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# Applications and generalizations of idempotents

**Liufeng Cao[a,*], Junchao Wei[b], Hua Yao[c]**

[a]*Department of Mathematics, Yancheng Institute of Technology*
[b]*School of Mathematical Sciences, Yangzhou University*
[c]*School of Mathematics and Statistics, Huanghuai University*

**Abstract.** In this paper, we introduce the concepts of one-sided $x$-idempotents, one-sided $x$-equal elements, one-sided $x$-projections, and list some properties of them. Furthermore, we apply these elements to describe generalized inverses in rings with involution.

## 1. Introduction

Throughout, all rings are associative and unital, the symbols $\mathbb{Z}$, $\mathbb{Z}_+$, $N(R)$, $E(R)$ and $Z(R)$ stand for the ring of integers, the set of positive integers, the set of all nilpotent elements, the set of all idempotents and the center of $R$, respectively. In the studies of ring theory, idempotents play an important role. For example, the definitions of clean rings [21], left quasi-duo rings [23], quasi-normal rings [26] are related to idempotents. Furthermore, idempotents are often used to describe rings satisfying given conditions. For instances, in [24], based on the works [7] and [5], Wei defined the generalized weakly symmetric rings, and use idempotents to describe generalized weakly symmetric rings. Then, Meng et al. in [11], [12] and [13] used idempotents to study $e$-symmetric rings and weak $e$-symmetric rings, where $e \in E(R)$. The studies of properties of idempotents in rings appear in [8], [9] and [10]. For other studies of idempotents in rings, one can refer [3], [4], [6], [25] and [29]. Motivated by the previous works, we give the definitions of one-sided $x$-idempotents, one-sided $x$-equal elements and one-sided $x$-projections in this article and study their properties. Moreover, we apply these elements to characterize EP and SEP elements in involution rings.

An element $e \in R$ is said to be anti-idempotent if $e^2 = -e$. We call an element $e \in E(R)$ left (resp. right) minimal idempotent of $R$ if $Re$ (resp. $eR$) is a minimal left (resp. right) ideal of $R$. Denote the set of all left (resp. right) minimal idempotents of $R$ by $ME_l(R)$ (resp. $ME_r(R)$). An idempotent $e \in R$ is called left (resp. right) semicentral if $ae = eae$ (resp. $ea = eae$) for any $a \in R$. Moreover, if $e$ is both left and right semicentral, then $e$ is a central idempotent. An element $a \in R$ is said to be regular if there exists $b \in R$ such that $a = aba$, where $b$ is called an inner inverse of $a$. The set of all regular elements of $R$ is denoted by $R^{reg}$. In general, the

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* Corresponding author: Liufeng Cao
*Email addresses:* 1204719495@qq.com (Liufeng Cao), jcweiyz@126.com (Junchao Wei), dalarstone@126.com (Hua Yao)

inner inverse of $a$ is not unique, we denote the set of all inner inverses of $a$ by $a\{1\}$, and $a^-$ stands for some fixed inner inverse of $a$. We say that an element $a \in R$ is group invertible if there exists $a^\# \in R$ satisfying

$$a = aa^\# a,\ a^\# = a^\# aa^\#,\ aa^\# = a^\# a,$$

where $a^\#$ is called the group inverse of $a$, and if $a^\#$ exists, then it is unique [22].

A map $* : R \to R$ is said to be an involution of $R$ if

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^* a^*.$$

A ring with an involution $*$ is an involution ring (or $*$-ring). We call an element $a \in R$ Hermitian if $a^* = a$ [14], and the set of all Hermitian elements of $R$ is denoted by $R^{Her}$. In particular, if $e \in E(R)$ is Hermitian, then $e$ is called a projection, and we write $R^{proj}$ for the set of all projections of $R$. Furthermore, if $e \in R^{proj}$ is central, then $e$ is a central projection. An element $a \in R$ is said to be a partial isometry if $a = aa^*a$ [19], and the set of all partial isometries of $R$ is denoted by $R^{PI}$. We call $a^+$ the Moore-Penrose inverse (MP-inverse) of $a$, if

$$a = aa^+a,\ a^+ = a^+aa^+,\ (aa^+)^* = aa^+,\ (a^+a)^* = a^+a.$$

$a^+$ is unique if it exists [22]. Denote the set of all MP-invertible elements of $R$ by $R^+$. In particular, if $a \in R^\# \cap R^+$ and $a^\# = a^+$, then $a$ is called EP [20]. $R^{EP}$ stands for the set of all EP elements of $R$. Moreover, $a$ is said to be SEP if $a \in R^\# \cap R^+$ and $a^\# = a^+ = a^*$ [14]. The set of all SEP elements of $R$ is denoted by $R^{SEP}$. In recent years, the studies of characterizations of EP and SEP elements in involution rings are popular [15–18, 27, 29]. $a_l^{\textcircled{\#}}$ is called a left core inverse of $a$ if $aa_l^{\textcircled{\#}}a = a$, $a_l^{\textcircled{\#}}aa_l^{\textcircled{\#}} = a_l^{\textcircled{\#}}$, $a_l^{\textcircled{\#}}a^2 = a$ and $(aa_l^{\textcircled{\#}})^* = aa_l^{\textcircled{\#}}$ [2]. $a^{\textcircled{\#}}$ is said to be a core inverse of $a$ if $a{a^{\textcircled{\#}}}^2 = a^{\textcircled{\#}}$, $a^{\textcircled{\#}}a^2 = a$ and $(aa^{\textcircled{\#}})^* = aa^{\textcircled{\#}}$ [1]. It is noted that $a^{\textcircled{\#}} = a{a^{\textcircled{\#}}}^2 = a^{\textcircled{\#}}a^2{a^{\textcircled{\#}}}^2 = (a^{\textcircled{\#}}a)(a{a^{\textcircled{\#}}}^2) = a^{\textcircled{\#}}aa^{\textcircled{\#}}$ and $a = a^{\textcircled{\#}}a^2 = a{a^{\textcircled{\#}}}^2a^2 = (aa^{\textcircled{\#}})(a^{\textcircled{\#}}a^2) = aa^{\textcircled{\#}}a$. In this paper, we will first define the one-sided $x$-idempotents, one-sided $x$-equal elements and one-sided $x$-projections, and then apply these elements to characterize EP and SEP elements.

