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Law of large numbers and central limit theorem for independent and non-identical distributed random variables under convex expectations dominated by sub-linear expectations

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Abstract. Motivated by some interesting problems in mathematical economics, quantum mechanics and finance, non-linear expectations have been used to describe the phenomena which have the stochastic characteristic of non-additivity. In this paper, we study two limit theorems for random variables under convex expectations, which are dominated by sub-linear expectations. Firstly, a central limit theorem (Theorem 3.1) is proved for independent and non-identical distributed random variables with only finite second order moments. Secondly, a law of large numbers (Theorem 4.1) is proved for independent and non-identical distributed random variables with only finite first order moments. These results include and extend some existing results. Furthermore, we give an example for the application of Theorem 4.1.

1. Introduction

The law of large numbers (LLN for short) and central limit theorem (CLT for short) as fundamental limit theorems in probability theory play a fruitful role in the development of probability theory and its applications. However, these kinds of limit theorems have always considered additive probabilities and additive expectations. In fact, the additivity of probabilities and expectations has been abandoned in some areas because many uncertain phenomena cannot be well modelled by using additive probabilities and additive expectations. In objective setting, non-additive probabilities have been used especially in Quantum Mechanics. Indeed, as a consequence of the famous wave-particle duality, the probabilities that describe quantum phenomena are generally non-additive, even though a frequentist interpretation is usually attached to them (see, e.g., [11]). In subjective setting, non-additive probabilities have been used because additivity prevents an effective analysis of the degree of confidence that decision makers have in their probability assessments.

Since the paper [1] on coherent risk measures, people are more and more interested in sub-linear expectations (or more generally, convex expectations, see [7, 12–14]). By [25], we know that a sub-linear expectation $\hat{E}$ can be represented as the upper expectation of a subset of linear expectations $\{E_\theta : \theta \in \Theta\}$, i.e., $\hat{E}[\cdot] = \sup_{\theta\in\Theta} E_\theta[\cdot]$. In most cases, this subset is often treated as an uncertain model of probabilities $\{P_\theta : \theta \in \Theta\}$

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and the notion of sub-linear expectation provides a robust way to measure a risk loss X. In fact, the non-linear expectation theory provides many rich, flexible and elegant tools. Up to now, many researchers have been working on this subject and related properties of the theory of non-linear expectation due to the connection of this subject with stochastic analysis, stochastic control, mathematical finance, partial differential equations, mathematical economics, and systems science (see, e.g., [8–10, 19, 20, 22, 27–31, 33]).

Motivated by modeling uncertainty in finance, Peng [23–26] initiated the notion of independent and identically distributed (IID for short) random variables and the definitions of G-normal distribution and maximal distribution under sub-linear expectations. He further obtained a new CLT and a new LLN under sub-linear expectations. Chen [3] firstly obtained a strong LLN in this framework. Later, the limit theorems such as LLN, CLT and the laws of the iterated logarithm (LIL for short) under sub-linear expectations have been studied by many researchers (see, e.g., [4, 16, 17, 21, 34–36]). In [18], the author firstly studied LLN and CLT under convex expectations. He needs the following assumptions: (i) Random variables are IID under a convex expectation. (ii) The convex expectation is dominated by a sub-linear expectation. (iii) Random variables are IID, and their all order moments are finite under the sub-linear expectation. In this paper, we also study LLN and CLT under convex expectations. Our results extend those that [18] yielded. We only need the following assumptions: (i) Random variables are independent under a convex expectation. (ii) The convex expectation is dominated by a sub-linear expectation. (iii) Random variables are independent, and only their first order moments are finite for LLN, only their second order moments are finite for CLT under the sub-linear expectation.

This paper is organized as follows: in Sec. 2, we recall some useful notions and propositions under sub-linear expectations and convex expectations. In Sec. 3, we give one of our main results: central limit theorem (Theorem 3.1). In Sec. 4, we give another main results: law of large numbers (Theorem 4.1) and the application of Theorem 4.1 (Example 4.7).

2. Preliminaries

In this section, we present some preliminaries in the theory of sub-linear expectations and convex expectations. For more details, we can see [18, 25, 26, 34].

Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$. We suppose that $\mathcal{H}$ satisfies $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. The space $\mathcal{H}$ can be considered as the space of random variables.

Definition 2.1. *[25, 34] A sub-linear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E}$: $\mathcal{H} \mapsto \overline{\mathbb{R}} := [-\infty, +\infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) ***Monotonicity:*** *If $X \geq Y$, then $\hat{E}[X] \geq \hat{E}[Y]$;*

(b) ***Constant preserving:*** *$\hat{E}[c] = c$, $\forall c \in \mathbb{R}$;*

(c) ***Sub-additivity:*** *$\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$, whenever $\hat{E}[X] + \hat{E}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;*

(d) ***Positive homogeneity:*** *$\hat{E}[\lambda X] = \lambda \hat{E}[X]$, $\forall \lambda \geq 0$.*

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sub-linear expectation space.

In this paper, we consider the following sub-linear expectation space $(\Omega, \mathcal{H}, \hat{E})$: if $X_1, \cdots, X_n \in \mathcal{H}$, then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l.Lip}(\mathbb{R}^n)$, where $C_{l.Lip}(\mathbb{R}^n)$ denotes the linear space of functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ. Let $C_{b.Lip}(\mathbb{R}^n)$ denote the linear space of bounded functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C|x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

for some $C > 0$ depending on φ.

Definition 2.2. *[25] **Identical distribution:** Let X_1 and X_2 be two n-dimensional random vectors defined in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$, respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if*

$$\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}^n),$$

whenever the sub-linear expectations are finite.

Independence: *In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is called independent to another random vector $X := (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}$, if for each test function $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$, we have*

$$\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}],$$

whenever $\overline{\varphi}(x) := \hat{E}[|\varphi(x, Y)|] < \infty$ for all x and $\hat{E}[\overline{\varphi}(X)] < \infty$.

IID random variables: *A sequence of random sequence $\{X_i\}_{i=1}^{\infty} \subset \mathcal{H}$ is called IID random variables, if $X_i \overset{d}{=} X_1$ and X_{i+1} is independent to $Y := (X_1, \cdots, X_i)$ for each $i = 1, 2, \cdots$.*

Definition 2.3. (G-normal distribution) *[25] A random variable X in a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{E})$ with $\overline{\sigma}^2 = \hat{E}[X^2]$, $\underline{\sigma}^2 = -\hat{E}[-X^2]$ is called G-normal distributed, denoted by $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$, if for each $Y \in \mathcal{H}$ which is independent to X such that $Y \overset{d}{=} X$, it holds that $aX + bY \overset{d}{=} \sqrt{a^2 + b^2}X$, $\forall a, b \geq 0$.*

Remark 2.4. [25] Let $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$ under $\hat{E}$. For each $\varphi \in C_{l.Lip}(\mathbb{R})$, we define a function

$$v(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)], \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$

Then v is the unique viscosity solution of the following parabolic partial differential equation (PDE for short):

$$\partial_t v - G(\partial_{xx}^2 v) = 0, \quad v(0, x) = \varphi(x),$$

where $G(\alpha) := \frac{1}{2}\hat{E}[\alpha X^2] = \frac{1}{2}(\overline{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-)$.

