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# On quotients of ideals of bounded holomorphic maps

**A. Jiménez-Vargas[a], D. Ruiz-Casternado[a,*]**

[a]*Departamento de Matemáticas, Universidad de Almería, Ctra. de Sacramento s/n, 04120 La Cañada de San Urbano, Almería, Spain*

**Abstract.** Based on the notion of left-hand quotient of operator ideals, we introduce and study the concept of bounded-holomorphic left-hand quotient $\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}$, where $\mathcal{I}$ is an operator ideal and $\mathcal{J}^{\mathcal{H}^\infty}$ is a bounded-holomorphic ideal. We show that such quotients are a method for generating new bounded-holomorphic ideals. In fact, if $\mathcal{J}^{\mathcal{H}^\infty}$ has the linearization property in an operator ideal $\mathcal{A}$, then $\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}$ is a composition ideal of the form $(\mathcal{I}^{-1} \circ \mathcal{A}) \circ \mathcal{H}^\infty$. We also introduce the notion of Grothendieck holomorphic map and prove that they form a bounded-holomorphic ideal which can be seen as a bounded-holomorphic left-hand quotient. In the same way, the ideal of holomorphic maps with Rosenthal range can be generated as a bounded-holomorphic left-hand quotient.

## 1. Introduction and preliminaries

Let $\mathcal{I}, \mathcal{J}$ be operator ideals and let $E, F$ be Banach spaces. Following [12, p. 132], a bounded linear operator $T : E \to F$ is said to belong to the *left-hand quotient* $\mathcal{I}^{-1} \circ \mathcal{J}$, and we write $T \in \mathcal{I}^{-1} \circ \mathcal{J}(E, F)$, if $S \circ T \in \mathcal{J}(E, G)$ for all $S \in \mathcal{I}(F, G)$, where $G$ is an arbitrary Banach space. The *right-hand quotient* $\mathcal{I} \circ \mathcal{J}^{-1}$ is defined in a similar way. Of course, the symbols $\mathcal{I}^{-1}$ and $\mathcal{J}^{-1}$ have no meaning. It is well known that $\mathcal{I}^{-1} \circ \mathcal{J}$ and $\mathcal{I} \circ \mathcal{J}^{-1}$ are operator ideals (see [11, 3.2.2]). We will say that $\mathcal{I}^{-1} \circ \mathcal{J}$ is the left-hand quotient ideal generated or induced by the ideals $\mathcal{I}$ and $\mathcal{J}$, and similarly for the right-hand quotient ideal $\mathcal{I} \circ \mathcal{J}^{-1}$.

Furthermore, if $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ and $[\mathcal{J}, \|\cdot\|_{\mathcal{J}}]$ are Banach operator ideals, and we set

$$\|T\|_{\mathcal{I}^{-1}\circ\mathcal{J}} = \sup\{\|S \circ T\|_{\mathcal{J}} : S \in \mathcal{I}(F, G), \|S\|_{\mathcal{I}} \leq 1\},$$

for every $T \in \mathcal{I}^{-1} \circ \mathcal{J}(E, F)$, where $G$ ranges over all Banach spaces, then $[\mathcal{I}^{-1} \circ \mathcal{J}, \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}}]$ is a Banach operator ideal by [11, 7.2.2].

Left-hand and right-hand quotients of operator ideals have been studied by some authors over time. For example, Johnson, Lillemets and Oja showed in [8] that completely continuous operators can be represented as a right-hand quotient generated by the ideals of weakly $\infty$-compact operators and weakly compact operators, and used it to show that only in Schur spaces the weak Grothendieck compactness principle is satisfied; Carl and Defant proved in [3] that the ideal of $(s, p)$-mixing operators is expressible as

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* Corresponding author: D. Ruiz-Casternado
*Email addresses:* ajimenez@ual.es (A. Jiménez-Vargas), drc446@ual.es (D. Ruiz-Casternado)

a left-hand quotient induced by the ideals of $s$-summing operators and $p$-summing operators; Causey and Navoyan generalised in [4] a result from Pietsch's book ([11, 3.2.3]) showing that the class of $\xi$-completely continuous operators can be seen as a right-hand quotient induced by the classes of compact operators and $\xi$-weakly compact operators; Kim proved in [9] that the class of operators which sends weakly $p$-summable sequences to unconditionally $p$-summable sequences is a right-hand quotient generated by the ideals of unconditionally $p$-compact operators and weakly $p$-compact operators.

Through this paper, $E$ and $F$ will be complex Banach spaces and $U$ will be an open subset of $E$. As usual, $B_E$ stands for the closed unit ball of $E$, $\mathcal{L}(E,F)$ for the space of all bounded linear operators from $E$ into $F$ endowed with the operator norm, and $E^*$ for the topological dual of $E$. Given a set $A \subseteq E$, $\mathrm{lin}(A)$ and $\overline{\mathrm{abco}}(A)$ represent the linear span and the norm-closed absolutely convex hull of $A$ in $E$, respectively.

