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On the special (α, β)-change of a Finsler space with m-th root metric

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Abstract. The present paper focuses on the theory of transformation of the m-th root metric. We find that the transformed m-th root metric and the m-th root metric are projectively related. Further, we establish a necessary and sufficient condition for a transformed m-th root metric to be locally dually flat and projectively flat.

1. Introduction

The m-th root metrics are studied in [11, 13, 14, 18]. Antonelli discussed their applications in Ecology in [2]. Randers change of a m-th root metric is studied in [17]. The m-th root metrics are an extension of Riemannian metrics (when $m = 2$). The fourth root metric, $F = \sqrt[4]{y^1 y^2 y^3 y^4}$, is known as Berwald-Moór metric, which finds an application in the theory of space-time [3]. Recently, Nekouee *et al.* [7] have investigated the applications of the Finsler-Randers metric in cosmology. Finsler geometry has also been applied to study various wormhole models [5, 8]. The Finslerian Schwarzschild-de sitter space-time is also recently investigated in [6]. Charged gravastars are discussed in [9]. m-th root metrics also found applications in general relativity and unified gauge field theory. Applications of conformal change are discussed in [16].

Shen and Li considered fourth root metrics in the form $F = \sqrt[4]{a_{ijkl}(x) y^i y^j y^k y^l}$, which are locally projectively flat and studied their geometric features [4]. Locally dually flat metrics are discussed in [15]. The Randers change of m-th root metric is also studied in [12], which discusses the relation between Finsler space with m-th root metric and the different tensors of the transformed Finsler space.

This paper discusses the conditions under which the given Finsler space and transformed Finsler space are projectively related. We study the conditions under which the transformed Finsler space is locally dually flat. And the conditions under which it is projectively flat.

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Keywords. Finsler space, Special (α, β)-metric, m-th root metric, Projectively related metrics, Locally dually flat metric, Projective flatness.

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2. Preliminaries

Let M^n denote an n-dimensional C^∞-manifold, and T_xM the tangent space of M^n at x. The tangent bundle TM is the union of tangent spaces, $TM := \bigcup_{x\in M} T_xM$. The element of TM is denoted by (x, y), where $x = (x^i)$ is a point of M^n and $y \in T_xM$ is a supporting element. We denote $TM_0 = TM\backslash\{0\}$.

Definition 2.1. *[2] A Finsler metric on M^n is a function $F : TM \to [0, \infty)$ with the following properties:*

1. *F is C^∞ on TM_0,*
2. *F is positively 1-homogeneous on the fibers of tangent bundle TM, and*
3. *the Hessian of F^2 with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.*

The pair $(M^n, F) = F^n$ is called the fundamental function, and g_{ij} is called the fundamental tensor of the Finsler space F^n. The angular metric tensor h_{ij}, normalized supporting element l_i, and metric tensor g_{ij} of F^n are defined as follows:

$$l_i = \frac{\partial F}{\partial y^i}, \quad h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}, \quad g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{1}$$

Let F be a Finsler metric defined by $F = \sqrt[m]{A}$, where A is denoted by
$A := a_{i_1 i_2 \ldots\ldots i_m}(x) y^{i_1} y^{i_2} \ldots\ldots y^{i_m}$, with $a_{i_1 i_2 \ldots\ldots i_m}$ symmetric in all its indices [11]. Then F is called an m-th root Finsler metric. A is homogeneous in y to the degree m. Let

$$A_i = a_{i i_2 \ldots i_m}(x) y^{i_2} \ldots y^{i_m} = \frac{1}{m}\frac{\partial A}{\partial y^i}, \tag{2}$$

$$A_{ij} = a_{i j i_3 \ldots i_m}(x) y^{i_3} \ldots y^{i_m} = \frac{1}{m(m-1)}\frac{\partial^2 A}{\partial y^i \partial y^j}, \tag{3}$$

$$A_{ijk} = a_{i j k i_4 \ldots i_m}(x) y^{i_4} \ldots y^{i_m} = \frac{1}{m(m-1)(m-2)}\frac{\partial^3 A}{\partial y^i \partial y^j \partial y^k}. \tag{4}$$

The normalized supporting element of F^n is denoted by

$$l_i := F_{y^i} = \frac{\partial F}{\partial y^i} = \frac{\partial \sqrt[m]{A}}{\partial y^i} = \frac{1}{m}\frac{\frac{\partial A}{\partial y^i}}{A^{\frac{m-1}{m}}} = \frac{A_i}{F^{m-1}}. \tag{5}$$

Consider the transformation

$$\bar{F} = F + \beta + \frac{F^2}{\beta}, \tag{6}$$

where $F = \sqrt[m]{A}$ is an m-th root metric and $\beta(x, y) = b_i(x) y^i$ is a one-form on the manifold M^n. $\bar{F}$ is clearly a Finsler metric on M^n. We refer to the Finsler metric $\bar{F}$ as a special (α, β)-transformed m-th root metric throughout the paper, and $(M^n, \bar{F}) = \bar{F}^n$ as a special (α, β)-transformed Finsler space. Throughout the paper, we limit ourselves to $m > 2$, and the quantities corresponding to $\bar{F}^n$ will be denoted by putting a bar on top of that quantity.

