $p-$numbers and gived some special cases and algebraic properties of the generalized hybrid Fibonacci and Lucas $p-$numbers [8]. Tan and Ait-Amrane introduced the bi–periodic Horadam hybrid numbers which generalize the classical Horadam hybrid numbers [16]. Finally, Dağlı et al. computed the Frobenius norm and established upper and lower

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bounds for the spectral norm of r-circulant matrices. They also presented explicit formulas for efficiently calculating the eigenvalues and determinants of these matrices [5].

Fibonacci and Lucas numbers, which are one of the most interesting topics throughout history, are used in many fields such as mathematics, geometry, physics and economics. So far, the generalization and application of Fibonacci and Lucas numbers have been studied by several authors. For example, Koshy obtained certain identities and significant applications of numbers such as Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, and so on [9]. Falcon and Plaza introduced and investigated newly generalized $k-$Fibonacci sequences. They demonstrated many properties of these sequences using elementary matrix algebra [6]. Civciv and Türkmen defined a new matrix generalization of Fibonacci numbers. Also, they demonstrated certain properties of this matrix sequence using matrix approaches [2]. Nalli and Haukkanen introduced $h(x)-$Fibonacci and $h(x)-$Lucas polynomials, which are generalizations of Fibonacci and Lucas polynomials, where $h(x)$ is a polynomial with real coefficients [10]. Yazlık et al. defined a new sequence that generalizes the $(s,t)-$Fibonacci and $(s,t)-$Lucas sequences. Also, they presented some significant relationships between the $(s,t)-$Fibonacci and $(s,t)-$Lucas sequences and their matrix sequences [18]. Coşkun and Taşkara focused on the matrix polynomial obtained using the bi-periodic Fibonacci matrix polynomial. In addition, they provided the Binet formula, generating function, certain properties, and binomial sums of these matrix polynomials [3]. Then the authors introduced the bi-periodic Lucas matrix sequence and presented some fundamental properties of this generalized matrix sequence. Also, the authors investigated important relationships between the bi-periodic Fibonacci and Lucas matrix sequences [4]. Verma and Bala defined the bi-variate bi-periodic Fibonacci polynomials. Furthermore, the authors obtained the Catalan, Cassini, d'Ocagne, and Gelin-Cesaro identities for these polynomials and presented their generating function and Binet formula [17]. Bala and Verma defined the bi-variate bi-periodic Lucas polynomials. Additionally, they derived the well-known properties of these polynomials, such as the Catalan, Cassini, and d'Ocagne identities [1].

Since hybrid numbers are a generalization of complex, dual and hyperbolic numbers, they are widely used in many fields such as mathematics, physics, computer science and engineering. Our study aims to investigate the field of generalized Fibonacci matrix hybrinomials using the bi–periodic Fibonacci matrix polynomials. Also, we explore the fascinating area of bi–periodic Lucas matrix polynomials, defining them, and proving their some identities. Later, we define the generalized Lucas matrix hybrinomials by utilizing bi– periodic Lucas polynomials. Finally, we obtain the generating function, Binet's formula and some summation formulas for the generalized Lucas matrix hybrinomials. With the help of this study, we hope to foster a deeper understanding of them, opening up new directions for research in a variety of fields of science and engineering. Also, in this paper presents a significant contribution to the field of matrix theory and its application in the study of bi–periodic Fibonacci and Lucas sequences.This innovative approach not only deepens our comprehension of these sequences but also offers a potent mathematical tool for solving challenging issues in other disciplines, including algebraic structures and number theory.

2. Preliminaries

In this section, we present some definitions and the preliminary facts which we use in this paper.

The set of hybrid numbers, denoted by $\mathbb{K}$, is defined as

$$\mathbb{K} = \left\{a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}\right\}. \tag{1}$$

The following table shows that the multiplication operation in the hybrid numbers is not commutative, but associative [11].

Table 1: Multiplication table with $\mathbf{i}$, ε, and $\mathbf{h}$.

$\times$	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$\mathbf{h}+1$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

Now, we give the Binet formulas, generating functions and some summation formulas of the bi-periodic Fibonacci polynomials, bi-periodic Lucas polynomials and bi-periodic Fibonacci matrix polynomials.

In [19], any variable x and a, b nonzero real numbers, the authors gave the bi–periodic Fibonacci polynomials and bi-periodic Lucas polynomials as

$$q_n(a,b,x) = \begin{cases} axq_{n-1}(a,b,x) + q_{n-2}(a,b,x), & \text{if } n \text{ is even} \\ bxq_{n-1}(a,b,x) + q_{n-2}(a,b,x), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{2}$$

$$l_n(a,b,x) = \begin{cases} bxl_{n-1}(a,b,x) + l_{n-2}(a,b,x), & \text{if } n \text{ is even} \\ axl_{n-1}(a,b,x) + l_{n-2}(a,b,x), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{3}$$

where $q_0(a,b,x) = 0$, $q_1(a,b,x) = 1$ and $l_0(a,b,x) = 2$, $l_1(a,b,x) = ax$. Also, the authors presented the generating functions of the bi-periodic Fibonacci polynomials and bi-periodic Lucas polynomials as

$$\sum_{n=0}^{\infty} q_n(a,b,x)t^n = \frac{t + axt^2 - t^3}{1 - (abx^2+2)\,t^2 + t^4}, \tag{4}$$

$$\sum_{n=0}^{\infty} l_n(a,b,x)t^n = \frac{2 + axt - \left(abx^2+2\right)t^2 + axt^3}{1 - (abx^2+2)\,t^2 + t^4}. \tag{5}$$

In [17], Verma and Bala obtained the Binet's formula for the bi–periodic Fibonacci polynomials as

$$q_n(a,b,x) = \frac{(ax)^{1-\xi(n)}}{(abx^2)^{\lfloor \frac{n}{2} \rfloor}} \left[\frac{\alpha^n - \beta^n}{\alpha - \beta}\right], \tag{6}$$

where α and β are the roots of the characteristic equation $\lambda^2 - \left(abx^2\right)\lambda - abx^2 = 0$ and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function. Furthermore, in [1], Bala and Verma presented the Binet's formula for the bi–periodic Lucas polynomials as

$$l_n(a,b,x) = \frac{(ax)^{\xi(n)}\left(\alpha^n + \beta^n\right)}{(abx^2)^{\lfloor \frac{n+1}{2} \rfloor}}. \tag{7}$$

In [3], for any variable x and a, b nonzero real numbers, the bi–periodic Fibonacci matrix polynomials $\mathcal{F}_n(a,b,x)$ are defined by

$$\mathcal{F}_n(a,b,x) = \begin{cases} ax\mathcal{F}_{n-1}(a,b,x) + \mathcal{F}_{n-2}(a,b,x), & \text{if } n \text{ is even} \\ bx\mathcal{F}_{n-1}(a,b,x) + \mathcal{F}_{n-2}(a,b,x), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{8}$$

with initial conditions $\mathcal{F}_0(a,b,x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathcal{F}_1(a,b,x) = \begin{pmatrix} bx & \frac{b}{a} \\ 1 & 0 \end{pmatrix}$. Then the following equality is valid for $n \in \mathbb{N}$:

$$\mathcal{F}_n(a,b,x) = \begin{pmatrix} \left(\frac{b}{a}\right)^{\xi(n)} q_{n+1}(a,b,x) & \frac{b}{a} q_n(a,b,x) \\ q_n(a,b,x) & \left(\frac{b}{a}\right)^{\xi(n)} q_{n-1}(a,b,x) \end{pmatrix} \tag{9}$$

Theorem 2.1. *[3] The generating function of the bi–periodic Fibonacci matrix polynomial is defined by*

$$\sum_{n=0}^{\infty}\mathcal{F}_n(a,b,x)t^n = \frac{1}{1-(abx^2+2)\,t^2+t^4}\begin{pmatrix} 1+bxt-t^2 & \frac{b}{a}\left(t+axt^2-t^3\right) \\ t+axt^2-t^3 & 1-\left(abx^2+1\right)t^2+bxt^3 \end{pmatrix}. \tag{10}$$

Theorem 2.2. *[3] For $n\in\mathbb{N}$, the Binet formula for the bi-periodic Fibonacci matrix polynomials is defined by*

$$\mathcal{F}_n(a,b,x) = A_{\xi(n)}\left(\alpha^n-\beta^n\right)+B_{\xi(n)}\left(\alpha^{2\lfloor\frac{n}{2}\rfloor+2}-\beta^{2\lfloor\frac{n}{2}\rfloor+2}\right), \tag{11}$$

where

$$A_{\xi(n)} = \frac{\{\mathcal{F}_1(a,b,x)-bx\mathcal{F}_0(a,b,x)\}^{\xi(n)}\left\{ax\mathcal{F}_1(a,b,x)-\mathcal{F}_0(a,b,x)-abx^2\mathcal{F}_0(a,b,x)\right\}^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)x^{2\lfloor\frac{n}{2}\rfloor}}$$

$$B_{\xi(n)} = \frac{(b)^{\xi(n)}\mathcal{F}_0(a,b,x)}{(ab)^{\lfloor\frac{n}{2}\rfloor+1}(\alpha-\beta)x^{n+2\xi(n+1)}},$$

where α,β are roots of the equation $\lambda^2-(abx^2)\lambda-abx^2=0$.

Corollary 2.3. *[3] For $k\geq 0$, the following statements are true:*
(i)

$$\sum_{k=0}^{n-1}\mathcal{F}_k(a,b,x) = \frac{a^{\xi(n)}b^{1-\xi(n)}\mathcal{F}_n(a,b,x)+a^{1-\xi(n)}b^{\xi(n)}\mathcal{F}_{n-1}(a,b,x)-a\mathcal{F}_1(a,b,x)}{abx} + \frac{abx\mathcal{F}_0(a,b,x)-b\mathcal{F}_0(a,b,x)}{abx}, \tag{12}$$

(ii)

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{F}_k(a,b,x)t^{-k} = \frac{1}{1-(abx^2+2)t^2+t^4}\Big(&\frac{\mathcal{F}_{n-1}(a,b,x)}{t^{n-1}}-\frac{\mathcal{F}_{n+1}(a,b,x)}{t^{n-3}}+\frac{\mathcal{F}_n(a,b,x)}{t^n}\\ &-\frac{\mathcal{F}_{n+2}(a,b,x)}{t^{n+2}}+t^4\mathcal{F}_0(a,b,x)+t^3\mathcal{F}_1(a,b,x)\\ &-t^2\left[\left(abx^2+1\right)\mathcal{F}_0(a,b,x)-ax\mathcal{F}_1(a,b,x)\right]\\ &-t\left(\mathcal{F}_1(a,b,x)-bx\mathcal{F}_0(a,b,x)\right)\Big),\end{aligned} \tag{13}$$

(iii)

$$\sum_{k=0}^{\infty}\mathcal{F}_k(a,b,x)t^{-k} = \frac{t}{1-(abx^2+2)\,t^2+t^4}\begin{pmatrix} t^3+bxt^2-t & \frac{b}{a}t^2+bxt-\frac{b}{a} \\ t^2+axt-1 & t^3-\left(abx^2+1\right)t+bx \end{pmatrix}. \tag{14}$$

3. Generalized Fibonacci Matrix Hybrinomials

In this section, we define the generalized Fibonacci matrix hybrinomials by virtue of the bi–periodic Fibonacci matrix polynomials. Furthermore, we give the Binet formula and generating function of these hybrinomials. Finally, we obtain some newly identities.