Let $\mathcal{H}(U,F)$ be the space of all holomorphic mappings from $U$ into $F$. Moreover, $\mathcal{H}^\infty(U,F)$ will be the subspace formed by all $f \in \mathcal{H}(U,F)$ for which $f(U)$ is a bounded subset of $F$. We will use the abbreviations $\mathcal{H}(U)$ and $\mathcal{H}^\infty(U)$ instead of $\mathcal{H}(U,\mathbb{C})$ and $\mathcal{H}^\infty(U,\mathbb{C})$, respectively. Let us recall that $\mathcal{H}^\infty(U,F)$ is a Banach space under the supremum norm

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in U\} \qquad (f \in \mathcal{H}^\infty(U,F)).$$

Our goal in this note is to present a holomorphic version of the notion of left-hand quotient of operator ideals, involving the concept of bounded-holomorphic ideal introduced in [2, Definition 2.1]. To our knowledge, nothing has been published so far about quotients of ideals in the setting of bounded holomorphic maps.

Let us recall that a normed (Banach) bounded-holomorphic ideal, denoted as $[\mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$, is a subclass $\mathcal{J}^{\mathcal{H}^\infty}$ equipped with a norm $\|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}$ of the class $\mathcal{H}^\infty$ of all bounded holomorphic mappings equipped with the norm $\|\cdot\|_\infty$ such that, for every open subset $U$ of a complex Banach space $E$ and every complex Banach space $F$, the components $\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ verify the following properties:

- (P1) $(\mathcal{J}^{\mathcal{H}^\infty}(U,F), \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}})$ is a normed (Banach) space and $\|f\|_\infty \leq \|f\|_{\mathcal{J}^{\mathcal{H}^\infty}}$ for $f \in \mathcal{J}^{\mathcal{H}^\infty}(U,F)$.
- (P2) For any $g \in \mathcal{H}^\infty(U)$ and $y \in F$, the map $g \cdot y \colon U \to F$, given by $(g \cdot y)(x) = g(x)y$ if $x \in U$, is in $\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ and $\|g \cdot y\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|g\|_\infty \|y\|$.
- (P3) The ideal property: Given two complex Banach spaces $H, G$, an open subset $V$ of $H$, $f \in \mathcal{J}^{\mathcal{H}^\infty}(U,F)$, $S \in \mathcal{L}(F,G)$ and $h \in \mathcal{H}(V,U)$, the map $S \circ f \circ h$ is in $\mathcal{J}^{\mathcal{H}^\infty}(V,G)$ and $\|S \circ f \circ h\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|S\| \|f\|_{\mathcal{J}^{\mathcal{H}^\infty}}$.

A normed bounded-holomorphic ideal $[\mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ is

- (S) *Surjective* if $f \in \mathcal{J}^{\mathcal{H}^\infty}(U,F)$ with $\|f\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|f \circ \pi\|_{\mathcal{J}^{\mathcal{H}^\infty}}$, whenever $f \in \mathcal{H}^\infty(U,F)$, $\pi \in \mathcal{H}(V,U)$ is a surjective map, where $V$ is an open subset of a complex Banach space $G$ and $f \circ \pi \in \mathcal{J}^{\mathcal{H}^\infty}(V,F)$.

Influenced by the notion of left-hand quotient of operator ideals (see, e.g., [11, 3.2.1]), we introduce the concept of left-hand quotient of an operator ideal and a bounded-holomorphic ideal.

**Definition 1.1.** *Let $\mathcal{I}$ be an operator ideal and let $\mathcal{J}^{\mathcal{H}^\infty}$ be a bounded-holomorphic ideal. A mapping $f \in \mathcal{H}^\infty(U,F)$ is said to belong to the bounded-holomorphic left-hand quotient $\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}$, and will be written as $f \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U,F)$, if $T \circ f \in \mathcal{J}^{\mathcal{H}^\infty}(U,G)$ for all $T \in \mathcal{I}(F,G)$, where $G$ is a complex Banach space.*

*If $\mathcal{I}$ is endowed with a complete norm $\|\cdot\|_{\mathcal{I}}$ and $\mathcal{J}^{\mathcal{H}^\infty}$ with a norm $\|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}$, we set*

$$\|f\|_{\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}} = \sup\{\|T \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} : T \in \mathcal{I}(F,G), \|T\|_{\mathcal{I}} \leq 1\}.$$

Our main tool in this paper is a method of linearization of bounded holomorphic mappings gathered in the following result due to Mujica [10].

