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Biquaternion Fourier transform and its applications

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Abstract. In this paper, based on the biquaternion algebra, we proposed three kinds of biquaternion Fourier transforms (BiQFTs). These transforms are the extension of the complex Fourier transform. Then, the relationships between the three kinds of transforms are obtained, and it is shown that the transform can be computed by four complex Fourier transforms. Next, the inversion transforms and Plancherel theorems of the BiQFTs are proved. Moreover, the convolution theorems of the BiQFTs are studied by new convolution operators of the biquaternion. Finally, according to the convolution operator and convolution theorem associated with the right-side BiQFT, the biquaternion linear time-invariant systems are analyzed, and the biquaternion linear time-invariant systems for the right-side BiQFT is verified by the actual signal.

1. Introduction

Recently, biquaternions have become a hot topic of research. As a generalization of the quaternions [3, 11, 17–19], biquaternions first discovered by Hamilton in 1853 [13]. Biquaternions are also known as quaternions with complex components, or complex numbers with quaternion real and imaginary parts and are another hypercomplex algebra [2, 12]. A biquaternion-valued signal [20], which is a quaternion with complex components signal, includes a scalar, pseudoscalar, vector and a bivector part. Based on biquaternions, Said et al. [20] studied the extension of the Fourier transform to discretized biquaternion-valued signals, and proposed the discrete forms of the biquaternion Fourier transforms (BiQFTs). Recently, Srivastava [21] introduced several integral transformations and obtained related results. Bi et al. [2] proposed the biquaternion Z transform, and using the transform to solve a class of biquaternion recurrence relations. The BiQFTs are different from other developed techniques[7, 8]. Felsberg [8] give a Clifford Fourier transform for N-dimensional scalar-valued signals. Ebling and Scheuermann [7] defined a Clifford Fourier transform (with bivector or pseudo-scalar exponential) to analyze vector-field images.

The BiQFTs are novel tools for harmonic analysis of biquaternion-valued signals, they have been attracted more and more attention. But the basic theoretical research of the BiQFTs is not perfect, especially some basic properties for the BiQFTs. This will also limit the applications in signal processing. In this paper, we propose the BiQFTs by substituting the quaternion Fourier transform kernel with the biquaternion Fourier transform kernel. According to the properties of biquaternion, the BiQFTs have three kinds forms:

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two-side BiQFT (TBiQFT), left-side BiQFT (LBiQFT), and right-side BiQFT (RBiQFT). Applying these definitions, the relations between the three kinds of transforms are obtained. Then, based on the definitions of the BiQFTs, the inversion formula and Plancherel theorem of the BiQFTs are analyzed. In addition, the biquaternion convolution and correlation operators are defined, and the convolution and correlation theorems of the BiQFTs are studied. Finally, as applications, according to the convolution operator and convolution theorem, the biquaternion linear time-invariant systems are analyzed, and they are implemented by actual signals. Some potential applications are also presented. The study of this paper is helpful to the study of Fourier transform from general Fourier transform to hypercomplex systems. Our proposed the BiQFTs will be applied to signal processing and signals with double quaternion numerical samples. This is quite different from previous work, where the components of the considered signal take values in real numbers. The BiQFTs generalizes some interesting properties of the general Fourier transform to biquaternion-valued signals. Moreover, it can generalize the concept of analytic signals to complex-valued signals. This helps to investigate the concept of hyperanalytic signals.

The paper is organized as follows: Section 2 gives a brief introduction to some general definitions and basic properties of biquaternions. We give the definition and study the properties of the BiQFTs in Section 3. Section 4 provides the convolution theorems associated with the BiQFTs. The correlation theorems of the BiQFTs are obtained in Section 5. Section 6 studies the biquaternion linear time-invariant systems. Some conclusions are drawn in Section 7.

2. Preliminary

2.1. Biquaternions

Biquaternions form an 8-dimensional algebra first discovered by Hamilton in 1853 [13]. In the following, we present their definitions and useful properties.

A biquaternion $q \in \mathbb{H}_{\mathbb{C}}$ can be written in the form [26]

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \tag{1}$$

where $q_0, q_1, q_2, q_3 \in \mathbb{C}$ are complex numbers, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are exactly the same in real quaternions [22]. If $q_0 = 0$ then biquaternion q is known as pure biquaternion. The complex numbers are written by $\mathbf{I}^2 = -1$.

q is also possible to write a biquaternion in the following form [17]:

$$q = S(q) + V(q), \tag{2}$$

where $S(q) = q_0$ is the scalar part of q and $V(q) = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is its vector part.

Moreover, the real and imaginary parts (with respect to $\mathbf{I}$) of a biquaternion are respectively defined as [20]:

$$\Re(q) = \Re(q_0) + \Re(q_1)\mathbf{i} + \Re(q_2)\mathbf{j} + \Re(q_3)\mathbf{k}, \tag{3}$$

$$\Im(q) = \Im(q_0) + \Im(q_1)\mathbf{i} + \Im(q_2)\mathbf{j} + \Im(q_3)\mathbf{k}, \tag{4}$$

where $\Re(q)$ and $\Im(q)$ are (real) quaternions. $\Re(q_i)$ is the real part and $\Im(q_i)$ (i=0,1,2,3) is the imaginary part of a complex number. So any biquaternion $q \in \mathbb{H}_{\mathbb{C}}$ can be written as $q = \Re(q) + \mathbf{I}\Im(q)$. The complex imaginary unit $\mathbf{I}$ commutes with the quaternion imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$, that is to say [17]

$$\mathbf{iI} = \mathbf{Ii}, \mathbf{jI} = \mathbf{Ij}, \mathbf{kI} = \mathbf{Ik}. \tag{5}$$

There are two basic ways of conjugating a biquaternion [17]. Quaternion conjugation is related to the imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and complex conjugation to $\mathbf{I}$.

The quaternion conjugate of a biquaternion $q \in \mathbb{H}_{\mathbb{C}}$ is $\overline{q} = S(q) - V(q)$. The complex conjugate of a biquaternion $q \in \mathbb{H}_{\mathbb{C}}$ is defined as $q^* = q_0^* + q_1^*\mathbf{i} + q_2^*\mathbf{j} + q_3^*\mathbf{k}$, and $q_0^*, q_1^*, q_2^*, q_3^*$ are the complex conjugates of the complex coefficients of q. Biquaternion conjugation is the combination of the two conjugations that have just been defined. The biquaternion conjugate of q is defined as [20]

$$\widetilde{q} = \overline{q^*} = \overline{q}^* = q_0^* - q_1^*\mathbf{i} - q_2^*\mathbf{j} - q_3^*\mathbf{k}. \tag{6}$$

Complex conjugation is multiplicative, i.e., $(pq)^* = p^*q^*$, while quaternion conjugation and biquaternion conjugation are involutive, that is, $\widetilde{pq} = \widetilde{q}\ \widetilde{p}$ [20].

The norm of a biquaternion q can be defined by $\|q\| = q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$. If $\|q\| = 1$, then q is called unit biquaternion. The modulus of a biquaternion q is $|q| = \sqrt{\|q\|}$. Biquaternions are not a normed algebra. So the norm is not multiplicative, $|pq| \neq |p||q|$ [22].