Definition 2.5. (Maximal distribution) *[25] A random variable η in a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called maximal distributed if*

$$\hat{E}[\varphi(\eta)] = \sup_{\underline{\mu} \leq y \leq \overline{\mu}} \varphi(y), \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}),$$

where $\overline{\mu} := \hat{E}[\eta]$ and $\underline{\mu} := -\hat{E}[-\eta]$.

Remark 2.6. [25] Let η be maximal distributed with $\overline{\mu} := \hat{E}[\eta]$, $\underline{\mu} := -\hat{E}[-\eta]$, the distribution of η is characterized by the following PDE:

$$\partial_t u - g(\partial_x u) = 0, \quad u(0, x) = \varphi(x),$$

where $u(t, x) := \hat{E}[\varphi(x + t\eta)]$, $(t, x) \in [0, \infty) \times \mathbb{R}$, $g(x) := \overline{\mu}x^+ - \underline{\mu}x^-$.

Definition 2.7. *[26] Let Q be a subset of $[0, \infty) \times \mathbb{R}$, we denote by $C(Q)$ all continuous functions V defined on Q, in the relative topology on Q, with a finite norm*

$$\|V\|_{C(Q)} = \sup_{(t,x) \in Q} |V(t, x)|.$$

Given $\alpha, \beta \in (0, 1)$, let $C^{\alpha,\beta}(Q)$ be the set of function in $C(Q)$ such that following norm is finite:

$$\|V\|_{C^{\alpha,\beta}(Q)} = \|V\|_{C(Q)} + \sup_{(t,x),(s,y) \in Q, (t,x) \neq (s,y)} \frac{|V(s, x) - V(t, y)|}{|s - t|^{\alpha} + |x - y|^{\beta}}.$$

We also introduce the norms

$$\|V\|_{C^{1+\alpha,1+\beta}(Q)} = \|V\|_{C^{\alpha,\beta}(Q)} + \|\partial_t V\|_{C^{\alpha,\beta}(Q)} + \|\partial_x V\|_{C^{\alpha,\beta}(Q)},$$
$$\|V\|_{C^{1+\alpha,2+\beta}(Q)} = \|V\|_{C^{1+\alpha,1+\beta}(Q)} + \left\|\partial^2_{xx} V\right\|_{C^{\alpha,\beta}(Q)}.$$

The corresponding subspaces of $C(Q)$ in which the correspondent derivatives exist and the above norms are finite are denoted respectively by $C^{1+\alpha,1+\beta}(Q)$, $C^{1+\alpha,2+\beta}(Q)$.

Definition 2.8. *[26] A convex expectation $\widetilde{E}$ on $\mathcal{H}$ is a functional $\widetilde{E} : \mathcal{H} \mapsto \overline{\mathbb{R}}$ satisfying (a) and (b) of Definition 2.1 and the following property: for all $X,\ Y \in \mathcal{H}$, we have*

*(e) **Convexity:** $\widetilde{E}[\lambda X + (1-\lambda)Y] \le \lambda\widetilde{E}[X] + (1-\lambda)\widetilde{E}[Y]$, $\forall \lambda \in [0,1]$, whenever $\lambda\widetilde{E}[X] + (1-\lambda)\widetilde{E}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$.*

The triple $(\Omega, \mathcal{H}, \widetilde{E})$ is called a convex expectation space.

Remark 2.9. The definitions of identical distribution and independence of random variables under convex expectations are similar to those of Definition 2.2 under sub-linear expectations and so we omit them.

Definition 2.10. *[26] Let $\widetilde{E}$ be a convex expectation and $\hat{E}$ be a sub-linear expectation on $(\Omega, \mathcal{H})$. $\widetilde{E}$ is said to be dominated by $\hat{E}$ if*

$$\widetilde{E}[X] - \widetilde{E}[Y] \le \hat{E}[X - Y], \quad \forall\, X,\ Y \in \mathcal{H}. \tag{1}$$

Proposition 2.11. *[18] Let $\widetilde{E}$ be a convex expectation on $(\Omega, \mathcal{H})$, which is dominated by a sub-linear expectation $\hat{E}$ in the sense of (2.1). If $\hat{E}[X] = -\hat{E}[-X]$, then we have $\widetilde{E}[X] = \hat{E}[X]$ and $\widetilde{E}[X + Y] = \widetilde{E}[X] + \widetilde{E}[Y]$ for all $Y \in \mathcal{H}$.*

Proposition 2.12. *[18] Suppose that $(\Omega, \mathcal{H}, \widetilde{E})$ is a convex expectation space. Let $X \in \mathcal{H}$, then*

(1) There exist two constant $\overline{\sigma}^2$, $\underline{\sigma}^2$ satisfying $\overline{\sigma}^2 \ge \underline{\sigma}^2 \ge 0$ such that

$$\lim_{\delta\downarrow 0} \delta^{-1}\widetilde{E}\left[\delta X^2\right] = \overline{\sigma}^2,\ \ \lim_{\delta\downarrow 0} \delta^{-1}\left(-\widetilde{E}\left[-\delta X^2\right]\right) = \underline{\sigma}^2;$$

(2) There exist two constant $\overline{\mu}$, $\underline{\mu}$ satisfying $\overline{\mu} \ge \underline{\mu}$ such that

$$\lim_{\delta\downarrow 0} \delta^{-1}\widetilde{E}\left[\delta X\right] = \overline{\mu},\ \ \lim_{\delta\downarrow 0} \delta^{-1}\left(-\widetilde{E}\left[-\delta X\right]\right) = \underline{\mu};$$

(3) If a is bounded, then $\delta^{-1}\widetilde{E}\left[\frac{\delta}{2}aX^2\right]$ converges uniformly to $G(a) := \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$;

(4) If a is bounded, then $\delta^{-1}\widetilde{E}\left[\delta a X\right]$ converges uniformly to $g(a) := \overline{\mu}a^+ - \underline{\mu}a^-$.

Lemma 2.13 (**Hölder's inequality**) *[26] Let X, Y be two random variables in sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, then for $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\hat{E}[|XY|] \le (\hat{E}[|X|^p])^{\frac{1}{p}} \cdot (\hat{E}[|Y|^q])^{\frac{1}{q}}.$$

3. Central Limit Theorem

In this section, we present a CLT for independent random variables under convex expectations dominated by sub-linear expectations.