**Theorem 1.2.** *[10, Theorem 2.1 and Remark 2.2] Let $U$ be an open subset of a complex Banach space $E$. Consider the Banach space*

$$\mathcal{G}^\infty(U) := \overline{\mathrm{lin}}(\{\delta_x : x \in U\}) \subseteq \mathcal{H}^\infty(U)^*,$$

*where $\delta_x \colon \mathcal{H}^\infty(U) \to \mathbb{C}$ is the functional defined by $\delta_x(f) = f(x)$ for all $f \in \mathcal{H}^\infty(U)$.*

(i) *The map* $g_U : U \to \mathcal{G}^\infty(U)$, *given by*

$$g_U(x) = \delta_x \qquad (x \in U),$$

*is in* $\mathcal{H}^\infty(U, \mathcal{G}^\infty(U))$ *and* $\|\delta_x\| = 1$ *for any* $x \in U$.

(ii) $B_{\mathcal{G}^\infty(U)} = \overline{\mathrm{abco}}(g_U(U))$.

(iii) *For each complex Banach space* $F$ *and each map* $f \in \mathcal{H}^\infty(U, F)$, *there exists a unique operator* $T_f \in \mathcal{L}(\mathcal{G}^\infty(U), F)$ *such that* $T_f \circ g_U = f$. *Moreover,* $\|T_f\| = \|f\|_\infty$.

Our study will depend essentially on a linearization property of the maps of the bounded-holomorphic ideal $\mathcal{J}^{\mathcal{H}^\infty}$.

**Definition 1.3.** *Let* $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ *be a normed operator ideal and let* $[\mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ *be a normed bounded-holomorphic ideal. We say that* $\mathcal{J}^{\mathcal{H}^\infty}$ *has the linearization property (LP, for short) in* $\mathcal{A}$ *if given* $f \in \mathcal{H}^\infty(U, F)$, *we have that* $f \in \mathcal{J}^{\mathcal{H}^\infty}(U, F)$ *if and only if* $T_f \in \mathcal{A}(\mathcal{G}^\infty(U), F)$, *in whose case* $\|f\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|T_f\|_{\mathcal{A}}$.

This paper has been divided into two sections. Section 2 gathers the first properties of the left-hand quotients $\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}$, where $\mathcal{I}$ is an operator ideal and $\mathcal{J}^{\mathcal{H}^\infty}$ is a bounded-holomorphic ideal. If both ideals are endowed with complete norms, we show that $\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}$ with the norm $\|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$ is a Banach bounded-holomorphic ideal which becomes surjective whenever $\mathcal{J}^{\mathcal{H}^\infty}$ is surjective. Thus, bounded-holomorphic left-hand quotients prove to be an interesting method of generating bounded-holomorphic ideals. There are already two well known ways to produce bounded-holomorphic ideals: by composition and by transposition (see [2, Theorems 2.4 and 4.3]).

We show that if $\mathcal{J}^{\mathcal{H}^\infty}$ has the linearization property in an operator ideal $\mathcal{A}$, then a map $f \in \mathcal{H}^\infty(U, F)$ belongs to the bounded-holomorphic left-hand quotient $\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U, F)$ if and only if its Mujica's linearization $T_f \in \mathcal{L}(\mathcal{G}^\infty(U), F)$ belongs to the operator left-hand quotient $\mathcal{I}^{-1} \circ \mathcal{A}(\mathcal{G}^\infty(U), F)$. In this case, we also prove that $\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}$ is a composition ideal of the form $(\mathcal{I}^{-1} \circ \mathcal{A}) \circ \mathcal{H}^\infty$.

Section 3 is devoted to two examples of bounded-holomorphic left-hand quotient ideals generated by an operator ideal and a bounded-holomorphic ideal: the spaces of bounded holomorphic maps with Grothendieck range and Rosenthal range.

## 2. Bounded-holomorphic left-hand quotient ideals

Our first aim is to justify the existence of the following supremum which appears in Definition 1.1. Our proof is based on [11, 7.2.2].

**Proposition 2.1.** *Let* $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ *be a Banach operator ideal and let* $[\mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ *be a normed bounded-holomorphic ideal. If* $f \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U, F)$, *then*

$$\sup\{\|T \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} : T \in \mathcal{I}(F, G), \|T\|_{\mathcal{I}} \leq 1\} < \infty.$$

*Proof.* Assume that this supremum is not finite. Then, for each $n \in \mathbb{N}$, we could find a complex Banach space $G_n$ and an operator $T_n \in \mathcal{I}(F, G_n)$ with $\|T_n\|_{\mathcal{I}} \leq 1/2^n$ such that $\|T_n \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} \geq n$.

