Filomat 38:28 (2024), 9813–9833
https://doi.org/10.2298/FIL2428813Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Convergence of densities of spatial averages of the linear stochastic heat equation

Wanying Zhang[a], Yong Zhang[a,*], Jingyu Li[a]

[a] *School of Mathematics, Jilin University, Changchun 130012, China*

Abstract. Let $\{u(t,x)\}_{t>0,x\in\mathbb{R}^d}$ denote the solution to the linear (fractional) stochastic heat equation. We establish convergence rates with respect to the uniform distance between the density of spatial averages of the solution and the density of the standard normal distribution in some different scenarios. We first consider the case when $u_0 \equiv 1$ and the stochastic fractional heat equation is driven by a space-time white noise. When $\alpha = 2$ (parabolic Anderson model, PAM for short), and the stochastic heat equation is driven by colored noise in space, we present the rates of convergence respectively for $u_0 \equiv 1$, $d \geq 1$ and $u_0 = \delta_0$, $d = 1$ under the additional condition $\hat{f}(\mathbb{R}^d) < \infty$. Our results are obtained by using a combination of the Malliavin calculus and Stein's method for normal approximations.

1. Introduction

Consider the following stochastic (fractional) heat equation:

$$\begin{cases} \partial_t u(t,x) = v \cdot (-(-\Delta)^{\frac{\alpha}{2}})u(t,x) + u(t,x)\eta(t,x) & \text{for } (t,x) \in (0,+\infty)\times\mathbb{R}^d, \\ \text{subject to} \quad u(0,x) = u_0(x), \end{cases} \tag{1}$$

where v is a positive constant, $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplace operator (see [21]) and η denotes a centered, generalized Gaussian random field such that

$$\mathrm{Cov}[\eta(t,x),\eta(s,y)] = \delta_0(t-s)f(x-y) \quad \text{for all } s,t \geq 0 \text{ and } x,y \in \mathbb{R}^d, \tag{2}$$

for a non-zero, nonnegative-definite, tempered Borel measure f on $\mathbb{R}^d$.

We are interested in the rates of convergence of the uniform norm of densities in the following three cases:
Case 1 ($d = 1$): $\alpha \in (1,2]$, $v = 1$, $u_0 \equiv 1$ and $f(x) = \delta_0(x)$ for all $x \in \mathbb{R}$.

2020 *Mathematics Subject Classification*. Primary 60H15; Secondary 60H07, 60F99
Keywords. stochastic fractional heat equation, parabolic Anderson model, Malliavin calculus, Stein's method
Received: 05 January 2024; Accepted: 30 August 2024
Communicated by Miljana Jovanović
Research supported by the Science and Technology Program of Jilin Educational Department during the 14th Five-Year Plan Period (Grant No. JJKH20241239KJ) and the National Natural Science Foundation of China (Grant No. 12171198).
* Corresponding author: Yong Zhang
Email addresses: wyzhang20@mails.jlu.edu.cn (Wanying Zhang), zyong2661@jlu.edu.cn (Yong Zhang), lijingyu@jlu.edu.cn (Jingyu Li)

Case 2 ($d \geq 1$): $\alpha = 2$, $v = \frac{1}{2}$, $u_0 \equiv 1$, $f(\mathbb{R}^d) < \infty$ and $\hat{f}(\mathbb{R}^d) < \infty$[1)].
Case 3 ($d = 1$): $\alpha = 2$, $v = \frac{1}{2}$, $u_0 = \delta_0$, $f(\mathbb{R}) < \infty$ and $\hat{f}(\mathbb{R}) < \infty$.
Following from Walsh [7], we can interpret (1) in the following mild form:
In case 1,

$$u(t,x) = 1 + \int_{(0,t)\times\mathbb{R}} G_\alpha(t-s, x-y)u(s,y)\eta(\mathrm{d}s,\mathrm{d}y) \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \tag{3}$$

where G_α denotes the Green kernel defined through its Fourier transform $\widehat{G_\alpha(t,\cdot)}(x) = e^{-t|x|^\alpha}$.
In case 2,

$$u(t,x) = 1 + \int_{(0,t)\times\mathbb{R}^d} p_{t-s}(x-y)u(s,y)\eta(\mathrm{d}s,\mathrm{d}y) \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^d, \tag{4}$$

where $p_t(x)$ denotes the heat kernel that satisfies $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{\|x\|^2}{2t}}$.
In case 3,

$$U(t,x) = 1 + \int_{(0,t)\times\mathbb{R}} p_{s(t-s)/t}\left(y - \frac{s}{t}x\right)U(s,y)\eta(\mathrm{d}s,\mathrm{d}y) \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \tag{5}$$

where $U(t,x) := u(t,x)/p_t(x)$ (see [2, 15]).

The existence and uniqueness problems for the solution to (1) have been studied extensively [2, 17, 19, 22, 23]. In the present setting, we must ensure that the Fourier transform $\hat{f}$ satisfies the integrability condition:

$$\Upsilon(\beta) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}y)}{\beta + \|y\|^\alpha} < \infty \quad \text{for all } \beta > 0. \tag{6}$$

Clearly, (6) holds in the cases mentioned above.

For any fixed $t > 0$ and $N > 0$, we introduce the centered and normalized spatial averages:
In case 1,

$$F_{N,1} := \frac{1}{\sigma_{N,1}} \left(\int_{[0,N]} (u(t,x) - 1)\,\mathrm{d}x \right), \text{ where } \sigma_{N,1}^2 := \mathrm{Var}\left(\int_{[0,N]} u(t,x)\mathrm{d}x \right). \tag{7}$$

In case 2,

$$F_{N,2} := \frac{1}{\sigma_{N,2}} \left(\int_{[0,N]^d} (u(t,x) - 1)\,\mathrm{d}x \right), \text{ where } \sigma_{N,2}^2 := \mathrm{Var}\left(\int_{[0,N]^d} u(t,x)\mathrm{d}x \right). \tag{8}$$

In case 3,

$$F_{N,3} := \frac{1}{\sigma_{N,3}} \left(\int_{[0,N]} (U(t,x) - 1)\,\mathrm{d}x \right), \text{ where } \sigma_{N,3}^2 := \mathrm{Var}\left(\int_{[0,N]} U(t,x)\mathrm{d}x \right). \tag{9}$$

There are many arguments for a quantitative central limit theorem (CLT for short) for spatial averages of the solution to (1). The CLT using techniques of the Malliavin-Stein method was first established by Huang et al [8] with $\alpha = 2$, $u_0 \equiv 1$ and $f = \delta_0$. Later, Chen et al derived the case that $d \geq 1$ in [13] and [14] under the condition $f(\mathbb{R}^d) < \infty$. As for the delta initial condition, Chen et al [15] proved the CLT for the PAM when η is a space-time white noise. After that, Khoshnevisan et al [2] extended the result to cover the scenario where η is colored in space. Recently, Assaad et al [22] studied the case of stochastic fractional

[1)]The Fourier transform of f is denoted by $\hat{f}$, that is, $\hat{f}(y) = \int_{\mathbb{R}^d} e^{ix\cdot y} f(dx)$. In general, $\hat{f}$ is a nonnegative definite measure.

heat equation with the initial condition $u_0 \equiv 1$. Other related limit theorems and their variations have been explored in [3, 5, 9, 11, 12, 28]. In our current context, the convergence rates for the total variation distance[2)] are precisely expressed as follows:

$$d_{\mathrm{TV}}\left(F_{N,1}, \mathrm{N}(0,1)\right) \leq \frac{C_t}{\sqrt{N}} \quad \text{for all } N \geq 1, \tag{10}$$

$$d_{\mathrm{TV}}\left(F_{N,2}, \mathrm{N}(0,1)\right) \leq \frac{C_t}{(\sqrt{N})^d} \quad \text{for all } N \geq 1, \tag{11}$$

and

$$d_{\mathrm{TV}}\left(F_{N,3}, \mathrm{N}(0,1)\right) \leq \frac{C_t\sqrt{\log N}}{\sqrt{N}} \quad \text{for all } N \geq e. \tag{12}$$

The above results describe the convergence rates for total variation distance between the spatial average and the standard normal random variable. When the density functions of the two random variables exist, the convergence rate for total variation distance is equal to the convergence rate described in the $L^1(\mathbb{R}, \mathrm{d}x)$ difference of the corresponding density functions. Motivated by this insight, we aim to explore the convergence rates in the $L^\infty(\mathbb{R})$ difference of two density functions.

The analysis of upper bounds for the uniform norm of densities, using Malliavin calculus techniques, was first introduced by Hu et al [27]. Recently, Kuzgun and Nualart [24] extended their results and established upper bounds for the uniform distance of densities between the spatial averages of the solution to the stochastic heat equation in two different cases and the standard normal random variable. Specifically, they deduced the convergence rates with respect to the nonlinear stochastic heat equation and the PAM under the condition $u_0 \equiv 1$, $f = \delta_0$ and $u_0 = \delta_0$, $f = \delta_0$, respectively. Later, Kuzgun and Nualart [25] derived the convergence rates when $u_0 \equiv 1$ and f is given by the Riesz kernel($f(\mathrm{d}x) = |x|^{-\beta}\mathrm{d}x$, $\beta \in (0, d \wedge 2)$).

In the three cases of interest in this paper, the convergence rates for the uniform distance of densities have not been studied. Therefore, we present the following results:

Theorem 1.1. *In case 1, let $F_{N,1}$ be the spatial average defined in (7). Then, for all $N \geq 1$,*

$$\sup_{x\in\mathbb{R}} |f_{F_{N,1}}(x) - \phi(x)| \leq \frac{C_t}{\sqrt{N}}, \tag{13}$$

where $f_{F_{N,1}}$ and ϕ denote the densities of $F_{N,1}$ and $\mathrm{N}(0,1)$, respectively.

It can be seen that (1) becomes the PAM when $\alpha = 2$. Therefore, Theorem 1.1 serves as an extension of the linear case in [24, Theorem 1.1]. The previous theorem is established for $f = \delta_0$. In the following, we shift our focus to the case that η is colored in space.

Theorem 1.2. *In case 2, let $F_{N,2}$ be the spatial average defined in (8). Then, for all $N \geq 1$,*

$$\sup_{x\in\mathbb{R}} |f_{F_{N,2}}(x) - \phi(x)| \leq \frac{C_t}{(\sqrt{N})^d}, \tag{14}$$

where $f_{F_{N,2}}$ and ϕ denote the densities of $F_{N,2}$ and $\mathrm{N}(0,1)$, respectively.

Theorem 1.3. *In case 3, let $F_{N,3}$ be the spatial average defined in (9). Fix $\beta > 21$. Then, for all $N \geq e$,*

$$\sup_{x\in\mathbb{R}} |f_{F_{N,3}}(x) - \phi(x)| \leq \frac{C_t(\sqrt{\log N})^\beta}{\sqrt{N}}, \tag{15}$$

where $f_{F_{N,3}}$ and ϕ denote the densities of $F_{N,3}$ and $\mathrm{N}(0,1)$, respectively.

[2)] The total variation distance between two random variables X and Y is defined as $d_{\mathrm{TV}}(X,Y) = \sup_{B\in\mathcal{B}(\mathbb{R})} |\mathrm{P}(X \in B) - \mathrm{P}(Y \in B)|$, where $\mathcal{B}(\mathbb{R})$ is the collection of all Borel sets in $\mathbb{R}$.