Theorem 3.1. *Suppose that* $(\Omega, \mathcal{H}, \hat{E})$ *is a sub-linear expectation space,* $(\Omega, \mathcal{H}, \widetilde{E})$ *is a convex expectation space, and* $\widetilde{E}$ *is dominated by* $\hat{E}$ *in the sense of (1). Let* $\{X_i\}_{i=1}^{\infty} \subset \mathcal{H}$ *be a sequence of random variables which satisfies the following conditions:*

(i) *Each* X_{i+1} *is independent to* $(X_1, \cdots, X_i)$ *under* $\widetilde{E}$ *and* $\hat{E}$*, for* $i = 1, 2, \cdots$*;*

(ii) $\hat{E}[X_i] = -\hat{E}[-X_i] = 0$*, for* $i = 1, 2, \cdots$*;*

(iii) *Denote* $\overline{\sigma}_i^2 := \lim_{\delta \downarrow 0} \delta^{-1} \widetilde{E}\left[\delta X_i^2\right]$, $\underline{\sigma}_i^2 := \lim_{\delta \downarrow 0} \delta^{-1} \left(-\widetilde{E}\left[-\delta X_i^2\right]\right)$*, there exist two positive constants* $\underline{\sigma}$ *and* $\overline{\sigma}$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left|\underline{\sigma}_i^2 - \underline{\sigma}^2\right| = 0, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left|\overline{\sigma}_i^2 - \overline{\sigma}^2\right| = 0;$$

(iv) *Denote* $G_i(a) := \lim_{\delta \downarrow 0} \delta^{-1} \widetilde{E}\left[\frac{\delta}{2} a X_i^2\right] = \frac{1}{2}\left(\overline{\sigma}_i^2 a^+ - \underline{\sigma}_i^2 a^-\right)$,

$$\lim_{\delta \downarrow 0} \sup_{i \geq 1} \left| \delta^{-1} \widetilde{E}\left[\frac{\delta}{2} a X_i^2\right] - G_i(a) \right| = 0;$$

(v) $\lim_{c\to\infty} \sup_{i\geq 1} \hat{E}\left[(X_i^2 - c)^+\right] = 0$*;*

(vi) $\sup_{i\geq 1} \hat{E}\left[X_i^2\right] < \infty$*. Then for any continuous function* φ *satisfying* $|\varphi(x)| \leq C(1 + x^2)$*, we have*

$$\lim_{n\to\infty} \widetilde{E}\left[\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right] = \hat{E}[\varphi(\xi)], \tag{2}$$

where $S_n = \sum_{i=1}^{n} X_i$, $\xi \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$ *under* $\hat{E}$*. Furthermore, if* $p > 2$ *and* $\sup_{i\geq 1} \hat{E}[|X_i|^p] < \infty$*, then* (2) *holds for any continuous function* φ *satisfying* $|\varphi(x)| \leq C(1 + |x|^p)$*.*

Proof. The main idea of our proof comes from the proofs of Theorem 2.4.11 in [18] and Theorem 3.5 in [35]. Let $Y_i = (-\sqrt{i}) \vee (X_i \wedge \sqrt{i})$, $T_n = \sum_{i=1}^{n} Y_i$. In order to prove Theorem 3.1, we need the following facts:

(A1) Suppose that the condition (v) is satisfied, then

$$\frac{\sum_{i=1}^{n} \hat{E}[|X_i - Y_i|]}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

(A2) Suppose that the conditions (v) and (vi) are satisfied, then

$$\frac{\sum_{i=1}^{n} \hat{E}[|Y_i|^{\alpha+2}]}{n^{\frac{\alpha}{2}+1}} \to 0 \quad \text{as} \quad n \to \infty, \quad \forall 0 < \alpha < 1.$$

(A3) Suppose that the conditions (i), (ii), (v) and (vi) are satisfied, then

$$\hat{E}[|T_n|^p] \leq C_p n^{\frac{p}{2}}, \quad \forall p \geq 2.$$

For (A1), note that

$$\sqrt{n}\hat{E}[|X_n - Y_n|] \leq \hat{E}[(X_n^2 - n)^+] \leq \sup_{i\geq 1} \hat{E}[(X_i^2 - n)^+].$$

So (A1) holds.

For (A2), note that

$$\begin{aligned}\hat{E}[|Y_n|^{\alpha+2}] &\le \hat{E}[X_n^2|Y_n|^\alpha] \le \hat{E}[(X_n^2-c+c)|Y_n|^\alpha]\\ &\le n^{\frac{\alpha}{2}}\hat{E}[(X_n^2-c)^+]+c\hat{E}[|Y_n|^\alpha]\\ &\le n^{\frac{\alpha}{2}}\hat{E}[(X_n^2-c)^+]+c\hat{E}[|X_n|^\alpha]\\ &\le n^{\frac{\alpha}{2}}\hat{E}[(X_n^2-c)^+]+c\left(\hat{E}[X_n^2]\right)^{\frac{\alpha}{2}}\\ &\le n^{\frac{\alpha}{2}}\cdot\sup_{i\ge1}\hat{E}[(X_i^2-c)^+]+c\left(\sup_{i\ge1}\hat{E}[X_i^2]\right)^{\frac{\alpha}{2}}\end{aligned}$$

for any $c>1$, where the fourth line of the right side is obtained by Hölder's inequality (Lemma 2.13). So (A2) is true.

For (A3), by the Rosenthal's inequality (2.4) in [34] and (A1), we have

$$\begin{aligned}\hat{E}[|T_n|^p] &\le C_p\sum_{i=1}^n\hat{E}[|Y_i|^p]+C_p\left(\sum_{i=1}^n\hat{E}[Y_i^2]\right)^{\frac{p}{2}}+C_p\left(\sum_{i=1}^n\left[(\hat{E}[Y_i])^++(\hat{E}[-Y_i])^+\right]\right)^p\\ &\le C_pn^{\frac{p}{2}-1}\sum_{i=1}^n\hat{E}[X_i^2]+C_p\left(\sum_{i=1}^n\hat{E}[X_i^2]\right)^{\frac{p}{2}}+C_p\left(\sum_{i=1}^n2\hat{E}[|X_i-Y_i|]\right)^p\\ &\le C_pn^{\frac{p}{2}-1}\cdot n\cdot\sup_{i\ge1}\hat{E}[X_i^2]+C_p\left(n\cdot\sup_{i\ge1}\hat{E}[X_i^2]\right)^{\frac{p}{2}}+C_p\left(\sum_{i=1}^n2\hat{E}[|X_i-Y_i|]\right)^p\\ &\le C_pn^{\frac{p}{2}}.\end{aligned}$$

So (A3) is true.