A biquaternion $\boldsymbol{\mu} \in \mathbb{H}_{\mathbb{C}}$ is a biquaternion root of -1 iff $\boldsymbol{\mu}^2 = -1$. Any three mutually orthogonal roots of can be used as a basis to decompose a biquaternion. Given any biquaternion root of -1, $\boldsymbol{\mu}$ and any biquaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, q can be rewritten as [20, 23]

$$q = (q_0' + q_1'\boldsymbol{\mu}) + (q_2' + q_3'\boldsymbol{\mu})\boldsymbol{\nu} = q_0' + q_1'\boldsymbol{\mu} + q_2'\boldsymbol{\nu} + q_3'\boldsymbol{\xi}, \tag{7}$$

where $\boldsymbol{\nu}$ is a biquaternion root of -1 orthogonal to $\boldsymbol{\mu}$, $\boldsymbol{\mu\nu} = -\boldsymbol{\nu\mu}$, $\boldsymbol{\xi} = \boldsymbol{\mu\nu}$, $\boldsymbol{\mu}\perp\boldsymbol{\xi}$, $\boldsymbol{\nu}\perp\boldsymbol{\xi}$ and q_0', q_1', q_2', q_3' are complex numbers. The relationship between (q_0, q_1, q_2, q_3) and (q_0', q_1', q_2', q_3') is equivalent to a change in basis from $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to $(\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\xi})$. Equation (7) allows the definition of a decomposition for any biquaternion q, with respect to $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, and $Simp(q) = (q_0' + q_1'\boldsymbol{\mu})$ is simplex part, $Perp(q) = (q_2' + q_3'\boldsymbol{\mu})\boldsymbol{\nu}$ is perplex part. So $q = Simp(q) + Perp(q)$. The exponential of a biquaternion q is defined by $\mathrm{e}^q = \Sigma_{n\in\mathbb{N}}\frac{q^n}{n!}$ [20].

A biquaternion-valued function $f(x, y)$ is given by [9]

$$\begin{aligned} f(x,y) &= f_0(x,y) + f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j} + f_3(x,y)\mathbf{k} \\ &= Simp(f) + Perp(f) \\ &= S(f) + V(f) \\ &= \Re(f) + \mathbf{I}\Im(f), \end{aligned} \tag{8}$$

where f_0, f_1, f_2, f_3 are complex-valued signals, $Simp(f) = (f_0' + f_1'\boldsymbol{\mu})$, $Perp(f) = (f_2' + f_3'\boldsymbol{\mu})\boldsymbol{\nu} = f_2'\boldsymbol{\nu} + f_3'\boldsymbol{\xi}$. These signals can represent a variety of physical quantities (such as dipoles and magnetic rings for recording electromagnetic signals) captured on sensors at the same location [20]. The significance of the biquaternion signal itself have been given in [10].

For any biquaternion-valued signal $f(x, y)$ over $L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$, the L^2-norm of f is defined by

$$\|f\|_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2 \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.$$

Now we introduce an inner product of biquaternion functions f, g defined on $L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$ given by

$$(f, g)_{L^2(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}})} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\overline{g(x,y)}dxdy.$$

3. Biquaternion Fourier transform

As a generalized transform of the QFT [1, 4–6, 14, 15, 24, 25], the study of the BiQFTs is not sufficient. Especially the basic properties of the BiQFTs haven't been studied yet, In the following, we will study some properties of BiQFTs, which also provide theoretical support for their applications. There are three types of BiQFTs:

Definition 3.1. *(TBiQFT) The TBiQFT of signal $f \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$ is defined by*

$$F_B^T(f)(\omega, \upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{e}^{-\boldsymbol{\mu}\omega x} f(x,y)\mathbf{e}^{-\boldsymbol{\nu}\upsilon y}\mathrm{d}x\mathrm{d}y, \tag{9}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in the TBiQFT are two pure unit biquaternions that are orthogonal to each other.

Definition 3.2. *(LBiQFT) The LBiQFT of signal $f \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$ is defined by*

$$F_B^L(f)(\omega, \upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{e}^{-\boldsymbol{\mu}(\omega x+\upsilon y)} f(x,y)\mathrm{d}x\mathrm{d}y. \tag{10}$$

Definition 3.3. *(RBiQFT) The RBiQFT of signal $f \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$ is defined by*

$$F_B^R(f)(\omega, \upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathbf{e}^{-\boldsymbol{\mu}(\omega x+\upsilon y)}\mathrm{d}x\mathrm{d}y. \tag{11}$$

Note that to every different chosen there corresponds a different transform.

3.1. Relationship between LBiQFT and RBiQFT

Theorem 3.4. *The LBiQFT and RBiQFT are related by the following equation*

$$F_B^R\{f\}(\omega, \upsilon) = F_B^L\{Simp(f)\}(\omega, \upsilon) + F_B^L\{Perp(f)\}(-\omega, -\upsilon). \tag{12}$$

Proof. According to the equation (24), we obtain

$$\begin{aligned} f(x,y)\mathbf{e}^{-\boldsymbol{\mu}(\omega x+\upsilon y)} &= (Simp(f) + Perp(f))\mathbf{e}^{-\boldsymbol{\mu}(\omega x+\upsilon y)} \\ &= \mathbf{e}^{-\boldsymbol{\mu}(\omega x+\upsilon y)}Simp(f) + \mathbf{e}^{-\boldsymbol{\mu}(-\omega x-\upsilon y)}Perp(f). \end{aligned} \tag{13}$$

Integrating both sides with respect to $\mathrm{d}x\mathrm{d}y$, we have

$$F_B^R\{f\}(\omega, \upsilon) = F_B^L\{Simp(f)\}(\omega, \upsilon) + F_B^L\{Perp(f)\}(-\omega, -\upsilon). \tag{14}$$

□

If $\omega = -\omega$ and $\upsilon = -\upsilon$, then the following relation can be obtained

$$F_B^R\{f\}(\omega, \upsilon) = F_B^L\{f\}(\omega, \upsilon). \tag{15}$$

3.2. Relationship between TBiQFT and LBiQFT

As we all know that

$$\begin{aligned} \mathbf{e}^{-\boldsymbol{\mu}\omega x}\mathbf{e}^{-\boldsymbol{\mu}\upsilon y}f(x,y) &= (\cos(\omega x) - \boldsymbol{\mu}\sin(\omega x))(\cos(\upsilon y) - \boldsymbol{\mu}\sin(\upsilon y))f(x,y) \\ &= \cos(\omega x)\cos(\upsilon y)f(x,y) - \boldsymbol{\mu}\cos(\omega x)\sin(\upsilon y)f(x,y) \\ &\quad - \boldsymbol{\mu}\sin(\omega x)\cos(\upsilon y)f(x,y) - \sin(\omega x)\sin(\upsilon y)f(x,y), \end{aligned} \tag{16}$$

then, the LBiQFT can be split into even and odd parts

$$F_B^L\{f\}(\omega, \upsilon) = F_{B,ee}^L\{f\}(\omega, \upsilon) - F_{B,eo}^L\{f\}(\omega, \upsilon) - F_{B,oe}^L\{f\}(\omega, \upsilon) - F_{B,oo}^L\{f\}(\omega, \upsilon), \tag{17}$$

where the even and odd parts of the LBiQFT are written by

$$F_{B,ee}^L\{f\}(\omega, \upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(\omega x)\cos(\upsilon y)f(x,y)\mathrm{d}x\mathrm{d}y = \mathrm{C}_x^f\mathrm{C}_y^f(\omega, \upsilon), \tag{18}$$

$$F_{B,eo}^L\{f\}(\omega, \upsilon) = \boldsymbol{\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(\omega x)\sin(\upsilon y)f(x,y)\mathrm{d}x\mathrm{d}y = \boldsymbol{\mu}\mathrm{C}_x^f\mathrm{S}_y^f(\omega, \upsilon), \tag{19}$$

$$F_{B,oe}^L\{f\}(\omega, \upsilon) = \boldsymbol{\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin(\omega x)\cos(\upsilon y)f(x,y)\mathrm{d}x\mathrm{d}y = \boldsymbol{\mu}\mathrm{S}_x^f\mathrm{C}_y^f(\omega, \upsilon), \tag{20}$$

$$F_{B,oo}^L\{f\}(\omega, \upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin(\omega x)\sin(\upsilon y)f(x,y)\mathrm{d}x\mathrm{d}y = \mathrm{S}_x^f\mathrm{S}_y^f(\omega, \upsilon), \tag{21}$$

where $S_x^f S_y^f$ is the sine transforms of $f(x,y)$ in the x and y directions; $S_x^f C_y^f$ indicates the sine transform of $f(x,y)$ in the x direction and cosine transform of $f(x,y)$ in the y direction, and so on.