The paper is organized as follows: In Section 2, we define one-sided $x$-idempotents and give some results. In Section 3, we give the definition of one-sided $x$-equal elements, and study the properties of them. In Section 4, we propose the concept of one-sided $x$-projections, and give some characterizations of them. In Section 5, we apply these elements to describe EP and SEP elements in involution rings.

## 2. One-sided $x$-idempotent

**Definition 2.1.** *Let $x \in R$. Then an element $a \in R$ is called a left (resp. right) $x$-idempotent if $a^2 = xa$ (resp. $a^2 = ax$).*

Consider the non-commutative polynomial ring $\mathbb{Z} < x, y > /(x^2 - yx, y^2 - xy)$. It is easy to check that $x^2 = yx \neq xy$ and $y^2 = xy$, which implies that one-sided $x$-idempotent is not unique and a left $x$-idempotent is not necessary a right $x$-idempotent. Furthermore, for the same element $0 \neq a \in R$, $a$ can both be a left (resp. right) $x$-idempotent and a left (resp. right) $y$-idempotent with $x \neq y$.

**Proposition 2.2.** *Let $x \in R$. Then*

*(1) $x$ is a left and right $x$-idempotent.*

*(2) $0 \neq x \in N(R)$ if and only if there exists $0 \leq n \in \mathbb{Z}$ such that $x$ is a left (right) $(x + x^n)$-idempotent.*

*(3) $x \in N(R)$ if and only if there exists some $k \in \mathbb{Z}_+$ such that $x^k$ is a left (right) 0-idempotent.*

*(4) $x$ is anti-idempotent if and only if $x$ is a left (right) $(2x+1)$-idempotent.*

*Proof.* (1) It is obvious.

(2) $\Rightarrow$ If $0 \neq x \in N(R)$, then there exists some $n \in \mathbb{Z}_+$ such that $x^n = 0$. Thus, $x^2 = (x + x^{n-1})x$.

$\Leftarrow$ Provided that $0 \leq n \in \mathbb{Z}$ such that $x^2 = (x + x^n)x$, then $x^{n+1} = 0$. It follows that $x \in N(R)$.

(3) $\Rightarrow$ If $x^n = 0$, then taking $k = n$, we have

$$(x^k)^2 = x^{2n} = 0 = 0x^k.$$

$\Leftarrow$ Assume that $(x^k)^2 = x^{2k} = 0x^k = 0$, then $x \in N(R)$.

(4) $\Rightarrow$ By $x^2 = -x$, then $(2x+1)x = x(2x+1) = 2x^2 + x = 2x^2 - x^2 = x^2$.

$\Leftarrow$ If $(2x+1)x = x(2x+1) = 2x^2 + x = x^2$, then $x^2 = -x$. □

**Proposition 2.3.** *Let $e \in R$. Then the following statements are equivalent:*

*(1) $e \in E(R)$;*

*(2) $e$ is a left 1-idempotent;*

*(3) $e$ is a right 1-idempotent;*

*(4) $e$ is a left $(2e-1)$-idempotent;*

*(5) $e$ is a right $(2e-1)$-idempotent.*

*Proof.* It follows from a straightforward verification. □

**Theorem 2.4.** *Let $a, x \in R$. Then $a$ is a left (resp. right) $x$-idempotent if and only if $x - a$ is a right (resp. left) $x$-idempotent.*

*Proof.* $\Rightarrow$ By $a^2 = xa$, we have $(x-a)^2 = x^2 - xa - ax + a^2 = x^2 - ax = (x-a)x$, which gives the desired result.

$\Leftarrow$ If $(x-a)^2 = x^2 - xa - ax + a^2 = (x-a)x = x^2 - ax$, then $a^2 - xa = 0$, i.e., $a^2 = xa$. It follows that $a$ is a left $x$-idempotent. □

**Theorem 2.5.** *Let $e \in E(R)$. Then $e$ is a left (resp. right) semicentral element if and only if $xe$ (resp. $ex$) is a left (resp. right) $x$-idempotent for each $x \in R$.*

*Proof.* $\Rightarrow$ If $exe = xe$, then for any $x \in R$, $(xe)^2 = x(exe) = x(xe)$. It follows that $xe$ is a left $x$-idempotent.

$\Leftarrow$ Taking $y, z \in R$, then

$$(ye)^2 = y^2e, \;\; (ze)^2 = z^2e, \;\; ((y+z)e)^2 = (y+z)^2e.$$

Hence, $yeze + zeye = yze + zye$. Setting $y = 1 - e$, we obtain $(1-e)ze = 0$. This shows that $e$ is left semicentral. □

**Corollary 2.6.** *Let $e \in E(R)$. Then $e$ is left (resp. right) semicentral if and only if $x - xe$ (resp. $x - ex$) is a right (resp. left) $x$-idempotent for every $x \in R$.*

*Proof.* It follows from Theorems 2.4 and 2.5. □

**Theorem 2.7.** *Let $e \in R$. Then the following statements are equivalent:*

*(1) $e \in E(R)$;*

*(2) $e$ is a left $(x+1)$-idempotent for each $x \in l(e)$, where $l(e) = \{y \in R | ye = 0\}$;*

*(3) $e$ is a right $(x+1)$-idempotent for every $x \in r(e)$, where $r(e) = \{y \in R | ey = 0\}$.*

*Proof.* (1)$\Rightarrow$(2) By $e^2 = e$, for any $x \in l(e)$, $(x+1)e = xe + e = e = e^2$.

(2)$\Rightarrow$(1) If $x \in l(e)$, then $e^2 = (x+1)e = e$.