Now, for a small but fixed $h>0$, let V be the unique viscosity solution of the following equation:

$$\partial_tV+G(\partial_{xx}^2V)=0,\quad (t,x)\in[0,1+h]\times\mathbb{R},\quad V|_{t=1+h}=\varphi(x),\tag{3}$$

where $G(a):=\frac{1}{2}(\overline{\sigma}^2a^+-\underline{\sigma}^2a^-)$ and $\varphi\in C_{b.Lip}(\mathbb{R})$. According to the definition of G-normal distribution, we have

$$V(t,x)=\hat{E}\left[\varphi\left(x+\sqrt{1+h-t}\xi\right)\right],\quad V(h,0)=\hat{E}[\varphi(\xi)],\quad V(1+h,x)=\varphi(x).\tag{4}$$

Since (3) is a uniformly parabolic PDE, by the interior regularity of V (see [32]), we have

$$\|V\|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,1]\times\mathbb{R})}<\infty,\ \text{ for some }\alpha\in(0,1).\tag{5}$$

First, let $\delta=\frac{1}{n}$, we show that

$$\lim_{n\to\infty}\widetilde{E}\left[V\left(1,\sqrt{\delta}T_n\right)\right]=V(0,0).\tag{6}$$

Let $T_0=0$, then

$$\begin{aligned}V(1,\sqrt{\delta}T_n)-V(0,0)&=\sum_{i=0}^{n-1}\left\{V((i+1)\delta,\sqrt{\delta}T_{i+1})-V(i\delta,\sqrt{\delta}T_i)\right\}\\ &=\sum_{i=0}^{n-1}\left\{\left[V((i+1)\delta,\sqrt{\delta}T_{i+1})-V(i\delta,\sqrt{\delta}T_{i+1})\right]\right.\\ &\quad\left.+\left[V(i\delta,\sqrt{\delta}T_{i+1})-V(i\delta,\sqrt{\delta}T_i)\right]\right\}\\ &=\sum_{i=0}^{n-1}\left\{I_\delta^i+J_\delta^i\right\},\end{aligned}$$

with, by Taylor's expansion,

$$\begin{aligned}
J_\delta^i &= \partial_t V(i\delta, \sqrt{\delta}T_i)\delta + \frac{1}{2}\partial_{xx}^2 V(i\delta, \sqrt{\delta}T_i)Y_{i+1}^2\delta + \partial_x V(i\delta, \sqrt{\delta}T_i)Y_{i+1}\sqrt{\delta} \\
&= \left(\partial_t V(i\delta, \sqrt{\delta}T_i)\delta + \frac{1}{2}\partial_{xx}^2 V(i\delta, \sqrt{\delta}T_i)X_{i+1}^2\delta + \partial_x V(i\delta, \sqrt{\delta}T_i)X_{i+1}\sqrt{\delta}\right) \\
&\quad + \left(\frac{1}{2}\partial_{xx}^2 V(i\delta, \sqrt{\delta}T_i)(Y_{i+1}^2 - X_{i+1}^2)\delta + \partial_x V(i\delta, \sqrt{\delta}T_i)(Y_{i+1} - X_{i+1})\sqrt{\delta}\right) \\
&= J_{\delta,1}^i + J_{\delta,2}^i,
\end{aligned}$$

$$\begin{aligned}
I_\delta^i &= \int_0^1 \left[\partial_t V((i+\beta)\delta, \sqrt{\delta}T_{i+1}) - \partial_t V(i\delta, \sqrt{\delta}T_{i+1})\right] d\beta\delta \\
&\quad + \left[\partial_t V(i\delta, \sqrt{\delta}T_{i+1}) - \partial_t V(i\delta, \sqrt{\delta}T_i)\right]\delta \\
&\quad + \int_0^1 \int_0^1 \left[\partial_{xx}^2 V(i\delta, \sqrt{\delta}T_i + \gamma\beta Y_{i+1}\sqrt{\delta}) - \partial_{xx}^2 V(i\delta, \sqrt{\delta}T_i)\right]\gamma d\beta d\gamma Y_{i+1}^2\delta.
\end{aligned}$$

It follows by Proposition 2.11 that

$$\widetilde{E}[V(1, \sqrt{\delta}T_n) - V(0,0)] = \widetilde{E}[V(1, \sqrt{\delta}T_n)] - V(0,0) = \widetilde{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^i + \sum_{i=0}^{n-1} J_{\delta,2}^i + \sum_{i=0}^{n-1} I_\delta^i\right].$$

From the fact that $\widetilde{E}$ is dominated by $\hat{E}$, we have

$$\begin{aligned}
&\widetilde{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^i\right] - \left(\sum_{i=0}^{n-1} \hat{E}[|J_{\delta,2}^i|] + \sum_{i=0}^{n-1} \hat{E}[|I_\delta^i|]\right) \\
&\leq \widetilde{E}[V(1, \sqrt{\delta}T_n)] - V(0,0) \leq \widetilde{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^i\right] + \left(\sum_{i=0}^{n-1} \hat{E}[|J_{\delta,2}^i|] + \sum_{i=0}^{n-1} \hat{E}[|I_\delta^i|]\right).
\end{aligned} \tag{7}$$

For $J_{\delta,1}^i$, from the conditions (i) and (ii) we have

$$\hat{E}\left[\partial_x V(i\delta, \sqrt{\delta}T_i)X_{i+1}\sqrt{\delta}\right] = \hat{E}\left[-\partial_x V(i\delta, \sqrt{\delta}T_i)X_{i+1}\sqrt{\delta}\right] = 0.$$

It yields that

$$\hat{E}\left[\sum_{i=0}^{n-1}\left(\partial_x V(i\delta, \sqrt{\delta}T_i)X_{i+1}\sqrt{\delta}\right)\right] = \hat{E}\left[-\sum_{i=0}^{n-1}\left(\partial_x V(i\delta, \sqrt{\delta}T_i)X_{i+1}\sqrt{\delta}\right)\right] = 0.$$

Hence from Proposition 2.11, we have

$$\widetilde{E}\left[\sum_{i=0}^{n-1}\left(\partial_x V(i\delta, \sqrt{\delta}T_i)X_{i+1}\sqrt{\delta}\right)\right] = 0. \tag{8}$$

Denote $G_i(a) := \lim_{\delta\downarrow 0}\delta^{-1}\widetilde{E}[\frac{\delta}{2}aX_i^2] = \frac{1}{2}(\overline{\sigma}_i^2 a^+ - \underline{\sigma}_i^2 a^-)$, $A_i := \partial_t V(i\delta, \sqrt{\delta}T_i)\delta + \frac{1}{2}\partial_{xx}^2 V(i\delta, \sqrt{\delta}T_i)X_{i+1}^2\delta$ and $B_i := \partial_{xx}^2 V(i\delta, \sqrt{\delta}T_i)$. By Definition 2.10, Propositions 2.11 and 2.12, and combining (8) with (3), (4) as well as the condition (i) and (iv), it follows that