Definition 3.1. *For $n\in\mathbb{N}$, any variable x and a,b nonzero real numbers, the bi–periodic Fibonacci matrix hybrinomial is defined by*

$$H\mathcal{F}_n(a,b,x) = \mathcal{F}_n(a,b,x)+\boldsymbol{i}\mathcal{F}_{n+1}(a,b,x)+\varepsilon\mathcal{F}_{n+2}(a,b,x)+\boldsymbol{h}\mathcal{F}_{n+3}(a,b,x), \tag{15}$$

where $\mathcal{F}_n(a,b,x)$ is the bi–periodic Fibonacci matrix polynomial.

Theorem 3.2. *The following equality is valid for $n \in \mathbb{N}$:*

$$H\mathcal{F}_n(a,b,x) = \begin{pmatrix} \tilde{\mathcal{F}}_{1,1}(n;a,b,x) & \tilde{\mathcal{F}}_{1,2}(n;a,b,x) \\ \tilde{\mathcal{F}}_{2,1}(n;a,b,x) & \tilde{\mathcal{F}}_{2,2}(n;a,b,x) \end{pmatrix}, \tag{16}$$

where

$$\begin{aligned}
\tilde{\mathcal{F}}_{1,1}(n;a,b,x) &= \left(\frac{b}{a}\right)^{\xi(n)} q_{n+1}(a,b,x) + \boldsymbol{i}\left(\frac{b}{a}\right)^{\xi(n+1)} q_{n+2}(a,b,x) + \boldsymbol{\varepsilon}\left(\frac{b}{a}\right)^{\xi(n)} q_{n+3}(a,b,x) + \boldsymbol{h}\left(\frac{b}{a}\right)^{\xi(n+1)} q_{n+4}(a,b,x),\\
\tilde{\mathcal{F}}_{1,2}(n;a,b,x) &= \frac{b}{a}\left(q_n(a,b,x) + \boldsymbol{i}q_{n+1}(a,b,x) + \boldsymbol{\varepsilon}q_{n+2}(a,b,x) + \boldsymbol{h}q_{n+3}(a,b,x)\right),\\
\tilde{\mathcal{F}}_{2,1}(n;a,b,x) &= q_n(a,b,x) + \boldsymbol{i}q_{n+1}(a,b,x) + \boldsymbol{\varepsilon}q_{n+2}(a,b,x) + \boldsymbol{h}q_{n+3}(a,b,x),\\
\tilde{\mathcal{F}}_{2,2}(n;a,b,x) &= \left(\frac{b}{a}\right)^{\xi(n)} q_{n-1}(a,b,x) + \boldsymbol{i}\left(\frac{b}{a}\right)^{\xi(n+1)} q_{n}(a,b,x) + \boldsymbol{\varepsilon}\left(\frac{b}{a}\right)^{\xi(n)} q_{n+1}(a,b,x) + \boldsymbol{h}\left(\frac{b}{a}\right)^{\xi(n+1)} q_{n+2}(a,b,x).
\end{aligned}$$

Proof. From (9) and (15), the proof can be clearly seen. □

Next, we give the generating function of the bi–periodic Fibonacci matrix hybrinomials.

Theorem 3.3. *The generating function of the bi-periodic Fibonacci matrix hybrinomials is defined by*

$$\begin{aligned}
\sum_{n=0}^{\infty} H\mathcal{F}_n(a,b,x)t^n =& \frac{1 + i\frac{1}{t} + \varepsilon\frac{1}{t^2} + \boldsymbol{h}\frac{1}{t^3}}{1-(abx^2+2)\,t^2+t^4}\begin{pmatrix} 1+bxt-t^2 & \frac{b}{a}\left(t+axt^2-t^3\right) \\ t+axt^2-t^3 & 1-\left(abx^2+1\right)t^2+bxt^3 \end{pmatrix}\\
&- \boldsymbol{i}\frac{1}{t}\mathcal{F}_0(a,b,x) - \boldsymbol{\varepsilon}\frac{1}{t^2}\mathcal{F}_0(a,b,x) - \boldsymbol{\varepsilon}\frac{1}{t}\mathcal{F}_1(a,b,x)\\
&- \boldsymbol{h}\frac{1}{t^3}\mathcal{F}_0(a,b,x) - \boldsymbol{h}\frac{1}{t^2}\mathcal{F}_1(a,b,x) - \boldsymbol{h}\frac{1}{t}\mathcal{F}_2(a,b,x).
\end{aligned} \tag{17}$$

Proof. From (10) and (15), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} H\mathcal{F}_n(a,b,x)t^n &= \sum_{n=0}^{\infty}\left(\mathcal{F}_n(a,b,x) + \boldsymbol{i}\mathcal{F}_{n+1}(a,b,x) + \boldsymbol{\varepsilon}\mathcal{F}_{n+2}(a,b,x) + \boldsymbol{h}\mathcal{F}_{n+3}(a,b,x)\right)t^n\\
&= \sum_{n=0}^{\infty}\mathcal{F}_n(a,b,x)t^n + \boldsymbol{i}\sum_{n=0}^{\infty}\mathcal{F}_{n+1}(a,b,x)t^n + \boldsymbol{\varepsilon}\sum_{n=0}^{\infty}\mathcal{F}_{n+2}(a,b,x)t^n + \boldsymbol{h}\sum_{n=0}^{\infty}\mathcal{F}_{n+3}(a,b,x)t^n\\
&= \sum_{n=0}^{\infty}\mathcal{F}_n(a,b,x)t^n + \boldsymbol{i}\frac{1}{t}\sum_{n=0}^{\infty}\mathcal{F}_n(a,b,x)t^n - \boldsymbol{i}\frac{1}{t}\mathcal{F}_0(a,b,x)\\
&\quad + \boldsymbol{\varepsilon}\frac{1}{t^2}\sum_{n=0}^{\infty}\mathcal{F}_n(a,b,x)t^n - \boldsymbol{\varepsilon}\frac{1}{t^2}\mathcal{F}_0(a,b,x) - \boldsymbol{\varepsilon}\frac{1}{t}\mathcal{F}_1(a,b,x)\\
&\quad + \boldsymbol{h}\frac{1}{t^3}\sum_{n=0}^{\infty}\mathcal{F}_n(a,b,x)t^n - \boldsymbol{h}\frac{1}{t^3}\mathcal{F}_0(a,b,x) - \boldsymbol{h}\frac{1}{t^2}\mathcal{F}_1(a,b,x) - \boldsymbol{h}\frac{1}{t}\mathcal{F}_2(a,b,x)\\
&= \left(1 + i\frac{1}{t} + \varepsilon\frac{1}{t^2} + \boldsymbol{h}\frac{1}{t^3}\right)\sum_{n=0}^{\infty}\mathcal{F}_n(a,b,x)t^n - \boldsymbol{i}\frac{1}{t}\mathcal{F}_0(a,b,x) - \boldsymbol{\varepsilon}\frac{1}{t^2}\mathcal{F}_0(a,b,x) - \boldsymbol{\varepsilon}\frac{1}{t}\mathcal{F}_1(a,b,x)\\
&\quad - \boldsymbol{h}\frac{1}{t^3}\mathcal{F}_0(a,b,x) - \boldsymbol{h}\frac{1}{t^2}\mathcal{F}_1(a,b,x) - \boldsymbol{h}\frac{1}{t}\mathcal{F}_2(a,b,x),
\end{aligned}$$

and consequently,

$$\begin{aligned}\sum_{n=0}^{\infty} H\mathcal{F}_n(a,b,x)t^n =& \frac{1+i\frac{1}{t}+\varepsilon\frac{1}{t^2}+\boldsymbol{h}\frac{1}{t^3}}{1-(abx^2+2)\,t^2+t^4}\begin{pmatrix} 1+bxt-t^2 & \frac{b}{a}\left(t+axt^2-t^3\right) \\ t+axt^2-t^3 & 1-\left(abx^2+1\right)t^2+bxt^3 \end{pmatrix} \\ & -\boldsymbol{i}\frac{1}{t}\mathcal{F}_0(a,b,x)-\boldsymbol{\varepsilon}\frac{1}{t^2}\mathcal{F}_0(a,b,x)-\boldsymbol{\varepsilon}\frac{1}{t}\mathcal{F}_1(a,b,x) \\ & -\boldsymbol{h}\frac{1}{t^3}\mathcal{F}_0(a,b,x)-\boldsymbol{h}\frac{1}{t^2}\mathcal{F}_1(a,b,x)-\boldsymbol{h}\frac{1}{t}\mathcal{F}_2(a,b,x).\end{aligned}$$

Thus, the proof is completed. □

In the following theorem, we give the Binet formula of the bi–periodic Fibonacci matrix hybrinomials.