3. Fundamental metric tensor of special (α, β)-transformed m-th root metric

Theorem 3.1. *The covariant metric tensor $\bar{g}_{ij}$ and contravariant metric tensor $\bar{g}^{ij}$ of special (α, β)-transformed m-th root Finsler space $\bar{F}^n$ are as follows:*

$$\bar{g}_{ij} = \rho A_{ij} + \rho_0 b_i b_j + \rho_1(A_i b_j + A_j b_i) + \rho_2 A_i A_j \tag{7}$$

and

$$\bar{g}^{ij} = \sigma A^{ij} + \sigma_0 b^i b^j - \sigma_1(b^i y^j + b^j y^i) + \sigma_2 y^i y^j. \tag{8}$$

Proof. The normalized supporting element l_i is obtained by differentiating (6) with respect to y^i,

$$\bar{l}_i = l_i + b_i. \tag{9}$$

Considering (5), we obtain

$$\bar{l}_i = \left[\frac{1}{F^{m-1}} + \frac{2}{\beta F^{m-2}}\right]A_i + \left[1 - \frac{F^2}{\beta^2}\right]b_i. \tag{10}$$

Differentiating (8) again with respect to y^i yields:

$$\bar{h}_{ij} = \frac{(m-1)\bar{F}}{F^{m-2}}\left[\frac{1}{F} + \frac{2}{\beta}\right]A_{ij} - \frac{\bar{F}}{F^{2(m-1)}}\left[\frac{(m-1)}{F} - \frac{2(m-2)}{\beta}\right]A_iA_j - \frac{2\bar{F}}{\beta^2 F^{m-2}}(A_ib_j + A_jb_i) + \frac{2\bar{F}}{\beta^2F^2}b_ib_j. \tag{11}$$

From (8) and (9), the fundamental metric tensor $\bar{g}_{ij}$ of Finsler space $\bar{F}^n$ is obtained as follows:

$$\bar{g}_{ij} = \bar{h}_{ij} + \bar{l}_i\bar{l}_j,$$

After simplification, we obtain

$$\bar{g}_{ij} = \rho A_{ij} + \rho_0 b_ib_j + \rho_1(A_ib_j + A_jb_i) + \rho_2 A_iA_j, \tag{12}$$

where

$$\rho = \frac{\tau(m-1)}{F^{m-1}}(1+2\tau); \quad \rho_0 = \left[\frac{\tau^4(2+F^4)}{F^4} + 1\right];$$
$$\rho_1 = \frac{1-4\tau^3}{F^{m-1}}; \quad \rho_2 = \frac{1}{F^{2(m-1)}}\left[(1-(m-1)\tau) - 2\tau^2(m-4)\right].$$

The contravariant metric tensor $\bar{g}^{ij}$ of Finsler space $\bar{F}^n$ is given by

$$\bar{g}^{ij} = \sigma A^{ij} + \sigma_0 b^ib^j - \sigma_1(b^iy^j + b^jy^i) + \sigma_2 y^iy^j. \tag{13}$$

where

$$\sigma = \frac{F^{m-2}(2\tau+1)}{2\tau^2(m-1)}; \quad \sigma_0 = \frac{4F^m[1 + p(p(1+w) - (3+w))]}{\beta^2[(m-4) - 8\tau^4d^2]};$$
$$\sigma_1 = \left[\frac{F(\beta^4(m-4) - 8F^4d^2) - 2\beta^3\tau(m-4)\bar{F}q_1}{\tau(m-1)F^2(\beta^4(m-4) - 8F^4d^2)}\right]; \ \sigma_2 = \left[\frac{b^2 - 1 + (m-1)\tau(1 + \bar{F}^2F^2q_2)}{\bar{F}^2\tau(m-1)F^2}\right];$$
$$\gamma = \frac{-8F^2}{\beta^4(m-4)}; \ p = \frac{\gamma v^2}{1+\gamma c^2}; \ v = \frac{F^{m-2}}{2\tau^2(m-1)}; \ c^2 = \frac{F^{m-3}\beta b^2}{2\tau(m-1)};$$
$$w = \frac{(m-4)\beta}{2F^m}; \ d^2 = v\left[w\beta + w^2F^m + (b^2 + w\beta)\left(1 - \frac{\gamma v(1+w)}{1+\gamma c^2}\right)\right];$$
$$q_1 = \frac{8(m-1)^2\tau^4 + 2\gamma F^{2(m-2)} + \gamma F^{m-4}(m-4)\beta}{4\tau^2(m-1)^2 + \gamma F^{m-2}b^2\tau^4}; \quad q_2 = \frac{(m-4)^2}{2F^6[(m-4) - 8\tau^4d^2]}. \tag{14}$$