$$\begin{aligned}
\widetilde{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^{i}\right] &= \widetilde{E}\left[\sum_{i=0}^{n-1} A_i\right] \\
&= \widetilde{E}\left[\sum_{i=0}^{n-2} A_i + \partial_t V\left((n-1)\delta,\ \sqrt{\delta}T_{n-1}\right)\delta + \frac{1}{2}\partial_{xx}^2 V\left((n-1)\delta,\ \sqrt{\delta}T_{n-1}\right) X_n^2\delta\right] \\
&= \widetilde{E}\left[\sum_{i=0}^{n-2} A_i - \delta G\left(B_{n-1}\right) + \delta G_n\left(B_{n-1}\right) + o(1)\delta\right] \\
&\le \widetilde{E}\left[\sum_{i=0}^{n-2} A_i\right] + \hat{E}\left[-\delta G(B_{n-1}) + \delta G_n(B_{n-1})\right] + o(1)\delta \\
&= \widetilde{E}\left[\sum_{i=0}^{n-2} A_i\right] + \frac{\delta}{2}\hat{E}\left[(B_{n-1})^+(\overline{\sigma}_n^2 - \overline{\sigma}^2) - (B_{n-1})^-(\underline{\sigma}_n^2 - \underline{\sigma}^2)\right] + o(1)\delta \\
&\le \widetilde{E}\left[\sum_{i=0}^{n-2} A_i\right] + \frac{\delta}{2}\hat{E}\left[|B_{n-1}|\right]\left(|\overline{\sigma}_n^2 - \overline{\sigma}^2| + |\underline{\sigma}_n^2 - \underline{\sigma}^2|\right) + o(1)\delta \\
&\le \cdots \\
&\le \frac{\delta}{2}\sum_{i=0}^{n-1}\hat{E}\left[|B_i|\right]\left(|\overline{\sigma}_{i+1}^2 - \overline{\sigma}^2| + |\underline{\sigma}_{i+1}^2 - \underline{\sigma}^2|\right) + o(1).
\end{aligned}$$

For B_i, by Definition 2.7 and (5), we have

$$\hat{E}\left[|B_i|\right] = \hat{E}\left[\left|\partial_{xx}^2 V\left(i\delta,\ \sqrt{\delta}T_i\right)\right|\right] \le C.$$

And by the condition (iii), it follows that

$$\widetilde{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^{i}\right] \le C\frac{1}{n}\sum_{i=0}^{n-1}\left(\left|\overline{\sigma}_{i+1}^2 - \overline{\sigma}^2\right| + \left|\underline{\sigma}_{i+1}^2 - \underline{\sigma}^2\right|\right) + o(1) \to 0 \quad \text{as} \quad n \to \infty.$$

In a similar manner as above, we also have

$$\widetilde{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^{i}\right] \ge -C\frac{1}{n}\sum_{i=0}^{n-1}\left(\left|\overline{\sigma}_{i+1}^2 - \overline{\sigma}^2\right| + \left|\underline{\sigma}_{i+1}^2 - \underline{\sigma}^2\right|\right) + o(1) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\lim_{n\to\infty}\widetilde{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^{i}\right] = 0. \tag{9}$$

For $J_{\delta,2}^{i}$, by Definition 2.7 and (5) again, we have

$$\hat{E}\left[\left|\partial_x V(i\delta,\ \sqrt{\delta}T_i)\right|\right] \le C.$$

By the conditions (i), (v), (A1) and Stolz theorem,

$$\begin{aligned}
&\sum_{i=0}^{n-1}\hat{E}\left[|J_{\delta,2}^{i}|\right]\\
&\leq \sum_{i=0}^{n-1}\left\{\frac{1}{2}\hat{E}\left[\left|\partial_{xx}^{2}V(i\delta,\sqrt{\delta}T_i)\right|\right]\hat{E}\left[\left|X_{i+1}^{2}-Y_{i+1}^{2}\right|\right]\delta+\hat{E}\left[\left|\partial_{x}V(i\delta,\sqrt{\delta}T_i)\right|\right]\hat{E}\left[|X_{i+1}-Y_{i+1}|\right]\sqrt{\delta}\right\}\\
&\leq C\frac{1}{n}\sum_{i=0}^{n-1}\hat{E}\left[\left(X_{i+1}^{2}-(i+1)\right)^{+}\right]+C\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\hat{E}\left[|X_{i+1}-Y_{i+1}|\right]\to 0 \quad\text{as}\quad n\to\infty.
\end{aligned}$$

Thus

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\hat{E}\left[|J_{\delta,2}^{i}|\right]=0. \tag{10}$$

For I_{δ}^{i}, since both $\partial_t V$ and $\partial_{xx}^2 V$ are uniformly $\frac{\alpha}{2}$-hölder continuous in t and α-hölder continuous in x on $[0,1]\times\mathbb{R}$, then we have

$$|I_{\delta}^{i}|\leq C\delta^{1+\frac{\alpha}{2}}(1+|Y_{i+1}|^{\alpha}+|Y_{i+1}|^{2+\alpha}).$$

From (A2), we have

$$\sum_{i=0}^{n-1}\hat{E}\left[|I_{\delta}^{i}|\right]\leq C\left(\frac{1}{n}\right)^{1+\frac{\alpha}{2}}\sum_{i=0}^{n-1}\left(1+\hat{E}[|Y_{i+1}|^{\alpha}]+\hat{E}[|Y_{i+1}|^{2+\alpha}]\right)\to 0 \quad\text{as}\quad n\to\infty.$$

Thus

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\hat{E}\left[|I_{\delta}^{i}|\right]=0. \tag{11}$$

Then combining (7), (9), (10) with (11), (6) holds.

Additionally, it is obvious that if $\varphi\in C_{b.Lip}(\mathbb{R})$, i.e., $|\varphi(x)-\varphi(y)|\leq C|x-y|$, then for each $t,s\in[0,1+h]$ and $x\in\mathbb{R}$,

$$|V(t,x)-V(s,x)|\leq C\hat{E}[|\xi|]\sqrt{|t-s|}\leq C\sqrt{|t-s|}. \tag{12}$$

In particular,

$$|V(0,0)-V(h,0)|\leq C\sqrt{h}. \tag{13}$$

By Definition 2.10, it is easy to obtain that for any $X,Y\in\mathcal{H}$, $|\widetilde{E}[X]-\widetilde{E}[Y]|\leq\hat{E}[|X-Y|]$. And then combining (4), (12), with (13), we have

$$\begin{aligned}
&\left|\widetilde{E}\left[\varphi\left(\sqrt{\delta}T_n\right)\right]-\hat{E}[\varphi(\xi)]\right|=\left|\widetilde{E}\left[V\left(1+h,\sqrt{\delta}T_n\right)\right]-V(h,0)\right|\\
&\leq\left|\widetilde{E}\left[V\left(1+h,\sqrt{\delta}T_n\right)\right]-\widetilde{E}\left[V\left(1,\sqrt{\delta}T_n\right)\right]\right|\\
&+\left|\widetilde{E}\left[V\left(1,\sqrt{\delta}T_n\right)\right]-V(0,0)\right|+|V(0,0)-V(h,0)|\\
&\leq\hat{E}\left[\left|V\left(1+h,\sqrt{\delta}T_n\right)-V\left(1,\sqrt{\delta}T_n\right)\right|\right]\\
&+\left|\widetilde{E}\left[V\left(1,\sqrt{\delta}T_n\right)\right]-V(0,0)\right|+|V(0,0)-V(h,0)|\\
&\leq 2C\sqrt{h}+\left|\widetilde{E}\left[V\left(1,\sqrt{\delta}T_n\right)\right]-V(0,0)\right|.
\end{aligned}$$