Theorem 3.5. *The TBiQFT and LBiQFT are related by the following equation*

$$F_B^T\{f\}(\omega,\upsilon) = F_B^L\{f\}(\omega,\upsilon) + \boldsymbol{\mu} C_x^f S_y^f(\omega,\upsilon) - C_x^f S_y^f(\omega,\upsilon)\boldsymbol{\nu} + \boldsymbol{\mu} S_x^f S_y^f(\omega,\upsilon)\boldsymbol{\nu} + S_x^f S_y^f(\omega,\upsilon). \tag{22}$$

Proof. The proof process is similar to Theorem 3.4. □

Figure 1 (a) illustrates the implementation blocks of the TBiQFT by the left-side BiQFT.

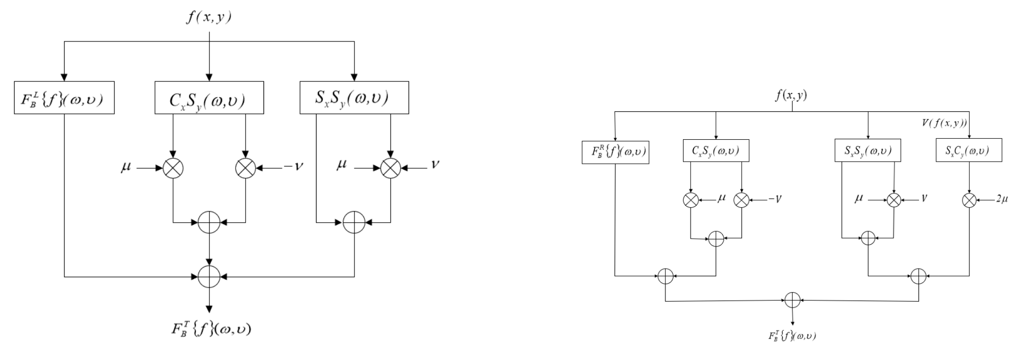
3.3. Relationship between TBiQFT and RBiQFT

Applying the method of the Theorem 3.5, the relationship between TBiQFT and RBiQFT can be obtained.

Theorem 3.6. *The TBiQFT and RBiQFT are related by the following equation*

$$\begin{aligned} F_B^T\{f\}(\omega,\upsilon) = {} & F_B^R\{f\}(\omega,\upsilon) + C_x^f S_y^f(\omega,\upsilon)\boldsymbol{\mu} - C_x^f S_y^f(\omega,\upsilon)\boldsymbol{\nu} + S_x^f S_y^f(\omega,\upsilon) + \boldsymbol{\mu} S_x^f S_y^f(\omega,\upsilon)\boldsymbol{\nu} \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\boldsymbol{\mu} V(f(x,y))\sin(\omega x)\cos(\upsilon y)\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{23}$$

where $V(f(x,y)) = f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j} + f_3(x,y)\mathbf{k}$.



(a) (b)

Figure 1: Block diagrams for the TBiQFT implemented via: (a) the LBiQFT; (b) the RBiQFT.

Figure 1 (b) illustrates the implementation blocks of the TBiQFT by the RBiQFT.

The relationships between these three transforms are very important in fast algorithms for calculating the BiQFTs [20]. For example, if we want to compute the LBiQFT and RBiQFT, we can first compute the LBiQFT and then use (12) to obtain the implementation of the RBiQFT. We can compute the LBiQFT with three steps as follows

(1) According to (24), decompose the input signal as follows

$$f(x,y) = f_a(x,y) + f_b(x,y) + f_c(x,y) + f_d(x,y). \tag{24}$$

where $f_a = \Re(f_0') + \Re(f_1')\boldsymbol{\mu}$, $f_b = \mathbf{I}(\Im(f_0') + \Im(f_1')\boldsymbol{\mu})$, $f_c = (\Re(f_2') + \Re(f_3')\boldsymbol{\mu})\boldsymbol{\nu}$, $f_d = \mathbf{I}(\Im(f_2') + \Im(f_3')\boldsymbol{\mu})\boldsymbol{\nu}$.

(2) Then, calculate the follow formulas

$$F_{a,B}^L(f_a)(\omega,\upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{e}^{-\boldsymbol{\mu}(\omega x+\upsilon y)} f_a(x,y)\mathrm{d}x\mathrm{d}y, \tag{25}$$

$$F_{b,B}^L(f_b)(\omega,\upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{e}^{-\boldsymbol{\mu}(\omega x+\upsilon y)} f_b(x,y)\mathrm{d}x\mathrm{d}y, \tag{26}$$

$$F_{c,B}^{L}(f_c)(\omega,\upsilon)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}(\omega x+\upsilon y)}f_c(x,y)\mathrm{d}x\mathrm{d}y, \tag{27}$$

$$F_{d,B}^{L}(f_d)(\omega,\upsilon)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}(\omega x+\upsilon y)}f_d(x,y)\mathrm{d}x\mathrm{d}y. \tag{28}$$

(3) Calculate the transform result of the left-side BiQFT by

$$F_{B}^{L}(f)(\omega,\upsilon)=F_{a,B}^{L}(f_a)(\omega,\upsilon)+F_{b,B}^{L}(f_b)(\omega,\upsilon)+F_{c,B}^{L}(f_c)(\omega,\upsilon)+F_{d,B}^{L}(f_d)(\omega,\upsilon). \tag{29}$$

The algorithm of the LBiQFT decomposed the LBiQFT into four complex FTs.

From (12), the RBiQFT can be represented in terms of the LBiQFT provided the signal is split into simplex and perplex components, and the formula $F_{B}^{L}\{Perp(f)\}(-\omega,-\upsilon)$ may be regarded as the algorithm for the LBiQFT of the perplex part with negated (ω,υ). So we can calculate the RBiQFT easily.

3.4. Inversion transforms of the BiQFTs

Next, we obtain the inversion transforms associate with the BiQFTs.

Theorem 3.7. *[Inversion transform of the RBiQFT] Let $f\in L^2\left(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}}\right)$. Then we have the inversion formula of the RBiQFT,*

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_{B}^{R}\{f\}(\omega,\upsilon)\mathbf{e}^{\boldsymbol{\mu}(\omega x+\upsilon y)}\mathrm{d}\omega\mathrm{d}\upsilon. \tag{30}$$

Proof. Based on the definition of the RBiQFT, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_{B}^{R}\{f\}(\omega,\upsilon)\mathbf{e}^{\boldsymbol{\mu}(\omega x+\upsilon y)}\mathrm{d}\omega\mathrm{d}\upsilon\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x',y')\mathbf{e}^{-\boldsymbol{\mu}(\omega x'+\upsilon y')}\mathrm{dx'dy'}\mathbf{e}^{\boldsymbol{\mu}(\omega\mathrm{x}+\upsilon\mathrm{y})}\mathrm{d}\omega\mathrm{d}\upsilon\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x',y')\int_{-\infty}^{\infty}\mathbf{e}^{\boldsymbol{\mu}(x-x')\omega}\mathrm{d}\omega\int_{-\infty}^{\infty}\mathbf{e}^{\boldsymbol{\nu}(\mathrm{y}-\mathrm{y}')\upsilon}\mathrm{d}\upsilon\mathrm{dx'dy'}\\
&=f(x,y).
\end{aligned} \tag{31}$$

□

According to the same method, we obtain the inversion transforms of the LBiQFT and TBiQFT.