Consider the sequence of Banach spaces $(G_i)$ with $i \in \mathbb{N}$, and the *Cartesian* $\ell_1$*-product* $\ell_1(\mathbb{N}, G_i)$ defined as the set of all sequences $(x_i)$, where $x_i \in G_i$ for each $i \in \mathbb{N}$, such that the sequence $(\|x_i\|) \in \ell_1(\mathbb{N}, \mathbb{R})$. By [11, C.4.1], $\ell_1(\mathbb{N}, G_i)$ is a Banach space with the norm

$$\|(x_i)\|_1 = \sum_{i=1}^{\infty} \|x_i\|.$$

For each $n \in \mathbb{N}$, let $J_n : G_n \to \ell_1(\mathbb{N}, G_i)$ and $Q_n : \ell_1(\mathbb{N}, G_i) \to G_n$ be the bounded linear operators given by

$$J_n(x) = (\delta_{in} x)_i \qquad (x \in G_n),$$
$$Q_n((x_i)) = x_n \qquad ((x_i) \in \ell_1(\mathbb{N}, G_i)),$$

where $\delta_{in}$ is the Kronecker delta. Notice that $\|J_n\| = 1$ and $\|Q_n\| \leq 1$. Since $(J_n \circ T_n)$ is a sequence of vectors of the Banach space $(\mathcal{I}(F, \ell_1(\mathbb{N}, G_i)), \|\cdot\|_{\mathcal{I}})$, and

$$\left\| \sum_{n=k+1}^{k+h} J_n \circ T_n \right\|_{\mathcal{I}} \leq \sum_{n=k+1}^{k+h} \|J_n \circ T_n\|_{\mathcal{I}} = \sum_{i=1}^{h} \|J_{k+i} \circ T_{k+i}\|_{\mathcal{I}}$$
$$\leq \sum_{i=1}^{h} \|T_{k+i}\|_{\mathcal{I}} \leq \sum_{i=1}^{\infty} \|T_{k+i}\|_{\mathcal{I}} \leq \sum_{i=1}^{\infty} \frac{1}{2^{k+i}} = \frac{1}{2^k},$$

for all $h, k \in \mathbb{N}$, then the series $\sum_{n\geq 1} J_n \circ T_n$ converges in the norm $\|\cdot\|_{\mathcal{I}}$ to $T := \sum_{n=1}^{\infty} J_n \circ T_n \in \mathcal{I}(F, \ell_1(\mathbb{N}, G_i))$. Thus we obtain

$$n \leq \|T_n \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|Q_n \circ T \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|T \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}},$$

which is a contradiction. □

In general, we can establish an inclusion property between bounded-holomorphic left-hand quotients through the inclusion of their associated bounded-holomorphic ideals. For two Banach operator ideals $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ and $[\mathcal{J}, \|\cdot\|_{\mathcal{J}}]$, we write $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}] \leq [\mathcal{J}, \|\cdot\|_{\mathcal{J}}]$ if $\mathcal{I} \subseteq \mathcal{J}$ and $\|f\|_{\mathcal{J}} \leq \|f\|_{\mathcal{I}}$ for all $f \in \mathcal{I}$.

**Proposition 2.2.** *Let* $[\mathcal{J}_1^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}_1^{\mathcal{H}^\infty}}]$ *and* $[\mathcal{J}_2^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}_2^{\mathcal{H}^\infty}}]$ *be normed bounded-holomorphic ideals such that*

$$[\mathcal{J}_1^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}_1^{\mathcal{H}^\infty}}] \leq [\mathcal{J}_2^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}_2^{\mathcal{H}^\infty}}].$$

*Then*

$$[\mathcal{I}^{-1} \circ \mathcal{J}_1^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}_1^{\mathcal{H}^\infty}}] \leq [\mathcal{I}^{-1} \circ \mathcal{J}_2^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}_2^{\mathcal{H}^\infty}}],$$

*for any Banach operator ideal* $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$. □

It is well known that $[\mathcal{H}^\infty, \|\cdot\|_\infty]$ is a Banach bounded-holomorphic ideal. Thus, as an immediate consequence of the previous result, we can ensure that $\mathcal{I}^{-1} \circ \mathcal{H}^\infty$ is the biggest bounded-holomorphic left-hand quotient for any Banach operator ideal $\mathcal{I}$ in the following sense.

**Corollary 2.3.** *Let* $[\mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ *be a normed bounded-holomorphic ideal. Then*

$$[\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}] \leq [\mathcal{I}^{-1} \circ \mathcal{H}^\infty, \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{H}^\infty}]$$

*for any Banach operator ideal* $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$. □

Closely related to Corollary 2.3, we have the following useful result.

**Proposition 2.4.** *Let* $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ *be a Banach operator ideal and* $[\mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ *be a normed bounded-holomorphic ideal. Then*

$$[\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}] \leq [\mathcal{H}^\infty, \|\cdot\|_\infty].$$

*Furthermore,*

$$[\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}] = [\mathcal{H}^\infty, \|\cdot\|_\infty]$$

*whenever* $\mathcal{J}^{\mathcal{H}^\infty}$ *has the LP in* $\mathcal{I}$.