Theorem 3.4. *For $n \in \mathbb{N}$, the Binet formula of the bi–periodic Fibonacci matrix hybrinomials is defined by*

$$\begin{aligned} H\mathcal{F}_n(a,b,x) =& A_{\xi(n)}\left(\alpha^n\underline{\alpha}-\beta^n\underline{\beta}\right)+B_{\xi(n)}\left(\alpha^{2\lfloor\frac{n}{2}\rfloor+2}\underline{\alpha}-\beta^{2\lfloor\frac{n}{2}\rfloor+2}\underline{\beta}\right) \\ &+A_{\xi(n+1)}\left(\alpha^{n+1}\underline{\underline{\alpha}}-\beta^{n+1}\underline{\underline{\beta}}\right)+B_{\xi(n+1)}\left(\alpha^{2\lfloor\frac{n+1}{2}\rfloor+2}\underline{\underline{\alpha}}-\beta^{2\lfloor\frac{n+1}{2}\rfloor+2}\underline{\underline{\beta}}\right) \end{aligned} \tag{18}$$

where, $A_{\xi(n)}$ and $B_{\xi(n)}$ are given in Theorem (2.2). *Also, $\underline{\alpha}$, $\underline{\beta}$, $\underline{\underline{\alpha}}$ and $\underline{\underline{\beta}}$ are given by*

$$\underline{\alpha}=1+\varepsilon\alpha^2, \quad \underline{\beta}=1+\varepsilon\beta^2, \quad \underline{\underline{\alpha}}=i+h\alpha^2, \quad \underline{\underline{\beta}}=i+h\beta^2.$$

Proof. For $k \in \mathbb{N}$, we will consider the theorem in two different cases.
Case $n = 2k$:

$$\begin{aligned} H\mathcal{F}_{2k}(a,b,x) &= \mathcal{F}_{2k}(a,b,x)+\boldsymbol{i}\mathcal{F}_{2k+1}(a,b,x)+\boldsymbol{\varepsilon}\mathcal{F}_{2k+2}(a,b,x)+\boldsymbol{h}\mathcal{F}_{2k+3}(a,b,x) \\ &= A_0\left(\alpha^{2k}-\beta^{2k}\right)+B_0\left(\alpha^{2k+2}-\beta^{2k+2}\right)+\boldsymbol{i}\left(A_1\left(\alpha^{2k+1}-\beta^{2k+1}\right)+B_1\left(\alpha^{2k+2}-\beta^{2k+2}\right)\right) \\ &\quad +\boldsymbol{\varepsilon}\left(A_0\left(\alpha^{2k+2}-\beta^{2k+2}\right)+B_0\left(\alpha^{2k+4}-\beta^{2k+4}\right)\right)+\boldsymbol{h}\left(A_1\left(\alpha^{2k+3}-\beta^{2k+3}\right)+B_1\left(\alpha^{2k+4}-\beta^{2k+4}\right)\right) \\ &= A_0\alpha^{2k}\left(1+\boldsymbol{\varepsilon}\alpha^2\right)-A_0\beta^{2k}\left(1+\boldsymbol{\varepsilon}\beta^2\right)+B_0\alpha^{2k+2}\left(1+\boldsymbol{\varepsilon}\alpha^2\right)-B_0\beta^{2k+2}\left(1+\boldsymbol{\varepsilon}\beta^2\right) \\ &\quad +A_1\alpha^{2k+1}\left(\boldsymbol{i}+\boldsymbol{h}\alpha^2\right)-A_1\beta^{2k+1}\left(\boldsymbol{i}+\boldsymbol{h}\beta^2\right)+B_1\alpha^{2k+2}\left(\boldsymbol{i}+\boldsymbol{h}\alpha^2\right)-B_1\beta^{2k+2}\left(\boldsymbol{i}+\boldsymbol{h}\beta^2\right). \end{aligned}$$

By defining $\underline{\alpha}, \underline{\beta}, \underline{\underline{\alpha}}$ and $\underline{\underline{\beta}}$ as follows:

$$\underline{\alpha}=1+\varepsilon\alpha^2, \quad \underline{\beta}=1+\varepsilon\beta^2, \quad \underline{\underline{\alpha}}=i+h\alpha^2, \quad \underline{\underline{\beta}}=i+h\beta^2,$$

we have,

$$H\mathcal{F}_{2k}(a,b,x) =A_0\left(\alpha^{2k}\underline{\alpha}-\beta^{2k}\underline{\beta}\right)+B_0\left(\alpha^{2k+2}\underline{\alpha}-\beta^{2k+2}\underline{\beta}\right)+A_1\left(\alpha^{2k+1}\underline{\underline{\alpha}}-\beta^{2k+1}\underline{\underline{\beta}}\right)+B_1\left(\alpha^{2k+2}\underline{\underline{\alpha}}-\beta^{2k+2}\underline{\underline{\beta}}\right). \tag{19}$$

Case $n = 2k + 1$:

$$\begin{aligned} H\mathcal{F}_{2k+1}(a,b,x) &= \mathcal{F}_{2k+1}(a,b,x)+\boldsymbol{i}\mathcal{F}_{2k+2}(a,b,x)+\boldsymbol{\varepsilon}\mathcal{F}_{2k+3}(a,b,x)+\boldsymbol{h}\mathcal{F}_{2k+4}(a,b,x) \\ &= A_1\left(\alpha^{2k+1}-\beta^{2k+1}\right)+B_1\left(\alpha^{2k+2}-\beta^{2k+2}\right)+\boldsymbol{i}\left(A_0\left(\alpha^{2k+2}-\beta^{2k+2}\right)+B_0\left(\alpha^{2k+4}-\beta^{2k+4}\right)\right) \\ &\quad +\boldsymbol{\varepsilon}\left(A_1\left(\alpha^{2k+3}-\beta^{2k+3}\right)+B_1\left(\alpha^{2k+4}-\beta^{2k+4}\right)\right)+\boldsymbol{h}\left(A_0\left(\alpha^{2k+4}-\beta^{2k+4}\right)+B_0\left(\alpha^{2k+6}-\beta^{2k+6}\right)\right) \\ &= A_1\alpha^{2k+1}\left(1+\boldsymbol{\varepsilon}\alpha^2\right)-A_1\beta^{2k+1}\left(1+\boldsymbol{\varepsilon}\beta^2\right)+B_1\alpha^{2k+2}\left(1+\boldsymbol{\varepsilon}\alpha^2\right)-B_1\beta^{2k+2}\left(1+\boldsymbol{\varepsilon}\beta^2\right) \\ &\quad +A_0\alpha^{2k+2}\left(\boldsymbol{i}+\boldsymbol{h}\alpha^2\right)-A_0\beta^{2k+2}\left(\boldsymbol{i}+\boldsymbol{h}\beta^2\right)+B_0\alpha^{2k+4}\left(\boldsymbol{i}+\boldsymbol{h}\alpha^2\right)-B_0\beta^{2k+4}\left(\boldsymbol{i}+\boldsymbol{h}\beta^2\right). \end{aligned}$$

Then, we have

$$
\begin{aligned}
H\mathcal{F}_{2k+1}(a,b,x) =& A_1\left(\alpha^{2k+1}\underline{\alpha}-\beta^{2k+1}\underline{\beta}\right)+B_1\left(\alpha^{2k+2}\underline{\alpha}-\beta^{2k+2}\underline{\beta}\right)\\
&+A_0\left(\alpha^{2k+2}\underline{\underline{\alpha}}-\beta^{2k+2}\underline{\underline{\beta}}\right)+B_0\left(\alpha^{2k+4}\underline{\underline{\alpha}}-\beta^{2k+4}\underline{\underline{\beta}}\right).
\end{aligned}
\tag{20}
$$

Finally, from the (19) and (20), we have,

$$
\begin{aligned}
H\mathcal{F}_{n}(a,b,x) =& A_{\xi(n)}\left(\alpha^{n}\underline{\alpha}-\beta^{n}\underline{\beta}\right)+B_{\xi(n)}\left(\alpha^{2\lfloor\frac{n}{2}\rfloor+2}\underline{\alpha}-\beta^{2\lfloor\frac{n}{2}\rfloor+2}\underline{\beta}\right)+A_{\xi(n+1)}\left(\alpha^{n+1}\underline{\underline{\alpha}}-\beta^{n+1}\underline{\underline{\beta}}\right)\\
&+B_{\xi(n+1)}\left(\alpha^{2\lfloor\frac{n+1}{2}\rfloor+2}\underline{\underline{\alpha}}-\beta^{2\lfloor\frac{n+1}{2}\rfloor+2}\underline{\underline{\beta}}\right).
\end{aligned}
$$

Hence, the proof is complete. □

Theorem 3.5. *For $k\in\mathbb{N}$, the following statements are true:*

(i)

$$
\begin{aligned}
\sum_{k=0}^{n-1}H\mathcal{F}_k(a,b,x) =& (1+\boldsymbol{i}+\boldsymbol{\varepsilon}+\boldsymbol{h})\Big[\frac{a^{\xi(n)}b^{1-\xi(n)}\mathcal{F}_n(a,b,x)+a^{1-\xi(n)}b^{\xi(n)}\mathcal{F}_{n-1}(a,b,x)-a\mathcal{F}_1(a,b,x)}{abx}\\
&+\frac{abx\mathcal{F}_0(a,b,x)-b\mathcal{F}_0(a,b,x)}{abx}\Big]-(\boldsymbol{i}+\boldsymbol{\varepsilon}+\boldsymbol{h})(\mathcal{F}_0(a,b,x)-\mathcal{F}_n(a,b,x))\\
&-(\boldsymbol{\varepsilon}+\boldsymbol{h})(\mathcal{F}_1(a,b,x)-\mathcal{F}_{n+1}(a,b,x))-\boldsymbol{h}(\mathcal{F}_2(a,b,x)-\mathcal{F}_{n+2}(a,b,x)),
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\sum_{k=0}^{n}H\mathcal{F}_k(a,b,x)t^{-k} =& \frac{(1+\boldsymbol{i}t+\boldsymbol{\varepsilon}t^2+\boldsymbol{h}t^3)}{1-(abx^2+2)t^2+t^4}\Big(\frac{\mathcal{F}_{n-1}(a,b,x)}{t^{n-1}}-\frac{\mathcal{F}_{n+1}(a,b,x)}{t^{n-3}}+\frac{\mathcal{F}_{n}(a,b,x)}{t^{n}}\\
&-\frac{\mathcal{F}_{n+2}(a,b,x)}{t^{n+2}}+t^4\mathcal{F}_0(a,b,x)+t^3\mathcal{F}_1(a,b,x)-t^2\left[\left(abx^2+1\right)\mathcal{F}_0(a,b,x)-ax\mathcal{F}_1(a,b,x)\right]\\
&-t\left(\mathcal{F}_1(a,b,x)-bx\mathcal{F}_0(a,b,x)\right)\Big)-\left(\boldsymbol{i}t+\boldsymbol{\varepsilon}t^2+\boldsymbol{h}t^3\right)\left(\mathcal{F}_0(a,b,x)-\mathcal{F}_{n+1}(a,b,x)t^{-(n+1)}\right)\\
&-\left(\boldsymbol{\varepsilon}t^2+\boldsymbol{h}t^3\right)\left(\mathcal{F}_1(a,b,x)t^{-1}-\mathcal{F}_{n+2}(a,b,x)t^{-(n+2)}\right)-\boldsymbol{h}t^3\left(\mathcal{F}_2(a,b,x)t^{-2}-\mathcal{F}_{n+3}(a,b,x)t^{-(n+3)}\right),
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\sum_{k=0}^{\infty}H\mathcal{F}_k(a,b,x)t^{-k} =& \frac{t(1+\boldsymbol{i}t+\boldsymbol{\varepsilon}t^2+\boldsymbol{h}t^3)}{1-(abx^2+2)\,t^2+t^4}\begin{pmatrix} t^3+bxt^2-t & \frac{b}{a}t^2+bxt-\frac{b}{a}\\ t^2+axt-1 & t^3-\left(abx^2+1\right)t+bx\end{pmatrix}\\
&-\left(\boldsymbol{i}t+\boldsymbol{\varepsilon}t^2+\boldsymbol{h}t^3\right)\mathcal{F}_0(a,b,x)-\left(\boldsymbol{\varepsilon}t+\boldsymbol{h}t^2\right)\mathcal{F}_1(a,b,x)-\boldsymbol{h}t\mathcal{F}_2(a,b,x).
\end{aligned}
$$