(1)$\Leftrightarrow$(3) The proof is similar. □

Note that if $e \in E(R)$, then $(1-e)e = e(1-e) = 0$. Thus, from Theorem 2.7 we infer the following corollary.

**Corollary 2.8.** *Let $e \in R$. Then the following statements are equivalent:*

*(1) $e \in E(R)$;*

*(2) $e$ is a left $(x(1-e)+1)$-idempotent for every $x \in R$;*

*(3) $e$ is a right $((1-e)x+1)$-idempotent for each $x \in R$.*

**Theorem 2.9.** *Let $e, x \in R$. Then the following statements are equivalent:*
*(1) $e$ is a left $x$-idempotent;*
*(2) $\begin{pmatrix} e & 0 \\ 0 & y \end{pmatrix}$ is a left $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$-idempotent for each $y \in R$;*
*(3) $\begin{pmatrix} e & y \\ 0 & 0 \end{pmatrix}$ is a left $\begin{pmatrix} x & u \\ 0 & v \end{pmatrix}$-idempotent for any $y \in r(e-x)$, $u, v \in R$;*
*(4) $\begin{pmatrix} e & y \\ 0 & e \end{pmatrix}$ is a left $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$-idempotent for every $y \in r(e-x)$.*

*Proof.* (1)⇒(2) By $e^2 = xe$,

$$\begin{pmatrix} e & 0 \\ 0 & y \end{pmatrix}^2 = \begin{pmatrix} e^2 & 0 \\ 0 & y^2 \end{pmatrix} = \begin{pmatrix} xe & 0 \\ 0 & y^2 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & y \end{pmatrix}.$$

(2)⇒(3) By assumption, one has $e^2 = xe$. By a straightforward computation,

$$\begin{pmatrix} e & y \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} e^2 & ey \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} x & u \\ 0 & v \end{pmatrix}\begin{pmatrix} e & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} xe & xy \\ 0 & 0 \end{pmatrix}.$$

Since $y \in r(e-x)$, $xy = ey$. Thus $\begin{pmatrix} e^2 & ey \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} xe & xy \\ 0 & 0 \end{pmatrix}$. This gives the desired result.

(3)⇒(4) By hypothesis, $e^2 = xe$. Moreover, $ey = xy$ by $y \in r(e-x)$. Thus,

$$\begin{pmatrix} e & y \\ 0 & e \end{pmatrix}^2 = \begin{pmatrix} e^2 & ey + ye \\ 0 & e^2 \end{pmatrix} = \begin{pmatrix} xe & xy + ye \\ 0 & xe \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\begin{pmatrix} e & y \\ 0 & e \end{pmatrix}.$$

This proves (4).

(4)⇒(1) By assumption, we have $e^2 = xe$. This shows (1). □

Similarly, we have the following proposition.

**Proposition 2.10.** *Let $e, x \in R$. Then the following statements are equivalent:*
*(1) $e$ is a right $x$-idempotent;*
*(2) $\begin{pmatrix} e & 0 \\ 0 & y \end{pmatrix}$ is a right $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$-idempotent for every $y \in R$;*
*(3) $\begin{pmatrix} e & 0 \\ y & 0 \end{pmatrix}$ is a right $\begin{pmatrix} x & 0 \\ u & v \end{pmatrix}$-idempotent for any $y \in l(e-x)$, $u, v \in R$;*
*(4) $\begin{pmatrix} e & 0 \\ y & e \end{pmatrix}$ is a right $\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}$-idempotent for each $y \in l(e-x)$.*

Notice that $x$ is a left and right $x$-idempotent, and in this case, $y \in l(0), r(0)$ for any $y \in R$. Hence, by Theorem 2.9 and Proposition 2.10, we have the following corollary.

**Corollary 2.11.** *Let $x, y \in R$. Then*
*(1) $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ is a left $\begin{pmatrix} x & u \\ 0 & v \end{pmatrix}$-idempotent for any $u, v \in R$.*
*(2) $\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$ is a right $\begin{pmatrix} x & 0 \\ u & v \end{pmatrix}$-idempotent for any $u, v \in R$.*

**Proposition 2.12.** *Let $a, x \in R$ and $a$ be a left $x$-idempotent. Then $axa$ is a left $ax^2$-idempotent.*

*Proof.* Assume that $a^2 = xa$, then $(axa)^2 = axa^2xa = ax^2axa = ax^2(axa)$. □

In general, the converse of Proposition 2.12 is not true. For example, in $(\mathbb{Z}_4, +, \times)$, we set $a = [1]$ and $x = [0]$, then $[0]$ is a left $[0]$-idempotent, however $[1]$ is not a left $[0]$-idempotent. The following result states when the converse of Proposition 2.12 holds.

**Proposition 2.13.** *Let $a, x \in R$ and $axa$ be a left $ax^2$-idempotent. Then $a$ be a left $x$-idempotent if one of the following statements holds:*

*(1) $x = a^{\#}$;*

*(2) $x_l^{\textcircled{\#}} = a$;*

*(3) $a^{\textcircled{\#}} = x$;*

*(4) $x^{\textcircled{\#}} = a$.*

*Proof.* By hypothesis, $axa^2xa = ax^2axa$.

(1) If $x = a^{\#}$, then $aa^{\#}a^2a^{\#}a = a^2a^{\#}a = a^2$ and $a(a^{\#})^2aa^{\#}a = a(a^{\#})^2a = a^{\#}a$. Then $a^2 = a^{\#}a = xa$ by assumption.

(2) By $x_l^{\textcircled{\#}} = a$, one has $xax = x$, $axa = a$ and $ax^2 = x$. Thus $axa^2xa = (axa)(axa) = a^2$ and $ax^2axa = xaxa = xa$. Hence, $a^2 = xa$.