Theorem 3.8 (Inversion transform of the LBiQFT). *Let $f\in L^2\left(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}}\right)$. Then we have the inversion transform of the LBiQFT,*

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{\boldsymbol{\mu}(\omega x+\upsilon y)}F_{B}^{L}\{f\}(\omega,\upsilon)\mathrm{d}\omega\mathrm{d}\upsilon. \tag{32}$$

Theorem 3.9 (Inversion transform of the TBiQFT). *Let $f\in L^2\left(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}}\right)$. Then we have the inversion transform of the TBiQFT,*

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{\boldsymbol{\mu}\omega x}F_{B}^{T}\{f\}(\omega,\upsilon)\mathbf{e}^{\boldsymbol{\nu}\upsilon y}\mathrm{d}\omega\mathrm{d}\upsilon. \tag{33}$$

3.5. Plancherel theorems of the BiQFTs

Firstly, let us prove Plancherel theorem of RBiQFT as follows:

Theorem 3.10 (Plancherel theorem of the RBiQFT). *Let* $f, g \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$. *Then we have*

$$(F_B^R\{f\}(\omega, \upsilon), F_B^R\{g\}(\omega, \upsilon)) = (f, g). \tag{34}$$

Proof. Applying the Theorem 3.7 and the definition of the RBiQFT, we obtain

$$\begin{aligned}
&(F_B^R\{f\}(\omega, \upsilon), F_B^R\{g\}(\omega, \upsilon)) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_B^R\{f\}(\omega, \upsilon)\overline{F_B^R\{g\}(\omega, \upsilon)}\mathrm{d}\omega\mathrm{d}\upsilon \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_B^R\{f\}(\omega, \upsilon)\mathbf{e}^{\boldsymbol{\mu}(\omega x+\upsilon y)}\mathrm{d}\omega\mathrm{d}\upsilon\overline{\mathrm{g}(\mathrm{x}, \mathrm{y})}\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\overline{g(x, y)}\mathrm{d}x\mathrm{d}y \\
&= (f, g).
\end{aligned} \tag{35}$$

□

Corollary 3.11. *If* $f = g$, *then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F_B^R\{f\}(\omega, \upsilon)|^2\mathrm{d}\omega\mathrm{d}\upsilon = \|f\|^2_{L^2(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}})}. \tag{36}$$

Similarly, Plancherel theorems of the LBiQFT and TBiQFT can be obtained.

Theorem 3.12 (Plancherel theorem of the LBiQFT). *Let* $f \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$. *Then we have the inversion formula of the LBiQFT,*

$$(F_B^L\{f\}(\omega, \upsilon), F_B^L\{g\}(\omega, \upsilon)) = (f, g). \tag{37}$$

Corollary 3.13. *If* $f = g$, *then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F_B^L\{f\}(\omega, \upsilon)|^2\mathrm{d}\omega\mathrm{d}\upsilon = \|f\|^2_{L^2(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}})}. \tag{38}$$

Theorem 3.14 (Plancherel theorem of the TBiQFT). *Let* $f \in L^2\left(\mathbb{R}^2, \mathbb{R}\right)$. *Then we have the inversion formula of the TBiQFT,*

$$(F_B^T\{f\}(\omega, \upsilon), F_B^T\{g\}(\omega, \upsilon)) = (f, g). \tag{39}$$

Corollary 3.15. *If* $f = g$, *then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F_B^T\{f\}(\omega, \upsilon)|^2\mathrm{d}\omega\mathrm{d}\upsilon = \|f\|^2_{L^2(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}})}. \tag{40}$$

The Plancherel theorem is important since it deals with the power of a signal in the spatial and frequency domains.

4. Convolution theorem

According to the convolution with quaternion or hypercomplex mask coefficients, Sangwine [?] designed the color edge detection filter. Then the author [?] extended the classical grayscale edge detecting filters attributed.

In this section, the convolution theorems of the BiQFTs are exploted.

Definition 4.1. *Let $f, g \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$, the convolution operator of the BiQFTs as follows:*

$$h(x,y) = (f *_B g)(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau,\eta)g(x-\tau, y-\eta)\mathrm{d}\tau\mathrm{d}\eta. \tag{41}$$

According to the convolution operator of the BiQFTs, we have the following convolution theorems of the BiQFTs.

Theorem 4.2 (Convolution theorem of the RBiQFT). *Assume $f, g \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$, then the RBiQFT of the convolution of f and g are provided by*

$$F_B^R\{h\}(\omega,\upsilon) = F_B^R(f)(\omega,\upsilon)F_B^R(Simp(g))(\omega,\upsilon) + F_B^R(f)(-\omega,-\upsilon)F_B^R(Perp(g))(\omega,\upsilon). \tag{42}$$

Proof. According to the definition of the RBiQFT, we have

$$\begin{aligned} F_B^R\{h\}(\omega,\upsilon) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (f *_B g)(x,y)\mathbf{e}^{-\boldsymbol{\mu}(x\omega+y\upsilon)}\mathrm{d}x\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau,\eta)g(x-\tau,y-\eta)\mathbf{e}^{-\boldsymbol{\mu}(x\omega+y\upsilon)}\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\mathrm{d}\eta. \end{aligned}$$

Let $\epsilon = x-\tau, \rho = y-\eta$, then the above formula becomes that

$$\begin{aligned} F_B^R\{h\}(\omega,\upsilon) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau,\eta)g(\epsilon,\rho)\mathbf{e}^{-\boldsymbol{\mu}((\epsilon+\tau)\omega+(\rho+\eta)\upsilon)}\mathrm{d}\tau\mathrm{d}\eta\mathrm{d}\epsilon\mathrm{d}\rho \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau,\eta)Simp(g)(\epsilon,\rho)\mathbf{e}^{-\boldsymbol{\mu}((\epsilon+\tau)\omega+(\rho+\eta)\upsilon)}\mathrm{d}\tau\mathrm{d}\eta\mathrm{d}\epsilon\mathrm{d}\rho \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau,\eta)Perp(g)(\epsilon,\rho)\mathbf{e}^{-\boldsymbol{\mu}((\epsilon+\tau)\omega+(\rho+\eta)\upsilon)}\mathrm{d}\tau\mathrm{d}\eta\mathrm{d}\epsilon\mathrm{d}\rho \\ &= F_B^R(f)(\omega,\upsilon)F_B^R(Simp(g))(\omega,\upsilon) + F_B^R(f)(-\omega,-\upsilon)F_B^R(Perp(g))(\omega,\upsilon). \end{aligned}$$

□

In the case where $f(x,y)$ is the even function, then by the definition of the RBiQFT, we have

$$F_B^R(f *_B g)(\omega,\upsilon) = F_B^R(f)(\omega,\upsilon)F_B^R(g)(\omega,\upsilon). \tag{43}$$

Hence, when $f(x,y)$ is even, the convolution operation of two biquaternion-valued functions in the time domain is equivalent to the product operation in the frequency domain.

When $f(x,y)$ is the odd function, then

$$F_B^R(f *_B g)(\omega,\upsilon) = F_B^R(f)(\omega,\upsilon)F_B^R(g')(\omega,\upsilon). \tag{44}$$

where $g'(x,y) = Simp(g) - Perp(g)$.

In general, if $f(x,y)$ is neither even nor odd, based on (43) and (44), we can conclude that the relation between the inputs and the output of the convolution for the RBiQFT in the frequency domain can be written as

$$F_B^R(f *_B g)(\omega,\upsilon) = F_{e,B}^R(f)(\omega,\upsilon)F_B^R(g)(\omega,\upsilon) + F_{o,B}^R(f)(\omega,\upsilon)F_B^R(g')(\omega,\upsilon), \tag{45}$$

where $F^R_{e,B}(f)(\omega, v)$ and $F^R_{o,B}(f)(\omega, v)$ are the even and odd parts of the RBiQFT $F^R_B(f)(\omega, v)$

$$F^R_{e,B}(f)(\omega, v) = \frac{\left[F^R_B(f)(\omega, v) + F^R_B(f)(-\omega, -v)\right]}{2}, \tag{46}$$

$$F^R_{o,B}(f)(\omega, v) = \frac{\left[F^R_B(f)(\omega, v) - F^R_B(f)(-\omega, -v)\right]}{2}. \tag{47}$$

Based on the definitions of the LBiQFT and TBiQFT, we also obtain their convolution theorems.