From (6), we obtain

$$\limsup_{n\to\infty}\left|\widetilde{E}\left[\varphi\left(\sqrt{\delta}T_n\right)\right]-\hat{E}[\varphi(\xi)]\right|\le 2C\sqrt{h},$$

so

$$\lim_{n\to\infty}\widetilde{E}\left[\varphi\left(\frac{T_n}{\sqrt{n}}\right)\right]=\hat{E}[\varphi(\xi)].$$

By the Lipschitz continuity of φ, Definition 2.10 and (A1), we have

$$\left|\widetilde{E}\left[\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right]-\widetilde{E}\left[\varphi\left(\frac{T_n}{\sqrt{n}}\right)\right]\right|\le \hat{E}\left[\left|\varphi\left(\frac{S_n}{\sqrt{n}}\right)-\varphi\left(\frac{T_n}{\sqrt{n}}\right)\right|\right]\le C\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\hat{E}\left[|X_i-Y_i|\right]\to 0 \text{ as } n\to\infty.$$

Thus

$$\lim_{n\to\infty}\widetilde{E}\left[\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right]=\hat{E}[\varphi(\xi)],\quad \forall\varphi\in C_{b.Lip}(\mathbb{R}).$$

The rest of the proof is very similar to that of Theorem 3.5 in [35] and so it is omitted. □

Remark 3.2. In the proof of Theorem 3.1, we mainly use the sub-additivity of sub-linear expectation $\hat{E}$, the convexity of convex expectation $\widetilde{E}$ and the assumption that convex expectation $\widetilde{E}$ is dominated by sub-linear expectation $\hat{E}$ in the sense of (1).

4. Law of Large Numbers

In this section, we give a LLN for independent random variables under convex expectations dominated by sub-linear expectations (Theorem 4.1). Furthermore, an example for the application of Theorem 4.1 (Example 4.7) is presented.

Theorem 4.1. *Suppose that $(\Omega,\mathcal{H},\hat{E})$ is a sub-linear expectation space, $(\Omega,\mathcal{H},\widetilde{E})$ is a convex expectation space, and $\widetilde{E}$ is dominated by $\hat{E}$ in the sense of (1). Let $\{X_i\}_{i=1}^{\infty}\subset\mathcal{H}$ be a sequence of random variables which satisfies the following conditions:*

(i) *Each X_{i+1} is independent to $(X_1,\cdots,X_i)$ under $\widetilde{E}$ and $\hat{E}$, for $i=1,2,\cdots$;*

(ii) *Denote $\overline{\mu}_i:=\lim_{\delta\downarrow 0}\delta^{-1}\widetilde{E}[\delta X_i]$, $\underline{\mu}_i:=\lim_{\delta\downarrow 0}\delta^{-1}(-\widetilde{E}[-\delta X_i])$, there exist two constants $\underline{\mu}$ and $\overline{\mu}$ such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left|\underline{\mu}_i-\underline{\mu}\right|=0,\quad \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left|\overline{\mu}_i-\overline{\mu}\right|=0;$$

(iii) *Denote $g_i(a):=\lim_{\delta\downarrow 0}\delta^{-1}\widetilde{E}[\delta aX_i]=\overline{\mu}_i a^+-\underline{\mu}_i a^-$,*

$$\lim_{\delta\downarrow 0}\sup_{i\ge 1}\left|\delta^{-1}\widetilde{E}\left[\delta aX_i\right]-g_i(a)\right|=0;$$

(iv) $\lim_{d\to\infty}\sup_{i\ge 1}\hat{E}[(|X_i|-d)^+]=0$;

(v) $\sup_{i\ge 1}\hat{E}[|X_i|]<\infty$. *Then for any continuous function φ satisfying $|\varphi(x)|\le C(1+|x|)$, we have*

$$\lim_{n\to\infty}\widetilde{E}\left[\varphi\left(\frac{S_n}{n}\right)\right]=\hat{E}[\varphi(\eta)],\tag{14}$$

where $S_n=\sum_{i=1}^{n}X_i$, η is maximal distributed under $\hat{E}$ with $\overline{\mu}=\hat{E}[\eta]$, $\underline{\mu}=-\hat{E}[-\eta]$. Furthermore, if $p>1$ and $\sup_{i\ge 1}\hat{E}\left[|X_i|^p\right]<\infty$, then (14) holds for any continuous function φ satisfying $|\varphi(x)|\le C(1+|x|^p)$.

Combining the proof method of Theorem 3.1 with that of Theorem 2.4.13 in [18], we can easily prove Theorem 4.1 and so it is omitted.

Remark 4.2. The result of Theorem 4.1 is very interesting. The limit distribution under convex expectation $\widetilde{E}$ dominated by sub-linear expectation $\hat{E}$ in the sense of (1) is $\hat{E}[\varphi(\eta)]$, where φ is a continuous function satisfying $|\varphi(x)| \le C(1+|x|)$. Next, we provide an example to illustrate application of Theorem 4.1.

In order to investigate the following example (Example 4.7), we need the following notations, notions and lemmas.

Denote
$L^2(\Omega, \mathcal{F}, P) := \{\xi : \xi$ is $\mathbb{R}$ valued and $\mathcal{F}$-measurable random variable such that $E[|\xi|^2] < \infty\}$,
$S^2(\Omega, (\mathcal{F}_t)_{t\ge0}, P; \mathbb{R}) := \{V : V_t$ is $\mathbb{R}$ valued and $\mathcal{F}_t$-adapted process such that $E[\sup_{t\ge0} |V_t|^2] < \infty\}$,
$L^2(\Omega, (\mathcal{F}_t)_{t\ge0}, P; \mathbb{R}) := \{V : V_t$ is $\mathbb{R}$ valued and $\mathcal{F}_t$-adapted process such that $E[(\int_0^\infty |V_s|^2 ds)] < \infty\}$.
Consider the following 1-dimensional infinite time interval backward stochastic differential equation (BSDE for short):

$$Y_t = \xi + \int_t^\infty g(s, Y_s, Z_s) ds - \int_t^\infty Z_s dB_s, \quad t \ge 0. \tag{15}$$

Let

$$g : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$$

such that for any $(y, z) \in \mathbb{R}\times\mathbb{R}$, $g(\cdot, y, z)$ is $\mathcal{F}_t$ -progressively measurable. We make the following assumptions:
(B.1) $E[(\int_0^\infty |g(t, 0, 0)|dt)^2] < \infty$;
(B.2) There exist two positive non-random functions α_t and β_t, such that for all $y_1, y_2,\ z_1, z_2 \in \mathbb{R}$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \le \beta_t|y_1 - y_2| + \alpha_t|z_1 - z_2|,$$

where α_t and β_t satisfy that $\int_0^\infty \alpha_t^2 dt < \infty$ and $\int_0^\infty \beta(t)dt < \infty$;
(B.3) $g(\cdot, y, 0) \equiv 0,\ \forall y \in \mathbb{R}$.