The proofs of (3) and (4) are similar to the proof of (2). □

**Proposition 2.14.** *Let $x, y, z \in R$ and $y, z$ be left (resp. right) $x$-idempotents. Then $y + z$ is a left (resp. right) $x$-idempotent if and only if $yz + zy = 0$.*

*Proof.* Since $y, z$ are left $x$-idempotents, $y^2 = xy$ and $z^2 = xz$. Then $(y+z)^2 = y^2+yz+zy+z^2 = xy+yz+zy+xz = x(y + z) + yz + zy$. Hence, $(y + z)^2 = x(y + z)$ if and only if $yz + zy = 0$. Thus the proof is completed. □

**Theorem 2.15.** *Let $e \in R$. Then $e \in E(R)$ if and only if $e$ is a left $(x-xe+e)$-idempotent (a right $(x-ex+e)$-idempotent) for any $x \in R$.*

*Proof.* ⇒ Suppose that $e^2 = e$, then

$$(x - xe + e)e = xe - xe^2 + e^2 = xe - xe + e = e.$$

⇐ By $(x - xe + e)e = xe - xe^2 + e^2 = e^2$, then $xe - xe^2 = 0$. In particular, taking $x = 1$, then $e = e^2$. Hence, $e \in E(R)$. □

**Proposition 2.16.** *Let $e \in ME_l(R)$ (resp. $e \in ME_r(R)$) and $e$ be left (resp. right) semicentral. Then for any $x \in R$ with $xe \neq 0$ (resp. $ex \neq 0$), there exists some $y \in R$ such that $e$ is a left $xy$-idempotent (resp. a right $yx$-idempotent).*

*Proof.* Since $xe \neq 0$, $Rxe = Re$. Assume that $e = yxe$, we obtain $e = yexe$, and so $ye \neq 0$. By $Rye = Re$, we suppose that $e = zye$. Hence, $ze = z(yexe) = (zye)(xe) = exe = xe$, and so $e = zye = zeye = xeye = xye$. It follows that $e^2 = e = xye$, and thus $e$ is a left $xy$-idempotent. □

**Proposition 2.17.** *For any $x \in R$ with $xe \neq 0$ (resp. $ex \neq 0$), if there always exists some $y \in R$ such that $e$ is a left $xy$-idempotent (resp. a right $yx$-idempotent), then $e$ is left (resp. right) semicentral.*

*Proof.* Taking any $x \in R$, if $(1 - e)xe \neq 0$, then by the assumption, there exists some $y \in R$ such that $e = e^2 = (1 - e)xye$, a contradiction. Hence, $(1 - e)xe = 0$ for any $x \in R$, which implies that $e$ is left semicentral. □

From Propositions 2.16 and 2.17, we infer the following corollary immediately.

**Corollary 2.18.** *Let $e \in ME_l(R)$ (resp. $e \in ME_r(R)$). Then $e$ is left (resp. right) semicentral if and only if for any $x \in R$ with $xe \neq 0$ (resp. $ex \neq 0$), there exists some $y \in R$ such that $e$ is a left $xy$-idempotent (resp. a right $yx$-idempotent).*

**Theorem 2.19.** *Let $x \in R$. Then $x \in Z(R)$ if and only if for each $a \in R$, $xa - ax$ is a left (resp. right) $xa$-idempotent and a left (resp. right) $(xa + a)$-idempotent.*

*Proof.* $\Rightarrow$ If $x \in Z(R)$, then it is obvious that $xa - ax = 0$ is a left $xa$-idempotent and a left $(xa + a)$-idempotent.
$\Leftarrow$ By $(xa - ax)^2 = xa(xa - ax)$, we have $ax(ax - xa) = 0$, i.e., $ax[a, x] = 0$. According to $(xa - ax)^2 = (xa + a)(xa - ax)$, then $a(xa - ax) = 0$, i.e., $a[a, x] = 0$. Since $a + 1 \in R$, $(a + 1)[a + 1, x] = 0$, thus $[a, x] = 0$. It follows that $x \in Z(R)$. □

**Theorem 2.20.** *Let $x \in R$. Then $x \in Z(R)$ if and only if $xa - ax$ is a left (right) $a$-idempotent for each $a \in R$.*

*Proof.* $\Rightarrow$ Note that $x \in Z(R)$, then $xa - ax = 0$, and so it is a left $a$-idempotent for any $a \in R$.
$\Leftarrow$ By a straightforward computation, $[a, x] = [a + 1, x]$. Thus, by assumption, we have

$$a[a, x] = (xa - ax)^2 = (x(a + 1) - (a + 1)x)^2 = (a + 1)[a, x].$$

Hence, $[a, x] = 0$ for any $a \in R$, and so $x \in Z(R)$. □

**Theorem 2.21.** *Let $e \in R$. Then $e \in R^{PI}$ if and only if $e$ is a left $(x - xee^* + e)$-idempotent (a right $(x - e^*ex + e)$-idempotent) for each $x \in R$.*

*Proof.* $\Rightarrow$ If $e = ee^*e$, then $(x - xee^* + e)e = xe - xee^*e + e^2 = e^2$.
$\Leftarrow$ By $(1 - ee^* + e)e = e^2$, one gets $e = ee^*e$. □

## 3. One-sided $x$-equal elements

**Definition 3.1.** *Let $a, b, x \in R$, $a$ and $b$ are called left (resp. right) $x$-equal if $xa = xb$ (resp. $ax = bx$).*

In particular, for the case of left $x$-equal in Definition 3.1, if $x = a$, then $a$ is a right $b$-idempotent; provided that $x = b$, then $b$ is a right $a$-idempotent.