Proof. Firstly, we proof the case (i). For convenience, the notation $\mathcal{F}_k(a,b,x)=\mathcal{F}_k$ is used only in this proof. From the definition of the bi–periodic Fibonacci matrix hybrinomial and doing some calculations, we have

$$
\begin{aligned}
\sum_{k=0}^{n-1}H\mathcal{F}_k(a,b,x) =& \sum_{k=0}^{n-1}\mathcal{F}_k+\boldsymbol{i}\sum_{k=0}^{n-1}\mathcal{F}_{k+1}+\boldsymbol{\varepsilon}\sum_{k=0}^{n-1}\mathcal{F}_{k+2}+\boldsymbol{h}\sum_{k=0}^{n-1}\mathcal{F}_{k+3}\\
=& \sum_{k=0}^{n-1}\mathcal{F}_k+\boldsymbol{i}\left(\sum_{k=0}^{n-1}\mathcal{F}_k-\mathcal{F}_0+\mathcal{F}_n\right)+\boldsymbol{\varepsilon}\left(\sum_{k=0}^{n-1}\mathcal{F}_k-\mathcal{F}_0-\mathcal{F}_1+\mathcal{F}_n+\mathcal{F}_{n+1}\right)\\
&+\boldsymbol{h}\left(\sum_{k=0}^{n-1}\mathcal{F}_k-\mathcal{F}_0-\mathcal{F}_1-\mathcal{F}_2+\mathcal{F}_n+\mathcal{F}_{n+1}+\mathcal{F}_{n+2}\right)\\
=&(1+\boldsymbol{i}+\boldsymbol{\varepsilon}+\boldsymbol{h})\sum_{k=0}^{n-1}\mathcal{F}_k-(\boldsymbol{i}+\boldsymbol{\varepsilon}+\boldsymbol{h})\left(\mathcal{F}_0-\mathcal{F}_n\right)-(\boldsymbol{\varepsilon}+\boldsymbol{h})\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)-\boldsymbol{h}\left(\mathcal{F}_2-\mathcal{F}_{n+2}\right).
\end{aligned}
$$

Then, from Eq. (12), we have the proof of (*i*). Now, we proof the (*ii*). From the definition of bi-periodic Fibonacci matrix hybrinomials, we have

$$\begin{aligned}\sum_{k=0}^{n} H\mathcal{F}_k(a,b,x)t^{-k} &= \sum_{k=0}^{n}\mathcal{F}_k t^{-k} + \boldsymbol{i}\sum_{k=0}^{n}\mathcal{F}_{k+1}t^{-k} + \boldsymbol{\varepsilon}\sum_{k=0}^{n}\mathcal{F}_{k+2}t^{-k} + \boldsymbol{h}\sum_{k=0}^{n}\mathcal{F}_{k+3}t^{-k}\\
&= \sum_{k=0}^{n}\mathcal{F}_k t^{-k} + \boldsymbol{i}t\left(\sum_{k=0}^{n}\mathcal{F}_k t^{-k} - \mathcal{F}_0 + \mathcal{F}_{n+1}t^{-(n+1)}\right) + \boldsymbol{\varepsilon}t^2\left(\sum_{k=0}^{n}\mathcal{F}_k t^{-k} - \mathcal{F}_0 - \mathcal{F}_1 t^{-1} + \mathcal{F}_{n+1}t^{-(n+1)}\right)\\
&\quad + \boldsymbol{\varepsilon}t^2\mathcal{F}_{n+2}t^{-(n+2)} + \boldsymbol{h}t^3\left(\sum_{k=0}^{n}\mathcal{F}_k t^{-k} - \mathcal{F}_0 - \mathcal{F}_1 t^{-1} - \mathcal{F}_2 t^{-2}\right)\\
&\quad + \boldsymbol{h}t^3\left(\mathcal{F}_{n+1}t^{-(n+1)} + \mathcal{F}_{n+2}t^{-(n+2)} + \mathcal{F}_{n+3}t^{-(n+3)}\right)\\
&= \left(1 + \boldsymbol{i}t + \boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3\right)\left(\sum_{k=0}^{n}\mathcal{F}_k t^{-k}\right) - \left(\boldsymbol{i}t + \boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3\right)\left(\mathcal{F}_0 - \mathcal{F}_{n+1}t^{-(n+1)}\right)\\
&\quad - \left(\boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3\right)\left(\mathcal{F}_1 t^{-1} - \mathcal{F}_{n+2}t^{-(n+2)}\right) - \boldsymbol{h}t^3\left(\mathcal{F}_2 t^{-2} - \mathcal{F}_{n+3}t^{-(n+3)}\right).\end{aligned}$$

Also, by virtue of Eq. (13), we prove (*ii*). Now, we need to proof the case (*iii*). From the definition of bi-periodic Fibonacci matrix hybrinomials, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} H\mathcal{F}_k(a,b,x)t^{-k} &= \sum_{k=0}^{\infty}\left(\mathcal{F}_k + \boldsymbol{i}\mathcal{F}_{k+1} + \boldsymbol{\varepsilon}\mathcal{F}_{k+2} + \boldsymbol{h}\mathcal{F}_{k+3}\right)t^{-k}\\
&= \sum_{k=0}^{\infty}\mathcal{F}_k t^{-k} + \boldsymbol{i}t\left(\sum_{k=0}^{\infty}\mathcal{F}_k t^{-k} - \mathcal{F}_0\right) + \boldsymbol{\varepsilon}t^2\left(\sum_{k=0}^{\infty}\mathcal{F}_k t^{-k} - \mathcal{F}_0 - \mathcal{F}_1 t^{-1}\right)\\
&\quad + \boldsymbol{h}t^3\left(\sum_{k=0}^{\infty}\mathcal{F}_k t^{-k} - \mathcal{F}_0 - \mathcal{F}_1 t^{-1} - \mathcal{F}_2 t^{-2}\right)\\
&= \left(1 + \boldsymbol{i}t + \boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3\right)\sum_{k=0}^{\infty}\mathcal{F}_k t^{-k} - \left(\boldsymbol{i}t + \boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3\right)\mathcal{F}_0 - \left(\boldsymbol{\varepsilon}t + \boldsymbol{h}t^2\right)\mathcal{F}_1 - \boldsymbol{h}t\mathcal{F}_2.\end{aligned}$$

Finally, by using Eq. (14), we prove (*iii*). Hence the proof is completed. □

4. Generalized Lucas Matrix Hybrinomials

In this section, we define the generalized Lucas matrix hybrinomials by virtue of the bi–periodic Lucas matrix polynomials. Furthermore, we give the Binet formula and generating function of these hybrinomials. Finally, we obtain some newly identities.

Firstly, we start with the following definition. The next definition presents the bi–periodic Lucas matrix polynomials.

Definition 4.1. *For any variable x and a, b nonzero real numbers, the bi–periodic Lucas matrix polynomials are defined by*

$$\mathcal{L}_n(a,b,x) = \begin{cases} bx\mathcal{L}_{n-1}(a,b,x) + \mathcal{L}_{n-2}(a,b,x), & \text{if } n \text{ is even} \\ ax\mathcal{L}_{n-1}(a,b,x) + \mathcal{L}_{n-2}(a,b,x), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{21}$$

with initial conditions $\mathcal{L}_0(a,b,x) = \begin{pmatrix} ax & 2 \\ 2\frac{a}{b} & -ax \end{pmatrix}$ *and* $\mathcal{L}_1(a,b,x) = \begin{pmatrix} a^2x^2 + 2\frac{a}{b} & ax \\ \frac{a^2x}{b} & 2\frac{a}{b} \end{pmatrix}$. *Then the following equality is valid for* $n \in \mathbb{N}$*:*

$$\mathcal{L}_n(a,b,x) = \begin{pmatrix} \left(\frac{a}{b}\right)^{\xi(n)} l_{n+1}(a,b,x) & l_n(a,b,x) \\ \frac{a}{b} l_n(a,b,x) & \left(\frac{a}{b}\right)^{\xi(n)} l_{n-1}(a,b,x) \end{pmatrix}. \tag{22}$$

Here, $l_n(a,b,x)$ *is bi-periodic Lucas polynomial given in* (3)*.*

Next, we give the generating function of the bi–periodic Lucas matrix polynomials $\mathcal{L}_n(a,b,x)$.