Theorem 4.3 (Convolution theorem of the LBiQFT). *Assume the functions $f, g \in L^2\left(\mathbb{R}^2, \mathbb{H}_\mathbb{C}\right)$, then the LBiQFT of the convolution of f and g are provided by*

$$F^L_B\{h\}(\omega, v) = F^R_B(Simp(f))(\omega, v)F^R_B(g)(\omega, v) + F^R_B(Perp(f))(\omega, v)F^R_B(g)(-\omega, -v). \tag{48}$$

In the case where $g(x, y)$ is the even function, then according to the definition of the LBiQFT,

$$F^L_B(f *_B g)(\omega, v) = F^L_B(f)(\omega, v)F^L_B(g)(\omega, v). \tag{49}$$

Hence, when $g(x, y)$ is even, the convolution operation of two biquaternion-valued functions in the time domain is equivalent to the product operation in the frequency domain.

When $g(x, y)$ is the odd function, then

$$F^L_B(f *_B g)(\omega, v) = F^L_B(f')(\omega, v)F^L_B(g)(\omega, v), \tag{50}$$

where $f'(x, y) = Simp(f) - Perp(f)$.

In general, if $g(x, y)$ is neither even nor odd, we can conclude that the relation between the inputs and the output of the convolution for the LBiQFT in the frequency domain can be written as

$$F^L_B(f *_B g)(\omega, v) = F^L_B(f)(\omega, v)F^L_{e,B}(g)(\omega, v) + F^L_B(f')(\omega, v)F^L_{o,B}(g)(\omega, v), \tag{51}$$

where $F^L_{e,B}(g)(\omega, v)$ and $F^L_{o,B}(g)(\omega, v)$ are the even and odd parts of the LBiQFT $F^L_B(g)(\omega, v)$

$$F^L_{e,B}(g)(\omega, v) = \frac{\left[F^L_B(g)(\omega, v) + F^L_B(g)(\omega, v)\right]}{2}; \;\; F^L_{o,B}(g)(\omega, v) = \frac{\left[F^L_B(g)(\omega, v) - F^L_B(g)(\omega, v)\right]}{2}. \tag{52}$$

According to (48), if $f(x, y)$ and $g(x, y)$ are even or odd, then $h(x, y)$ is even, and if $f(x, y)$ is even and $g(x, y)$ is odd (or $f(x, y)$ is odd and $g(x, y)$ is even), then $h(x, y)$ is odd. So we obtain

$$F^L_{e,B}\{h\}(\omega, v) = F^L_{e,B}\{f\}(\omega, v)F^L_{e,B}(g)(\omega, v) + F^L_{o,B}\{f'\}(\omega, v)F^L_{o,B}(g)(\omega, v),$$

$$F^L_{o,B}\{h\}(\omega, v) = F^L_{o,B}\{f\}(\omega, v)F^L_{e,B}(g)(\omega, v) + F^L_{e,B}\{f'\}(\omega, v)F^L_{o,B}(g)(\omega, v).$$

Theorem 4.4 (Convolution theorem of the TBiQFT). *Assume the functions $f, g \in L^2\left(\mathbb{R}^2, \mathbb{H}_\mathbb{C}\right)$, $g = Simp(g) + Perp(g)$, $Simp(g) = g'_0 + g'_1\boldsymbol{\mu}$, $Perp(f) = (g'_2 + g'_3\boldsymbol{\mu})\boldsymbol{\nu} = g'_2\boldsymbol{\nu} + g'_3\boldsymbol{\xi}$ and g'_0, g'_1, g'_2, g'_3 are complex-valued signals, then the TBiQFT of the convolution of f and g are provided by*

$$F^T_B\{h\}(\omega, v) = F^{Simp(f)}_{g,01} + F^{Simp(f)}_{g,23} + F^{Perp(f)}_{g,01} + F^{Perp(f)}_{g,01}, \tag{53}$$

where

$$\begin{aligned} F^{Simp(f)}_{g,01} &= F^T_{B,\omega e}\{Simp(f)\}(\omega, v)F^T_B\{g'_0\}(\omega, v) + F^T_{B,\omega e}\{Simp(f)\}(\omega, v)F^T_B\{g'_1\}(\omega, -v)\cdot\boldsymbol{\mu} \\ &\quad - F^T_{B,\omega o}\{Simp(f)\}(\omega, v)F^T_B\{g'_0\}(\omega, v)\cdot\boldsymbol{\nu} - F^T_{B,\omega o}\{Simp(f)\}(\omega, v)F^T_B\{g'_1\}(\omega, -v)\cdot\boldsymbol{\xi}, \end{aligned}$$

$$\begin{aligned}F_{g,23}^{Simp(f)} &= F_{B,\omega o}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_2'\}(\omega,\upsilon) + F_{B,\omega o}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_3'\}(\omega,-\upsilon)\cdot\boldsymbol{\mu}\\&+ F_{B,\omega e}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_2'\}(\omega,\upsilon)\cdot\boldsymbol{\nu} + F_{B,\omega e}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_3'\}(\omega,-\upsilon)\cdot\boldsymbol{\xi},\end{aligned}$$

$$\begin{aligned}F_{g,01}^{Perp(f)} &= F_{B,\omega e}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{g_0'\}(-\omega,\upsilon) + F_{B,\omega e}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{g_1'\}(-\omega,-\upsilon)\cdot\boldsymbol{\mu}\\&- F_{B,\omega o}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{g_0'\}(-\omega,\upsilon)\cdot\boldsymbol{\nu} - F_{B,\omega o}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{g_1'\}(-\omega,-\upsilon)\cdot\boldsymbol{\xi},\end{aligned}$$

$$\begin{aligned}F_{g,23}^{Perp(f)} &= F_{B,\omega o}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{g_2'\}(-\omega,\upsilon) + F_{B,\omega o}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{g_3'\}(-\omega,-\upsilon)\cdot\boldsymbol{\mu}\\&+ F_{B,\omega e}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{g_2'\}(-\omega,\upsilon)\cdot\boldsymbol{\nu} + F_{B,\omega e}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{g_3'\}(-\omega,-\upsilon)\cdot\boldsymbol{\xi},\end{aligned}$$

and

$$F_{B,\omega e}^T\{Simp(f)\}(\omega,\upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}\tau\omega}Simp(f)(\tau,\eta)\cos(\upsilon\eta)\mathrm{d}\tau\mathrm{d}\eta,$$

$$F_{B,\omega o}^T\{Simp(f)\}(\omega,\upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}\tau\omega}Simp(f)(\tau,\eta)\sin(\upsilon\eta)\mathrm{d}\tau\mathrm{d}\eta,$$

$$F_{B,\omega e}^T\{Perp(f)\}(\omega,\upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}\tau\omega}Perp(f)(\tau,\eta)\cos(\upsilon\eta)\mathrm{d}\tau\mathrm{d}\eta,$$

$$F_{B,\omega o}^T\{Perp(f)\}(\omega,\upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}\tau\omega}Perp(f)(\tau,\eta)\sin(\upsilon\eta)\mathrm{d}\tau\mathrm{d}\eta.$$

Proof. From the definition of the TBiQFT, we obtain

$$F_B^T\{h\}(\omega,\upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}x\omega}f(\tau,\eta)g(x-\tau,y-\eta)\mathrm{d}\tau\mathrm{d}\eta\mathbf{e}^{-\boldsymbol{\nu}y\upsilon}\mathrm{d}x\mathrm{d}y.$$