Following the method in [24] (see Lemma 2.1), the estimates on p-norm of the second Malliavin derivative and $\|(D_vF)^{-1}\|_4$ are crucial components in discussing convergence rates. It is essential to note that, when dealing with $\|(D_vF)^{-1}\|_4$, we must ensure that $\mathrm{E}\left[|u(t,x)|^{-p}\right] < \infty$ for any fixed $(t,x) \in (0,+\infty) \times \mathbb{R}^d$. In our settings, the non-negativity of $u(t,x)$ is guaranteed almost surely thanks to the comparison principle or Feynman-Kac formula; see, for example, [10, 17, 18]. Therefore, we use $\mathrm{E}\left[(u(t,x))^{-p}\right]$ instead of $\mathrm{E}\left[|u(t,x)|^{-p}\right]$. For the case of $d = 1$, $f = \delta_0$, Chen et al [20] have proved that $\mathrm{E}\left[(u(t,x))^{-p}\right] < \infty$ based on small-ball probability estimate. When η is colored in space, small-ball probability estimate was provided by Chen and Huang [17]. However, the negative moment of the solution exists only when $\hat{f}(\mathbb{R}^d) < \infty$. (see Lemma 2.3 part (4) and Lemma 2.4 part (4)).

Remark 1.4. *The collection of measures under the condition $\hat{f}(\mathbb{R}^d) < \infty$ is massive. For example, f is given by a Gaussian kernel ($f(\mathrm{d}x) = p_1(x)\mathrm{d}x$ and $\hat{f}(\mathrm{d}x) = e^{-\|x\|^2/2}\mathrm{d}x$) or a Cauchy kernel ($f(\mathrm{d}x) = \left[\prod_{j=1}^d (1+|x_j|^2)\right]^{-1}\mathrm{d}x$ and $\hat{f}(\mathrm{d}x) = \pi^d \prod_{j=1}^d e^{-|x_j|}\mathrm{d}x$).*

Remark 1.5. *Unfortunately, the multidimensional situation of case 3 has not been investigated so far. It has been proved in [2, Theorem 1.3] that the convergence rate for CLT in terms of total variation is $N^{-1/2}$, implying a corresponding convergence rate of $\|1 - D_vF_N\|_2$ in Lemma 2.1 as $N^{-1/2}$. However, the control of $\|(D_vF_N)^{-1}\|_4$ by N^a for any $a < 1/2$ remains elusive through our current methodology. This is an issue for future research to explore.*

The organization of this paper is as follows. In Section 3, we derive moment estimates of the second Malliavin derivative of $u(t,x)$. Notably, we obtain a more general result for the nonlinear stochastic fractional heat equation in case 1. Section 4 is devoted to analyzing the negative moments of D_vF. Furthermore, we prove the convergence rates on uniform distance in Theorems 1.1-1.3 in Section 5, based on Malliavin-Stein method and Fourier analysis. Finally, in the Appendix, we introduce some technical lemmas used along the paper.

Throughout this paper, we write $\|Z\|_p$ instead of $(\mathrm{E}|Z|^p)^{1/p}$ for any $Z \in L^p$ and we denote the generic nonnegative constant by C, which can take different values and depend on different variables.

2. Preliminaries

2.1. The BDG inequality

For every continuous $L^2(\Omega)$-martingale $\{M_t\}_{t\geq 0}$, we have the following Burkholder-Davis-Gundy (BDG for short) inequality:

$$\mathrm{E}\left(|M_t|^k\right) \leq z_k^k \mathrm{E}\left(\langle M\rangle_t^{k/2}\right) \quad \text{for all } t \geq 0 \text{ and } k \geq 2,$$

where $\{z_k\}_{k\geq 2}$ are the optimal constants. Moreover, the method in [1] and [6] together implies that

$$z_2 = 1, \quad \text{and} \quad \sup_{k\geq 2} \frac{z_k}{\sqrt{k}} = \lim_{k\to\infty} \frac{z_k}{\sqrt{k}} = 2,$$

which means z_k is bounded from above by the multiples of $\sqrt{k}$, uniformly for all $k \geq 2$.

2.2. Malliavin calculus and Stein's method

Let $\mathcal{H}_0$ be the reproducing kernel Hilbert space spanned by all real-valued functions on $\mathbb{R}^d$, with respect to the scalar product $\langle \phi, \psi \rangle := \langle \phi, \psi * f \rangle_{L^2(\mathbb{R}^d)}$, and let $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathcal{H}_0)$. The Gaussian random field $\{W(h)\}_{h\in\mathcal{H}}$ formed by such Wiener integrals

$$W(h) := \int_{\mathbb{R}^+\times\mathbb{R}^d} h(s,y)\eta(\mathrm{d}s,\mathrm{d}y) \tag{16}$$

defines an isonormal Gaussian process on the Hilbert space $\mathcal{H}$. On the basis of this, we can develop the Malliavin calculus (see, for instance, [4]).

Let $\mathcal{S}$ denote the space of simple random variables of the form

$$F = f(W(h_1), \ldots, W(h_d)),$$

where $f \in C_p^\infty(\mathbb{R}^d)$, that is, f is a smooth function and all its partial derivatives have at most polynomial growth at infinity, and $h_1, \ldots, h_n \in \mathcal{H}$. Then the Malliavin derivative DF is defined as $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} (W(h_1), \ldots, W(h_d))\, h_i. \tag{17}$$

For any $p \geq 1$ and $k \geq 1$, we denote the completion of $\mathcal{S}$ by $\mathbb{D}^{k,p}$ with respect to the norm

$$\|F\|_{k,p} = \left(\mathrm{E}\left[|F|^p\right] + \sum_{j=1}^{k} \mathrm{E}\left[\|D^j F\|^p_{\mathcal{H}^{\otimes j}} \right] \right)^{1/p},$$

where D^j denotes the j-th iterated Malliavin derivative and $\otimes$ denotes the tensor product. Similarly, we can introduce the spaces $\mathbb{D}^{k,p}(V)$ for some real separable Hilbert space V. The adjoint operator δ of the derivative D is characterized by the duality formula

$$\mathrm{E}[F\delta(v)] = E[\langle DF, v\rangle_{\mathcal{H}}],$$

which is valid for any $F \in \mathbb{D}^{1,2}$. An important property of δ is that any predictable and square integrable random field v belongs to the domain of δ and $\delta(v)$ coincides with the Walsh integral, that is,

$$\delta(v) = \int_{\mathbb{R}^+ \times \mathbb{R}^d} v(s, x) \eta(\mathrm{d}s, \mathrm{d}x).$$

For an $\mathcal{H}$-valued random variable v and $F \in \mathbb{D}^{1,1}$, define

$$D_v F := \langle DF, v\rangle_{\mathcal{H}}. \tag{18}$$

The following lemma, which characterizes the uniform distance of densities, plays an important role in proving Theorems 1.1-1.3.

Lemma 2.1. [24, Theorem 3.2] *Assume that* $v \in \mathbb{D}^{1,6}(\mathcal{H})$, $F = \delta(v) \in \mathbb{D}^{2,6}$ *with* $\mathrm{E}(F) = 0$ *and* $\mathrm{E}(F^2) = 1$, *and* $\|(D_v F)^{-1}\|_4 < \infty$. *Then F admits a density* $f_F(x)$ *and the following inequality holds true,*

$$\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq \left(\|F\|_4 \|(D_v F)^{-1}\|_4 + 2 \right) \|1 - D_v F\|_2 + \|(D_v F)^{-1}\|_4^2 \|D_v(D_v F)\|_2, \tag{19}$$

where $\phi(x)$ *denotes the density of* $\mathrm{N}(0,1)$.

2.3. Some properties of $u(t, x)$

We first introduce some properties of the moments and Malliavin derivative of $u(t, x)$ in the following three lemmas.

Lemma 2.2. *Let* $u(t, x)$ *be the solution to (1) in case 1, we have*
(1) [22, Theorem 2.1] *The process* $u(t) := \{u(t, x)\}_{x \in \mathbb{R}}$ *is stationary. Moreover, for any* $p \geq 1$ *and any* $T > 0$,

$$\sup_{t \in (0,T], x \in \mathbb{R}} \mathrm{E}[|u(t, x)|^p] < \infty.$$

(2) [22, Propositions 5.1-5.2] *For almost all* $0 < s < t < T$, $x, y \in \mathbb{R}$,

$$D_{s,y} u(t, x) = G_\alpha(t - s, x - y) u(s, y) + \int_s^t \int_{\mathbb{R}} G_\alpha(t - r, x - z) D_{s,y} u(r, z) \eta(\mathrm{d}r, \mathrm{d}z). \tag{20}$$

Moreover, for all $p \geq 2$,

$$\|D_{s,y}u(t,x)\|_p \leq C_{T,p}(t-s)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(t-s, x-y).$$

(3) [20, Theorem 1.5] *Fix* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, *for all* $p > 0$,

$$\mathrm{E}\left([u(t,x)]^{-p}\right) < \infty.$$

Lemma 2.3. *Let* $u(t,x)$ *be the solution to (1) in case 2, we have*
(1) [16, Theorem 1.1] *The random field* $u(t) := \{u(t,x)\}_{x\in\mathbb{R}^d}$ *is stationary.*
(2) [13, Lemma 4.2] *For almost all* $0 < s < t < T$, $x, y \in \mathbb{R}^d$,

$$D_{s,y}u(t,x) = p_{t-s}(x-y)u(s,y) + \int_s^t \int_{\mathbb{R}^d} p_{t-r}(x-z) D_{s,y}u(r,z)\eta(\mathrm{d}r,\mathrm{d}z). \tag{21}$$

Moreover, for all $p \geq 2$,

$$\sup_{t\in(0,T],x\in\mathbb{R}^d} \mathrm{E}[|u(t,x)|^p] < \infty \textit{ and } \|D_{s,y}u(t,x)\|_p \leq C_{T,p}p_{t-s}(x-y).$$

(3) [14, Proposition 3.4] *For all* $0 < r < s < t$, $x, y, z \in \mathbb{R}^d$ *and for every* $p \geq 2$,

$$\|D_{r,z}D_{s,y}u(t,x)\|_p \leq C_{t,p}p_{t-s}(x-y)p_{s-r}(y-z).$$

(4) *Fix* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, *for all* $p > 0$,

$$\mathrm{E}\left([u(t,x)]^{-p}\right) < \infty.$$

Proof. (4). From [17, Theorem 1.6], since $\hat{f}(\mathbb{R}^d) < \infty$, we have that, for any fixed $(t,x) \in (0,+\infty)\times\mathbb{R}^d$, there exists a finite constant $A = A_{t,x} > 0$ such that for all $\varepsilon > 0$ small enough,

$$\mathrm{P}(u(t,x) < \varepsilon) \leq A\exp(-A|\log\varepsilon|(\log|\log\varepsilon|)^2). \tag{22}$$