Theorem 4.2. *The generating function of the bi–periodic Lucas matrix polynomial is defined by*

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{L}_n(a,b,x)t^n &= \frac{1}{1-(abx^2+2)\,t^2+t^4} \\ &\times \begin{pmatrix} ax + \left(a^2x^2 + 2\frac{a}{b}\right)t + axt^2 - 2\frac{a}{b}t^3 & 2 + axt - \left(abx^2+2\right)t^2 + axt^3 \\ 2\frac{a}{b} + \frac{a^2x}{b}t - \left(a^2x^2 + 2\frac{a}{b}\right)t^2 + \frac{a^2x}{b}t^3 & -ax + 2\frac{a}{b}t + \left(3ax + a^2bx^3\right)t^2 - \left(a^2x^2 + 2\frac{a}{b}\right)t^3 \end{pmatrix}. \end{aligned} \tag{23}$$

Proof. Assume that $F(t)$ is the generating function for the polynomial $\{\mathcal{L}_n(a,b,x)\}_{n\in\mathbb{N}}$. Then, we have

$$F(t) = \sum_{i=0}^{\infty} \mathcal{L}_i(a,b,x)t^i = \mathcal{L}_0(a,b,x) + \mathcal{L}_1(a,b,x)t + \sum_{i=2}^{\infty} \mathcal{L}_i(a,b,x)t^i$$

Thus, we can write

$$\left(1 - axt - t^2\right)F(t) = \mathcal{L}_0(a,b,x) + \left(\mathcal{L}_1(a,b,x) - ax\mathcal{L}_0(a,b,x)\right)t + \sum_{i=2}^{\infty}\left(\mathcal{L}_i(a,b,x) - ax\mathcal{L}_{i-1}(a,b,x) - \mathcal{L}_{i-2}(a,b,x)\right)t^i.$$

Since $\mathcal{L}_{2i+1}(a,b,x) = ax\mathcal{L}_{2i}(a,b,x) + \mathcal{L}_{2i-1}(a,b,x)$, we get

$$\begin{aligned} \left(1 - axt - t^2\right)F(t) =& \mathcal{L}_0(a,b,x) + \left(\mathcal{L}_1(a,b,x) - ax\mathcal{L}_0(a,b,x)\right)t \\ &+ \sum_{i=1}^{\infty}\left(\mathcal{L}_{2i}(a,b,x) - ax\mathcal{L}_{2i-1}(a,b,x) - \mathcal{L}_{2i-2}(a,b,x)\right)t^{2i} \\ =& \mathcal{L}_0(a,b,x) + \left(\mathcal{L}_1(a,b,x) - ax\mathcal{L}_0(a,b,x)\right)t + (b-a)xt\sum_{i=1}^{\infty}\mathcal{L}_{2i-1}(a,b,x)t^{2i-1}. \end{aligned}$$

Now, let

$$f(t) = \sum_{i=1}^{\infty}\mathcal{L}_{2i-1}(a,b,x)t^{2i-1}.$$

Since

$$\begin{aligned} \mathcal{L}_{2i+1}(a,b,x) &= ax\mathcal{L}_{2i}(a,b,x) + \mathcal{L}_{2i-1}(a,b,x) \\ &= ax\left(bx\mathcal{L}_{2i-1}(a,b,x) + \mathcal{L}_{2i-2}(a,b,x)\right) + \mathcal{L}_{2i-1}(a,b,x) \\ &= \left(abx^2+1\right)\mathcal{L}_{2i-1}(a,b,x) + ax\mathcal{L}_{2i-2}(a,b,x) \\ &= \left(abx^2+1\right)\mathcal{L}_{2i-1}(a,b,x) + \mathcal{L}_{2i-1}(a,b,x) - \mathcal{L}_{2i-3}(a,b,x) \\ &= \left(abx^2+2\right)\mathcal{L}_{2i-1}(a,b,x) - \mathcal{L}_{2i-3}(a,b,x), \end{aligned}$$

we have

$$\left(1-\left(abx^2+2\right)t^2+t^4\right)f(t)=\mathcal{L}_1(a,b,x)t+\mathcal{L}_3(a,b,x)t^3-\left(abx^2+2\right)\mathcal{L}_1(a,b,x)t^3.$$

Therefore,

$$\begin{aligned} f(t) &= \frac{\mathcal{L}_1(a,b,x)t+\mathcal{L}_3(a,b,x)t^3-\left(abx^2+2\right)\mathcal{L}_1(a,b,x)t^3}{1-(abx^2+2)\,t^2+t^4} \\ &= \frac{\mathcal{L}_1(a,b,x)t+\left(ax\mathcal{L}_0(a,b,x)-\mathcal{L}_1(a,b,x)\right)t^3}{1-(abx^2+2)\,t^2+t^4} \end{aligned}$$

and as a result, we get

$$F(t)=\frac{\mathcal{L}_0(a,b,x)+\mathcal{L}_1(a,b,x)t+\left(bx\mathcal{L}_1(a,b,x)-\mathcal{L}_0(a,b,x)-abx^2\mathcal{L}_0(a,b,x)\right)t^2+\left(ax\mathcal{L}_0(a,b,x)-\mathcal{L}_1(a,b,x)\right)t^3}{1-(abx^2+2)\,t^2+t^4}$$

where $\mathcal{L}_0(a,b,x)=\begin{pmatrix} ax & 2 \\ 2\frac{a}{b} & -ax \end{pmatrix}$ and $\mathcal{L}_1(a,b,x)=\begin{pmatrix} (ax)^2+2\frac{a}{b} & ax \\ \frac{a^2x}{b} & 2\frac{a}{b} \end{pmatrix}$. Thus, the proof is completed. □

In the following theorem, we give the Binet formula of the bi–periodic Lucas matrix polynomials.

Theorem 4.3. *For $n\in\mathbb{N}$, the Binet formula of the bi–periodic Lucas matrix polynomials is defined by*

$$\mathcal{L}_n(a,b,x)=\frac{\alpha^{n-1}\overline{A}_{\xi(n)}+\beta^{n-1}\overline{B}_{\xi(n)}}{(abx^2)^{\lfloor\frac{n}{2}\rfloor}}, \tag{24}$$

where

$$\overline{A}_{\xi(n)}=\begin{pmatrix} (ax)^{\xi(n+1)}\left(\frac{a}{b}\right)^{\xi(n)}\frac{\alpha^2}{abx^2} & \left(\frac{ax}{abx^2}\right)^{\xi(n)}\alpha \\ \left(\frac{a}{b}\right)\left(\frac{ax}{abx^2}\right)^{\xi(n)}\alpha & (ax)^{\xi(n+1)}\left(\frac{a}{b}\right)^{\xi(n)} \end{pmatrix}$$

and

$$\overline{B}_{\xi(n)}=\begin{pmatrix} (ax)^{\xi(n+1)}\left(\frac{a}{b}\right)^{\xi(n)}\frac{\beta^2}{abx^2} & \left(\frac{ax}{abx^2}\right)^{\xi(n)}\beta \\ \left(\frac{a}{b}\right)\left(\frac{ax}{abx^2}\right)^{\xi(n)}\beta & (ax)^{\xi(n+1)}\left(\frac{a}{b}\right)^{\xi(n)} \end{pmatrix}.$$

Proof. For $k\in\mathbb{N}$, we will prove the theorem in two different cases.
Case $n=2k$:

$$\mathcal{L}_{2k}(a,b,x)=\begin{pmatrix} \left(\frac{a}{b}\right)^{\xi(2k)}l_{2k+1}(a,b,x) & l_{2k}(a,b,x) \\ \left(\frac{a}{b}\right)l_{2k}(a,b,x) & \left(\frac{a}{b}\right)^{\xi(2k)}l_{2k-1}(a,b,x) \end{pmatrix}=\begin{pmatrix} \frac{(ax)^{\xi(2k+1)}\left(\alpha^{2k+1}+\beta^{2k+1}\right)}{(abx^2)^{\lfloor\frac{2k+2}{2}\rfloor}} & \frac{(ax)^{\xi(2k)}\left(\alpha^{2k}+\beta^{2k}\right)}{(abx^2)^{\lfloor\frac{2k+1}{2}\rfloor}} \\ \left(\frac{a}{b}\right)\frac{(ax)^{\xi(2k)}\left(\alpha^{2k}+\beta^{2k}\right)}{(abx^2)^{\lfloor\frac{2k+1}{2}\rfloor}} & \frac{(ax)^{\xi(2k-1)}\left(\alpha^{2k-1}+\beta^{2k-1}\right)}{(abx^2)^{\lfloor\frac{2k}{2}\rfloor}} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{(ax)\left(\alpha^{2k+1}+\beta^{2k+1}\right)}{(abx^2)^{k+1}} & \frac{\left(\alpha^{2k}+\beta^{2k}\right)}{(abx^2)^{k}} \\ \left(\frac{a}{b}\right)\frac{\left(\alpha^{2k}+\beta^{2k}\right)}{(abx^2)^{k}} & \frac{(ax)\left(\alpha^{2k-1}+\beta^{2k-1}\right)}{(abx^2)^{k}} \end{pmatrix}=\frac{\alpha^{2k-1}}{(abx^2)^k}\begin{pmatrix} \frac{(ax)\alpha^2}{abx^2} & \alpha \\ \left(\frac{a}{b}\right)\alpha & ax \end{pmatrix}+\frac{\beta^{2k-1}}{(abx^2)^k}\begin{pmatrix} \frac{(ax)\beta^2}{abx^2} & \beta \\ \left(\frac{a}{b}\right)\beta & ax \end{pmatrix}$$

By defining $\overline{A}_0$ and $\overline{B}_0$ as follows:

$\overline{A}_0=\begin{pmatrix} \frac{(ax)\alpha^2}{abx^2} & \alpha \\ \left(\frac{a}{b}\right)\alpha & ax \end{pmatrix}$ and $\overline{B}_0=\begin{pmatrix} \frac{(ax)\beta^2}{abx^2} & \beta \\ \left(\frac{a}{b}\right)\beta & ax \end{pmatrix}$,

we have

$$\mathcal{L}_{2k}(a,b,x)=\frac{\alpha^{2k-1}\overline{A}_0+\beta^{2k-1}\overline{B}_0}{(abx^2)^k}. \tag{25}$$

Case $n = 2k+1$:

$$\mathcal{L}_{2k+1}(a,b,x) = \begin{pmatrix} \left(\frac{a}{b}\right)^{\xi(2k+1)} l_{2k+2}(a,b,x) & l_{2k+1}(a,b,x) \\ \left(\frac{a}{b}\right) l_{2k+1}(a,b,x) & \left(\frac{a}{b}\right)^{\xi(2k+1)} l_{2k}(a,b,x) \end{pmatrix} = \begin{pmatrix} \left(\frac{a}{b}\right) \frac{(ax)^{\xi(2k+2)}\left(\alpha^{2k+2}+\beta^{2k+2}\right)}{(abx^2)^{\lfloor \frac{2k+3}{2} \rfloor}} & \frac{(ax)^{\xi(2k+1)}\left(\alpha^{2k+1}+\beta^{2k+1}\right)}{(abx^2)^{\lfloor \frac{2k+2}{2} \rfloor}} \\ \left(\frac{a}{b}\right) \frac{(ax)^{\xi(2k+1)}\left(\alpha^{2k+1}+\beta^{2k+1}\right)}{(abx^2)^{\lfloor \frac{2k+2}{2} \rfloor}} & \left(\frac{a}{b}\right) \frac{(ax)^{\xi(2k)}\left(\alpha^{2k}+\beta^{2k}\right)}{(abx^2)^{\lfloor \frac{2k+1}{2} \rfloor}} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{a}{b}\right) \frac{\left(\alpha^{2k+2}+\beta^{2k+2}\right)}{(abx^2)^{k+1}} & \frac{(ax)\left(\alpha^{2k+1}+\beta^{2k+1}\right)}{(abx^2)^{k+1}} \\ \left(\frac{a}{b}\right) \frac{(ax)\left(\alpha^{2k+1}+\beta^{2k+1}\right)}{(abx^2)^{k+1}} & \left(\frac{a}{b}\right) \frac{\left(\alpha^{2k}+\beta^{2k}\right)}{(abx^2)^{k}} \end{pmatrix} = \frac{\alpha^{2k}}{(abx^2)^k} \begin{pmatrix} \left(\frac{a}{b}\right) \frac{\alpha^2}{abx^2} & \frac{(ax)\alpha}{abx^2} \\ \left(\frac{a}{b}\right) \frac{(ax)\alpha}{abx^2} & \left(\frac{a}{b}\right) \end{pmatrix} + \frac{\beta^{2k}}{(abx^2)^k} \begin{pmatrix} \left(\frac{a}{b}\right) \frac{\beta}{abx^2} & \frac{(ax)\beta}{abx^2} \\ \left(\frac{a}{b}\right) \frac{(ax)\beta}{abx^2} & \left(\frac{a}{b}\right) \end{pmatrix}$$