Here we have used $A^{ij}A_j = A^i = y^i$ and $A_jb^j = \beta$. □

4. Spray coefficients of the Finsler space given by special (α, β)-change of m-th root metric

The following system of equations gives the geodesics of a Finsler space F^n:

$$\frac{d^2x^i}{dt^2} + G^i\left(x, \frac{dx}{dt}\right),$$

where

$$G^i = \frac{1}{4} g^{il}\left\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\right\}. \tag{15}$$

The global vector field $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ is defined by the local functions $G^i = G^i(x,y)$ is known as the spray of F, and G^i is known as the spray coefficient.
Two Finsler metrics F and $\bar{F}$ on a manifold M^n are called projectively related if there is a scalar function $P(x,y)$ defined on TM_0 such that $\bar{G}^i = G^i + Py^i$, where $\bar{G}^i$ and G^i are the geodesic spray coefficients of F^n and $\bar{F}^n$ respectively.

Theorem 4.1. *The special (α,β)-transformed m-th root Finsler metric $\bar{F}$ and m-th root Finsler metric F, on an open subset $\cup \subset R^n$, are projectively related if satisfies the following equation:*

$$\begin{aligned} &\frac{1}{4} b^i(\phi y^l - \sigma_0 b^l) \times \left[\tau(1+2\tau)\left(\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right) + 2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\right] y^j y^k \\ &= \frac{F^{m-2}(2\tau+1)A^{il}}{8\tau(m-1)} y^j y^k \left[2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\right]. \end{aligned}$$

Proof. Considering (10) the metric tensor $\bar{g}_{ij}$ of $\bar{F}^n$ can be rewritten as:

$$\bar{g}_{ij} = \tau(1+2\tau)g_{ij} + \left[\frac{1-4\tau^3}{F^{m-1}}\right](A_i b_j + A_j b_i) + \left[\frac{\beta^4 + 2 + F^4}{\beta^4}\right] b_i b_j + \left[\frac{1-\tau(1-4\tau)}{F^{2(m-1)}}\right] A_i A_j. \tag{16}$$

where

$$g_{ij} = (m-1)\frac{A_{ij}}{F^{m-2}} - (m-2)\frac{A_i A_j}{F^{2(m-1)}} \tag{17}$$

Further considering (11) contravariant metric tensor $\bar{g}^{ij}$ can be rewritten as:

$$\bar{g}^{ij} = \left[\frac{2\tau+1}{2\tau^2}\right] g^{ij} + \sigma_0 b^i b^j - \phi(b^i y^j + b^j y^i) + \varphi y^i y^j, \tag{18}$$

and

$$g^{ij} = \frac{F^{m-2}}{(m-1)} A^{ij} + \frac{(m-2)}{(m-1)}\frac{y^i y^j}{F^2},$$

$$\begin{aligned} \phi &= \left[\frac{1}{\tau(m-1)\bar{F}} - \frac{2\beta^3\tau(m-4)\bar{F}q_1}{F^2(m-1)[\beta^4(m-4) - 8F^4 d^2 \bar{F}]}\right] \\ \varphi &= \left[\frac{b^2 - (m-1)\tau - 1}{\tau(m-1)\bar{F}^2} - \frac{(m-2)}{\tau(m-2)F^2} + q_3\right] \\ q_3 &= \frac{(m-4)^2(m-1)\tau^2 - (m-2)[(m-4) - 8\tau^4 d^2]\beta^4}{2F^2\tau^2\beta^4(m-1)[(m-4) - 8\tau^4 d^2]}. \end{aligned} \tag{19}$$

where σ_0, q_1, and d^2 are expressed in (12). The spray coefficients of special (α,β)-transformed Finsler space $\bar{F}^n$ are given by

$$\bar{G}^i = \frac{1}{4} g^{il}\left\{[\bar{F}^2]_{x^k y^l} y^k - [\bar{F}^2]_{x^l}\right\}.$$