According to [20, Lemma A.1], it suffices to show that, for all $p > 0$, there exists some finite constant $C_p > 0$ such that

$$\mathrm{P}(u(t,x) < \varepsilon) \leq C_p\varepsilon^p. \tag{23}$$

For any $p > 0$, choose $C_p = A \vee \exp(p\exp(\sqrt{p/A}))$. Then, for any $0 < \varepsilon < \exp(-\exp(\sqrt{p/A}))$,

$$\mathrm{P}(u(t,x) < \varepsilon) \leq A\exp(-A|\log\varepsilon|(\log|\log\varepsilon|)^2) \leq C_p\varepsilon^p, \tag{24}$$

and for any $\varepsilon \geq \exp(-\exp(\sqrt{p/A}))$,

$$\mathrm{P}(u(t,x) < \varepsilon) \leq 1 \leq \exp(p\exp(\sqrt{p/A}))\varepsilon^p \leq C_p\varepsilon^p. \tag{25}$$

Combining (24) and (25), we prove the result. □

Lemma 2.4. *Let* $u(t,x)$ *be the solution to (1) in case 3,* $U(t,x) = u(t,x)/p_t(x)$, *we have*
(1) [2, Theorem 1.1] *The random field* $U(t) := \{u(t,x)\}_{x\in\mathbb{R}}$ *is stationary. Moreover, for any* $p \geq 2$ *and any* $T > 0$,

$$\sup_{t\in(0,T],x\in\mathbb{R}} \mathrm{E}[|U(t,x)|^p] < \infty.$$

(2) [2, Proposition 4.1] *For all* $0 < r < s < t$, $x, y, z \in \mathbb{R}$,

$$D_{s,y}U(t,x) = p_{\frac{s(t-s)}{t}}\left(y - \frac{s}{t}x\right)U(s,y) + \int_s^t \int_{\mathbb{R}} p_{\frac{r(t-r)}{t}}\left(z - \frac{r}{t}x\right) D_{s,y}U(r,z)\eta(\mathrm{d}r,\mathrm{d}z). \tag{26}$$

(3) [26, Corallary 1.2] *Suppose that* $0 < r_i < r_j < t$ *for all* $1 \le i < j \le k$ *and* $z_i \in \mathbb{R}$ *for* $1 \le i \le k$, *let* $D^k_{r_k,z_k}U(t,x)$ *denote the k-th iterated Malliavin derivative of* $U(t,x)$, *i.e.,* $D^k_{r_k,z_k}U(t,x) = D_{r_1,z_1}\ldots D_{r_k,z_k}U(t,x)$, *then for all* $p \ge 2$,

$$\|D^k_{r_k,z_k}U(t,x)\|_p \le C_{t,p}\left(\prod_{m=1}^{k-1} p_{\frac{r_m(r_{m+1}-r_m)}{r_{m+1}}}\left(z_m - \frac{r_m}{r_{m+1}}z_{m+1}\right)\right)p_{\frac{r_k(t-r_k)}{t}}\left(z_k - \frac{r_k}{t}x\right).$$

(4) *Fix* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, *for all* $p > 0$,

$$\mathrm{E}\left([u(t,x)]^{-p}\right) < \infty, \text{ hence, } \mathrm{E}\left([U(t,x)]^{-p}\right) < \infty.$$

Proof. (4). The argument is the same as the proof of Lemma 2.3 part (4). □

The following lemma describes the asymptotic behavior of the variance functions and upper bounds for the moment of spatial averages.

Lemma 2.5. *Let* $F_{N,i}$ *and* $\sigma^2_{N,i}$ $(i = 1,2,3.)$ *be as defined in (7)-(9). Then, we have*
(1) *For any* $p \ge 2$, $\sup_{N\ge1}\|F_{N,i}\|_p \le C_t$ $(i = 1,2)$, *and* $\sup_{N\ge e}\|F_{N,3}\|_p \le C_t$.
(2) [22, Theorem 5.6] $\lim_{N\to\infty}\frac{\sigma^2_{N,1}}{N} = t$.
(3) [14, Proposition 5.2] $\lim_{N\to\infty}\frac{\sigma^2_{N,2}}{N^d} = \int_{\mathbb{R}^d}\mathrm{Cov}[u(t,x),u(t,0)]\mathrm{d}x < \infty$.
(4) [2, Theorems 5.1-5.2] $\lim_{N\to\infty}\frac{\sigma^2_{N,3}}{N\log N} = tf(\mathbb{R})$.

Proof. (1). The upper bounds of the moments follow easily from the BDG inequality; see, for instance, [15, Lemma 2.4]. □

3. Second Malliavin derivative

This section aims to estimate the moment bounds of the second Malliavin derivative of $u(t,x)$. The estimates for cases 2 and 3 can be found in Lemmas 2.3-2.4. Therefore, we will only prove the result for case 1. Now, consider the more general setting,

$$\begin{cases} \partial_t u(t,x) = -(-\Delta)^{\frac{\alpha}{2}}u(t,x) + \sigma(u(t,x))\eta(t,x) & \text{for } (t,x) \in (0,+\infty)\times\mathbb{R}, \\ \text{subject to} \quad u(0,x) = 1, \end{cases} \tag{27}$$

where σ denotes the Lipschitz function satisfying $\sigma(1) \neq 0$. In order to obtain the following result, we further assume that σ is twice continuously differentiable, σ' is bounded and $|\sigma''(x)| \le C(1+|x|^m)$ for some $m > 0$.

Proposition 3.1. *Let* $u(t,x)$ *denote the solution to (27). Then,* $u(t,x) \in \cap_{p\ge2}\mathbb{D}^{2,p}$ *and for almost all* $0 < r < s < t$, $y,z \in \mathbb{R}$, *we have*

$$\begin{aligned} D_{r,z}D_{s,y}u(t,x) =& G_\alpha(t-s,x-y)\sigma'(u(s,y))D_{r,z}u(s,y) \\ &+ \int_{[s,t]\times\mathbb{R}} G_\alpha(t-\tau,x-\xi)\sigma''(u(\tau,\xi))D_{r,z}u(\tau,\xi)D_{s,y}u(\tau,\xi)\eta(\mathrm{d}\tau,\mathrm{d}\xi) \\ &+ \int_{[s,t]\times\mathbb{R}} G_\alpha(t-\tau,x-\xi)\sigma'(u(\tau,\xi))D_{r,z}D_{s,y}u(\tau,\xi)\eta(\mathrm{d}\tau,\mathrm{d}\xi). \end{aligned}$$

Moreover, for all $0 \le r < s < t \le T$ *and* $x,y,z \in \mathbb{R}$,

$$\left\|D_{r,z}D_{s,y}u(t,x)\right\|^2_p \le C_{T,p}K^2_{r,z,s,y}(t,x), \tag{28}$$

where

$$K^2_{r,z,s,y}(t,x) = (t-s)^{-\frac{1}{\alpha}} G_\alpha(t-s, x-y)(s-r)^{-\frac{1}{\alpha}} G_\alpha(s-r, y-z)$$
$$+ \int_s^t \int_{\mathbb{R}} [(t-\theta)(\theta-r)(\theta-s)]^{-\frac{1}{\alpha}} G_\alpha(t-\theta, x-w) G_\alpha(\theta-r, w-z) G_\alpha(\theta-s, w-y) \mathrm{d}\theta \mathrm{d}w.$$

In particular, if $\sigma(x) = x$, then

$$\left\| D_{r,z} D_{s,y} u(t,x) \right\|_p \leq C_{T,p} (t-s)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(t-s, x-y)(s-r)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(s-r, y-z). \tag{29}$$

Proof. First, we define the Picard iteration for the solution to (27). Let $u_0(t,x) = 1$, and for $n \in \mathbb{N}$,

$$u_{n+1}(t,x) = 1 + \int_{(0,t)\times\mathbb{R}} G_\alpha(t-s, x-y) \sigma(u_n(s,y)) \eta(\mathrm{d}s, \mathrm{d}y) \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}. \tag{30}$$

Applying the properties of the divergence operator [4, Proposition 1.3.8], we deduce, for almost all $(s,y) \in (0,t) \times \mathbb{R}$, that

$$D_{s,y} u_{n+1}(t,x) = G_\alpha(t-s, x-y)\sigma(u_n(s,y)) + \int_{[s,t]\times\mathbb{R}} G_\alpha(t-\tau, x-\xi) \sigma'(u_n(\tau,\xi)) D_{s,y} u_n(\tau,\xi) \eta(\mathrm{d}\tau, \mathrm{d}\xi), \tag{31}$$

and for almost all $s > t$, $D_{s,y} u_{n+1}(t,x) = 0$. Using the properties of the divergence operator again, we obtain, for almost all $0 < r < s < t$ and $y, z \in \mathbb{R}$,

$$\begin{aligned} D_{r,z} D_{s,y} u_{n+1}(t,x) &= G_\alpha(t-s, x-y) \sigma'(u_n(s,y)) D_{r,z} u_n(s,y) \\ &+ \int_{[s,t]\times\mathbb{R}} G_\alpha(t-\tau, x-\xi) \sigma''(u_n(\tau,\xi)) D_{r,z} u_n(\tau,\xi) D_{s,y} u_n(\tau,\xi) \eta(\mathrm{d}\tau, \mathrm{d}\xi) \\ &+ \int_{[s,t]\times\mathbb{R}} G_\alpha(t-\tau, x-\xi) \sigma'(u_n(s,y)) D_{r,z} D_{s,y} u_n(\tau,\xi) \eta(\mathrm{d}\tau, \mathrm{d}\xi). \end{aligned} \tag{32}$$

Moreover, applying the BDG inequality, Minkowski inequality and Lemma 2.2 (Lemma 2.2 still holds for the nonlinear case, see [22]), we have, for almost all $(t,x) \in [0,T] \times \mathbb{R}$,

$$\begin{aligned} &\left\| D_{r,z} D_{s,y} u_{n+1}(t,x) \right\|_p^2 \leq C_{T,p} G_\alpha^2(t-s, x-y)(s-r)^{-\frac{1}{\alpha}} G_\alpha(s-r, y-z) \\ &+ C_{T,p} \int_s^t \int_{\mathbb{R}} G_\alpha^2(t-\tau, x-\xi)(\tau-r)^{-\frac{1}{\alpha}} G_\alpha(\tau-r, \xi-z)(\tau-s)^{-\frac{1}{\alpha}} G_\alpha(\tau-s, \xi-y) \mathrm{d}\xi \mathrm{d}\tau \\ &+ C_{T,p} \int_s^t \int_{\mathbb{R}} G_\alpha^2(t-\tau, x-\xi) \left\| D_{r,z} D_{s,y} u_n(\tau,\xi) \right\|_p^2 \mathrm{d}\xi \mathrm{d}\tau. \end{aligned} \tag{33}$$

To simplify the expression, we define the measure on $[s,t] \times \mathbb{R}$ such that

$$\begin{aligned} J(\mathrm{d}\tau, \mathrm{d}\xi) :=& (\tau-r)^{-\frac{1}{\alpha}} G_\alpha(\tau-r, \xi-z) \delta_{s,y}(\mathrm{d}\tau, \mathrm{d}\xi) \\ &+ (\tau-r)^{-\frac{1}{\alpha}} G_\alpha(\tau-r, \xi-z)(\tau-s)^{-\frac{1}{\alpha}} G_\alpha(\tau-s, \xi-y) \mathrm{d}\tau \mathrm{d}\xi. \end{aligned}$$