**Proposition 3.2.** *Let $a, x \in R$. Then $a$ is a left (resp. right) $x$-idempotent if and only if $a$ and $x$ are right (resp. left) $a$-equal.*

*Proof.* It follows from a straightforward verification. □

**Proposition 3.3.** *Let $a, b, x \in R$. Then $a$ and $b$ are left (resp. right) $x$-equal if and only if $a - x$ and $b - x$ are left (resp. right) $x$-equal.*

*Proof.* $\Rightarrow$ If $xa = xb$, then $x(a - x) = x(b - x)$.
$\Leftarrow$ By $x(a - x) = x(b - x)$, one gets $xa = xb$. □

**Proposition 3.4.** *Let $a, b, x \in R$, and $a$ and $b$ be left (resp. right) $x$-equal. Then if $ab = ba$, we have $a^2$ and $b^2$ are left (resp. right) $x$-equal.*

*Proof.* Since $xa = xb$ and $ab = ba$, we have

$$xa^2 = (xa)a = (xb)a = (xa)b = (xb)b = xb^2.$$

□

**Proposition 3.5.** *Let $a, b, x \in R$, and $a$ and $b$ be left (resp. right) $x$-equal. Then $a$ and $b$ are left (resp. right) $x^n$-equal, where $n \in \mathbb{Z}_+$.*

*Proof.* By $xa = xb$, we have

$$x^n a = x^{n-1}(xa) = x^{n-1}(xb) = x^n b.$$

□

**Theorem 3.6.** *Let $e \in R$. Then the following statements are equivalent:*
*(1) $e \in E(R)$;*
*(2) $1-e$ and 0 are left e-equal;*
*(3) $1-e$ and 0 are right e-equal;*
*(4) e and 0 are left $(1-e)$-equal;*
*(5) e and 0 are right $(1-e)$-equal;*
*(6) $2e-1$ and e are left e-equal;*
*(7) $2e-1$ and e are right e-equal.*

*Proof.* (1)⇒(2) By $e^2 = e$, we have $e(1-e) = e^2 - e = 0$.
(2)⇒(3) It follows from $e(1-e) = (1-e)e$.
(3)⇒(4) It is obvious.
(4)⇒(5) It follows from $(1-e)e = e(1-e)$.
(5)⇒(6) By $(1-e)e = 0$, $e^2 = e$. Thus,

$$e(2e-1) = 2e^2 - e = e = e^2.$$

(6)⇒(7) It follows from $e(2e-1) = (2e-1)e$.
(7)⇒(1) By $e^2 = (2e-1)e = 2e^2 - e$, one gets $e^2 = e$. □

**Theorem 3.7.** *Let $e \in E(R)$. Then e is left (resp. right) semicentral if and only if for each $x \in E(R)$, if e and x are left (resp. right) e-equal, then e and x are right (resp. left) e-equal.*

*Proof.* ⇒ By $e^2 = e$, $exe = xe$ and $e^2 = ex$, then

$$xe = exe = e^3 = e^2.$$

⇐ For any $y \in R$, let $x = e - (1-e)ye$. Then $xe = x$, $ex = e^2$, and $x^2 = (xe)x = x(ex) = xe^2 = xe = x$. Note that $ex = e^2$, hence $xe = e^2$ by hypothesis, and so $x = xe = e$. It follows that $(1-e)ye = 0$, i.e., $e$ is a left semicentral. □

**Theorem 3.8.** *Let $a, b, x, y, u \in R$, and $x = uyx$ (resp. $x = xyu$). Then a and b are left (resp. right) x-equal if and only if a and b are left yx-equal (resp. right xy-equal).*

*Proof.* ⇒ If $xa = xb$, then $y(xa) = yxb$.
⇐ By $yxa = yxb$, we have

$$xa = uyxa = uyxb = xb.$$

□

**Theorem 3.9.** *Let $a, b, x \in R$. Then a and b are left (resp. right) x-equal if and only if $a+b$ and $2a$ are left (resp. right) x-equal.*

*Proof.* ⇒ If $xa = xb$, then $x(a+b) = xa + xb = 2xa = x(2a)$.
⇐ Assume that $x(a+b) = x(2a) = 2xa$, then $xa = xb$. □

**Theorem 3.10.** *Let $e \in R$. Then e is left (resp. right) semicentral if and only if x and ex (resp. x and xe) are right (resp. left) e-equal for each $x \in R$.*

*Proof.* ⇒ Since $e$ is left semicentral, $exe = xe$ for any $x \in R$.
⇐ By assumption, $exe = xe$ for every $x \in R$. In particular, taking $x = 1$, then $e^2 = e$. Hence, $e$ is left semicentral. □

**Theorem 3.11.** *Let $e \in R$. Then e is left (resp. right) semicentral if and only if xy and xey are right (resp. left) e-equal for any $x, y \in R$.*

*Proof.* ⇒ By $exe = xe$ and $eye = ye$, then $xeye = xye$.

⇐ By hypothesis, $xeye = xye$ for each $x, y \in R$. Taking $x = 1$, then $eye = ye$. Furthermore, setting $x = y = 1$, then $e^2 = e$. Thus $e$ is left semicentral. □

**Theorem 3.12.** *Let $x \in R$. Then $x \in Z(R)$ if and only if $xy$ and $yx$ are left (right) $y$-equal for any $y \in R$.*

*Proof.* ⇒ If $x \in Z(R)$, then $yxy = y^2x$.

⇐ By $yxy = y^2x$ and $(y+1)x(y+1) = (y+1)^2x$, then $xy = yx$ for any $y \in R$. Hence, $x \in Z(R)$. □

**Theorem 3.13.** *Let $x \in R$ and $n \in \mathbb{Z}_+$. Then $x \in Z(R)$ if and only if $xy$ and $yx$ are left (right) $y^n$-equal for each $y \in R$.*

*Proof.* ⇒ If $x \in Z(R)$, then $y^nxy = y^{n+1}x$.

⇐ By $y^n(xy - yx) = 0$ and $(y+1)^n(x(y+1) - (y+1)x) = 0$, i.e., $(y+1)^n(xy - yx) = 0$. Then,

$$(1 + \binom{n}{n-1}y + \cdots + \binom{n}{2}y^{n-2} + \binom{n}{1}y^{n-1})[x, y] = 0. \tag{1}$$

Multiplying (1) on the left by $y^{n-1}$, then by $y^n(xy - yx) = 0$, one gets $y^{n-1}[x, y] = 0$. Repeating the above procedures, one has $[x, y] = 0$, and hence $x \in Z(R)$. □

## 4. One-sided $x$-projections

**Definition 4.1.** *Let $a, x \in R$. Then $a$ is called a left (resp. right) $x$-projection if $a^2 = xa$ (resp. $a^2 = ax$) and $a^* = a$.*

From Definitions 4.1 and 2.1, we get the following conclusion.