It can also be written as

$$\bar{G}^i = \frac{1}{4}\bar{g}^{il}\left[\left(2\frac{\partial \bar{g}_{jl}}{\partial x^k} - \frac{\partial \bar{g}_{jk}}{\partial x^l}\right) y^j y^k\right]. \tag{20}$$

From (16), (18) and (20), we get

$$\bar{G}^i = \frac{\bar{g}^{il}}{4}\Bigg[\Bigg(\frac{\partial}{\partial x^k}\Big\{\tau(1+2\tau)g_{jl} + \Big[\frac{1-4\tau^3}{F^{m-1}}\Big](A_jb_l + A_lb_j) + \Big[\frac{\beta^4+2+F^4}{\beta^4}\Big]b_lb_j + \Big[\frac{1-\tau(1-4\tau)}{F^{2(m-1)}}\Big]A_lA_j\Big\}$$
$$- \frac{\partial}{\partial x^l}\Big\{\tau(1+2\tau)g_{jk} + \Big[\frac{1-4\tau^3}{F^{m-1}}\Big](A_jb_k + A_kb_j) + \Big[\frac{\beta^4+2+F^4}{\beta^4}\Big]b_kb_j + \Big[\frac{1-\tau(1-4\tau)}{F^{2(m-1)}}\Big]A_kA_j\Big\}\Bigg)y^jy^k\Bigg],$$

which implies that

$$\bar{G}^i = \frac{\bar{g}^{il}}{4}\Bigg[\Bigg(2\Big\{2\tau^2\frac{\partial g_{jl}}{\partial x^k} + g_{jl}\frac{\partial}{\partial x^k}\Big(2\tau^2\Big) + \frac{\partial X_{jl}}{\partial x^k}\Big\} - \Big\{2\tau^2\frac{\partial g_{kl}}{\partial x^l} + g_{jk}\frac{\partial}{\partial x^l}(2\tau^2) + \frac{\partial X_{jk}}{\partial x^l}\Big\}\Bigg)y^jy^k\Bigg],$$

where

$$X_{jl} = \Big[\frac{1-4\tau^3}{F^{m-1}}\Big](A_jb_l + A_lb_j) + \Big[\frac{\beta^4+2+F^4}{\beta^4}\Big]b_lb_j + \Big[\frac{1-\tau(1-4\tau)}{F^{2(m-1)}}\Big]A_lA_j.$$

Now

$$\bar{G}^i = \frac{1}{4}\Bigg[\Big(\frac{2\tau+1}{2\tau^2}\Big)g^{il} + y^i(\varphi y^l - \phi b^l) - b^i(\phi y^l - \sigma_0 b^l)\Bigg]$$
$$\times\Bigg[\tau(1+2\tau)\Big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big) + 2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Bigg]y^jy^k, \tag{21}$$

where $\omega_k = \frac{\partial}{\partial x^k}\Big(\tau(1+2\tau)\Big)$. Further simplification gives

$$\bar{G}^i = \Big(\frac{2\tau+1}{2\tau}\Big)g^{il}\Big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)(2\tau+1)y^jy^k + \Big(\frac{(2\tau+1)^2}{8\tau}\Big)g^{il}\Big(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big)y^jy^k$$
$$+ \frac{F^{m-2}(2\tau+1)}{8\tau(m-1)}A^{il}\Bigg[2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Bigg]y^jy^k + \frac{(m-2)(2\tau+1)}{4\tau(m-1)F^2}y^iy^l$$
$$\times\Bigg[2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Bigg] + \frac{1}{4}\Big[y^i(\varphi y^l - \phi b^l) - b^i(\phi y^l - \sigma_0 b^l)\Big]$$
$$\times\Bigg[\tau(2\tau+1)\Big(\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big) + 2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x_k} - \frac{\partial X_{jk}}{\partial x^l}\Bigg]y^jy^k. \tag{22}$$

The equation (22) may be written as

$$\bar{G}^i = G^i + Py^i + Q^i,$$

where

$$P = \frac{1}{4}y^i(\varphi y^l - \phi b^l)\times\Bigg[\tau(1+2\tau)\Big(\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big) + 2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Bigg]y^jy^k$$
$$+ \frac{(m-2)(2\tau+1)}{4\tau(m-1)F^2}y^l\Bigg[2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Bigg]y^jy^k$$

and

$$Q^i = -\frac{1}{4}b^i(\phi y^l - \sigma_0 b^l)\times\Bigg[\tau(1+2\tau)\Big(\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big) + 2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Bigg]y^jy^k$$
$$+ \frac{F^{m-2}(2\tau+1)}{8\tau(m-1)}A^{il}\Bigg[2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\Bigg]y^jy^k$$