*Proof.* Let $f \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U, F)$. Then $f \in \mathcal{H}^\infty(U, F)$ and $T \circ f \in \mathcal{J}^{\mathcal{H}^\infty}(U, G)$ for all $T \in \mathcal{I}(F, G)$, where $G$ is a complex Banach space. For each $x \in U$, we can take a functional $\phi \in B_{F^*}$ such that $\left\|f(x)\right\| = \left|\phi(f(x))\right|$. Since $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is a Banach operator ideal, it follows that the functional $\phi \otimes 1 \colon F \to \mathbb{C}$, defined by $(\phi \otimes 1)(y) = \phi(y)$ if $y \in F$, is in $\mathcal{I}(F, \mathbb{C})$ with $\left\|\phi \otimes 1\right\|_{\mathcal{I}} = \left\|\phi\right\| \leq 1$ (see, for example, [5, p. 131]). Hence we can write

$$\left\|f(x)\right\| = \left|((\phi \otimes 1) \circ f)(x)\right| \leq \left\|(\phi \otimes 1) \circ f\right\|_\infty \leq \left\|(\phi \otimes 1) \circ f\right\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \left\|f\right\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}},$$

and taking supremum over all $x \in U$, we conclude that $\|f\|_\infty \leq \|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$.

Assume now that $\mathcal{J}^{\mathcal{H}^\infty}$ has the LP in $\mathcal{I}$. Let $f \in \mathcal{H}^\infty(U,F)$ and let $T \in \mathcal{I}(F,G)$, where $G$ is a complex Banach space. Clearly, $T \circ f \in \mathcal{H}^\infty(U,G)$. By Theorem 1.2, we can find operators $T_{T\circ f} \in \mathcal{L}(\mathcal{G}^\infty(U),G)$ and $T_f \in \mathcal{L}(\mathcal{G}^\infty(U),F)$ with $\|T_f\| = \|f\|_\infty$ verifying

$$T_{T\circ f} \circ g_U = T \circ f = T \circ T_f \circ g_U.$$

Hence $T_{T\circ f} = T \circ T_f$ by the norm-density of $g_U(U)$ in $\mathcal{G}^\infty(U)$. Now the ideal property of $\mathcal{I}$ yields that $T_{T\circ f} \in \mathcal{I}(\mathcal{G}^\infty(U),G)$ with $\|T_{T\circ f}\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}\|f\|_\infty$. Since $\mathcal{J}^{\mathcal{H}^\infty}$ has the LP in $\mathcal{I}$, it follows that $T\circ f \in \mathcal{J}^{\mathcal{H}^\infty}(U,G)$ with $\|T\circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|T_{T\circ f}\|_{\mathcal{I}}$. By the arbitrariness of $T \in \mathcal{I}(F,G)$, we conclude that $f \in \mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ with $\|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} \leq \|f\|_\infty$. □

Next, we show that bounded-holomorphic left-hand quotients are a method for generating bounded-holomorphic ideals.

**Theorem 2.5.** *Let $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ be a Banach operator ideal and let $[\mathcal{J}^{\mathcal{H}^\infty},\|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ be a normed (Banach) bounded-holomorphic ideal. Then $[\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty},\|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}]$ is a normed (Banach) bounded-holomorphic ideal. In addition, $[\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty},\|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}]$ is surjective whether $[\mathcal{J}^{\mathcal{H}^\infty},\|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ is surjective.*

*Proof.* (P1): It is easy to see that $\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ is a linear space. We will now show that $\|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$ is a norm on $\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$. Let $f \in \mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ and assume that $\|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} = 0$. Since $\|f\|_\infty \leq \|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$ by Proposition 2.4, we deduce that $f = 0$.

Given $\alpha \in \mathbb{C}$ and $f,g \in \mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$, it is immediate that $\|T\circ(\alpha f)\|_{\mathcal{J}^{\mathcal{H}^\infty}} = |\alpha|\|T\circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}}$ and $\|T\circ(f+g)\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|T\circ f + T\circ g\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} + \|g\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$ for all $T \in \mathcal{I}(F,G)$ with $\|T\|_{\mathcal{I}} \leq 1$, and therefore $\|\alpha f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} = |\alpha|\,\|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$ and $\|f+g\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} \leq \|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} + \|g\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$.

Let us suppose now that the norm $\|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}$ on $\mathcal{J}^{\mathcal{H}^\infty}$ is complete. Let $(f_n)$ be a Cauchy sequence in $(\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F),\|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}})$. Let $T \in \mathcal{I}(F,G)$, where $G$ is a complex Banach space. On a hand, since $\|\cdot\|_\infty \leq \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$ on $\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ by Proposition 2.4, there exists a map $f \in \mathcal{H}^\infty(U,F)$ such that $\|f_n - f\|_\infty \to 0$ as $n\to\infty$, and this implies that $\|T\circ f_n - T\circ f\|_\infty \to 0$ as $n \to \infty$. On the other hand, the inequality