**Proposition 4.2.** *Let $a, x \in R$. Then $a$ is a left (resp. right) $x$-projection if and only if $a \in R^{Her}$ and $a$ is a left (resp. right) $x$-idempotent.*

By Proposition 4.2, one gets the following corollary.

**Corollary 4.3.** *Let $x \in R$. Then $x$ is a left (right) $x$-projection if and only if $x \in R^{Her}$.*

**Corollary 4.4.** *Let $x \in R$ and $n \in \mathbb{Z}_+$. Then*

*(1) $(xx^*)^n$ is a left (right) $(xx^*)^n$-projection.*

*(2) $(x^*x)^n$ is a left (right) $(x^*x)^n$-projection.*

*(3) $x + x^*$ is a left (right) $(x + x^*)$-projection.*

*Proof.* It follows from Corollary 4.3. □

**Lemma 4.5.** *Let $a$ be a left $x$-projection and $m, n \in \mathbb{Z}_+$. Then*

*(1) $ax^* = xa$.*

*(2) $x^ma^n = a^{m+n}$.*

*(3) $a^n(x^*)^m = a^{m+n}$.*

*(4) $(ax^*)^n = (xa)^n = a^{2n}$.*

*(5) $(ax)^n = a^{2n-1}x$.*

*(6) $(x^*a)^n = x^*a^{2n-1}$.*

*Proof.* (1) $ax^* = a^*x^* = (xa)^* = (a^2)^* = a^2 = xa$.

(2) By $xa = a^2$, then

$$\begin{aligned} x^m a^n &= x^{m-1}(xa)a^{n-1} \\ &= x^{m-1}a^{n+1} \\ &= x^{m-2}(xa)a^n \\ &= x^{m-2}a^{n+2} \\ &= \cdots \\ &= a^{m+n}. \end{aligned}$$

(3) By $ax^* = a^2$, then

$$\begin{aligned} a^n(x^*)^m &= a^{n-1}(ax^*)(x^*)^{m-1} \\ &= a^{n+1}(x^*)^{m-1} \\ &= a^n(ax^*)(x^*)^{m-2} \\ &= a^{n+2}(x^*)^{m-2} \\ &= \cdots \\ &= a^{m+n}. \end{aligned}$$

(4) By $ax^* = xa = a^2$, then

$$(ax^*)^n = (xa)^n = (a^2)^n = a^{2n}.$$

(5) By $xa = a^2$, then

$$\begin{aligned} (ax)^n &= (ax)^{n-2}a(xa)x \\ &= (ax)^{n-2}a^3x \\ &= (ax)^{n-3}a(xa)a^2x \\ &= (ax)^{n-3}a^5x \\ &= \cdots \\ &= a^{2n-1}x. \end{aligned}$$

(6) By $ax^* = a^2$, then

$$\begin{aligned} (x^*a)^n &= (x^*a)^{n-2}x^*(ax^*)a \\ &= (x^*a)^{n-2}x^*a^3 \\ &= (x^*a)^{n-3}x^*(ax^*)a^3 \\ &= (x^*a)^{n-3}x^*a^5 \\ &= \cdots \\ &= x^*a^{2n-1}. \end{aligned}$$

□

**Theorem 4.6.** *Let $a$ be a left $x$-projection and $n, p, q \in \mathbb{Z}_+$. Then*
*(1) $a^n$ is a left $x^n$-projection.*
*(2) $a^n$ is a left $a^p x^q$-projection, where $p + q = n$.*
*(3) $a^n$ is a right $(x^*)^n$-projection.*
*(4) $a^n$ is a right $(x^*)^p a^q$-projection, where $p + q = n$.*

*Proof.* (1) By Lemma 4.5 (2), then $x^n a^n = a^{2n}$.
(2) By Lemma 4.5 (2), $(a^p x^q)a^n = a^p(x^q a^n) = a^{p+q+n} = a^{2n}$.
(3) By Lemma 4.5 (3), $a^n(x^*)^n = a^{2n}$.
(4) By Lemma 4.5 (3), $a^n((x^*)^p a^q) = (a^n(x^*)^p)a^q = a^{n+p+q} = a^{2n}$. □

By Theorem 4.6, one has the following conclusion.