The metrics $\bar{F}$ and F are projectively related if $Q^i = 0$, which implies

$$\frac{1}{4}b^i(\phi y^l - \sigma_0 b^l) \times \left[\tau(1+2\tau)\left(\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right) + 2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\right] y^j y^k$$
$$= \frac{F^{m-2}(2\tau+1)A^{il}}{8\tau(m-1)} y^j y^k \left[2\omega_k g_{jl} - \omega_l g_{jk} + 2\frac{\partial X_{jl}}{\partial x^k} - \frac{\partial X_{jk}}{\partial x^l}\right]. \tag{23}$$

□

5. Locally dually flatness of a Finsler space with special (α, β)-changed m-th root metric

The notion of dually flat metrics was introduced by Amari and Nagaoka [1] when they studied the information geometry on Riemannian spaces. Later Shen extended the notion of dually flatness to Finsler metrics [10]. If there is a standard coordinate system (x^i, y^i) in TM at any point such that $[\bar{F}^2]_{x^k y_{y^l}} y^k = 2[\bar{F}^2]_{x^l}$, then the Finsler space on manifold M^n is said to be locally dual flat. The coordinate system (x^i) is an adapted local coordinate system. It is well-known that every locally Minkowskian metric is locally flat.

Theorem 5.1. *Let $\bar{F}$ be a special (α, β)-changed m-th root Finsler metric on a Finsler manifold M^n. Then, $\bar{F}$ is locally dually flat metric if and only if it satisfies the following condition:*

$$A_{x^l} = \frac{A_0 A_{y^l}}{2}\left[\frac{F}{mA\bar{F}} - \frac{(1-m)}{mA} + \frac{(4-m)}{\beta F^2}\right] + A_{0l} + A_{y^l} y^k \left[\frac{\beta_k}{2\bar{F}} - \frac{1}{\beta}\right] + A_{0l}\left[\frac{1}{2\bar{F}} - \frac{b_l}{\beta}\right]$$
$$+ \frac{m\beta_k y^k b_l}{2}\left[\frac{1}{\bar{F}A^{\frac{1-m}{m}}} + \frac{3A}{2\beta^2}\right] - \frac{m\beta_l}{2}\left[\frac{1}{A^{\frac{1-m}{m}}} - \frac{A}{\beta}\right] - A_{x^l} + \frac{m}{2\beta}Ab_l. \tag{24}$$

Proof. Consider the special (α, β)-changed m-th root Finsler metric $\bar{F} = F + \beta + \frac{F^2}{\beta}$, where F is an m-th root metric. Then, we have

$$[\bar{F}^2]_{x^l} = \left[F + \beta + \frac{F^2}{\beta}\right]^2_{x^l} = 2\left[F + \beta + \frac{F^2}{\beta}\right]\left[\frac{1}{m}A^{\frac{1-m}{m}}A_{x^l} + \beta_l + \frac{(\frac{2}{m})A^{\frac{1-m}{m}}A_{x^l}}{\beta} - \frac{F^2\beta_l}{\beta^2}\right], \tag{25}$$

From (25), we get

$$[\bar{F}^2]_{x^l} = \frac{2}{m}\left[A^{\frac{2-m}{m}} + A^{\frac{1-m}{m}} + \frac{2}{\beta^2}A^{\frac{4-m}{m}}\right]A_{x^l} + 2\left[A^{\frac{1}{m}} + \beta - \frac{A^{\frac{4}{m}}}{\beta^3}\right]\beta_l. \tag{26}$$

If we put $b_{ij} = \frac{\partial b_i}{\partial x^j}$, then $\beta_j = \frac{\partial \beta}{\partial x^j} = b_{ij}y^j$. Furthermore, from (26), we obtain

$$[\bar{F}^2]_{x^k} = \frac{2}{m}\left[A^{\frac{2-m}{m}} + A^{\frac{1-m}{m}} + \frac{2}{\beta^2}A^{\frac{4-m}{m}}\right]A_{x^k} + 2\left[A^{\frac{1}{m}} + \beta - \frac{A^{\frac{4}{m}}}{\beta^3}\right]\beta_k. \tag{27}$$