Making a substitution, let $\epsilon = x-\tau, \rho = y-\eta$, then put them into the above formula becomes that

$$\begin{aligned}F_B^T\{h\}(\omega,\upsilon) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}(\epsilon+\tau)\omega}f(\tau,\eta)g(\epsilon,\rho)\mathbf{e}^{-\boldsymbol{\nu}(\rho+\eta)\upsilon}\mathrm{d}\epsilon\mathrm{d}\rho\mathrm{d}\tau\mathrm{d}\eta\\&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}(\epsilon+\tau)\omega}\Big[Simp(f)(\tau,\eta)Simp(g)(\epsilon,\rho)\\&\quad+ Simp(f)(\tau,\eta)Perp(g)(\epsilon,\rho) + Perp(f)(\tau,\eta)Simp(g)(\epsilon,\rho)\\&\quad+ Perp(f)(\tau,\eta)Perp(g)(\epsilon,\rho)\Big]\mathbf{e}^{-\boldsymbol{\nu}(\rho+\eta)\upsilon}\mathrm{d}\epsilon\mathrm{d}\rho\mathrm{d}\tau\mathrm{d}\eta.\end{aligned} \tag{54}$$

Based on the following fact

$$\mathbf{e}^{-\boldsymbol{\mu}\omega\epsilon}\mathbf{e}^{-\boldsymbol{\nu}\upsilon\eta} = \cos(\upsilon\eta)\mathbf{e}^{-\boldsymbol{\mu}\omega\epsilon} - \mathbf{e}^{-\boldsymbol{\mu}\omega\epsilon}\boldsymbol{\nu}\sin(\upsilon\eta), \tag{55}$$

then, the formula (54) can be broken down into the following four formulas

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}(\epsilon+\tau)\omega}Simp(f)(\tau,\eta)Simp(g)(\epsilon,\rho)\mathbf{e}^{-\boldsymbol{\nu}(\rho+\eta)\upsilon}\mathrm{d}\epsilon\mathrm{d}\rho\mathrm{d}\tau\mathrm{d}\eta\\&= F_{B,\omega e}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_0'\}(\omega,\upsilon) - F_{B,\omega o}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_0'\}(\omega,\upsilon)\cdot\boldsymbol{\nu}\\&\quad+ F_{B,\omega e}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_1'\}(\omega,-\upsilon)\cdot\boldsymbol{\mu} - F_{B,\omega o}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_1'\}(\omega,-\upsilon)\cdot\boldsymbol{\xi},\end{aligned}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}(\epsilon+\tau)\omega}Simp(f)(\tau,\eta)Perp(g)(\epsilon,\rho)\mathbf{e}^{-\boldsymbol{\nu}(\rho+\eta)\upsilon}\mathrm{d}\epsilon\mathrm{d}\rho\mathrm{d}\tau\mathrm{d}\eta$$
$$= F^T_{B,\omega e}\{Simp(f)\}(\omega,\upsilon)F^T_B\{g'_2\}(\omega,\upsilon)\cdot\boldsymbol{\nu} + F^T_{B,\omega o}\{Simp(f)\}(\omega,\upsilon)F^T_B\{g'_2\}(\omega,\upsilon)$$
$$+ F^T_{B,\omega e}\{Simp(f)\}(\omega,\upsilon)F^T_B\{g'_3\}(\omega,-\upsilon)\cdot\boldsymbol{\xi} + F^T_{B,\omega o}\{Simp(f)\}(\omega,\upsilon)F^T_B\{g'_3\}(\omega,-\upsilon)\cdot\boldsymbol{\mu},$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}(\epsilon+\tau)\omega}Perp(f)(\tau,\eta)Simp(g)(\epsilon,\rho)\mathbf{e}^{-\boldsymbol{\nu}(\rho+\eta)\upsilon}\mathrm{d}\epsilon\mathrm{d}\rho\mathrm{d}\tau\mathrm{d}\eta$$
$$= F^T_{B,\omega e}\{Perp(f)\}(\omega,\upsilon)F^T_B\{g'_0\}(-\omega,\upsilon) - F^T_{B,\omega o}\{Perp(f)\}(\omega,\upsilon)F^T_B\{g'_0\}(-\omega,\upsilon)\cdot\boldsymbol{\nu}$$
$$+ F^T_{B,\omega e}\{Perp(f)\}(\omega,\upsilon)F^T_B\{g'_1\}(-\omega,-\upsilon)\cdot\boldsymbol{\mu} - F^T_{B,\omega o}\{Perp(f)\}(\omega,\upsilon)F^T_B\{g'_1\}(-\omega,-\upsilon)\cdot\boldsymbol{\xi},$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{e}^{-\boldsymbol{\mu}(\epsilon+\tau)\omega}Perp(f)(\tau,\eta)Perp(g)(\epsilon,\rho)\mathbf{e}^{-\boldsymbol{\nu}(\rho+\eta)\upsilon}\mathrm{d}\epsilon\mathrm{d}\rho\mathrm{d}\tau\mathrm{d}\eta$$
$$= F^T_{B,\omega e}\{Perp(f)\}(\omega,\upsilon)F^T_B\{g'_2\}(-\omega,\upsilon)\cdot\boldsymbol{\nu} + F^T_{B,\omega o}\{Perp(f)\}(\omega,\upsilon)F^T_B\{g'_2\}(-\omega,\upsilon)$$
$$+ F^T_{B,\omega e}\{Perp(f)\}(\omega,\upsilon)F^T_B\{g'_3\}(-\omega,-\upsilon)\cdot\boldsymbol{\xi} + F^T_{B,\omega o}\{Perp(f)\}(\omega,\upsilon)F^T_B\{g'_3\}(-\omega,-\upsilon)\cdot\boldsymbol{\mu}.$$

□

We find that the convolution theorem of the TBiQFT is more complicated than others. It can only be analyzed under very specific conditions.

5. Correlation theorem

In this section, the correlation theorems of the BiQFTs are derived. First, we present the definition of correlation operator for the BiQFTs.

Definition 5.1. *Let $f, g \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$, the correlation operator of the RBiQFT as follows:*

$$(f \star_B g)(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+\tau, y+\eta)\widetilde{g}(\tau,\eta)\mathrm{d}\tau\mathrm{d}\eta. \tag{56}$$

Then, we have the following correlation theorems of the BiQFTs.

Theorem 5.2 (Correlation theorem of the RBiQFT). *Assume the functions $f, g \in L^2\left(\mathbb{R}^2, \mathbb{H}_{\mathbb{C}}\right)$, then the RBiQFT of the correlation of f and g are provided by*

$$F^R_B\{f \star_B g\}(\omega,\upsilon) = F^R_B(f)(\omega,\upsilon)F^R_B(\widetilde{Simp(g)})(-\omega,-\upsilon) + F^R_B(f)(-\omega,-\upsilon)F^R_B(\widetilde{Perp(g)})(-\omega,-\upsilon). \tag{57}$$

Proof. According to the correlation operator of the BiQFTs, we obtain

$$F^R_B\{f \star_B g\}(\omega,\upsilon) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+\tau, y+\eta)\widetilde{g}(\tau,\eta)\mathbf{e}^{-\boldsymbol{\mu}(x\omega+y\upsilon)}\mathrm{d}\tau\mathrm{d}\eta\mathrm{d}x\mathrm{d}y. \tag{58}$$

Let $\varepsilon = x+\tau, \zeta = y+\eta$, then the above formula becomes that

$$\begin{aligned}F^R_B\{f \star_B g\}(\omega,\upsilon) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\varepsilon,\zeta)\widetilde{g}(\tau,\eta)\mathbf{e}^{-\boldsymbol{\mu}((\varepsilon-\tau)\omega+(\zeta-\eta)\upsilon)}\mathrm{d}\tau\mathrm{d}\eta\mathrm{d}\varepsilon\mathrm{d}\zeta \\ &= F^R_B(f)(\omega,\upsilon)F^R_B(\widetilde{Simp(g)})(-\omega,-\upsilon) + F^R_B(f)(-\omega,-\upsilon)F^R_B(\widetilde{Perp(g)})(-\omega,-\upsilon).\end{aligned}$$

□

In general, if $f(x,y)$ is neither even nor odd, we obtain

$$F_B^R\{f\star_B g\}(\omega,\upsilon)=F_{e,B}^R(f)(\omega,\upsilon)F_B^R(\widetilde{g})(\omega,\upsilon)+F_{o,B}^R(f)(\omega,\upsilon)F_B^R\{\widetilde{Simp(g)}-\widetilde{Perp(g)}\}(\omega,\upsilon). \tag{59}$$

From this result, the correlation in the time domain corresponds to the product operation in the frequency domain. This is helpful to color image processing [16].