By defining $\overline{A}_1$ and $\overline{B}_1$ as follows:
$\overline{A}_1 = \begin{pmatrix} \left(\frac{a}{b}\right) \frac{\alpha^2}{abx^2} & \frac{(ax)\alpha}{abx^2} \\ \left(\frac{a}{b}\right) \frac{(ax)\alpha}{abx^2} & \left(\frac{a}{b}\right) \end{pmatrix}$ and $\overline{B}_1 = \begin{pmatrix} \left(\frac{a}{b}\right) \frac{\beta}{abx^2} & \frac{(ax)\beta}{abx^2} \\ \left(\frac{a}{b}\right) \frac{(ax)\beta}{abx^2} & \left(\frac{a}{b}\right) \end{pmatrix}$
we have,

$$\mathcal{L}_{2k+1}(a,b,x) = \frac{\alpha^{2k}\overline{A}_1 + \beta^{2k}\overline{B}_1}{(abx^2)^k}. \tag{26}$$

Finally, from the (25) and (26), we have,

$$\mathcal{L}_n(a,b,x) = \frac{\alpha^{n-1}\overline{A}_{\xi(n)} + \beta^{n-1}\overline{B}_{\xi(n)}}{(abx^2)^{\lfloor \frac{n}{2} \rfloor}}.$$

Hence, the proof is complete. □

Theorem 4.4. *For $k \in \mathbb{N}$, the following statements are true:*
(i)

$$\sum_{k=0}^{n-1} \mathcal{L}_k(a,b,x) = \frac{a^{\xi(n)}b^{1-\xi(n)}\mathcal{L}_{n-1}(a,b,x) + a^{1-\xi(n)}b^{\xi(n)}\mathcal{L}_n(a,b,x) - b\mathcal{L}_1(a,b,x) + abx\mathcal{L}_0(a,b,x) - a\mathcal{L}_0(a,b,x)}{abx},$$

(ii)

$$\sum_{k=0}^{n} \mathcal{L}_k(a,b,x)t^{-k} = \frac{1}{1-(abx^2+2)t^2+t^4}\Big(\frac{\mathcal{L}_{n-1}(a,b,x)}{t^{n-1}} - \frac{\mathcal{L}_{n+1}(a,b,x)}{t^{n-3}} + \frac{\mathcal{L}_n(a,b,x)}{t^n} - \frac{\mathcal{L}_{n+2}(a,b,x)}{t^{n-2}}$$
$$+ \mathcal{L}_0(a,b,x)t^4 + \mathcal{L}_1(a,b,x)t^3 - \left[\left(abx^2+1\right)\mathcal{L}_0(a,b,x) - bx\mathcal{L}_1(a,b,x)\right]t^2$$
$$- \left(\mathcal{L}_1(a,b,x) - ax\mathcal{L}_0(a,b,x)\right)t\Big),$$

(iii)

$$\sum_{k=0}^{\infty} \mathcal{L}_k(a,b,x)t^{-k} = \frac{t}{1-(abx^2+2)\,t^2+t^4}$$
$$\begin{pmatrix} axt^3 + \left(a^2x^2 + 2\frac{a}{b}\right)t^2 + axt - 2\frac{a}{b} & 2t^3 + axt^2 - \left(abx^2+2\right)t + ax \\ 2\frac{a}{b}t^3 + \frac{a^2x}{b}t^2 - \left(a^2x^2 + 2\frac{a}{b}\right)t + \frac{a^2x}{b} & -axt^3 + 2\frac{a}{b}t^2 + \left(3ax + a^2bx^3\right)t - \left(a^2x^2 + 2\frac{a}{b}\right) \end{pmatrix}.$$

Proof. Firstly, we proof the case (*i*). From the definition of the bi–periodic Lucas matrix poynomials and doing some calculations, we have

$$\begin{aligned}\sum_{k=0}^{n-1}\mathcal{L}_k(a,b,x) =&\mathcal{L}_0(a,b,x)+\mathcal{L}_1(a,b,x)+\mathcal{L}_2(a,b,x)+\cdots+\mathcal{L}_{n-2}(a,b,x)+\mathcal{L}_{n-1}(a,b,x)\\ =&\mathcal{L}_0(a,b,x)+\frac{\mathcal{L}_2(a,b,x)-\mathcal{L}_0(a,b,x)}{bx}+\frac{\mathcal{L}_3(a,b,x)-\mathcal{L}_1(a,b,x)}{ax}+\cdots\\ &+\frac{\mathcal{L}_{n-1}(a,b,x)-\mathcal{L}_{n-3}(a,b,x)}{a^{1-\xi(n)}b^{\xi(n)}x}+\frac{\mathcal{L}_n(a,b,x)-\mathcal{L}_{n-2}(a,b,x)}{a^{\xi(n)}b^{1-\xi(n)}x}\\ =&\mathcal{L}_0(a,b,x)-\frac{\mathcal{L}_0(a,b,x)}{bx}-\frac{\mathcal{L}_1(a,b,x)}{ax}+\frac{\mathcal{L}_{n-1}(a,b,x)}{a^{1-\xi(n)}b^{\xi(n)}x}+\frac{\mathcal{L}_n(a,b,x)}{a^{\xi(n)}b^{1-\xi(n)}x}\\ =&\frac{a^{\xi(n)}b^{1-\xi(n)}\mathcal{L}_{n-1}(a,b,x)+a^{1-\xi(n)}b^{\xi(n)}\mathcal{L}_n(a,b,x)-b\mathcal{L}_1(a,b,x)+abx\mathcal{L}_0(a,b,x)-a\mathcal{L}_0(a,b,x)}{abx}.\end{aligned}$$

Thus, the proof of (*i*) is completed. Now, we need to proof the case (*iii*). From the definition of bi-periodic Lucas matrix hybrinomials, we define

$$E_0(t)=\sum_{k=0}^{\infty}\mathcal{L}_{2k}(a,b,x)t^{-2k}\quad\text{and}\quad E_1(t)=\sum_{k=0}^{\infty}\mathcal{L}_{2k+1}(a,b,x)t^{-2k-1}.$$

So that

$$E(t)=E_0(t)+E_1(t).$$

We have

$$\begin{aligned}E_0(t)&=\sum_{k=0}^{\infty}\mathcal{L}_{2k}(a,b,x)t^{-2k}\\ &=\mathcal{L}_0(a,b,x)t^0+\mathcal{L}_2(a,b,x)t^{-2}+\sum_{k=2}^{\infty}\mathcal{L}_{2k}(a,b,x)t^{-2k}\\ &=\mathcal{L}_0(a,b,x)+\mathcal{L}_2(a,b,x)t^{-2}+\sum_{k=2}^{\infty}\left[\left(abx^2+2\right)\mathcal{L}_{2k-2}(a,b,x)-\mathcal{L}_{2k-4}(a,b,x)\right]t^{-2k}\\ &=\mathcal{L}_0(a,b,x)+\mathcal{L}_2(a,b,x)t^{-2}+\left(abx^2+2\right)t^{-2}\sum_{k=2}^{\infty}\mathcal{L}_{2k-2}(a,b,x)t^{-2k+2}-t^{-4}\sum_{k=2}^{\infty}\mathcal{L}_{2k-4}(a,b,x)t^{-2k+4}\\ &=\mathcal{L}_0(a,b,x)+\mathcal{L}_2(a,b,x)t^{-2}\\ &\quad+\left(abx^2+2\right)t^{-2}\left[\sum_{k=2}^{\infty}\mathcal{L}_{2k-2}(a,b,x)t^{-2k+2}+\mathcal{L}_0(a,b,x)t^0-\mathcal{L}_0(a,b,x)t^0\right]-t^{-4}E_0(t)\\ &=\mathcal{L}_0(a,b,x)+\mathcal{L}_2(a,b,x)t^{-2}+\left(abx^2+2\right)t^{-2}E_0(t)-\mathcal{L}_0(a,b,x)\left(abx^2+2\right)t^{-2}-t^{-4}E_0(t)\\ E_0(t)[1-\left(abx^2+2\right)t^{-2}+t^{-4}]&=\mathcal{L}_0(a,b,x)+\mathcal{L}_2(a,b,x)t^{-2}-\mathcal{L}_0(a,b,x)\left(abx^2+2\right)t^{-2}.\end{aligned}$$

Thus, we get

$$E_0(t)=\frac{\mathcal{L}_0(a,b,x)+\mathcal{L}_2(a,b,x)t^{-2}-\mathcal{L}_0(a,b,x)\left(abx^2+2\right)t^{-2}}{1-(abx^2+2)\,t^{-2}+t^{-4}}. \tag{27}$$

Similarly, we find

$$E_1(t)=\frac{\mathcal{L}_1(a,b,x)t^{-1}+\mathcal{L}_3(a,b,x)t^{-3}-\mathcal{L}_1(a,b,x)\left(abx^2+2\right)t^{-3}}{1-(abx^2+2)\,t^{-2}+t^{-4}}. \tag{28}$$

By virtue of (27) and (28), we can obtain

$$\begin{aligned} E(t) &= E_0(t) + E_1(t) \\ &= \sum_{k=0}^{\infty} \mathcal{L}_k(a,b,x)t^k \\ &= \frac{\left(\mathcal{L}_3(a,b,x) - \mathcal{L}_1(a,b,x)\left(abx^2+2\right)\right)t + \left(\mathcal{L}_2(a,b,x) - \mathcal{L}_0(a,b,x)\left(abx^2+2\right)\right)t^2 + \mathcal{L}_1(a,b,x)t^3 + \mathcal{L}_0(a,b,x)t^4}{1-(abx^2+2)\,t^2+t^4}. \end{aligned}$$

Hence, the proof of (*iii*) is completed. Finally, the proof of (*ii*) can be done in a similar way to the proof of (*iii*). □

The next definition presents the bi–periodic Lucas matrix hybrinomials.