and

$$[\bar{F}^2]_{x^k y^l} = \frac{2}{m}A^{\frac{2-m}{m}}A_{x^k y^l} + \frac{2}{m}\left(\frac{2-m}{m}\right)A^{\frac{2-2m}{m}}A_{y^l}A_{x^k} + \frac{1}{m}2\beta_k A^{\frac{1-m}{m}}A_{y^l} + 2A^{\frac{1}{m}}b_{lk} + \frac{2}{m}\beta A_{x^k}\left(\frac{1-m}{m}\right)A^{\frac{1-2m}{m}}A_{y^l}$$
$$+ \frac{2}{m}\beta A^{\frac{1-m}{m}}A_{x^k y^l} + \frac{2}{m}A^{\frac{1-m}{m}}A_{x^k}b_l + 2b_l\beta_k + 2\beta b_{lk} + \frac{4}{m\beta^2}A^{\frac{4-m}{m}}A_{x^k y^l} + \frac{A}{m\beta^2}\left(\frac{4-m}{m}\right)A^{\frac{4-2m}{m}}A_{y^l}$$
$$- \frac{8}{m\beta^3}A^{\frac{4-m}{m}}A_{x^k b_l} - \frac{8}{m\beta^3}A^{\frac{4-m}{m}}A_{y^l}\beta_l + \frac{6}{\beta^4}A^{\frac{4}{m}}\beta_k b_l + \frac{2}{\beta^3}A^{\frac{4}{m}}b_{lk}. \tag{28}$$

For the Finsler metric $\bar{F}$ to be locally dually flat, we need

$$[\bar{F}^2]_{x^k y^l} y^k - 2[\bar{F}^2]_{x^l} = 0. \tag{29}$$

Therefore, from (26)-(29), we have

$$[\bar{F}^2]_{x^k y^l} y^k - 2[\bar{F}^2]_{x^l} = \Big[\frac{2}{m} A^{\frac{2-m}{m}} A_{x^k y^l} + \frac{2}{m}\Big(\frac{2-m}{m}\Big) A^{\frac{2-2m}{m}} A_{y^l} A_{x^k} + \frac{1}{m} 2\beta_k A^{\frac{1-m}{m}} A_{y^l} + 2A^{\frac{1}{m}} b_{lk}$$
$$+ \frac{2}{m}\beta A_{x^k}\Big(\frac{1-m}{m}\Big) A^{\frac{1-2m}{m}} A_{y^l} + \frac{2}{m}\beta A^{\frac{1-m}{m}} A_{x^k y^l} + \frac{2}{m} A^{\frac{1-m}{m}} A_{x^k} b_l + 2b_l\beta_k + 2\beta b_{lk}$$
$$+ \frac{4}{m\beta^2} A^{\frac{4-m}{m}} A_{x^k y^l} + \frac{A}{m\beta^2}\Big(\frac{4-m}{m}\Big) A^{\frac{4-2m}{m}} A_{y^l} - \frac{8}{m\beta^3} A^{\frac{4-m}{m}} A_{x^k b_l} - \frac{8}{m\beta^3} A^{\frac{4-m}{m}} A_{y^l}\beta_l$$
$$+ \frac{6}{\beta^4} A^{\frac{4}{m}} \beta_k b_l + \frac{2}{\beta^3} A^{\frac{4}{m}} b_{lk}\Big] y^k - \Big[\frac{2}{m} A^{\frac{2-m}{m}} A_{x^l} + 2A^{\frac{1}{m}}\beta_l + \frac{2}{m} A^{\frac{2-m}{m}} \beta A_{x^l} + 2\beta\beta_l$$
$$+ \frac{4}{m\beta^2} A^{\frac{4-m}{m}} A_{x^l} - \frac{2}{\beta^3} A^{\frac{4}{m}} \beta_l\Big] = 0, \tag{30}$$

which implies that

$$A_{x^l}\frac{4}{m}\Big[A^{\frac{1-m}{m}} + 2A^{\frac{3-m}{m}}\Big]\Big(F + \beta + \frac{F^2}{\beta}\Big) = \frac{2}{m} A_0 A_{y^l} A^{\frac{1-2m}{m}}\Big[\Big(\frac{2-m}{m}\Big)F + \beta\Big(\frac{1-m}{m}\Big) + \frac{2F}{\beta^2}\Big(\frac{4-m}{m}\Big) A^{\frac{2-2m}{m}}\Big]$$
$$+ \frac{2}{m} A_0 A_{y^l} A^{\frac{1-2m}{m}}\Big[F + \beta + \frac{2F}{\beta^2} A^{\frac{2-m}{m}}\Big] + \frac{2}{m} A_0 b_l A^{\frac{1-m}{m}}\Big[1 - \frac{4}{\beta^3} F A^{\frac{2-m}{m}}\Big]$$
$$+ \frac{2}{m} A^{\frac{1-m}{m}} A_{y^l} y^k\Big[\beta_k - \frac{4}{\beta^3} F A^{\frac{2-m}{m}}\Big] - 2A^{\frac{1}{m}}\beta_l\Big[1 + \frac{3}{\beta^3} F A^{\frac{3}{m}}\Big] - 2\beta\beta_l$$
$$+ 2b_l\beta_k y^k\Big[1 + \frac{3}{\beta^3} F A^{\frac{3}{m}}\Big] - \frac{8}{m\beta^2} A^{\frac{4-m}{m}} A_{x^l}, \tag{31}$$