$$\|T\circ f_p - T\circ f_q\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|T\circ(f_p - f_q)\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|T\|_{\mathcal{I}}\|f_p - f_q\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} \qquad (p,q\in\mathbb{N})$$

shows that $(T\circ f_n)$ is a Cauchy sequence in $(\mathcal{J}^{\mathcal{H}^\infty}(U,G),\|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}})$. Hence we can take a map $g \in \mathcal{J}^{\mathcal{H}^\infty}(U,G)$ so that $\|T\circ f_n - g\|_{\mathcal{J}^{\mathcal{H}^\infty}} \to 0$ as $n\to\infty$. Taking into account that $\|\cdot\|_\infty \leq \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}$ on $\mathcal{J}^{\mathcal{H}^\infty}(U,G)$, we obtain that $T\circ f = g$, and thus $f \in \mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ and $\|T\circ f_n - T\circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} \to 0$ as $n\to\infty$.

To prove that $(f_n)$ converges to $f$ in $(\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F),\|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}})$, let $\varepsilon > 0$. Then there exists $m\in\mathbb{N}$ such that $\|f_p - f_q\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} < \varepsilon/2$ for all $p,q \geq m$. Hence we have that

$$\|T\circ f_p - T\circ f_{p+n}\|_{\mathcal{J}^{\mathcal{H}^\infty}} < \frac{\varepsilon}{2}$$

for all $p \geq m$, $n\in\mathbb{N}$ and $T\in\mathcal{I}(F,G)$ with $\|T\|_{\mathcal{I}} \leq 1$. Taking limits with $n\to\infty$, it follows that

$$\|T\circ f_p - T\circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \frac{\varepsilon}{2}$$

for all $p\geq m$ and $T\in\mathcal{I}(F,G)$ with $\|T\|_{\mathcal{I}}\leq 1$. Taking supremum over all such $T$, we get that $\|f_p - f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} < \varepsilon$ for all $p\geq m$, as desired.

(P2): Let $g\in\mathcal{H}^\infty(U)$ and $y\in F$. Since $\mathcal{J}^{\mathcal{H}^\infty}$ is a normed bounded-holomorphic ideal, we have that $g\cdot y\in\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ with $\|g\cdot y\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|g\|_\infty\|y\|$. Let $T\in\mathcal{I}(F,G)$, where $G$ is a complex Banach space, and note that $T \circ (g \cdot y) = g \cdot T(y)$. Hence $T \circ (g \cdot y) \in \mathcal{J}^{\mathcal{H}^\infty}(U,G)$ with $\|T \circ (g \cdot y)\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|g\|_\infty \|T(y)\|$, and thus $g \cdot y \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U,F)$ with

$$\begin{aligned}\|g \cdot y\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} &= \|g\|_\infty \sup\{\|T(y)\| : T \in \mathcal{I}(F,G), \|T\|_{\mathcal{I}} \leq 1\} \\ &\geq \|g\|_\infty \sup\{|\phi(y)| : \phi \in B_{F^*}\} = \|g\|_\infty \|y\|.\end{aligned}$$

To get the converse inequality, note that

$$\|T \circ (g \cdot y)\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|T\| \, \|g \cdot y\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|T\|_{\mathcal{I}} \|g \cdot y\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|g\|_\infty \|y\|$$

for all $T \in \mathcal{I}(F,G)$ with $\|T\|_{\mathcal{I}} \leq 1$, and so $\|g \cdot y\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} \leq \|g\|_\infty \|y\|$.

(P3): Let $H$ and $G$ be complex Banach spaces, let $V$ be an open subset of $H$, $f \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U,F)$, $h \in \mathcal{H}(V,U)$ and $S \in \mathcal{L}(F,G)$. Let $T \in \mathcal{I}(G,G_0)$, where $G_0$ is a complex Banach space. Then $T \circ S \in \mathcal{I}(F,G_0)$ with $\|T \circ S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}} \|S\|$ by the ideal property of $\mathcal{I}$, and $T \circ S \circ f \in \mathcal{J}^{\mathcal{H}^\infty}(U,G_0)$ with $\|T \circ S \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|T \circ S\|_{\mathcal{I}} \|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$ by the definitions of $\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}$ and $\|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$. Hence $T \circ S \circ f \circ h \in \mathcal{J}^{\mathcal{H}^\infty}(V,G_0)$ with $\|T \circ S \circ f \circ h\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|T \circ S \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}}$ by the ideal property of $\mathcal{J}^{\mathcal{H}^\infty}$. Consequently, $S \circ f \circ h \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(V,G)$ and since $\|T \circ S \circ f \circ h\|_{\mathcal{J}^{\mathcal{H}^\infty}} \leq \|S\| \|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$ for all $T \in \mathcal{I}(G,G_0)$ with $\|T\|_{\mathcal{I}} \leq 1$, we deduce that $\|S \circ f \circ h\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} \leq \|S\| \|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$.