Then, we can rewrite the inequality (33) as follows:

$$\left\| D_{r,z} D_{s,y} u_{n+1}(t,x) \right\|_p^2 \leq C_{T,p} \int_s^t \int_{\mathbb{R}} G_\alpha^2(t-\tau, x-\xi) J(\mathrm{d}\tau, \mathrm{d}\xi) + C_{T,p} \int_s^t \int_{\mathbb{R}} G_\alpha^2(t-\tau, x-\xi) \left\| D_{r,z} D_{s,y} u_n(\tau,\xi) \right\|_p^2 \mathrm{d}\xi \mathrm{d}\tau.$$

Notice that $\left\|D_{r,z}D_{s,y}u_1(t,x)\right\|_p^2 = 0$, then we perform $n-1$ iterations to obtain that

$$
\begin{aligned}
&\left\|D_{r,z}D_{s,y}u_{n+1}(t,x)\right\|_p^2 \le C_{T,p}\int_s^t\int_{\mathbb{R}} G_\alpha^2(t-s_1,x-y_1)J(\mathrm{d}s_1,\mathrm{d}y_1) \\
&+\sum_{k=1}^{n-1} C_{T,p}^{k+1}\int_s^t\int_{\mathbb{R}}\int_s^{s_1}\int_{\mathbb{R}}\cdots\int_s^{s_k}\int_{\mathbb{R}} G_\alpha^2(t-s_1,x-y_1)G_\alpha^2(s_1-s_2,y_1-y_2)\cdots \\
&\times G_\alpha^2(s_k-s_{k+1},y_k-y_{k+1})J(\mathrm{d}s_{k+1},\mathrm{d}y_{k+1})\,\mathrm{d}y_k\mathrm{d}s_k\cdots\mathrm{d}y_1\mathrm{d}s_1.
\end{aligned}
$$

Let $K^2(t,x)$ be defined by $K^2(t,x) = K^2_{r,z,s,y}(t,x) := \int_s^t\int_{\mathbb{R}}(t-\tau)^{-\frac{1}{\alpha}}G_\alpha(t-\tau,x-\xi)J(\mathrm{d}\tau,\mathrm{d}\xi)$. Using Lemma A.1 part (1) we can write

$$
\begin{aligned}
&\left\|D_{r,z}D_{s,y}u_{n+1}(t,x)\right\|_p^2 \le C_{T,p}K^2(t,x)+\sum_{k=1}^{n-1} C_{T,p}^{k+1}\int_s^t\int_{\mathbb{R}}\int_s^{s_1}\int_{\mathbb{R}}\cdots\int_s^{s_k}\int_{\mathbb{R}} \\
&\times\prod_{j=0}^{k}\left(s_j-s_{j+1}\right)^{-\frac{1}{\alpha}}G_\alpha(s_j-s_{j+1},y_j-y_{j+1})J(\mathrm{d}s_{k+1},\mathrm{d}y_{k+1})\,\mathrm{d}y_k\mathrm{d}s_k\cdots\mathrm{d}y_1\mathrm{d}s_1,
\end{aligned}
$$

where $s_0 = t$ and $y_0 = x$. Then, apply the semigroup property of the Green kernel to find that

$$
\begin{aligned}
&\left\|D_{r,z}D_{s,y}u_{n+1}(t,x)\right\|_p^2 \le C_{T,p}K^2(t,x)+\sum_{k=1}^{n-1} C_{T,p}^{k+1}\int_s^t\int_s^{s_1}\cdots\int_s^{s_{k-1}}\int_{\mathbb{R}}\int_s^{s_k} \\
&\times\left[(t-s_1)(s_1-s_2)\cdots(s_k-s_{k+1})\right]^{-\frac{1}{\alpha}}G_\alpha(t-s_{k+1},x-y_{k+1})J(\mathrm{d}s_{k+1},\mathrm{d}y_{k+1})\,\mathrm{d}s_k\cdots\mathrm{d}s_1 \\
&= C_{T,p}K^2(t,x)+\sum_{k=1}^{n-1}\left[C_{T,p}^{k+1}\int_{\mathbb{R}}\int_s^t (t-s_{k+1})^{\frac{(\alpha-1)k}{\alpha}}G_\alpha(t-s_{k+1},x-y_{k+1})J(\mathrm{d}s_{k+1},\mathrm{d}y_{k+1})\right. \\
&\left.\times\int_{0<r_k<\cdots<r_2<r_1<1}\left[(1-r_1)(r_1-r_2)\cdots r_k\right]^{-\frac{1}{\alpha}}\mathrm{d}r_k\cdots\mathrm{d}r_1\right] \\
&\le C_{T,p}K^2(t,x)+\sum_{k=1}^{n-1} C_{T,p}^{k+1}\frac{T^{\frac{(\alpha-1)k}{\alpha}}\Gamma\left(1-\frac{1}{\alpha}\right)^{k+1}}{\Gamma\left((k+1)\left(1-\frac{1}{\alpha}\right)\right)}\int_{\mathbb{R}}\int_s^t (t-\tau)^{-\frac{1}{\alpha}}G_\alpha(t-\tau,x-\xi)J(\mathrm{d}\tau,\mathrm{d}\xi) \\
&\le\left(C_{T,p}+\sum_{k=1}^{\infty} C_{T,p}^{k+1}\frac{T^{\frac{(\alpha-1)k}{\alpha}}\Gamma\left(1-\frac{1}{\alpha}\right)^{k+1}}{\Gamma\left((k+1)\left(1-\frac{1}{\alpha}\right)\right)}\right)\int_{\mathbb{R}}\int_s^t (t-\tau)^{-\frac{1}{\alpha}}G_\alpha(t-\tau,x-\xi)J(\mathrm{d}\tau,\mathrm{d}\xi) \\
&\le C_{T,p}K^2(t,x),
\end{aligned}
$$

where in the second inequality, we use the following identity:

$$
\int_{0<r_k<\cdots<r_2<r_1<1}\left[(1-r_1)(r_1-r_2)\cdots r_k\right]^{-\frac{1}{\alpha}}\mathrm{d}r_k\cdots\mathrm{d}r_1 = \frac{\Gamma\left(1-\frac{1}{\alpha}\right)^{k+1}}{\Gamma\left((k+1)\left(1-\frac{1}{\alpha}\right)\right)}. \tag{34}
$$

Hence, we obtain the moment estimate

$$
\sup_{n\in\mathbb{N}}\|D_{r,z}D_{s,y}u_n(t,x)\|_p^2 \le C_{T,p}K^2_{r,z,s,y}(t,x). \tag{35}
$$

In particular, if $\sigma(x) = x$, the second part of (32) vanishes. According to the preceding arguments, we have

$$
\sup_{n\in\mathbb{N}}\|D_{r,z}D_{s,y}u_n(t,x)\|_p \le C_{T,p}(t-s)^{-\frac{1}{2\alpha}}G_\alpha^{\frac{1}{2}}(t-s,x-y)(s-r)^{-\frac{1}{2\alpha}}G_\alpha^{\frac{1}{2}}(s-r,y-z).
$$

Moreover, using Minkowski inequality and Lemma A.2 we derive that

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mathrm{E}\left[\|D^2 u_n(t,x)\|^p_{\mathcal{H}\otimes\mathcal{H}}\right] &\le \sup_{n\in\mathbb{N}} \left(\int_{[0,t]^2}\int_{\mathbb{R}^2} \left\|D_{r,z}D_{s,y}u_n(t,x)\right\|_p^2 \mathrm{d}y\mathrm{d}z\mathrm{d}r\mathrm{d}s\right)^{\frac{p}{2}} \\ &\le C_{T,p}\left(2\int_0^t\int_0^s\int_{\mathbb{R}^2} K^2_{r,z,s,y}(t,x)\mathrm{d}z\mathrm{d}y\mathrm{d}r\mathrm{d}s\right)^{\frac{p}{2}} < \infty. \qquad (36)\end{aligned}$$

Finally, since $u_n(t,x)$ converges to $u(t,x)$ in $L^p(\Omega)$ for $p \ge 2$ (see in the proof of [22, Theorem 2.1]), we deduce that $u(t,x) \in \cap_{p\ge 2}\mathbb{D}^{2,p}$ by [4, Lemma 1.5.3]. Following an approximation argument similar to the proof of Theorem 6.4 in [16], we prove the results. □

4. Negative moments

In this section, we will give estimates for the negative moments of $D_v F$ in this section. The following lemmas play an important role in proving Theorems 1.1-1.3.

Proposition 4.1. (Case 1). *Let $F_{N,1}$ denote the spatial average defined in (7). Then, for any $p \ge 2$,*

$$\sup_{N\ge 1} \mathrm{E}\left[\left|D_{v_{N,1}} F_{N,1}\right|^{-p}\right] < \infty. \qquad (37)$$

Proposition 4.2. (Case 2). *Let $F_{N,2}$ denote the spatial average defined in (8). Then, for any $p \ge 2$,*

$$\sup_{N\ge 1} \mathrm{E}\left[\left|D_{v_{N,2}} F_{N,2}\right|^{-p}\right] < \infty. \qquad (38)$$

Proposition 4.3. (Case 3). *Let $F_{N,3}$ denote the spatial average defined in (9) Then, for any $p \ge 2$, $\gamma > 5p$,*

$$\sup_{N\ge e} \mathrm{E}\left[\left|D_{v_{N,3}} F_{N,3}\right|^{-p}\right] \le C_{t,p,\gamma}(\log N)^{\gamma}. \qquad (39)$$

For the sake of simplicity, we put (3)-(5) together and recast the mild solution in cases 1-3 as

$$\widetilde{u}(t,x) = 1 + \int_{(0,t)\times\mathbb{R}^d} \widetilde{G}(t-s,x-y)\widetilde{u}(s,y)\eta(\mathrm{d}s,\mathrm{d}y) \quad \text{for } t>0 \text{ and } x\in\mathbb{R}^d, \qquad (40)$$

where $\widetilde{u}(t,x)$, $\widetilde{G}(t-s,x-y)$ and η depend on the situation of cases 1-3. Similarly, (7)-(9) can be recast as

$$F_N := \frac{1}{\sigma_N}\left(\int_{[0,N]^d} (\widetilde{u}(t,x)-1)\,\mathrm{d}x\right), \text{ where } \sigma_N^2 := \mathrm{Var}\left(\int_{[0,N]^d}\widetilde{u}(t,x)\mathrm{d}x\right). \qquad (41)$$

Substituting (40) into (41), we have

$$\begin{aligned}F_N &= \frac{1}{\sigma_N}\left(\int_{[0,N]^d}\int_{(0,t)\times\mathbb{R}^d}\widetilde{G}(t-s,x-y)\widetilde{u}(s,y)\eta(\mathrm{d}s,\mathrm{d}y)\mathrm{d}x\right)\\ &= \int_0^t\int_{\mathbb{R}^d}\frac{1}{\sigma_N}\left(\int_{[0,N]^d}\widetilde{G}(t-s,x-y)\widetilde{u}(s,y)\mathrm{d}x\right)\eta(\mathrm{d}s,\mathrm{d}y) = \delta(v_N),\end{aligned}$$

where

$$v_N(s,y) = \mathbf{1}_{[0,t]}(s)\frac{1}{\sigma_N}\int_{[0,N]^d}\widetilde{G}(t-s,x-y)\widetilde{u}(s,y)\mathrm{d}x. \qquad (42)$$