Definition 4.5. *For $n \in \mathbb{N}$, any variable x and a, b nonzero real numbers, the bi–periodic Lucas matrix hybrinomial is defined by*

$$H\mathcal{L}_n(a,b,x) = \mathcal{L}_n(a,b,x) + \boldsymbol{i}\mathcal{L}_{n+1}(a,b,x) + \varepsilon\mathcal{L}_{n+2}(a,b,x) + \boldsymbol{h}\mathcal{L}_{n+3}(a,b,x), \tag{29}$$

where $\mathcal{L}_n(a,b,x)$ is the bi–periodic Lucas matrix polnomial.

Theorem 4.6. *The following equality is valid for $n \in \mathbb{N}$:*

$$H\mathcal{L}_n(a,b,x) = \begin{pmatrix} \tilde{\mathcal{L}}_{1,1}(n;a,b,x) & \tilde{\mathcal{L}}_{1,2}(n;a,b,x) \\ \tilde{\mathcal{L}}_{2,1}(n;a,b,x) & \tilde{\mathcal{L}}_{2,2}(n;a,b,x) \end{pmatrix},$$

where

$$\begin{aligned} \tilde{\mathcal{L}}_{1,1}(n;a,b,x) &= \left(\frac{a}{b}\right)^{\xi(n)} l_{n+1}(a,b,x) + \boldsymbol{i}\left(\frac{a}{b}\right)^{\xi(n+1)} l_{n+2}(a,b,x) + \boldsymbol{\varepsilon}\left(\frac{a}{b}\right)^{\xi(n)} l_{n+3}(a,b,x) + \boldsymbol{h}\left(\frac{a}{b}\right)^{\xi(n+1)} l_{n+4}(a,b,x), \\ \tilde{\mathcal{L}}_{1,2}(n;a,b,x) &= l_n(a,b,x) + \boldsymbol{i}l_{n+1}(a,b,x) + \boldsymbol{\varepsilon}l_{n+2}(a,b,x) + \boldsymbol{h}l_{n+3}(a,b,x), \\ \tilde{\mathcal{L}}_{2,1}(n;a,b,x) &= \frac{a}{b}\left(l_n(a,b,x) + \boldsymbol{i}l_{n+1}(a,b,x) + \boldsymbol{\varepsilon}l_{n+2}(a,b,x) + \boldsymbol{h}l_{n+3}(a,b,x)\right), \\ \tilde{\mathcal{L}}_{2,2}(n;a,b,x) &= \left(\frac{a}{b}\right)^{\xi(n)} l_{n-1}(a,b,x) + \boldsymbol{i}\left(\frac{a}{b}\right)^{\xi(n+1)} l_n(a,b,x) + \boldsymbol{\varepsilon}\left(\frac{a}{b}\right)^{\xi(n)} l_{n+1}(a,b,x) + \boldsymbol{h}\left(\frac{a}{b}\right)^{\xi(n+1)} l_{n+2}(a,b,x). \end{aligned}$$

Proof. From (22) and (29), the proof can be clearly seen. □

Next, we give the generating function of the bi–periodic Lucas matrix hybrinomials.

Theorem 4.7. *The generating function of the bi–periodic Lucas matrix hybrinomials is defined by*

$$\begin{aligned} \sum_{n=0}^{\infty} H\mathcal{L}_n(a,b,x)t^n &= \frac{1+i\frac{1}{t}+\varepsilon\frac{1}{t^2}+\boldsymbol{h}\frac{1}{t^3}}{1-(abx^2+2)\,t^2+t^4} \\ &\quad \begin{pmatrix} ax+\left(a^2x^2+2\frac{a}{b}\right)t+axt^2-2\frac{a}{b}t^3 & 2+axt-\left(abx^2+2\right)t^2+axt^3 \\ 2\frac{a}{b}+\frac{a^2x}{b}t-\left(a^2x^2+2\frac{a}{b}\right)t^2+\frac{a^2x}{b}t^3 & -ax+2\frac{a}{b}t+\left(3ax+a^2bx^3\right)t^2-\left(a^2x^2+2\frac{a}{b}\right)t^3 \end{pmatrix} \\ &\quad -\boldsymbol{i}\frac{1}{t}\mathcal{L}_0(a,b,x)-\boldsymbol{\varepsilon}\frac{1}{t^2}\mathcal{L}_0(a,b,x)-\boldsymbol{\varepsilon}\frac{1}{t}\mathcal{L}_1(a,b,x) \\ &\quad -\boldsymbol{h}\frac{1}{t^3}\mathcal{L}_0(a,b,x)-\boldsymbol{h}\frac{1}{t^2}\mathcal{L}_1(a,b,x)-\boldsymbol{h}\frac{1}{t}\mathcal{L}_2(a,b,x). \end{aligned} \tag{30}$$

Proof. From (23) and (29), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} H\mathcal{L}_n(a,b,x)t^n &= \sum_{n=0}^{\infty}\Big(\mathcal{L}_n(a,b,x) + \boldsymbol{i}\mathcal{L}_{n+1}(a,b,x) + \boldsymbol{\varepsilon}\mathcal{L}_{n+2}(a,b,x) + \boldsymbol{h}\mathcal{L}_{n+3}(a,b,x)\Big)t^n \\ &= \sum_{n=0}^{\infty}\mathcal{L}_n(a,b,x)t^n + \boldsymbol{i}\sum_{n=0}^{\infty}\mathcal{L}_{n+1}(a,b,x)t^n + \boldsymbol{\varepsilon}\sum_{n=0}^{\infty}\mathcal{L}_{n+2}(a,b,x)t^n + \boldsymbol{h}\sum_{n=0}^{\infty}\mathcal{L}_{n+3}(a,b,x)t^n \\ &= \sum_{n=0}^{\infty}\mathcal{L}_n(a,b,x)t^n + \boldsymbol{i}\frac{1}{t}\sum_{n=0}^{\infty}\mathcal{L}_n(a,b,x)t^n - \boldsymbol{i}\frac{1}{t}\mathcal{L}_0(a,b,x) + \boldsymbol{\varepsilon}\frac{1}{t^2}\sum_{n=0}^{\infty}\mathcal{L}_n(a,b,x)t^n \\ &\quad - \boldsymbol{\varepsilon}\frac{1}{t^2}\mathcal{L}_0(a,b,x) - \boldsymbol{\varepsilon}\frac{1}{t}\mathcal{L}_1(a,b,x) + \boldsymbol{h}\frac{1}{t^3}\sum_{n=0}^{\infty}\mathcal{L}_n(a,b,x)t^n - \boldsymbol{h}\frac{1}{t^3}\mathcal{L}_0(a,b,x) \\ &\quad - \boldsymbol{h}\frac{1}{t^2}\mathcal{L}_1(a,b,x) - \boldsymbol{h}\frac{1}{t}\mathcal{L}_2(a,b,x) \\ &= \Big(1 + i\frac{1}{t} + \varepsilon\frac{1}{t^2} + \boldsymbol{h}\frac{1}{t^3}\Big)\sum_{n=0}^{\infty}\mathcal{L}_n(a,b,x)t^n - \boldsymbol{i}\frac{1}{t}\mathcal{L}_0(a,b,x) - \boldsymbol{\varepsilon}\frac{1}{t^2}\mathcal{L}_0(a,b,x) - \boldsymbol{\varepsilon}\frac{1}{t}\mathcal{L}_1(a,b,x) \\ &\quad - \boldsymbol{h}\frac{1}{t^3}\mathcal{L}_0(a,b,x) - \boldsymbol{h}\frac{1}{t^2}\mathcal{L}_1(a,b,x) - \boldsymbol{h}\frac{1}{t}\mathcal{L}_2(a,b,x),\end{aligned}$$

and consequently, from Eq. (23), the proof is completed. □

In the following theorem, we give the Binet formula of the bi–periodic Lucas matrix hybrinomials.

Theorem 4.8. *For $n \in \mathbb{N}$, the Binet formula of the bi–periodic Lucas matrix hybrinomials is defined by*

$$H\mathcal{L}_n(a,b,x) = \frac{\overline{A}_{\xi(n)}\alpha^{n-1}\overline{\alpha} + \overline{B}_{\xi(n)}\beta^{n-1}\overline{\beta}}{(abx^2)^{\lfloor \frac{n}{2}\rfloor}} + \frac{\overline{A}_{\xi(n+1)}\alpha^{n}\overline{\overline{\alpha}} + \overline{B}_{\xi(n+1)}\beta^{n}\overline{\overline{\beta}}}{(abx^2)^{\lfloor \frac{n+1}{2}\rfloor}}. \tag{31}$$

where, $\overline{A}_{\xi(n)}$ and $\overline{B}_{\xi(n)}$ are given in Theorem (4.3)*. Also, $\overline{\alpha}$, $\overline{\beta}$, $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ are given by*

$$\overline{\alpha} = 1 + \frac{\varepsilon\alpha^2}{abx^2}, \quad \overline{\beta} = 1 + \frac{\varepsilon\beta^2}{abx^2}, \quad \overline{\overline{\alpha}} = i + \frac{h\alpha^2}{abx^2}, \quad \overline{\overline{\beta}} = i + \frac{h\beta^2}{abx^2}.$$