Lemma 4.3. *[6] Let $\xi \in L^2(\Omega, \mathcal{F}, P)$ be given. Suppose that (B.1) and (B.2) hold for g, then BSDE (15) has a unique solution $(Y, Z) \in S^2(\Omega, (\mathcal{F}_t)_{t\ge0}, P; \mathbb{R}) \times L^2(\Omega, (\mathcal{F}_t)_{t\ge0}, P; \mathbb{R})$.*

Definition 4.4. *[6] Suppose that g satisfies (B.2) and (B.3). For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, let (Y, Z) be the solution of BSDE (15). Consider the mapping $\mathcal{E}_g[\cdot] : L^2(\Omega, \mathcal{F}, P) \mapsto \mathbb{R}$ denoted by $\mathcal{E}_g[\xi] := Y_0$. We call $\mathcal{E}_g[\xi]$ g-expectation of ξ.*

Definition 4.5. *[6] Suppose that g satisfies (B.2) and (B.3). Conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by $\mathcal{E}_g[\xi|\mathcal{F}_t] := Y_t$.*

From [6], we know that g-expectation has the following property: $\mathcal{E}_g[\xi|\mathcal{F}_t]$ is the unique random variable η in $L^2(\Omega, \mathcal{F}_t, P)$ such that

$$\mathcal{E}_g[1_A\xi] = \mathcal{E}_g[1_A\eta], \quad \forall A \in \mathcal{F}_t.$$

Now we consider the following three BSDEs:

$$Y_t^1 = \xi + \int_t^\infty \alpha_s|Z_s^1| ds - \int_t^\infty Z_s^1 dB_s, \quad t \ge 0,$$

$$Y_t^2 = \xi - \int_t^\infty \alpha_s|Z_s^2| ds - \int_t^\infty Z_s^2 dB_s, \quad t \ge 0$$

and

$$Y_t^3 = \xi + \int_t^{\infty} \tilde{g}(s, Y_s^3, Z_s^3)\mathrm{d}s - \int_t^{\infty} Z_s^3 \mathrm{d}B_s, \quad t \geq 0.$$

For notational simplification, we shall write in the sequel $\mathcal{E}^{\alpha}[\cdot|\mathcal{F}_t] \equiv \mathcal{E}_g[\cdot|\mathcal{F}_t]$ for $g = \alpha_t|z|$ and $\mathcal{E}^{-\alpha}[\cdot|\mathcal{F}_t] \equiv \mathcal{E}_g[\cdot|\mathcal{F}_t]$ for $g = -\alpha_t|z|$.

Lemma 4.6. *Suppose that $\tilde{g}$ satisfies (B.2) and (B.3), and is convex with respect to y and z, then $\mathcal{E}_{\tilde{g}}[\cdot]$ is a convex expectation.*

Lemma 4.6 is the direct consequence of Proposition 4.3 in [15].

Example 4.7. Let $(\Omega, \mathcal{F}, P)$ be a completed probability space, $(B_t)_{t\geq 0}$ be a 1-dimensional standard Brownian motion defined on this space and $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by Brownian motion $(B_t)_{t\geq 0}$, that is,

$$\mathcal{F}_t := \sigma(B_s; s \leq t) \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets. Furthermore, we assume $\mathcal{F} := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$.

Consider the following family of probability measures:

$$\mathbb{P} := \left\{ Q^v : \frac{\mathrm{d}Q^v}{\mathrm{d}P} = \mathrm{e}^{-\frac{1}{2}\int_0^{\infty} |v_s|^2 \mathrm{d}s + \int_0^{\infty} v_s \mathrm{d}B_s}, \ \sup_{t\geq 0} |v_t| \leq \alpha_t \right\},$$

where α_t is a positive non-random function satisfying $\int_0^{\infty} \alpha_t^2 dt < \infty$. Denote $\overline{\mathcal{E}}[\cdot] := \sup_{Q^v \in \mathbb{P}} E_{Q^v}[\cdot]$. Obviously, $\overline{\mathcal{E}}[\cdot]$ is a sub-linear expectation. Suppose that $\tilde{g}(t, y, z)$ is deterministic, convex with respect to y and z, i.e., for any $y_1, y_2, z_1, z_2 \in \mathbb{R}$, $\alpha \in [0, 1]$,

$$\tilde{g}(t, \alpha y_1 + (1-\alpha)y_2, \alpha z_1 + (1-\alpha)z_2) \leq \alpha\tilde{g}(t, y_1, z_1) + (1-\alpha)\tilde{g}(t, y_2, z_2), \quad \mathrm{d}P \times \mathrm{d}t - \text{a.s.},$$

and satisfies the conditions (B.2), (B.3). By Lemma 4.6, we know that $\mathcal{E}_{\tilde{g}}[\cdot]$ is a convex expectation.

Let $X_i := a_i(B_i - B_{i-1})$, $i = 1, 2, \cdots$, where $a_i = 1/i^2$, $i = 1, 2, \cdots$, then for any continuous function φ satisfying $|\varphi(x)| \leq C(1 + |x|)$, we have

$$\lim_{n\to\infty} \mathcal{E}_{\tilde{g}}\left[\varphi\left(\frac{\sum_{i=1}^n X_i}{n}\right)\right] = \overline{\mathcal{E}}[\varphi(\eta)], \tag{16}$$

where η is maximal distributed under $\overline{\mathcal{E}}$ with $\overline{\mu} = \overline{\mathcal{E}}[\eta]$, $\underline{\mu} = -\overline{\mathcal{E}}[-\eta]$.

In the following, we give some lemmas for proving Example 4.7.

Lemma 4.8. *For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, we have*

$$\mathcal{E}^{\alpha}[\xi] = \overline{\mathcal{E}}[\xi], \ \ \mathcal{E}^{-\alpha}[\xi] = -\overline{\mathcal{E}}[-\xi].$$

The proof of Lemma 4.8 is very similar to that of Lemma 2 in [5]. So we omit it.