(S): Let $f \in \mathcal{H}^\infty(U,F)$ and assume that $f \circ \pi \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(V,F)$, where $V$ is an open subset of a complex Banach space $G$ and $\pi \in \mathcal{H}(V,U)$ is a surjective map. Then $T \circ f \circ \pi \in \mathcal{J}^{\mathcal{H}^\infty}(V,H)$ for all $T \in \mathcal{I}(F,H)$, being $H$ a complex Banach space. Since the normed bounded-holomorphic ideal $[\mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ is surjective and $T \circ f \in \mathcal{H}^\infty(U,H)$, it follows that $T \circ f \in \mathcal{J}^{\mathcal{H}^\infty}(U,H)$ with $\|T \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} = \|T \circ f \circ \pi\|_{\mathcal{J}^{\mathcal{H}^\infty}}$. By the arbitrariness of $T \in \mathcal{I}(F,H)$, we can ensure that $f \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U,F)$. Moreover, notice that

$$\begin{aligned}\|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} &= \sup\{\|T \circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} : T \in \mathcal{I}(F,H), \|T\|_{\mathcal{I}} \leq 1\} \\ &= \sup\{\|T \circ f \circ \pi\|_{\mathcal{J}^{\mathcal{H}^\infty}} : T \in \mathcal{I}(F,H), \|T\|_{\mathcal{I}} \leq 1\} \\ &= \|f \circ \pi\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}.\end{aligned}$$

Hence $[\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}]$ is surjective. □

The following result allows us to establish a relationship between left-hand quotients of operator ideals and left-hand quotients of an operator ideal and a bounded-holomorphic ideal with the LP.

**Theorem 2.6.** *Let $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ be a Banach operator ideal, $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a normed operator ideal and $[\mathcal{J}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ be a normed bounded-holomorphic ideal with the LP in $\mathcal{A}$. For every $f \in \mathcal{H}^\infty(U,F)$, the following are equivalent:*

*(i)* $f \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U,F)$.

*(ii)* $T_f \in \mathcal{I}^{-1} \circ \mathcal{A}(\mathcal{G}^\infty(U),F)$.

*In this case, $\|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} = \|T_f\|_{\mathcal{I}^{-1}\circ\mathcal{A}}$. In addition, the correspondence $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U,F), \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}})$ onto $(\mathcal{I}^{-1} \circ \mathcal{A}(\mathcal{G}^\infty(U),F), \|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{A}})$.*

*Proof.* (i) ⇒ (ii): Let $f \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U,F)$. Then, for all $T \in \mathcal{I}(F,G)$, being $G$ a complex Banach space, we have that $T \circ f \in \mathcal{J}^{\mathcal{H}^\infty}(U,G)$. As in the proof of Proposition 2.4, by using Theorem 1.2 we can ensure the existence of two operators $T_f \in \mathcal{L}(\mathcal{G}^\infty(U),F)$ and $T_{T\circ f} \in \mathcal{L}(\mathcal{G}^\infty(U),G)$ so that $T_{T\circ f} = T \circ T_f$. Since $\mathcal{J}^{\mathcal{H}^\infty}$ has the LP in $\mathcal{A}$, we deduce that $T_{T\circ f} \in \mathcal{A}(\mathcal{G}^\infty(U),G)$, and by the arbitrariness of $T \in \mathcal{I}(F,G)$, we conclude that $T_f \in \mathcal{I}^{-1} \circ \mathcal{A}(\mathcal{G}^\infty(U),F)$.

(ii) ⇒ (i): Assume that $T_f \in \mathcal{I}^{-1} \circ \mathcal{A}(\mathcal{G}^\infty(U),F)$. Then $T \circ T_f \in \mathcal{A}(\mathcal{G}^\infty(U),G)$ for all $T \in \mathcal{I}(F,G)$, and thus $T_{T\circ f} \in \mathcal{A}(\mathcal{G}^\infty(U),G)$ because $T_{T\circ f} = T \circ T_f$. Since $\mathcal{J}^{\mathcal{H}^\infty}$ has the LP in $\mathcal{A}$, it follows that $T \circ f \in \mathcal{J}^{\mathcal{H}^\infty}(U,G)$ and then $f \in \mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}(U,F)$.

In this case, we have

$$\begin{aligned}\|f\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}} &= \sup\{\|T\circ f\|_{\mathcal{J}^{\mathcal{H}^\infty}} : T\in\mathcal{I}(F,G), \|T\|_{\mathcal{I}}\le 1\}\\ &= \sup\{\|T_{T\circ f}\|_{\mathcal{A}} : T\in\mathcal{I}(F,G), \|T\|_{\mathcal{I}}\le 1\}\\ &= \sup\{\|T\circ T_f\|_{\mathcal{A}} : T\in\mathcal{I}(F,G), \|T\|_{\mathcal{I}}\le 1\}\\ &= \|T_f\|_{\mathcal{I}^{-1}\circ\mathcal{A}},\end{aligned}$$

where the second equality is due to the LP of $\mathcal{J}^{\mathcal{H}^\infty}$ in $\mathcal{A}$.