Consider the Malliavin derivative of F_N,

$$D_{s,y}F_N = \frac{1}{\sigma_N}\int_{[0,N]^d} D_{s,y}\widetilde{u}(t,x)\mathrm{d}x.$$

Since $\widetilde{u}(t,x) \geq 0$ and $D_{s,y}\widetilde{u}(t,x) \geq 0$ for almost all $0 < s < t$ and $x, y \in \mathbb{R}^d$ (see [14, Theorem 3.2] for case 2, others can be obtained similarly), together with (42), we have

$$\begin{aligned} D_{v_N}F_N &= \int_0^t \int_{\mathbb{R}^{2d}} v_N(s, y+y')D_{s,y}F_N \mathrm{d}y f(\mathrm{d}y')\mathrm{d}s \\ &= \frac{1}{\sigma_N^2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 \\ &\qquad \times D_{s,y}\widetilde{u}(t,x_1)\widetilde{G}(t-s, x_2-(y+y'))\widetilde{u}(s,y+y') \\ &\geq \frac{1}{\sigma_N^2}\int_{t_\alpha}^t \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 D_{s,y}\widetilde{u}(t,x_1)\widetilde{G}(t-s, x_2-(y+y'))\widetilde{u}(s,y+y'), \end{aligned} \tag{43}$$

where $t_\alpha := t - \varepsilon^\alpha$, for any $0 < \alpha < 1$ and $0 < \varepsilon^\alpha < \frac{t}{2}$. Recall (20), (21) and (26), thanks to a stochastic Fubini argument, we obtain

$$\begin{aligned} &\frac{1}{\sigma_N^2}\int_{t_\alpha}^t \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 D_{s,y}\widetilde{u}(t,x_1)\widetilde{G}(t-s, x_2-(y+y'))\widetilde{u}(s,y+y') \\ =& I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 :=& \frac{1}{\sigma_N^2}\int_{t_\alpha}^t \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 \\ &\qquad \times \widetilde{G}(t-s, x_1-y)\widetilde{G}(t-s, x_2-(y+y'))\widetilde{u}(s,y)\widetilde{u}(s,y+y'), \end{aligned} \tag{44}$$

$$\begin{aligned} I_2 :=& \frac{1}{\sigma_N^2}\int_{[t_\alpha,t]\times\mathbb{R}^d} \eta(\mathrm{d}r,\mathrm{d}z) \int_{t_\alpha}^r \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 \\ &\qquad \times \widetilde{G}(t-r, x_1-z)\widetilde{G}(t-s, x_2-(y+y'))\widetilde{u}(s,y+y')D_{s,y}\widetilde{u}(r,z). \end{aligned} \tag{45}$$

Hence, by Chebyshev's inequality, for any $q \geq 2$, we have

$$\begin{aligned} \mathrm{P}\left(D_{v_N}F_N < \varepsilon\right) &\leq \mathrm{P}\left(I_1 + I_2 < \varepsilon\right) \leq \mathrm{P}\left(I_1 < 2\varepsilon\right) + \mathrm{P}\left(|I_2| > \varepsilon\right) \\ &\leq (2\varepsilon)^q\, \mathrm{E}\left[|I_1|^{-q}\right] + \varepsilon^{-q}\mathrm{E}\left[|I_2|^q\right]. \end{aligned} \tag{46}$$

Now we begin to prove Propositions 4.1-4.3 by estimating $\mathrm{E}\left[|I_1|^{-q}\right]$ and $\mathrm{E}\left[|I_2|^q\right]$.

Proof. [Proof of Proposition 4.1] In case 1, we define

$$\phi_N(s,y) = \int_{[0,N]} G_\alpha(t-s, x-y)\mathrm{d}x.$$

Recall the definition of I_1 in (44), thanks to Jensen's inequality we have

$$\begin{aligned}
\mathrm{E}\left[|I_1|^{-q}\right] &= \mathrm{E}\left[\left|\frac{1}{\sigma_{N,1}^2}\int_{t_\alpha}^t\int_{\mathbb{R}}\phi_N^2(s,y)u^2(s,y)\mathrm{d}y\mathrm{d}s\right|^{-q}\right]\\
&= \left(\int_{t_\alpha}^t M_1(s,N)\mathrm{d}s\right)^{-q}\mathrm{E}\left[\left|\frac{\int_{t_\alpha}^t\int_{\mathbb{R}}\phi_N^2(s,y)u^2(s,y)\mathrm{d}y\mathrm{d}s}{\sigma_{N,1}^2\int_{t_\alpha}^t M_1(s,N)\mathrm{d}s}\right|^{-q}\right]\\
&\le \left(\int_{t_\alpha}^t M_1(s,N)\mathrm{d}s\right)^{-q-1}\frac{1}{\sigma_{N,1}^2}\int_{t_\alpha}^t\int_{\mathbb{R}}\phi_N^2(s,y)\mathrm{E}\left[u^{-2q}(s,y)\right]\mathrm{d}y\mathrm{d}s,
\end{aligned}$$

where

$$M_1(s,N) := \frac{1}{\sigma_{N,1}^2}\int_{\mathbb{R}}\phi_N^2(s,y)\mathrm{d}y. \tag{47}$$

From Lemma 2.2 part (1) and part (3),

$$\sup_{s\in[t/2,t]}\mathrm{E}\left[u^{-2q}(s,y)\right] = \sup_{s\in[t/2,t]}\mathrm{E}\left[u^{-2q}(s,0)\right] \le C_{t,q} < \infty.$$

Hence, we derive that

$$\mathrm{E}\left[|I_1|^{-q}\right] \le C_{t,q}\left(\int_{t_\alpha}^t M_1(s,N)\mathrm{d}s\right)^{-q}. \tag{48}$$

For every real number $N > 0$, define the following functions:

$$I_N(x) := N^{-d}\mathbf{1}_{[0,N]^d}(x), \quad \widetilde{I}_N(x) := I_N(-x) \quad \text{for } x \in \mathbb{R}^d. \tag{49}$$

Then, we obtain that the Fourier transform of $I_N * \widetilde{I}_N$ is $2^{-d}\prod_{j=1}^d \frac{1-\cos(Nz_j)}{(Nz_j)^2}$. Note that in this setting, the functions $I_N * \widetilde{I}_N$ and $G_\alpha(t,\bullet)$ belong to $L^2(\mathbb{R})$. By the semigroup property of G_α and Parseval's identity, we have that for any $N \ge 1$,

$$\begin{aligned}
M_1(s,N) &= \frac{1}{\sigma_{N,1}^2}\int_{[0,N]^2} G_\alpha(2(t-s),x_2-x_1)\mathrm{d}x_1\mathrm{d}x_2\\
&= \frac{N^2}{\sigma_{N,1}^2}\int_{\mathbb{R}}\left(I_N * \widetilde{I}_N\right)(x)G_\alpha(2(t-s),x)\mathrm{d}x\\
&= \frac{N^2}{\pi\sigma_{N,1}^2}\int_{\mathbb{R}}\frac{1-\cos(Nz)}{(Nz)^2}e^{-2(t-s)z^\alpha}\mathrm{d}z\\
&= \frac{N}{\pi\sigma_{N,1}^2}\int_{\mathbb{R}}\frac{1-\cos z}{z^2}\exp\left(-\frac{2(t-s)z^\alpha}{N^\alpha}\right)\mathrm{d}z \qquad (50)\\
&\ge \frac{N}{\pi\sigma_{N,1}^2}\int_{[1,2]}\frac{1-\cos z}{z^2}\exp\left(-\frac{2(t-s)z^\alpha}{N^\alpha}\right)\mathrm{d}z \ge C_t, \qquad (51)
\end{aligned}$$

where we use Lemma 2.5 part (2) in the last inequality. Then, from (48) and (51), we conclude that

$$\mathrm{E}\left[|I_1|^{-q}\right] \le C_{t,q}\left(\int_{t_\alpha}^t \mathrm{d}s\right)^{-q} \le C_{t,q}\varepsilon^{-\alpha q}. \tag{52}$$

Now, we estimate the term $\mathrm{E}[|I_2|^q]$. Recall the definition of I_2 in (45), we apply the BDG inequality and Minkowski inequality to find that

$$\begin{aligned}
\mathrm{E}[|I_2|^q] &\leq C_q \mathrm{E}\left[\left|\int_{t_\alpha}^t \int_{\mathbb{R}} \left(\frac{1}{\sigma_{N,1}^2}\int_{t_\alpha}^r \int_{\mathbb{R}} \phi_N(s,y)\phi_N(r,z)u(s,y)D_{s,y}u(r,z)\mathrm{d}y\mathrm{d}s\right)^2 \mathrm{d}z\mathrm{d}r\right|^{\frac{q}{2}}\right] \\
&= C_q \mathrm{E}\left[\left|\frac{1}{\sigma_{N,1}^4}\int_{t_\alpha}^t \mathrm{d}r \int_{[t_\alpha,r]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_{\mathbb{R}^3} \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z\phi_N(s_1,y_1)\phi_N(s_2,y_2)\phi_N^2(r,z)\right.\right. \\
&\quad \left.\left.\times u(s_1,y_1)u(s_2,y_2)D_{s_1,y_1}u(r,z)D_{s_2,y_2}u(r,z)\right|^{\frac{q}{2}}\right] \\
&\leq C_q \left(\frac{1}{\sigma_{N,1}^4}\int_{t_\alpha}^t \mathrm{d}r \int_{[t_\alpha,r]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_{\mathbb{R}^3} \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z\phi_N(s_1,y_1)\phi_N(s_2,y_2)\phi_N^2(r,z)\right. \\
&\quad \left.\times \left\|u(s_1,y_1)u(s_2,y_2)D_{s_1,y_1}u(r,z)D_{s_2,y_2}u(r,z)\right\|_{\frac{q}{2}}\right)^{\frac{q}{2}} \\
&\leq C_{t,q} \left(\frac{1}{\sigma_{N,1}^4}\int_{t_\alpha}^t \mathrm{d}r \int_{[t_\alpha,r]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_{\mathbb{R}^3} \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z\phi_N(s_1,y_1)\phi_N(s_2,y_2)\phi_N^2(r,z)\right. \\
&\quad \left.\times (r-s_1)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(r-s_1,z-y_1)(r-s_2)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(r-s_2,z-y_2)\right)^{\frac{q}{2}},
\end{aligned} \tag{53}$$

where we use Lemma 2.2 part (1) and part (2) in the last inequality. Notice that $\phi_N(s,y) \leq 1$ by Lemma A.1 part (3). Thanks to Lemma A.1 part (4), we obtain that for any $N \geq 1$,

$$\begin{aligned}
&\frac{1}{\sigma_{N,1}^4}\int_{\mathbb{R}^3} \phi_N^2(r,z)\prod_{i=1,2}\phi_N(s_i,y_i)(r-s_i)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(r-s_i,z-y_i)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z \\
&\leq \frac{1}{\sigma_{N,1}^4}\int_{\mathbb{R}^3} \phi_N^2(r,z)\prod_{i=1,2}(r-s_i)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(r-s_i,z-y_i)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z \\
&\leq \frac{C}{\sigma_{N,1}^4}\int_{\mathbb{R}} \phi_N^2(r,z)\mathrm{d}z = \frac{CN}{\pi\sigma_{N,1}^4}\int_{\mathbb{R}} \frac{1-\cos z}{z^2}\exp\left(-\frac{2(t-r)z^\alpha}{N^\alpha}\right)\mathrm{d}z \leq C_t,
\end{aligned} \tag{54}$$

where the equality holds by (50), and we use $\int_{\mathbb{R}} \frac{1-\cos z}{z^2}\mathrm{d}z = \pi$ and Lemma 2.5 part (2) in the last inequality. Then, substituting (54) into (53), we conclude that

$$\mathrm{E}[|I_2|^q] \leq C_{t,q}\left(\int_{t_\alpha}^t \mathrm{d}r \int_{[t_\alpha,r]^2} \mathrm{d}s_1\mathrm{d}s_2\right)^{\frac{q}{2}} \leq C_{t,q}\varepsilon^{\frac{3\alpha q}{2}}. \tag{55}$$