where $A_0 = A_{x^k} y^k$, and $A_{0l} = A_{x^k y^l} y^k$. Therefore, $\bar{F}$ is a locally dually flat metric if and only if

$$A_{x^l} = \frac{A_0 A_{y^l}}{2}\Big[\frac{F}{mA\bar{F}} - \frac{(1-m)}{mA} + \frac{(4-m)}{\beta F^2}\Big] + A_{0l} + A_{y^l} y^k\Big[\frac{\beta_k}{2\bar{F}} - \frac{1}{\beta}\Big] + A_{0l}\Big[\frac{1}{2\bar{F}} - \frac{b_l}{\beta}\Big]$$
$$+ \frac{m\beta_k y^k b_l}{2}\Big[\frac{1}{\bar{F} A^{\frac{1-m}{m}}} + \frac{3A}{2\beta^2}\Big] - \frac{m\beta_l}{2}\Big[\frac{1}{A^{\frac{1-m}{m}}} - \frac{A}{\beta}\Big] - A_{x^l} + \frac{m}{2\beta} A b_l, \tag{32}$$

which completes the proof. □

6. Projectively flatness of a Finsler space with special (α, β)-changed m-th root metric

Theorem 6.1. *Let $\bar{F}$ be a special (α, β)-changed m-th root Finsler metric on a Finsler manifold M^n. Then, $\bar{F}$ is a projectively flat metric if and only if it satisfies the following condition:*

$$A_{x^l} = \Big[\Big(\frac{1-m}{m}\Big) A^2 + \Big(\frac{1-m}{m}\Big)\frac{\beta}{2} A^{-1}\Big] A_{y^l} A_0 + 2\Big(\frac{1-m}{m}\Big) A^2\Big[A + \frac{1}{m\beta}\Big] A_{0l} + A_{y^l} A_0$$
$$- \Big[\frac{2mA + m\beta}{2A^{\frac{2-m}{m}}} - \frac{2A}{\beta^2} - \frac{1}{\beta}\Big](A_0 b_l - A_{y^l}\beta_k y^k) + \frac{mA}{\beta^3}(\beta + 2A)\beta_k b_l y^k. \tag{33}$$

Proof. A Finsler metric $\bar{F} = \bar{F}(x, y)$ on an open subset $U \subset R^n$ is projectively flat if and only if it satisfies the following equation:

$$[\bar{F}]_{x^k y^l} y^k - 2[\bar{F}]_{x^l} = 0. \tag{34}$$

Since we have $\bar{F} = F + \beta + \frac{F^2}{\beta}$, where $F = \sqrt[m]{A}$, we have

$$[\bar{F}]_{x^l} = \Big[\frac{1}{m} A^{\frac{1-m}{m}} + \frac{2}{m\beta} A^{\frac{2-m}{m}}\Big] A_{x^l} + \Big[1 - \frac{A^{\frac{2}{m}}}{\beta^2}\Big]\beta_l. \tag{35}$$

From (34) we get

$$[\bar{F}]_{x^k} = \left[\frac{1}{m}A^{\frac{1-m}{m}} + \frac{2}{m\beta}A^{\frac{2-m}{m}}\right]A_{x^k} + \left[1 - \frac{A^{\frac{2}{m}}}{\beta^2}\right]\beta_k, \tag{36}$$

which implies

$$\begin{aligned}[\bar{F}]_{x^k y^l} &= \left[\frac{1}{m}\left(\frac{1-m}{m}\right)A^{\frac{2(1-m)}{m}} + \frac{2}{m\beta}\left(\frac{2-m}{m}\right)A^{\frac{2(2-m)}{m}}\right]A_{y^l}A_{x^k} + \left[\frac{1}{m}A^{\frac{1-m}{m}} + \frac{2}{m\beta}A^{\frac{2-m}{m}}\right]A_{x^k y^l} \\ &+ \left[1 - \frac{2}{m\beta^2}A^{\frac{2-m}{m}}\right]A_{x^k}b_l + \left[1 - \frac{2}{m\beta^2}A^{\frac{2-m}{m}}\right]A_{y^l}\beta_k + \frac{2}{\beta^3}A^{\frac{2}{m}}\beta_k b_l - \frac{F^2}{\beta^2}b_{lk}.\end{aligned} \tag{37}$$