**Corollary 4.7.** *Let $a$ be a left $x$-projection and $m, n, s, t, p, q \in \mathbb{Z}_+$. Then*
*(1) $a^n(x^*)^t x^s a^m$ is a left $a^{n+m+s+t}$-projection.*
*(2) $a^n(x^*)^t x^s a^m$ is a left $a^n(x^*)^t x^{s+m}$-projection.*
*(3) $(ax^*)^n(xa)^s$ is a left $a^{2n+2s}$-projection.*
*(4) $(ax^*)^n(xa)^s$ is a left $(ax^*)^n x^{2s}$-projection.*
*(5) $a^n x^s(x^*)^t a^m$ is a left $a^n x^s(x^*)^t x^m$-projection.*
*(6) $x^s a^m(x^*)^t$ is a left $a^{m+s+t}$-projection.*
*(7) $x^s a^m(x^*)^t$ is a left $x^s a^{m+t}$-projection.*
*(8) $x^s(ax^*)^t$ is a left $a^{s+2t}$-projection.*
*(9) $x^s(ax^*)^t$ is a left $x^s a^{2t}$-projection.*
*(10) $x^s a^m(x^*)^t$ is a left $a^{m+s+t}$-projection.*
*(11) $x^s(ax^*)^t$ is a left $a^{s+2t}$-projection.*
*(12) $(xax^*)^s$ is a left $a^{3s}$-projection.*
*(13) $a^n(x^*)^t x^s a^m$ is a left $a^{m+n+s+t}$-projection.*
*(14) $a^n(x^*)^t x^s a^m$ is a left $a^p x^q$-projection, where $p + q = m + n + s + t$.*
*(15) $(ax^*)^n x^s a^m$ is a left $a^{2n+m+s}$-projection.*
*(16) $(ax^*)^n x^s a^m$ is a left $a^p x^q$-projection, where $p + q = 2n + m + s$.*
*(17) $(ax^*)^n(xa)^s$ is a left $a^{2n+2s}$-projection.*
*(18) $(ax^*)^n(xa)^s$ is a left $a^p x^q$-projection, where $p + q = 2n + 2s$.*
*(19) $a^n(x^*)^t(xa)^s$ is a left $a^{n+2s+t}$-projection.*
*(20) $a^n(x^*)^t(xa)^s$ is a left $a^p x^q$-projection, where $p + q = n + 2s + t$.*
*(21) $a^n(x^*)^t x^s a^m$ is a right $(x^*)^{n+m+s+t}$-projection.*
*(22) $a^n(x^*)^t x^s a^m$ is a right $(x^*)^p a^q$-projection, where $p + q = n + m + s + t$.*
*(23) $(ax^*)^n x^s a^m$ is a right $(x^*)^{2n+m+s}$-projection.*
*(24) $(ax^*)^n x^s a^m$ is a right $(x^*)^p a^q$-projection, where $p + q = 2n + m + s$.*
*(25) $a^n(x^*)^t(xa)^s$ is a right $(x^*)^{n+2s+t}$-projection.*
*(26) $a^n(x^*)^t(xa)^s$ is a right $(x^*)^p a^q$-projection, where $p + q = n + 2s + t$.*
*(27) $(ax^*)^n(xa)^s$ is a right $(x^*)^{2n+2s}$-projection.*
*(28) $(ax^*)^n(xa)^s$ is a right $(x^*)^p a^q$-projection, where $p + q = 2n + 2s$.*
*(29) $a^n(x^*)^t x^s a^m$ is a right $(x^*)^{n+m+s+t}$-projection.*
*(30) $a^n(x^*)^t x^s a^m$ is a right $(x^*)^p a^q$-projection, where $p + q = n + m + s + t$.*
*(31) $(ax^*)^n x^s a^m$ is a right $(x^*)^{2n+m+s}$-projection.*
*(32) $(ax^*)^n x^s a^m$ is a right $(x^*)^p a^q$-projection, where $p + q = 2n + m + s$.*
*(33) $a^n(x^*)^t(xa)^s$ is a right $(x^*)^{n+2s+t}$-projection.*
*(34) $a^n(x^*)^t(xa)^s$ is a right $(x^*)^p a^q$-projection, where $p + q = n + 2s + t$.*
*(35) $(ax^*)^n(xa)^s$ is a right $(x^*)^{2n+2s}$-projection.*
*(36) $(ax^*)^n(xa)^s$ is a right $(x^*)^p a^q$-projection, where $p + q = 2n + 2s$.*

**Proposition 4.8.** *Let $e, a \in R$. Then*

*(1) $e \in R^{proj}$ if and only if $e$ is a left (right) 1-projection.*

*(2) Assume that $x \in R^{Her}$, then $a$ is a left $x$-projection if and only if $a - x$ is a right $(-x)$-projection.*

*(3) $a$ is a left $x$-projection if and only if $a$ is a right $x^*$-projection.*

*Proof.* (1) $\Rightarrow$ If $e \in R^{proj}$, then $e^2 = e$ and $e^* = e$, and hence $e$ is a left 1-projection.

$\Leftarrow$ By $e^2 = e$ and $e^* = e$, we have $e \in R^{proj}$.

(2) $\Rightarrow$ If $x^* = x$, $a^* = a$ and $a^2 = xa$, then

$$(a - x)^2 = a^2 - ax - xa + x^2 = -ax + x^2 = (a - x)(-x), \text{ and } (a - x)^* = a^* - x^* = a - x.$$

It follows that $a - x$ is a right $(-x)$-projection.

$\Leftarrow$ By hypothesis, $x \in R^{Her}$, $(a - x)^* = a - x$, and $(a - x)^2 = (a - x)(a - x)$. Thus, one gets $a^* = a$ and $a^2 = xa$, which implies that $a$ is a left $x$-projection.

(3) It follows from Lemma 4.5 (1). □

**Proposition 4.9.** *Let $e \in R$. Then the following statements are equivalent:*

*(1) $e \in R^{proj}$;*

*(2) $e$ is a left $(2e - 1)$-projection;*

*(3) $e$ is a right $(2e - 1)$-projection.*

*Proof.* (1)$\Rightarrow$(2) Since $e^2 = e$ and $e^* = e$, we obtain that $e$ is a left $(2e - 1)$-projection.

(2)$\Rightarrow$(3) It follows from $e(2e - 1) = (2e - 1)e$.

(3)$\Rightarrow$(1) By $e^2 = e(2e - 1) = 2e^2 - e$, one gets $e = e^2$. Thus, $e \in R^{proj}$ by $e = e^*$. □

**Proposition 4.10.** *Let $a, x \in R$ and $a$ be a left $x$-projection. Then*

*(1) $x^2a = axa$.*

*(2) if $a^2 = 1$, then $x = a$.*

*(3) $x \in a\{1\}$ if and only if $a^3 = a$.*

*(4) $a$ is a left $(x + ax - x^2)$-projection.*

*Proof.* By assumption, $a = a^*$ and $a^2 = xa$.

(1) $x^2a = x(xa) = xa^2 = (xa)a = a^2a = aa^2 = axa$.

(2) If $a^2 = 1$, then $x = xa^2 = (xa)a = a^3 = a$.

(3) $\Rightarrow$ If $a = axa$, then $a^3 = aa^2 = axa = a$.

$\Leftarrow$ By $a^3 = a$, then $a(xa) = a^3 = a$.