Proof. For $k \in \mathbb{N}$, we will consider the theorem in two different cases.
Case $n = 2k$:

$$\begin{aligned}H\mathcal{L}_{2k}(a,b,x) &= \mathcal{L}_{2k}(a,b,x) + \boldsymbol{i}\mathcal{L}_{2k+1}(a,b,x) + \boldsymbol{\varepsilon}\mathcal{L}_{2k+2}(a,b,x) + \boldsymbol{h}\mathcal{L}_{2k+3}(a,b,x) \\ &= \frac{\overline{A}_0\alpha^{2k-1} + \overline{B}_0\beta^{2k-1}}{(abx^2)^k} + \boldsymbol{i}\frac{\overline{A}_1\alpha^{2k} + \overline{B}_1\beta^{2k}}{(abx^2)^k} + \boldsymbol{\varepsilon}\frac{\overline{A}_0\alpha^{2k+1} + \overline{B}_0\beta^{2k+1}}{(abx^2)^{k+1}} + \boldsymbol{h}\frac{\overline{A}_1\alpha^{2k+2} + \overline{B}_1\beta^{2k+2}}{(abx^2)^{k+1}}x \\ &= \frac{\overline{A}_0\alpha^{2k-1}}{(abx^2)^k}\left(1 + \frac{\varepsilon\alpha^2}{abx^2}\right) + \frac{\overline{B}_0\beta^{2k-1}}{(abx^2)^k}\left(1 + \frac{\boldsymbol{\varepsilon}\beta^2}{abx^2}\right) + \frac{\overline{A}_1\alpha^{2k}}{(abx^2)^k}\left(\boldsymbol{i} + \frac{\boldsymbol{h}\alpha^2}{abx^2}\right) + \frac{\overline{B}_1\beta^{2k}}{(abx^2)^k}\left(\boldsymbol{i} + \frac{\boldsymbol{h}\beta^2}{abx^2}\right)\end{aligned}$$

By defining $\overline{\alpha}, \overline{\beta}, \overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ as follows:

$$\overline{\alpha} = 1 + \frac{\varepsilon\alpha^2}{abx^2}, \quad \overline{\beta} = 1 + \frac{\varepsilon\beta^2}{abx^2}, \quad \overline{\overline{\alpha}} = i + \frac{h\alpha^2}{abx^2}, \quad \overline{\overline{\beta}} = i + \frac{h\beta^2}{abx^2},$$

we have,

$$H\mathcal{L}_{2k}(a,b,x) = \frac{\overline{A}_0\alpha^{2k-1}\overline{\alpha} + \overline{B}_0\beta^{2k-1}\overline{\beta} + \overline{A}_1\alpha^{2k}\overline{\overline{\alpha}} + \overline{B}_1\beta^{2k}\overline{\overline{\beta}}}{(abx^2)^k}. \tag{32}$$

Case $n = 2k+1$:

$$\begin{aligned} H\mathcal{L}_{2k+1}(a,b,x) &= \mathcal{L}_{2k+1}(a,b,x) + \boldsymbol{i}\mathcal{L}_{2k+2}(a,b,x) + \boldsymbol{\varepsilon}\mathcal{L}_{2k+3}(a,b,x) + \boldsymbol{h}\mathcal{L}_{2k+4}(a,b,x) \\ &= \frac{\overline{A}_1\alpha^{2k} + \overline{B}_1\beta^{2k}}{(abx^2)^k} + \boldsymbol{i}\frac{\overline{A}_0\alpha^{2k+1} + \overline{B}_0\beta^{2k+1}}{(abx^2)^{k+1}} + \boldsymbol{\varepsilon}\frac{\overline{A}_1\alpha^{2k+2} + \overline{B}_1\beta^{2k+2}}{(abx^2)^{k+1}} + \boldsymbol{h}\frac{\overline{A}_0\alpha^{2k+3} + \overline{B}_0\beta^{2k+3}}{(abx^2)^{k+2}} \\ &= \frac{\overline{A}_1\alpha^{2k}}{(abx^2)^k}\left(1 + \frac{\boldsymbol{\varepsilon}\alpha^2}{abx^2}\right) + \frac{\overline{B}_1\beta^{2k}}{(abx^2)^k}\left(1 + \frac{\boldsymbol{\varepsilon}\beta^2}{abx^2}\right) + \frac{\overline{A}_0\alpha^{2k+1}}{(abx^2)^{k+1}}\left(\boldsymbol{i} + \frac{\boldsymbol{h}\alpha^2}{abx^2}\right) + \frac{\overline{B}_0\beta^{2k+1}}{(abx^2)^{k+1}}\left(\boldsymbol{i} + \frac{\boldsymbol{h}\beta^2}{abx^2}\right) \end{aligned}$$

By defining $\overline{\alpha}, \overline{\beta}, \overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ as follows:

$$\overline{\alpha} = 1 + \frac{\varepsilon\alpha^2}{abx^2}, \quad \overline{\beta} = 1 + \frac{\varepsilon\beta^2}{abx^2}, \quad \overline{\overline{\alpha}} = i + \frac{h\alpha^2}{abx^2}, \quad \overline{\overline{\beta}} = i + \frac{h\beta^2}{abx^2},$$

we have,

$$H\mathcal{L}_{2k+1}(a,b,x) = \frac{\overline{A}_1\alpha^{2k}\overline{\alpha} + \overline{B}_1\beta^{2k}\overline{\beta}}{(abx^2)^k} + \frac{\overline{A}_0\alpha^{2k+1}\overline{\overline{\alpha}} + \overline{B}_0\beta^{2k+1}\overline{\overline{\beta}}}{(abx^2)^{k+1}}. \tag{33}$$

Finally, from the (32) and (33), we have,

$$H\mathcal{L}_n(a,b,x) = \frac{\overline{A}_{\xi(n)}\alpha^{n-1}\overline{\alpha} + \overline{B}_{\xi(n)}\beta^{n-1}\overline{\beta}}{(abx^2)^{\lfloor \frac{n}{2} \rfloor}} + \frac{\overline{A}_{\xi(n+1)}\alpha^{n}\overline{\overline{\alpha}} + \overline{B}_{\xi(n+1)}\beta^{n}\overline{\overline{\beta}}}{(abx^2)^{\lfloor \frac{n+1}{2} \rfloor}}.$$

Hence, the proof is completed. □

Theorem 4.9. *For $k \geq 0$, the following statements are true:*

(i)

$$\begin{aligned} \sum_{k=0}^{n-1} H\mathcal{L}_k(a,b,x) &= (1 + \boldsymbol{i} + \boldsymbol{\varepsilon} + \boldsymbol{h})\Big[\frac{a^{\xi(n)}b^{1-\xi(n)}\mathcal{L}_{n-1}(a,b,x) + a^{1-\xi(n)}b^{\xi(n)}\mathcal{L}_n(a,b,x)}{abx} \\ &\quad + \frac{-b\mathcal{L}_1(a,b,x) + abx\mathcal{L}_0(a,b,x) - a\mathcal{L}_0(a,b,x)}{abx}\Big] \\ &\quad - (\boldsymbol{i} + \boldsymbol{\varepsilon} + \boldsymbol{h})(\mathcal{L}_0(a,b,x) - \mathcal{L}_n(a,b,x)) \\ &\quad - (\boldsymbol{\varepsilon} + \boldsymbol{h})(\mathcal{L}_1(a,b,x) - \mathcal{L}_{n+1}(a,b,x)) - \boldsymbol{h}(\mathcal{L}_2(a,b,x) - \mathcal{L}_{n+2}(a,b,x)), \end{aligned}$$

(ii)

$$\begin{aligned} \sum_{k=0}^{n} H\mathcal{L}_k(a,b,x)t^{-k} &= \frac{(1 + \boldsymbol{i}t + \boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3)}{1 - (abx^2+2)t^2 + t^4}\Big(\frac{\mathcal{L}_{n-1}(a,b,x)}{t^{n-1}} - \frac{\mathcal{L}_{n+1}(a,b,x)}{t^{n-3}} + \frac{\mathcal{L}_n(a,b,x)}{t^n} - \frac{\mathcal{L}_{n+2}(a,b,x)}{t^{n+2}} \\ &\quad + t^4\mathcal{L}_0(a,b,x) + t^3\mathcal{L}_1(a,b,x) - t^2\left[\left(abx^2+1\right)\mathcal{L}_0(a,b,x) - bx\mathcal{L}_1(a,b,x)\right] \\ &\quad - t\left(\mathcal{L}_1(a,b,x) - ax\mathcal{L}_0(a,b,x)\right)\Big) - \left(\boldsymbol{i}t + \boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3\right)\left(\mathcal{L}_0(a,b,x) - \mathcal{L}_{n+1}(a,b,x)t^{-(n+1)}\right) \\ &\quad - \left(\boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3\right)\left(\mathcal{L}_1(a,b,x)t^{-1} - \mathcal{L}_{n+2}(a,b,x)t^{-(n+2)}\right) - \boldsymbol{h}t^3\left(\mathcal{L}_2(a,b,x)t^{-2} - \mathcal{L}_{n+3}(a,b,x)t^{-(n+3)}\right), \end{aligned}$$

(iii)

$$\begin{aligned} &\sum_{k=0}^{\infty} H\mathcal{L}_k(a,b,x)t^{-k} = \frac{t\left(1 + \boldsymbol{i}t + \boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3\right)}{1 - (abx^2+2)\,t^2 + t^4} \\ &\begin{pmatrix} axt^3 + \left(a^2x^2 + 2\frac{a}{b}\right)t^2 + axt - 2\frac{a}{b} & 2t^3 + axt^2 - \left(abx^2+2\right)t + ax \\ 2\frac{a}{b}t^3 + \frac{a^2x}{b}t^2 - \left(a^2x^2 + 2\frac{a}{b}\right)t + \frac{a^2x}{b} & -axt^3 + 2\frac{a}{b}t^2 + \left(3ax + a^2bx^3\right)t - \left(a^2x^2 + 2\frac{a}{b}\right) \end{pmatrix} \\ &- (\boldsymbol{i}t + \boldsymbol{\varepsilon}t^2 + \boldsymbol{h}t^3)\mathcal{L}_0(a,b,x) - (\boldsymbol{\varepsilon}t + \boldsymbol{h}t^2)\mathcal{L}_1(a,b,x) - \boldsymbol{h}t\mathcal{L}_2(a,b,x) \end{aligned}$$

Proof. This proof can be done in a similar way to the proof of bi–periodic Fibonacci matrix hybrinomials. □

5. Conclusion

Research on hybrid numbers has been increased in recent years, indicating a growing interest and importance of this mathematical concept. Studies on Fibonacci and Lucas matrix hybrinomials, which combine properties of Fibonacci and Lucas sequences with matrix polynomials and hybrid numbers, have provided valuable insight into the interplay of these mathematical structures. In this paper, we focused on the study of bi–periodic Fibonacci matrix hybrinomials using the existing literature on bi–periodic Fibonacci matrix polynomials. Additionally, by virtue of the concept of bi–periodic Lucas polynomials, the work is extended to define bi–periodic Lucas matrix polynomials, revealing their generating functions, Binet formulas, and some summation formulas. These definitions and properties obtained contribute to a deeper understanding of hybrid numbers and their applications in various fields. Also, as research in this area continues to evolve, the possibilities for further exploration and practical application are endless.

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7. Declarations

Conflict of interest The authors declare no conflict of interest.

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