For the Finsler metric $\bar{F}$ to be projectively flat, we must have (33). Therefore, from (33)-(35), we get

$$\begin{aligned}[\bar{F}]_{x^k y^l}y^k - F_{x^l} &= \left[\frac{1}{m}\left(\frac{1-m}{m}\right)A^{\frac{2(1-m)}{m}} + \frac{2}{m\beta}\left(\frac{2-m}{m}\right)A^{\frac{2(2-m)}{m}}\right]A_{y^l}A_{x^k}y^k + \left[\frac{1}{m}A^{\frac{1-m}{m}} + \frac{2}{m\beta}A^{\frac{2-m}{m}}\right]A_{x^k y^l}y^k \\ &+ \left[1 - \frac{2}{m\beta^2}A^{\frac{2-m}{m}}\right]A_{x^k}b_l y^k + \left[1 - \frac{2}{m\beta^2}A^{\frac{2-m}{m}}\right]A_{y^l}\beta_k y^k + \frac{2}{\beta^3}A^{\frac{2}{m}}\beta_k b_l y^k - \frac{F^2}{\beta^2}b_{lk}y^k \\ &- \left[\frac{1}{m}A^{\frac{1-m}{m}} + \frac{2}{m\beta}A^{\frac{2-m}{m}}\right]A_{x^l}y^k + \left[1 + \frac{A^{\frac{2}{m}}}{\beta^2}\right]\beta_l y^k = 0,\end{aligned} \tag{38}$$

which implies that

$$\begin{aligned}&A_{x^l}\left[\frac{1}{m}A^{\frac{1-m}{m}} + \frac{2}{m\beta}A^{\frac{2-m}{m}}\right] = \left[\frac{1}{m}\left(\frac{1-m}{m}\right)A^{\frac{2(1-m)}{m}}\right]A_{y^l}A_0 + \left[\frac{2}{m\beta}\left(\frac{2-m}{m}\right)A^{\frac{2(2-m)}{m}}\right]A_{0l} + \left[\frac{1}{m}A^{\frac{1-m}{m}} + \frac{2}{m\beta}A^{\frac{2-m}{m}}\right] \\ &A_{y^l}A_0 + \left[1 - \frac{2}{m\beta^2}A^{\frac{2-m}{m}}\right]A_0 b_l + \left[1 - \frac{2}{m\beta^2}A^{\frac{2-m}{m}}\right]A_{y^l}\beta_k y^k + \frac{2}{\beta^3}A^{\frac{2}{m}}\beta_k b_l y^k = 0,\end{aligned} \tag{39}$$

where $A_0 = A_{x^k}y^k$, and $A_{0l} = A_{x^k y^l}y^k$. Therefore $\bar{F}$ is a projectively flat metric if and only if

$$\begin{aligned}A_{x^l} &= \left[\left(\frac{1-m}{m}\right)A^2 + \left(\frac{1-m}{m}\right)\frac{\beta}{2}A^{-1}\right]A_{y^l}A_0 + 2\left(\frac{1-m}{m}\right)A^2\left[A + \frac{1}{m\beta}\right]A_{0l} + A_{y^l}A_0 \\ &- \left[\frac{2mA + m\beta}{2A^{\frac{2-m}{m}}} - \frac{2A}{\beta^2} - \frac{1}{\beta}\right](A_0 b_l - A_{y^l}\beta_k y^k) + \frac{mA}{\beta^3}(\beta + 2A)\beta_k b_l y^k,\end{aligned} \tag{40}$$

which completes the proof. □

7. Conclusions

The m-th root metric is considered a direct generalization of the Riemannian metric in a view that the m-th root metric becomes Riemannian if $m = 2$. It founds a lot of applications recently in physics and biology. In this paper, we have obtained the conditions under which the transformed Finsler space and the original Finsler space with m-th root metric are projectively related. Further, the necessary and sufficient conditions under which the transformed Finsler space is projectively flat and locally dually flat are derived. Also, we have studied a sufficient condition under which both the transformed Finsler space and the given Finsler space reduce to Riemannian.

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