Lemma 4.9 (**Comparison Theorem**) *[6] Let $\xi \in L^2(\Omega, \mathcal{F}, P)$ be given, and $g(t, y, z)$ satisfy (B.1) and (B.2). Suppose that (Y, Z) be the solution of (15) and $(\overline{Y}, \overline{Z})$ be the solution of the following BSDE:*

$$\overline{Y}_t = \overline{\xi} + \int_t^{\infty} \overline{g}(s, \overline{Y}_s, \overline{Z}_s)\mathrm{d}s - \int_t^{\infty} \overline{Z}_s \mathrm{d}B_s, \quad t \geq 0,$$

where $\overline{g}(t,y,z)$ *satisfies (B.1) and (B.2),* $\overline{\xi} \in L^2(\Omega,\mathcal{F},P)$. *If*

$$\hat{\xi} := \xi - \overline{\xi} \geq 0, \quad \hat{g}_t := g(t,\overline{Y}_t,\overline{Z}_t) - \overline{g}(t,\overline{Y}_t,\overline{Z}_t) \geq 0, \ a.s.,$$

then we have $Y_t \geq \overline{Y}_t$, *a.s.,* $\forall t \in [0,\infty)$.

Lemma 4.10. *Suppose that g is deterministic, and satisfies (B.2) and (B.3). For any* $\xi \in L^2(\Omega,\mathcal{F},P)$, *we have* $\mathcal{E}_g[\xi|\mathcal{F}_t] = \mathcal{E}_g[\xi]$ *as soon as* ξ *is independent of* $\mathcal{F}_t$.

The proof of Lemma 4.10 is very similar to that of Proposition 3.1 in [2]. So we omit it.

Lemma 4.11. *Let* $\xi \in L^2(\Omega,\mathcal{F},P)$ *be given, and* $g(t,y,z)$ *satisfy (B.2) and (B.3). Then there exists a constant* $C > 0$ *such that*

$$|\mathcal{E}_g[\xi]| \leq C\left(E\left[|\xi|^2\right]\right)^{\frac{1}{2}}.$$

Lemma 4.11 is the direct consequence of Remark 4.1 in [37].

Proof of Example 4.7. By Lemmas 4.8 and 4.9, we know that $\mathcal{E}_{\tilde{g}}[\cdot] \leq \overline{\mathcal{E}}[\cdot]$. Hence we only need to check that $\{X_i\}_{i=1}^{\infty}$ satisfies the conditions (i)-(v) of Theorem 4.1.

First, we show that $\{X_i\}_{i=1}^{\infty}$ satisfies the condition (i) of Theorem 4.1, i.e., $\{X_i\}_{i=1}^{\infty}$ is independent under $\mathcal{E}_{\tilde{g}}$ and $\overline{\mathcal{E}}$. Let $X^i = (X_1,\cdots,X_i)$, $i = 1,2,\cdots$. By Lemma 4.10, for each $\varphi \in C_{l.Lip}(\mathbb{R}^{i+1})$, we have

$$\begin{aligned}\mathcal{E}[\varphi(X^i,X_{i+1})] &= \mathcal{E}[\mathcal{E}[\varphi(X^i,X_{i+1})|\mathcal{F}_i]]\\ &= \mathcal{E}[\mathcal{E}[\varphi(x,X_{i+1})|\mathcal{F}_i]_{x=X^i}]\\ &= \mathcal{E}[\mathcal{E}[\varphi(x,X_{i+1})]_{x=X^i}],\end{aligned}$$

where $\mathcal{E} = \mathcal{E}_{\tilde{g}}$ (or $\overline{\mathcal{E}}$).

Next, we show that $\{X_i\}_{i=1}^{\infty}$ satisfies the conditions (ii), (iii), (iv) and (v) of Theorem 4.1. By Lemma 4.11, we can obtain that

$$\left|\delta^{-1}\mathcal{E}_{\tilde{g}}[\delta a X_i]\right| \leq C|a|\left(E\left[|X_i|^2\right]\right)^{\frac{1}{2}} \leq C|a|(1/i^2), \tag{17}$$

$$\sup_{i\geq 1}\overline{\mathcal{E}}[(|X_i|-d)^+] \leq C\left(E\left[\left|(|X_1|-d)^+\right|^2\right]\right)^{\frac{1}{2}} \to 0, \text{ as } d \to \infty, \tag{18}$$

$$\sup_{i\geq 1}\overline{\mathcal{E}}[|X_i|] \leq C\sup_{i\geq 1}\left(E\left[|X_i|^2\right]\right)^{\frac{1}{2}} \leq C < \infty. \tag{19}$$

For the condition (ii) of Theorem 4.1, by (17), we have

$$\frac{1}{n}\sum_{i=1}^{n}\left|\underline{\mu}_i\right| \leq C\frac{1}{n}\sum_{i=1}^{n}(1/i^2) \to 0, \ \ as \ \ n \to \infty$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\left|\overline{\mu}_i\right| \leq C\frac{1}{n}\sum_{i=1}^{n}(1/i^2) \to 0, \ \ as \ \ n \to \infty.$$

For the condition (iii) of Theorem 4.1, by (17), it follows that for any given $\varepsilon > 0$, there exist a a positive integer n_0, such that

$$\left|\delta^{-1}\mathcal{E}_{\tilde{g}}\left[\delta a X_i\right] - g_i(a)\right| \leq 2C|a|(1/i^2) < \varepsilon$$

for all $i \geq n_0$. And for the above given $\varepsilon > 0$ and for any $i \in \{1, 2, \cdots, n_0 - 1\}$, there exists $\delta_i > 0$, such that $\left|\delta^{-1}\mathcal{E}_{\tilde{g}}[\delta a X_i] - g_i(a)\right| < \varepsilon$. Then choosing $\delta := \max\{\delta_1, \delta_2 \cdots, \delta_{n_0-1}\}$, we have $\sup\limits_{i\geq 1}\left|\delta^{-1}\mathcal{E}_{\tilde{g}}[\delta a X_i] - g_i(a)\right| < \varepsilon$, i.e.,

$$\lim_{\delta\downarrow 0}\sup_{i\geq 1}\left|\delta^{-1}\mathcal{E}_{\tilde{g}}[\delta a X_i] - g_i(a)\right| = 0.$$

Obviously, by (18) and (19), the conditions (iv), (v) of Theorem 4.1 hold.

Therefore, we have verified that $\{X_i\}_{i=1}^{\infty}$ satisfies all the conditions of Theorem 4.1. Thus, for any continuous function φ satisfying $|\varphi(x)| \leq C(1 + |x|)$, we have

$$\lim_{n\to\infty}\mathcal{E}_{\tilde{g}}\left[\varphi\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)\right] = \overline{\mathcal{E}}[\varphi(\eta)], \tag{20}$$

where η is maximal distributed under $\overline{\mathcal{E}}$ with $\overline{\mu} = \overline{\mathcal{E}}[\eta]$, $\underline{\mu} = -\overline{\mathcal{E}}[-\eta]$. So the proof of Example 4.7 is completed.

Declarations

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

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Authors contributions

Feng Hu prepared the original draft and Miaomiao Gao, Zhaojun Zong checked, edited and prepared the final draft of the manuscript. All authors read and approved the final manuscript.

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