Similarly, we can also obtain the following correlation theorems.

Theorem 5.3 (Correlation theorem of the LBiQFT). *Assume the functions $f,g\in L^2\left(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}}\right)$, then the LBiQFT of the correlation of f and g are provided by*

$$F_B^L\{f\star_B g\}(\omega,\upsilon)=F_B^L(\widetilde{Simp(f)})(-\omega,-\upsilon)F_B^L(g)(\omega,\upsilon)+F_B^L(\widetilde{Perp(f)})(-\omega,-\upsilon)F_B^L(g)(-\omega,-\upsilon). \tag{60}$$

Theorem 5.4 (Correlation theorem of the TBiQFT). *Assume the functions $f,g\in L^2\left(\mathbb{R}^2,\mathbb{H}_{\mathbb{C}}\right)$, $\widetilde{g}=\widetilde{Simp(g)}+\widetilde{Perp(g)}$, $\widetilde{Simp(g)}=(g_0')^*+(g_1')^*\boldsymbol{\mu}^*$, $\widetilde{Perp(f)}=(g_2')^*\boldsymbol{\nu}^*+(g_3')^*\boldsymbol{\xi}^*$, then the TBiQFT of the correlation of f and g are provided by*

$$F_B^T\{f\star_B g\}(\omega,\upsilon)=F_{g,01*}^{Simp(f)}+F_{g,23*}^{Simp(f)}+F_{g,01*}^{Perp(f)}+F_{g,01*}^{Perp(f)}, \tag{61}$$

where

$$\begin{aligned}F_{g,01*}^{Simp(f)}&=F_{B,\omega e}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{(g_0')^*\}(\omega,\upsilon)+F_{B,\omega e}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{(g_1')^*\}(\omega,-\upsilon)\cdot\boldsymbol{\mu}^*\\&-F_{B,\omega o}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_0'\}(\omega,\upsilon)\cdot\boldsymbol{\nu}-F_{B,\omega o}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{g_1'\}(\omega,-\upsilon)\cdot\boldsymbol{\nu}\boldsymbol{\mu}^*,\end{aligned}$$

$$\begin{aligned}F_{g,23*}^{Simp(f)}&=-F_{B,\omega o}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{(g_2')^*\}(\omega,\upsilon)+F_{B,\omega o}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{(g_3')^*\}(\omega,-\upsilon)\cdot\boldsymbol{\nu}\boldsymbol{\xi}^*\\&+F_{B,\omega e}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{(g_2')^*\}(\omega,\upsilon)\cdot\boldsymbol{\nu}^*\\&+F_{B,\omega e}^T\{Simp(f)\}(\omega,\upsilon)F_B^T\{(g_3')^*\}(\omega,-\upsilon)\cdot\boldsymbol{\xi}^*,\end{aligned}$$

$$\begin{aligned}F_{g,01*}^{Perp(f)}&=F_{B,\omega e}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{(g_0')^*\}(-\omega,\upsilon)+F_{B,\omega e}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{(g_1')^*\}(-\omega,-\upsilon)\cdot\boldsymbol{\mu}^*\\&-F_{B,\omega o}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{(g_0')^*\}(-\omega,\upsilon)\cdot\boldsymbol{\nu}\\&+F_{B,\omega o}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{(g_1')^*\}(-\omega,-\upsilon)\cdot\boldsymbol{\nu}\boldsymbol{\mu}^*,\end{aligned}$$

$$\begin{aligned}F_{g,23*}^{Perp(f)}&=-F_{B,\omega o}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{(g_2')^*\}(-\omega,\upsilon)+F_{B,\omega o}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{(g_3')^*\}(-\omega,-\upsilon)\cdot\boldsymbol{\nu}\boldsymbol{\xi}^*\\&+F_{B,\omega e}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{(g_2')^*\}(-\omega,\upsilon)\cdot\boldsymbol{\nu}^*\\&+F_{B,\omega e}^T\{Perp(f)\}(\omega,\upsilon)F_B^T\{(g_3')^*\}(-\omega,-\upsilon)\cdot\boldsymbol{\xi}^*.\end{aligned}$$

6. Applications

In this section, based on the relations between the convolution and the RBiQFT, the usages of the BiQFTs in the analysis of BiQLTI systems are explored.

Next, we limit the transform considered to the case of the RBiQFT. The BiQLTI in two variables can be represented in terms of a convolution operator, which relates the output of the system to its input as

$$h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau,\eta)g(x-\tau,y-\eta)\mathrm{d}\tau\mathrm{d}\eta, \tag{62}$$

where $f(x, y)$ is the input, $h(x, y)$ is the output, and $g(x, y)$ is the impulse response.

Next, we will discuss how to use the RBiQFT to analyze the combination of BiQLTI systems. Case 1, when the BiQLTI systems are combined in parallel, the relation between the input and the output can be expressed as

$$h_n(x, y) = f(x, y) *_B g_n(x, y), \tag{63}$$

where $g_n(x, y) = \sum_{m=1}^{M} g_m(x, y)$. In the frequency domain, based on (42), we obtain the following relation

$$F_B^R\{h_n\}(\omega, \upsilon) = \begin{bmatrix} F_B^R(f)(\omega, \upsilon) & F_B^R(f)(-\omega, -\upsilon) \end{bmatrix} \left\{ \Sigma_m^M \begin{bmatrix} F_B^R\{Simp(g_m)\}(\omega, \upsilon) \\ F_B^R\{Perp(g_m)\}(\omega, \upsilon) \end{bmatrix} \right\}. \tag{64}$$

Case 2, when the BiQLTI systems are combined in series, the relation between the input and the output can be expressed as

$$h_M(x, y) = f(x, y) *_B g_1(x, y) *_B g_2(x, y) *_B \cdots *_B g_{M-1}(x, y) *_B g_M(x, y). \tag{65}$$

Then, in the frequency domain, we obtain the following relation

$$\begin{aligned} F_B^R\{h_M\}(\omega, \upsilon) = \begin{bmatrix} F_{e,B}^R(f)(\omega, \upsilon) & F_{o,B}^R(f)(\omega, \upsilon) \end{bmatrix} & \begin{bmatrix} F_{e,B}^R\{g_1\}(\omega, \upsilon) & F_{o,B}^R\{g_1\}(\omega, \upsilon) \\ F_{o,B}^R\{g_1'\}(\omega, \upsilon) & F_{e,B}^R\{g_1'\}(\omega, \upsilon) \end{bmatrix} \\ \cdots \begin{bmatrix} F_{e,B}^R\{g_{M-1}\}(\omega, \upsilon) & F_{o,B}^{R,}\{g_{M-1}\}(\omega, \upsilon) \\ F_{o,B}^R\{g_{M-1}'\}(\omega, \upsilon) & F_{e,B}^R\{g_{M-1}'\}(\omega, \upsilon) \end{bmatrix} & \cdot \begin{bmatrix} F_B^R\{g_M\}(\omega, \upsilon) \\ F_B^R\{g_M'\}(\omega, \upsilon) \end{bmatrix}. \end{aligned} \tag{66}$$

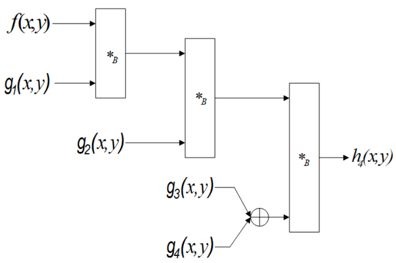


Figure 2: Combination of BiQLTI system: In parallel and series.