Choose $\alpha = 4/5$. (46), (52) and (55) together imply that

$$\sup_{N\geq 1} \mathrm{P}\left(D_{v_{N,1}}F_{N,1} < \varepsilon\right) \leq C_{t,q}\varepsilon^{\frac{q}{5}}. \tag{56}$$

Therefore, we finally get

$$\begin{aligned}
\sup_{N\geq 1} \mathrm{E}\left[\left(D_{v_{N,1}}F_{N,1}\right)^{-p}\right] &= \sup_{N\geq 1}\int_0^\infty p\varepsilon^{-p-1}\mathrm{P}\left(D_{v_{N,1}}F_{N,1} < \varepsilon\right)\mathrm{d}\varepsilon \\
&\leq 1 + C_{t,q}p\int_0^1 \varepsilon^{-p-1+q/5}\mathrm{d}\varepsilon < \infty,
\end{aligned} \tag{57}$$

for all $q > 5p$. This proves our result. □

Proof. [Proof of Proposition 4.2] In case 2, recall the definition of I_1 in (44), thanks to Jensen's inequality, we have

$$\begin{aligned}\mathrm{E}\left[|I_1|^{-q}\right] &= \mathrm{E}\Bigg[\Bigg|\frac{1}{\sigma_{N,2}^2}\int_{t_\alpha}^t \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 \\ &\qquad \times p_{t-s}(x_1-y)p_{t-s}(x_2-(y+y'))u(s,y)u(s,y+y')\Bigg|^{-q}\Bigg] \\ &\le \left(\int_{t_\alpha}^t M_2(s,N)\mathrm{d}s\right)^{-q-1} \int_{t_\alpha}^t \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 \\ &\qquad \times \frac{1}{\sigma_{N,2}^2} p_{t-s}(x_1-y)p_{t-s}(x_2-(y+y'))\mathrm{E}\left[u^{-q}(s,y)u^{-q}(s,y+y')\right],\end{aligned}$$

where

$$M_2(s,N) := \frac{1}{\sigma_{N,2}^2}\int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 p_{t-s}(x_1-y)p_{t-s}(x_2-(y+y')). \tag{58}$$

Thanks to Hölder's inequality, Lemma 2.3 part (1) and part (4), we can see that

$$\begin{aligned}\sup_{s\in[t/2,t]} \mathrm{E}\left[u^{-q}(s,y)u^{-q}(s,y+y')\right] &\le \sup_{s\in[t/2,t]} \left[\mathrm{E}\left(u^{-2q}(s,y)\right)\mathrm{E}\left(u^{-2q}(s,y+y')\right)\right]^{\frac{1}{2}} \\ &\le \sup_{s\in[t/2,t]} \mathrm{E}\left[u^{-2q}(s,0)\right] \le C_{t,q} < \infty.\end{aligned} \tag{59}$$

This, together with Lemma A.3 part (1), concludes that

$$\mathrm{E}\left[|I_1|^{-q}\right] \le C_{t,q}\left(\int_{t_\alpha}^t M_2(s,N)\mathrm{d}s\right)^{-q} \le C_{t,q}\left(\int_{t_\alpha}^t \mathrm{d}s\right)^{-q} \le C_{t,q}\varepsilon^{-\alpha q}, \tag{60}$$

for any $N \ge 1$. As for $\mathrm{E}\left[|I_2|^q\right]$, recall (45), using the BDG inequality and Minkowski inequality, we can write

$$\begin{aligned}\mathrm{E}\left[|I_2|^q\right] &\le C_q\Bigg(\int_{t_\alpha}^t \mathrm{d}r \int_{[t_\alpha,r]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_{\mathbb{R}^{6d}} \mathrm{d}y_1 f(\mathrm{d}y_1')\mathrm{d}y_2 f(\mathrm{d}y_2')\mathrm{d}z f(\mathrm{d}z') \int_{[0,N]^{4d}} \mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2' \\ &\qquad \times \frac{1}{\sigma_{N,2}^4} p_{t-r}(x_1-z)p_{t-r}(x_1'-(z+z'))p_{t-s_1}(x_2-(y_1+y_1'))p_{t-s_2}(x_2'-(y_2+y_2')) \\ &\qquad \times \|u(s_1,y_1+y_1')u(s_2,y_2+y_2')D_{s_1,y_1}u(r,z)D_{s_2,y_2}u(r,z+z')\|_{\frac{q}{2}}\Bigg)^{\frac{q}{2}} \\ &\le C_{t,q}\Bigg(\int_{t_\alpha}^t \mathrm{d}r \int_{[t_\alpha,r]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_{\mathbb{R}^{6d}} \mathrm{d}y_1 f(\mathrm{d}y_1')\mathrm{d}y_2 f(\mathrm{d}y_2')\mathrm{d}z f(\mathrm{d}z') \int_{[0,N]^{4d}} \mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2' \\ &\qquad \times \frac{1}{\sigma_{N,2}^4} p_{t-r}(x_1-z)p_{t-r}(x_1'-(z+z'))p_{t-s_1}(x_2-(y_1+y_1'))p_{t-s_2}(x_2'-(y_2+y_2')) \\ &\qquad \times p_{r-s_1}(z-y_1)p_{r-s_2}(z+z'-y_2)\Bigg)^{\frac{q}{2}},\end{aligned} \tag{61}$$

where we use Lemma 2.3 part (2) in the last inequality. Then, we proceed in the following order: integrating in y_1, y_2 and using the semigroup property of the heat kernel; integrating x_2, x_2' on $\mathbb{R}^d$, to obtain that for any $N \geq 1$,

$$\mathrm{E}\left[|I_2|^q\right] \leq C_{t,q}\left(\frac{1}{\sigma_{N,2}^4}\int_{t_\alpha}^t \mathrm{d}r \int_{[t_\alpha,r]^2}\mathrm{d}s_1\mathrm{d}s_2\int_{\mathbb{R}^{4d}} f(\mathrm{d}y_1')f(\mathrm{d}y_2')\mathrm{d}zf(\mathrm{d}z')\int_{[0,N]^{2d}}\mathrm{d}x_1\mathrm{d}x_1' \times p_{t-r}(x_1-z)p_{t-r}(x_1'-(z+z'))\right)^{\frac{q}{2}}$$

$$\leq C_{t,q}\left(\frac{\varepsilon^{2\alpha}(f(\mathbb{R}^d))^2}{\sigma_{N,2}^2}\int_{t_\alpha}^t M_2(r,N)\mathrm{d}r\right)^{\frac{q}{2}} \leq C_{t,q}\varepsilon^{\frac{3\alpha q}{2}}, \tag{62}$$

where we use Lemma A.3 part (2) and Lemma 2.5 part (3) in the last inequality. Choose $\alpha = 4/5$. Similar to the argument in (57), from (46), (60) and (62), we finally prove the result. □

Proof. [Proof of Proposition 4.3] In case 3, we first estimate the term $\mathrm{E}\left[|I_1|^{-q}\right]$. Similar to the proof of Proposition 4.2, we apply Jensen's inequality, Lemma 2.4 part (1) and part (4) to see that

$$\mathrm{E}\left[|I_1|^{-q}\right] \leq C_{t,q}\left(\int_{t_\alpha}^t M_3(s,N)\mathrm{d}s\right)^{-q}, \tag{63}$$

where

$$M_3(s,N) := \frac{1}{\sigma_{N,3}^2}\int_{\mathbb{R}^2}\mathrm{d}yf(\mathrm{d}y')\int_{[0,N]^2}\mathrm{d}x_1\mathrm{d}x_2 p_{\frac{s(t-s)}{t}}(y-\frac{s}{t}x_1)p_{\frac{s(t-s)}{t}}(y+y'-\frac{s}{t}x_2). \tag{64}$$

From Lemma A.4 part (1), we conclude that

$$\mathrm{E}\left[|I_1|^{-q}\right] \leq C_{t,q}(\log N)^q\left(\int_{t_\alpha}^t \mathrm{d}s\right)^{-q} \leq C_{t,q}\varepsilon^{-\alpha q}(\log N)^q, \tag{65}$$

for any $N \geq e$. Next, recall (45), Lemma 2.4 part (1) and part (3). By the BDG inequality and Minkowski inequality,

$$\mathrm{E}[|I_2|^q] \leq C_{t,q}\Bigg(\int_{t_\alpha}^t \mathrm{d}r\int_{[t_\alpha,r]^2}\mathrm{d}s_1\mathrm{d}s_2\int_{\mathbb{R}^6}\mathrm{d}y_1f(\mathrm{d}y_1')\mathrm{d}y_2f(\mathrm{d}y_2')\mathrm{d}zf(\mathrm{d}z')\int_{[0,N]^4}\mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'$$
$$\times\frac{1}{\sigma_{N,3}^4}p_{\frac{r(t-r)}{t}}\left(z-\frac{r}{t}x_1\right)p_{\frac{r(t-r)}{t}}\left(z+z'-\frac{r}{t}x_1'\right)p_{\frac{s_1(t-s_1)}{t}}\left(y_1+y_1'-\frac{s_1}{t}x_2\right)$$
$$\times p_{\frac{s_2(t-s_2)}{t}}\left(y_2+y_2'-\frac{s_2}{t}x_2'\right)p_{\frac{s_1(r-s_1)}{r}}\left(y_1-\frac{s_1}{r}z\right)p_{\frac{s_2(r-s_2)}{r}}\left(y_2-\frac{s_2}{r}(z+z')\right)\Bigg)^{\frac{q}{2}}. \tag{66}$$

Then, we use the same arguments in proving (62) and apply the following identity in integrating x_2, x_2':

$$p_t(\theta x) = \theta^{-d}p_{t/\theta^2}(x), \text{ for all } x \in \mathbb{R} \text{ and } t, \sigma > 0. \tag{67}$$

As a consequence, for any $N \geq e$,

$$\mathrm{E}\left[|I_2|^q\right] \leq C_{t,q}\left(\frac{(f(\mathbb{R}))^2}{\sigma_{N,3}^2}\int_{t_\alpha}^t M_3(r,N)\mathrm{d}r\int_{[t_\alpha,r]^2}\frac{1}{s_1s_2}\mathrm{d}s_1\mathrm{d}s_2\right)^{\frac{q}{2}} \leq C_{t,q}\varepsilon^{\frac{3\alpha q}{2}}, \tag{68}$$

where the last inequality holds by $t_\alpha > t/2$ and Lemma A.4 part (2). Choose $\alpha = 4/5$. Combining (46), (65) and (68), we finally get

$$\mathrm{E}\left[\left(D_{v_{N,3}}F_{N,3}\right)^{-p}\right] \leq 1 + C_{t,q}(\log N)^q p\int_0^1 \varepsilon^{-p-1+q/5}\mathrm{d}\varepsilon \leq C_{t,q}(\log N)^q, \tag{69}$$

for all $q > 5p$ and $N \geq e$. This proves the result. □

5. Proofs of Theorems 1.1-1.3

In this section, we will establish upper bounds for the uniform distance of densities and prove Theorems 1.1-1.3 by analyzing the behavior of $\|D_{v_N}(D_{v_N}F_N)\|_2$.