(4) $(x + ax - x^2)a = xa + axa - x^2a = a^2 + a^3 - xa^2 = a^2 + a^3 - a^3 = a^2$. Then by $a^* = a$, $a$ is a left $(x + ax - x^2)$-projection. □

**Proposition 4.11.** *Let $e \in R$. Then the following statements are equivalent:*

*(1) $e \in R^{proj}$;*

*(2) $1 - e$ is a left $(1 - 2e)$-projection;*

*(3) $1 - e$ is a right $(1 - 2e)$-projection;*

*(4) $e$ is a left $(e + e^* - 1)$-projection;*

*(5) $e$ is a right $(e + e^* - 1)$-projection.*

*Proof.* (1)$\Rightarrow$(2) If $e^2 = e = e^*$, then $(1 - e)^2 = 1 - 2e + e^2 = 1 - e$ and $(1 - 2e)(1 - e) = 1 - 3e + 2e^2 = 1 - e$. Hence, $(1 - e)^2 = (1 - 2e)(1 - e)$, and so $1 - e$ is a left $(1 - 2e)$-projection.

(2)$\Rightarrow$(3) It follows from $(1 - 2e)(1 - e) = (1 - e)(1 - 2e)$.

(3)$\Rightarrow$(4) By $1 - e = 1 - e^*$ and $(1 - e)^2 = (1 - e)(1 - 2e)$, one gets $e = e^*$ and $e = e^2$. Hence,

$$(e + e^* - 1)e = e^2 + ee^* - e = e = e^2.$$

(4)$\Rightarrow$(5) By hypothesis, $e = e^*$ and $e^2 = (e + e^* - 1)e$, then $e^*e = e$. Thus $e(e + e^* - 1) = e^2 + ee^* - e = e^2$.

(5)$\Rightarrow$(1) If $e = e^*$ and $e^2 = e(e + e^* - 1)$, then $ee^* = e$, and so $e^2 = ee^* = e$. □

**Proposition 4.12.** *Let $a \in R^{reg}$ and $a$ be a left (resp. right) $x$-projection. Then $a$ is a left $(x + u - uaa^-)$-projection (resp. a right $(x + u - a^-au)$-projection) for any $u \in R$.*

*Proof.* If $a = a^*$ and $a^2 = xa$, then $(x + u - uaa^-)a = xa + ua - uaa^-a = xa + ua - ua = xa$. □

**Theorem 4.13.** *Let $e \in E(R)$ and $ea^*$ be a left (right) $a$-idempotent for each $a \in R$. Then $e$ is a central projection.*

*Proof.* By hypothesis, for any $a \in R$,

$$ea^*ea^* = aea^*, \text{ and } e(a + 1)^*e(a + 1)^* = (a + 1)e(a + 1)^*.$$

Then $ea^*e = ae$. Taking $a = e^*$, then $e = e^*e$, which implies that $e \in R^{proj}$. For any $x \in R$, let $g = e + (1 - e)xe$. Then

$$eg = e, \quad ge = g, \text{ and } g^2 = g.$$

By hypothesis, $eg^*$ is a left $g$-idempotent. Then $eg^*eg^* = geg^*$, and hence $gg^* = geg^* = (geg^*)^* = (eg^*eg^*)^* = gege = g$. It follows that $g \in R^{proj}$, and so $g = g^* = (ge)^* = e^*g^* = eg = e$. Hence, $(1 - e)xe = 0$ for any $x \in R$, which shows that $e$ is left semicentral, and so $e^*$ is right semicentral. Notice that $e \in R^{proj}$, hence $e$ is central. □

**Remark 4.14.** *In Theorem 4.13, if $ea^*$ is replaced by $a^*e$ or $a - ea^*$ or $a - a^*e$, then the assertion also holds.*

**Proposition 4.15.** *Let $a, x, y \in R$ and $x$ and $y$ be left (resp. right) $a$-projections. Then $x + y$ is a left (resp. right) $a$-projection if and only if $xy + yx = 0$.*

*Proof.* By assumption, $x^* = x$, $y^* = y$, $x^2 = ax$ and $y^2 = ay$.
⇒ If $(x + y)^2 = a(x + y)$, then by $x^2 = ax$ and $y^2 = ay$, one gets $xy + yx = 0$.
⇐ By $xy + yx = 0$, then $(x + y)^2 = x^2 + xy + yx + y^2 = ax + ay = a(x + y)$. □

**Theorem 4.16.** *Let $e \in R$. Then the following statements are equivalent:*
*(1) $e \in R^{proj}$;*
*(2) $e$ is a left $(a + 1 - xe^*e + xe)$-projection for each $a \in l(e)$ and $x \in R$;*
*(3) $e$ is a right $(a + 1 - ee^*y + ey)$-projection for every $a \in r(e)$ and $y \in R$.*

*Proof.* (1)⇒(2) If $e^2 = e = e^*$, then

$$\begin{aligned}(a + 1 - xe^*e + xe)e &= ae + e - xe^*e^2 + xe^2 \\ &= ae + e \\ &= e \\ &= e^2.\end{aligned}$$

(2)⇒(3) By hypothesis, $e^* = e$ and $e^2 = (a + 1 - xe^*e + xe)e$, then one gets $e^2 = e$. Thus,

$$\begin{aligned}e(a + 1 - ee^*y + ey) &= ea + e - e^2e^*y + e^2y \\ &= e \\ &= e^2.\end{aligned}$$

(3)⇒(1) It follows from a straightforward verification. □

Notice that if $e \in E(R)$, then $(1 - e)e = e(1 - e) = 0$. Thus, we have the following corollary of Theorem 4.16.

**Corollary 4.17.** *Let $e \in R$. Then the following statements are equivalent:*
*(1) $e \in R^{proj}$;*
*(2) $e$ is a left $(a(1 - e) + 1 - xe^*e + xe)$-projection for any $a, x \in R$;*
*(3) $e$ is a right $((1 - e)a + 1 - ee^*y + ey)$-projection for any $a, y \in R$.*

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## Conflict of Interest

The authors declared that they have no conflict of interest.

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