To obtain some insight into the BiQLTI systems corresponding to the above contents, let us give an example.

Example

Applying (64) and (66), we can use the RBiQFT to represent many different combinations of BiQLTI systems. For example, for the BiQLTI systems combined as Figure 2:

Let the input signal $f(x, y)$ is Gaussian signal, defined by $f(x, y) = \mathbf{e}^{-\boldsymbol{\mu}(x^2+xy)}$. Hence, from (46) and (47), we obtain

$$F_{e,B}^R\{f\}(\omega, \upsilon) = F_B^R\{f\}(\omega, \upsilon); \quad F_{o,B}^R\{f\}(\omega, \upsilon) = 0. \tag{67}$$

Let the input signal $g_1(x,y)$ is defined by $g_1(x,y) = \mathbf{e}^{-\mathbf{I}(\frac{1}{4}x^2+y^2)}\boldsymbol{\mu} + \mathbf{e}^{-\mathbf{I}(x^2+y^2)}\boldsymbol{\nu}$. According to the definition of the RBiQFT, we have

$$F_B^R\{g_1\}(\omega,\upsilon) = -2\pi\mathbf{I}\mathbf{e}^{\mathbf{I}(\omega^2+\upsilon^2)}\boldsymbol{\mu} - \pi\mathbf{I}\mathbf{e}^{\mathbf{I}(\frac{\omega^2+\upsilon^2}{4})}\boldsymbol{\nu};\ \ F_B^R\{g_1'\}(\omega,\upsilon) = -2\pi\mathbf{I}\mathbf{e}^{\mathbf{I}(\omega^2+\upsilon^2)}\boldsymbol{\mu} + \pi\mathbf{I}\mathbf{e}^{\mathbf{I}(\frac{\omega^2+\upsilon^2}{4})}\boldsymbol{\nu},$$

and

$$F_B^R\{g_1\}(-\omega,-\upsilon) = F_B^R\{g_1\}(\omega,\upsilon);\ \ F_B^R\{g_1'\}(-\omega,-\upsilon) = F_B^R\{g_1'\}(\omega,\upsilon),$$

Hence, from (46) and (47), we obtain

$$F_{e,B}^R\{g_1\}(\omega,\upsilon) = F_B^R\{g_1\}(\omega,\upsilon);\ \ F_{o,B}^R\{g_1\}(\omega,\upsilon) = 0,$$

and

$$F_{e,B}^R\{g_1'\}(\omega,\upsilon) = F_B^R\{g_1'\}(\omega,\upsilon);\ \ F_{o,B}^R\{g_1'\}(\omega,\upsilon) = 0.$$

Let the input signal $g_2(x,y)$ is the Dirac-delta signal $g_2(x,y) = \delta(x)\delta(y)$, then, according to the definition of the RBiQFT, we have

$$F_B^R\{g_2\}(\omega,\upsilon) = F_B^R\{g_2'\}(\omega,\upsilon) = 1. \tag{68}$$

Let the input signal $g_3(x,y)$ is defined by $g_3(x,y) = \mathbf{e}^{-\boldsymbol{\mu}(\frac{1}{2}x^2+2xy+1)}$. According to the definition of the RBiQFT, we have

$$F_B^R\{Simp(g_3)\}(\omega,\upsilon) = \pi\mathbf{e}^{-\boldsymbol{\mu}(\frac{\upsilon^2-4\omega\upsilon}{8}+1)};\ \ F_B^R\{Perp(g_3)\}(\omega,\upsilon) = 0.$$

Let the input signal $g_4(x,y)$ is defined by $g_4(x,y) = \mathbf{e}^{\boldsymbol{\mu}a(x+y)},\ \ a \in \mathbb{R}$. According to the definition of the RBiQFT, we have

$$F_B^R\{Simp(g_4)\}(\omega,\upsilon) = 4\pi^2\delta(\omega-a)\delta(\upsilon-a);\ \ F_B^R\{Perp(g_4)\}(\omega,\upsilon) = 0.$$

According to the Figure 2, we can obtain the RBiQFT of the output signal h_4

$$F_B^R\{h_4\}(\omega,\upsilon) = \begin{bmatrix} F_B^R(J)(\omega,\upsilon) & F_B^R(J)(-\omega,-\upsilon)\end{bmatrix}\begin{bmatrix} F_B^R\{Simp(g_3)\}(\omega,\upsilon) + F_B^R\{Simp(g_4)\}(\omega,\upsilon) \\ F_B^R\{Perp(g_3)\}(\omega,\upsilon) + F_B^R\{Perp(g_4)\}(\omega,\upsilon)\end{bmatrix},$$

where

$$F_B^R\{J\}(\omega,\upsilon) = \begin{bmatrix} F_{e,B}^R(f)(\omega,\upsilon) & F_{o,B}^R(f)(\omega,\upsilon)\end{bmatrix}\begin{bmatrix} F_{e,B}^R\{g_1\}(\omega,\upsilon) & F_{o,B}^R\{g_1\}(\omega,\upsilon) \\ F_{o,B}^R\{g_1'\}(\omega,\upsilon) & F_{e,B}^R\{g_1'\}(\omega,\upsilon)\end{bmatrix}\begin{bmatrix} F_B^R\{g_2\}(\omega,\upsilon) \\ F_B^R\{g_2'\}(\omega,\upsilon)\end{bmatrix}.$$

Then,

$$F_B^R\{J\}(\omega,\upsilon) = F_B^R\{J\}(-\omega,-\upsilon) = -2\pi^2\mathbf{I}\mathbf{e}^{-\boldsymbol{\mu}(\upsilon^2-\omega\upsilon)}(2\mathbf{e}^{\mathbf{I}(\omega^2+\upsilon^2)}\boldsymbol{\mu} + \mathbf{e}^{\mathbf{I}(\frac{\omega^2+\upsilon^2}{4})}\boldsymbol{\nu});$$

$$\begin{bmatrix} F_B^R\{Simp(g_3)\}(\omega,\upsilon) + F_B^R\{Simp(g_4)\}(\omega,\upsilon) \\ F_B^R\{Perp(g_3)\}(\omega,\upsilon) + F_B^R\{Perp(g_4)\}(\omega,\upsilon)\end{bmatrix} = \begin{bmatrix} \pi\mathbf{e}^{-\boldsymbol{\mu}(\frac{\upsilon^2-4\omega\upsilon}{8}+1)} + 4\pi^2\delta(\omega-a)\delta(\upsilon-a) \\ 0 \end{bmatrix},$$

Hence, in the frequency domain, we obtain the following result

$$F_B^R\{h_4\}(\omega,\upsilon) = -2\pi^3\mathbf{I}\mathbf{e}^{-\boldsymbol{\mu}(\upsilon^2-\omega\upsilon)}(2\mathbf{e}^{\mathbf{I}(\omega^2+\upsilon^2)}\boldsymbol{\mu} + \mathbf{e}^{\mathbf{I}(\frac{\omega^2+\upsilon^2}{4})}\boldsymbol{\nu})(\mathbf{e}^{-\boldsymbol{\mu}(\frac{\upsilon^2-4\omega\upsilon}{8}+1)} + 4\pi\delta(\omega-a)\delta(\upsilon-a)).$$

7. Conclusions

In this paper, from the biquaternion algebra, we proposed a new transform tool-the BiQFTs. There are three types of BiQFTs: TBiQFT, LBiQFT and RBiQFT. The relationships between the three BiQFTs are explored, and then the effective algorithms of the BiQFTs are introduced. Some general properties of the BiQFTs are proved. In addition, the convolution and correlation theorems associated with the BiQFT are studied. These conclusions provides theoretical support for filter design. Finally, based on the convolution operator and convolution theorem of the RBiQFT, the biquaternion linear time-invariant systems are analyzed. The research of this paper enriches the theoretical system of the BiQFT, and also provides a theoretical basis for the application of the BiQFT in signal processing.

Conflict of interests:

The authors declare that they have no conflict of interest.

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