In cases 1-3, recall (43) that

$$D_{v_N}F_N = \frac{1}{\sigma_N^2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 \times D_{s,y}\widetilde{u}(t,x_1)\widetilde{G}(t-s,x_2-(y+y'))\widetilde{u}(s,y+y'). \tag{70}$$

Applying the Malliavin derivative operator, we have

$$D_{r,z}(D_{v_N}F_N) = \frac{1}{\sigma_N^2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 \widetilde{G}(t-s,x_2-(y+y')) \times \left(D_{s,y}\widetilde{u}(t,x_1)D_{r,z}\widetilde{u}(s,y+y') + \widetilde{u}(s,y+y')D_{r,z}D_{s,y}\widetilde{u}(t,x_1)\right).$$

Recall (42), we obtain

$$D_{v_N}(D_{v_N}F_N) = \frac{1}{\sigma_N^3}\int_0^t \mathrm{d}r \int_r^t \mathrm{d}s \int_{\mathbb{R}^{4d}} \mathrm{d}z f(\mathrm{d}z')\mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{3d}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \times \widetilde{G}(t-s,x_2-(y+y'))\widetilde{G}(t-r,x_3-(z+z'))\widetilde{u}(r,z+z') \times \left(D_{s,y}\widetilde{u}(t,x_1)D_{r,z}\widetilde{u}(s,y+y') + 2\widetilde{u}(s,y+y')D_{r,z}D_{s,y}\widetilde{u}(t,x_1)\right). \tag{71}$$

Now we begin to prove Theorems 1.1-1.3.

Proof. [Proof of Theorem 1.1.] In case 1, according to the proof of [22, Theorem 2.3], it is easy to see that

$$\left\|1 - D_{v_{N,1}}F_{N,1}\right\|_2 \le \frac{C_t}{\sqrt{N}}. \tag{72}$$

Recall Lemma 2.1 and Lemma 2.5 part (1), it remains to estimate the term $\|D_{v_{N,1}}(D_{v_{N,1}}F_{N,1})\|_2$. According to Hölder's inequality, Proposition 3.1, Lemma 2.2 part (1) and part (2), we have

$$\left\|u(r,z)\left(D_{s,y}u(t,x_1)D_{r,z}u(s,y) + 2u(s,y)D_{r,z}D_{s,y}u(t,x_1)\right)\right\|_2 \le C_t(t-s)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(t-s,x_1-y)(s-r)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(s-r,y-z). \tag{73}$$

Hence, recall (71), thanks to Minkowski inequality,

$$\|D_{v_{N,1}}(D_{v_{N,1}}F_{N,1})\|_2 \le \frac{C_t}{\sigma_{N,1}^3}\int_0^t \mathrm{d}r \int_r^t \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}z\mathrm{d}y \int_{[0,N]^3} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \times G_\alpha(t-s,x_2-y)G_\alpha(t-r,x_3-z)(t-s)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(t-s,x_1-y)(s-r)^{-\frac{1}{2\alpha}} G_\alpha^{\frac{1}{2}}(s-r,y-z).$$

Then, we proceed in the following order: integrating x_3 on $\mathbb{R}$ by using Lemma A.1 part (3); integrating x_1 and z on $\mathbb{R}$ by using Lemma A.1 part (4); integrating y on $\mathbb{R}$, to obtain that for any $N \ge 1$,

$$\|D_{v_{N,1}}(D_{v_{N,1}}F_{N,1})\|_2 \le \frac{C_t}{\sigma_{N,1}^3}\int_0^t \mathrm{d}r \int_r^t \mathrm{d}s \int_{[0,N]} \mathrm{d}x_2 \le \frac{C_t}{\sqrt{N}}, \tag{74}$$

where the last inequality holds by Lemma 2.5 part (2). This proves Theorem 1.1. □

Proof. [Proof of Theorem 1.2.] In case 2, from the proof of [14, Theorem 2.5], we have

$$\left\|1 - D_{v_{N,2}} F_{N,2}\right\|_2 \le \frac{C_t}{(\sqrt{N})^d}. \tag{75}$$

Thanks to Lemma 2.1 and Lemma 2.5 part (1), we estimate the term $\|D_{v_{N,2}}(D_{v_{N,2}} F_{N,2})\|_2$ in the following. According to Hölder's inequality, Lemma 2.3 part (2) and part (3), we have

$$\begin{aligned}&\left\|u(r, z+z')\left(D_{s,y} u(t, x_1) D_{r,z} u(s, y+y') + 2u(s, y+y') D_{r,z} D_{s,y} u(t, x_1)\right)\right\|_2 \\ &\quad \le C_t p_{t-s}(x_1 - y) p_{s-r}(y + y' - z) + C_t p_{t-s}(x_1 - y) p_{s-r}(y - z).\end{aligned} \tag{76}$$

Then, recall (71), we have

$$\|D_{v_{N,2}}(D_{v_{N,2}} F_{N,2})\|_2 \le \Phi_{N,1} + \Phi_{N,2}, \tag{77}$$

where

$$\begin{aligned}\Phi_{N,1} = &\frac{C_t}{\sigma_{N,2}^3} \int_0^t \mathrm{d}r \int_r^t \mathrm{d}s \int_{\mathbb{R}^{4d}} \mathrm{d}z f(\mathrm{d}z') \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{3d}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\ &\times p_{t-s}(x_2 - (y+y')) p_{t-r}(x_3 - (z+z')) p_{t-s}(x_1 - y) p_{s-r}(y + y' - z), \\ \Phi_{N,2} = &\frac{C_t}{\sigma_{N,2}^3} \int_0^t \mathrm{d}r \int_r^t \mathrm{d}s \int_{\mathbb{R}^{4d}} \mathrm{d}z f(\mathrm{d}z') \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{3d}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\ &\times p_{t-s}(x_2 - (y+y')) p_{t-r}(x_3 - (z+z')) p_{t-s}(x_1 - y) p_{s-r}(y - z).\end{aligned}$$

Next, for both $\Phi_{N,1}$ and $\Phi_{N,2}$, we first integrate in z and use the semigroup property of the heat kernel, then integrate x_3 on $\mathbb{R}^d$ to obtain that for $i = 1, 2$,

$$\begin{aligned}\Phi_{N,i} &\le \frac{C_t}{\sigma_{N,2}^3} \int_0^t \mathrm{d}r \int_r^t \mathrm{d}s \int_{\mathbb{R}^{3d}} f(\mathrm{d}z') \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 p_{t-s}(x_2 - (y+y')) p_{t-s}(x_1 - y) \\ &= \frac{C_t f(\mathbb{R}^d)}{\sigma_{N,2}} \int_0^t \mathrm{d}r \int_r^t M_2(s, N) \mathrm{d}s \le \frac{C_t}{(\sqrt{N})^d},\end{aligned} \tag{78}$$

where $M_2(s, N)$ is defined in (58), and we use Lemma A.3 part (2) and Lemma 2.5 part (3) in the last inequality. Hence, recall (77), we finish the proof. □

Proof. [Proof of Theorem 1.3.]In case 3, it follows from the proof of [2, Theorem 1.3] that

$$\left\|1 - D_{v_{N,3}} F_{N,3}\right\|_2 \le \frac{C_t \sqrt{\log N}}{\sqrt{N}}. \tag{79}$$

Since Lemma 2.4 part (1) and part (3) hold, we repeat the computation in the proof of Theorem 1.2 to find that

$$\|D_{v_{N,3}}(D_{v_{N,3}} F_{N,3})\|_2 \le \Psi_{N,1} + \Psi_{N,2}, \tag{80}$$

where

$$\begin{aligned}\Psi_{N,1} = &\frac{C_t}{\sigma_{N,3}^3} \int_0^t \mathrm{d}r \int_r^t \mathrm{d}s \int_{\mathbb{R}^4} \mathrm{d}z f(\mathrm{d}z') \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^3} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 p_{\frac{s(t-s)}{t}}\left(y + y' - \frac{s}{t} x_2\right) \\ &\times p_{\frac{r(t-r)}{t}}\left(z + z' - \frac{r}{t} x_3\right) p_{\frac{s(t-s)}{t}}\left(y - \frac{s}{t} x_1\right) p_{\frac{r(s-r)}{s}}\left(z - \frac{r}{s}(y + y')\right), \\ \Psi_{N,2} = &\frac{C_t}{\sigma_{N,3}^3} \int_0^t \mathrm{d}r \int_r^t \mathrm{d}s \int_{\mathbb{R}^4} \mathrm{d}z f(\mathrm{d}z') \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^3} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 p_{\frac{s(t-s)}{t}}\left(y + y' - \frac{s}{t} x_2\right) \\ &\times p_{\frac{r(t-r)}{t}}\left(z + z' - \frac{r}{t} x_3\right) p_{\frac{s(t-s)}{t}}\left(y - \frac{s}{t} x_1\right) p_{\frac{r(s-r)}{s}}\left(z - \frac{r}{s} y\right).\end{aligned}$$

Since $1/s$ is not integrable on $(0,t)$, we can not apply the same argument as Theorem 1.2 in the following estimate. Therefore, we first integrate x_1 and x_2 on $\mathbb{R}$ for $\Psi_{N,1}$ and $\Psi_{N,2}$, respectively, by using (67). Then for both $\Psi_{N,1}$ and $\Psi_{N,2}$, owing to the semigroup property and (67), we integrate in the variables y and z to obtain that for $i = 1,2$,

$$\begin{aligned}
\Psi_{N,i} &\le \frac{C_t}{\sigma_{N,3}^3}\int_0^t \mathrm{d}r \int_r^t \frac{1}{s}\mathrm{d}s \int_{\mathbb{R}^2} f(\mathrm{d}z')f(\mathrm{d}y') \int_{[0,N]^2} \mathrm{d}x_1\mathrm{d}x_2 \\
&\quad \times p_{[(r(s-r)/s)(s^2/r^2)+s(t-s)/t](r^2/s^2)+r(t-r)/t}\left(z' - \frac{r}{t}(x_2-x_1)\right) \\
&= \frac{C_t f(\mathbb{R})}{\sigma_{N,3}^3}\int_0^t \mathrm{d}r \int_r^t \frac{1}{s}\mathrm{d}s \int_{\mathbb{R}} f(\mathrm{d}z') \int_{[0,N]^2} \mathrm{d}x_1\mathrm{d}x_2 p_{\frac{2r(t-r)}{t}}\left(z' - \frac{r}{t}(x_2-x_1)\right) \\
&= \frac{C_t N^2 f(\mathbb{R})}{\sigma_{N,3}^3}\int_0^t \mathrm{d}r \int_r^t \frac{1}{s}\mathrm{d}s \int_{\mathbb{R}} \mathrm{d}x \left(I_N * \widetilde{I}_N\right)(x)\left(p_{2r(t-r)/t} * f\right)\left(\frac{r}{t}x\right) \\
&\le \frac{C_t N (f(\mathbb{R}))^2}{\sigma_{N,3}^3}\int_0^t \frac{1}{r}\mathrm{d}r \int_r^t \frac{1}{s}\mathrm{d}s \int_{\mathbb{R}} \frac{1-\cos z}{z^2} e^{-\frac{t(t-r)z^2}{rN^2}}\mathrm{d}z, \qquad (81)
\end{aligned}$$

where the last inequality holds by the proof of Lemma A.4. Then, integrating in the variable s and making a change of variable $\sigma = (t-r)/r$ yields

$$\begin{aligned}
\Psi_{N,i} &\le \frac{C_t N}{\sigma_{N,3}^3}\int_{\mathbb{R}} \frac{1-\cos z}{z^2}\int_0^\infty \frac{\log(1+\sigma)}{1+\sigma} e^{-\frac{t\sigma z^2}{N^2}}\mathrm{d}\sigma \mathrm{d}z \\
&= \frac{C_t N}{2\sigma_{N,3}^3}\int_{\mathbb{R}} \frac{1-\cos z}{z^2}\left(\int_0^\infty e^{-\frac{t\sigma z^2}{N^2}}\mathrm{d}(\log(1+\sigma))^2\right)\mathrm{d}z \\
&= \frac{C_t N}{2\sigma_{N,3}^3}\int_{\mathbb{R}} \frac{1-\cos z}{z^2}\left(\int_0^\infty \frac{tz^2}{N^2} e^{-\frac{t\sigma z^2}{N^2}}(\log(1+\sigma))^2\mathrm{d}\sigma\right)\mathrm{d}z,
\end{aligned}$$

where we integrate by parts in the second equality. Next, we make a change of variable $\theta = tz^2\sigma/N^2$ to obtain that

$$\begin{aligned}
\Psi_{N,i} &\le \frac{C_t N}{\sigma_{N,3}^3}\int_{\mathbb{R}} \frac{1-\cos z}{z^2}\left(\int_0^\infty e^{-\theta}\left(\log\left(1+\frac{N^2\theta}{tz^2}\right)\right)^2\mathrm{d}\theta\right)\mathrm{d}z \\
&\le \frac{C_t N(\log N)^2}{\sigma_{N,3}^3} \le \frac{C_t\sqrt{\log N}}{\sqrt{N}}, \qquad (82)
\end{aligned}$$

where we use Lemma A.5 in the second inequality, and the last inequality holds by Lemma 2.5 part (4). Finally, recall Lemma 2.1, we combine (79), (80), Proposition 4.3 and Lemma 2.5 part (1) to finish the proof. □

Remark 5.1. *Unlike the proof of nonlinear case [24, Proof of Theorem 1.1], when dealing with (71), we apply moment inequalities of Malliavin derivative of the solutions directly rather than using the expansion of the Malliavin derivative as in (20) and estimating the stochastic integral. The similar technique can also be used to simplify computations in proving Theorem 1.2 in [24].*

6. Appendix

First, We introduce some properties of the Green kernel $G_\alpha(t,x)$ that could be found in [22].

Lemma A.1. *Let $G_\alpha(t,x)$ denote the Green kernel defined in* (3). *Then,*
(1) $G_\alpha^2(t,x) \le Ct^{-\frac{1}{\alpha}} G_\alpha(t,x)$, *for all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$.
(2) (Semigroup property.) $\int_{\mathbb{R}} G_\alpha(t,z)G_\alpha(s,x-z)\mathrm{d}z = G_\alpha(t+s,x)$, *for* $t,s>0$ *and* $x \in \mathbb{R}$.
(3) $\int_{\mathbb{R}} G_\alpha(t,x)\mathrm{d}x = 1$, *for every* $t>0$.
(4) $\int_{\mathbb{R}} G_\alpha^{\frac{1}{2}}(t,x)\mathrm{d}x = Ct^{\frac{1}{2\alpha}}$, *for every* $t>0$.

Lemma A.2. *Let $K^2_{r,z,s,y}(t,x)$ be defined in (28), then for any fixed* $0<r<s<t$, *we have*

$$\int_0^t \int_0^s \int_{\mathbb{R}^2} K^2_{r,z,s,y}(t,x)\mathrm{d}z\mathrm{d}y\mathrm{d}r\mathrm{d}s \le C_t$$

Proof.

$$\begin{aligned}
&\int_0^t \int_0^s \int_{\mathbb{R}^2} K^2_{r,z,s,y}(t,x)\mathrm{d}z\mathrm{d}y\mathrm{d}r\mathrm{d}s \\
\le &\int_0^t \int_0^s \int_{\mathbb{R}^2} (t-s)^{-\frac{1}{\alpha}} G_\alpha(t-s,x-y)(s-r)^{-\frac{1}{\alpha}} G_\alpha(s-r,y-z)\mathrm{d}z\mathrm{d}y\mathrm{d}r\mathrm{d}s \\
&+ \int_0^t \int_0^s \int_s^t \int_{\mathbb{R}^3} [(t-\theta)(\theta-r)(\theta-s)]^{-\frac{1}{\alpha}} \\
&\quad \times G_\alpha(t-\theta,x-w)G_\alpha(\theta-r,w-z)G_\alpha(\theta-s,w-y)\mathrm{d}z\mathrm{d}y\mathrm{d}w\mathrm{d}\theta\mathrm{d}r\mathrm{d}s \\
= &\int_0^t \int_0^s [(t-s)(s-r)]^{-\frac{1}{\alpha}}\, \mathrm{d}r\mathrm{d}s + \int_0^t \int_0^s \int_s^t [(t-\theta)(\theta-r)(\theta-s)]^{-\frac{1}{\alpha}}\, \mathrm{d}\theta\mathrm{d}r\mathrm{d}s \\
\le &C_t < \infty,
\end{aligned}$$

where we integrate y, z, w in order and use Lemma A.1 part (2) and part(3) in the equality. □

Lemma A.3. *Let $M_2(s,N)$ be defined in (58). Then,*
(1) $M_2(s,N) \ge C_t$, *for all* $t/2<s<t$, $N \ge 1$.
(2) $M_2(s,N) \le C_t$, *for all* $0<s<t$, $N \ge 1$.

Proof. Recall the definition of I_N, $\widetilde{I_N}$ in (49),

$$\begin{aligned}
M_2(s,N) &= \frac{1}{\sigma^2_{N,2}} \int_{\mathbb{R}^2} \mathrm{d}y f(\mathrm{d}y') \int_{[0,N]^{2d}} \mathrm{d}x_1\mathrm{d}x_2 p_{t-s}(x_1-y)p_{t-s}(x_2-(y+y')) \\
&= \frac{N^{2d}}{\pi^d \sigma^2_{N,2}} \int_{\mathbb{R}^d} \mathrm{d}x \left(I_N * \widetilde{I_N}\right)(x) \left(p_{2(t-s)} * f\right)(x) \\
&= \frac{N^{2d}}{\pi^d \sigma^2_{N,2}} \int_{\mathbb{R}^d} \prod_{j=1}^d \frac{1-\cos(Nz_j)}{(Nz_j)^2} e^{-(t-s)\|z\|^2} \hat{f}(\mathrm{d}z) \\
&= \frac{N^{d}}{\pi^d \sigma^2_{N,2}} \int_{\mathbb{R}^d} \prod_{j=1}^d \frac{1-\cos z_j}{z_j^2} e^{-\frac{(t-s)\|z\|^2}{N^2}} \hat{f}(\mathrm{d}z).
\end{aligned}$$

(1). $f(\mathbb{R}^d) < \infty$ implies that $\hat{f}$ is a bounded and continuous function, then for $t/2<s<t$, choose $0<a<b<1$ such that $\inf_{z\in[a,b]} \hat{f}(z) > 0$, we have

$$M_2(s,N) \ge \frac{N^d}{\pi^d \sigma^2_{N,2}} \int_{[a,b]^d} \prod_{j=1}^d \frac{1-\cos z}{z^2} e^{-\frac{(t-s)\|z\|^2}{N^2}} \hat{f}(z)\mathrm{d}z \ge C_t,$$

where we use Lemma 2.5 part (3) in the last inequality.
(2). Notice that $\hat{f}(x) \le \hat{f}(0) = f(\mathbb{R}^d)$, then for all $0 < s < t$,

$$M_2(s,N) \le \frac{N^d \pi^d f(\mathbb{R}^d)}{\pi^d \sigma_{N,2}^2} \le C_t.$$

□

Lemma A.4. *Let $M_3(s,N)$ be defined in (64). Then,*
(1) $M_3(s,N) \ge \frac{C_t}{\log N}$, *for all $t/2 < s < t$, $N \ge e$.*
(2) $M_3(s,N) \le \frac{C_t}{s \log N}$, *for all $0 < s < t$, $N \ge e$.*

Proof. According to the proof of Lemma A.3, we have

$$\begin{aligned} M_3(s,N) &= \frac{N^2}{\sigma_{N,3}^2} \int_{\mathbb{R}} \mathrm{d}x \left(I_N * \widetilde{I}_N\right)(x) \left(p_{2s(t-s)/t} * f\right)\left(\frac{s}{t}x\right) \\ &= \frac{N^2}{\pi \sigma_{N,3}^2} \int_{\mathbb{R}} \frac{1-\cos(Nzs/t)}{(Nzs/t)^2} e^{-s(t-s)\|z\|^2} \hat{f}(\mathrm{d}z) \\ &= \frac{Nt}{s\pi\sigma_{N,2}^2} \int_{\mathbb{R}} \frac{1-\cos z}{z^2} e^{-\frac{t(t-s)\|z\|^2}{sN^2}} \hat{f}\left(\frac{tz}{s}\right) \mathrm{d}z. \end{aligned}$$

Then thanks to Lemma 2.5 part (4), by a similar argument in proving Lemma A.3, we prove the result. □

Lemma A.5. *For any $N \ge e$,*

$$\int_{\mathbb{R}} \frac{1-\cos z}{z^2} \left(\int_0^\infty e^{-\theta} \left(\log\left(1 + \frac{N^2\theta}{tz^2}\right)\right)^2 \mathrm{d}\theta \right) \mathrm{d}z \le C_t (\log N)^2.$$

Proof. Since

$$1 + \frac{N^2\theta}{tz^2} \le N^2 + N^2\theta \cdot \frac{1}{t} \cdot \frac{1}{z^2} \le N^2(\theta+1)\left(\frac{1}{t}+1\right)\left(\frac{1}{z^2}+1\right),$$

then, we have

$$\log\left(1 + \frac{N^2\theta}{tz^2}\right) \le \left(2\log N + \log\left(\frac{1}{t}+1\right)\right) \cdot \left(1 + \log(\theta+1) + \log\left(\frac{1}{z^2}+1\right)\right).$$

Notice that

$$\int_{\mathbb{R}} \frac{1-\cos z}{z^2} \left(\int_0^\infty e^{-\theta} \left(1 + \log(\theta+1) + \log\left(\frac{1}{z^2}+1\right)\right)^2 \mathrm{d}\theta \right) \mathrm{d}z < \infty,$$

we finish the proof. □

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