For the last assertion of the statement, it suffices to prove the surjectivity of the map $f\mapsto T_f$ from $\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ into $\mathcal{I}^{-1}\circ\mathcal{A}(\mathcal{G}^\infty(U),F)$. Towards this end, let $S\in\mathcal{I}^{-1}\circ\mathcal{A}(\mathcal{G}^\infty(U),F)$. We have $T\circ S\in\mathcal{A}(\mathcal{G}^\infty(U),G)$ for all $T\in\mathcal{I}(F,G)$, being $G$ a complex Banach space. By applying Theorem 1.2, $T\circ S=T_g$ for some $g\in\mathcal{H}^\infty(U,G)$. Hence $g\in\mathcal{J}^{\mathcal{H}^\infty}(U,G)$ by the LP of $\mathcal{J}^{\mathcal{H}^\infty}$ in $\mathcal{A}$. Consider the map $f=S\circ g_U\colon U\to F$. Clearly, $f\in\mathcal{H}^\infty(U,F)$ and $T\circ f=T\circ S\circ g_U=T_g\circ g_U=g\in\mathcal{J}^{\mathcal{H}^\infty}(U,G)$. Hence $f\in\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ and $T_f=S$. □

Let us recall now the composition method for generating bounded-holomorphic ideals. Given an operator ideal $\mathcal{I}$, a map $f\in\mathcal{H}^\infty(U,F)$ is in the composition ideal $\mathcal{I}\circ\mathcal{H}^\infty$, and it is written as $f\in\mathcal{I}\circ\mathcal{H}^\infty(U,F)$, if there are a complex Banach space $G$, an operator $T\in\mathcal{I}(G,F)$ and a map $g\in\mathcal{H}^\infty(U,G)$ so that $f=T\circ g$. If $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ is a normed operator ideal and $f\in\mathcal{I}\circ\mathcal{H}^\infty$, we set

$$\left\|f\right\|_{\mathcal{I}\circ\mathcal{H}^\infty}=\inf\left\{\|T\|_{\mathcal{I}}\left\|g\right\|_\infty\right\},$$

being the infimum taken over all factorizations of $f$ as above.

Left-hand quotients of an operator ideal and a bounded-holomorphic ideal with the LP in a certain operator ideal can be seen as a composition ideal as the following result reflects.

**Proposition 2.7.** *Let $[\mathcal{I},\|\cdot\|_{\mathcal{I}}]$ be a Banach operator ideal, $[\mathcal{A},\|\cdot\|_{\mathcal{A}}]$ be a normed operator ideal and $[\mathcal{J}^{\mathcal{H}^\infty},\|\cdot\|_{\mathcal{J}^{\mathcal{H}^\infty}}]$ be a normed bounded-holomorphic ideal with the LP in $\mathcal{A}$. Then*

$$[\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty},\|\cdot\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}]=[(\mathcal{I}^{-1}\circ\mathcal{A})\circ\mathcal{H}^\infty,\|\cdot\|_{(\mathcal{I}^{-1}\circ\mathcal{A})\circ\mathcal{H}^\infty}].$$

*Proof.* Let $f\in\mathcal{H}^\infty(U,F)$. Applying [1, Theorem 3.2] and Theorem 2.6, we have

$$\begin{aligned}f\in(\mathcal{I}^{-1}\circ\mathcal{A})\circ\mathcal{H}^\infty(U,F) &\Leftrightarrow T_f\in\mathcal{I}^{-1}\circ\mathcal{A}(\mathcal{G}^\infty(U),F)\\ &\Leftrightarrow f\in\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F),\end{aligned}$$

with $\left\|f\right\|_{(\mathcal{I}^{-1}\circ\mathcal{A})\circ\mathcal{H}^\infty}=\left\|T_f\right\|_{\mathcal{I}^{-1}\circ\mathcal{A}}=\left\|f\right\|_{\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}}$ for all $f\in\mathcal{I}^{-1}\circ\mathcal{J}^{\mathcal{H}^\infty}(U,F)$. □

## 3. Examples of bounded-holomorphic left-hand quotient ideals

An operator $T\in\mathcal{L}(E,F)$ is called *compact (respectively, weakly compact, separable, Rosenthal, Grothendieck)* if $T(B_E)$ is a relatively compact (respectively, relatively weakly compact, separable, Rosenthal, Grothendieck) subset of $F$. The ideals of compact operators, weakly compact operators, separable bounded operators, Rosenthal operators, and Grothendieck operators from $E$ into $F$ will be denoted as $\mathcal{K}(E,F)$, $\mathcal{W}(E,F)$, $\mathcal{S}(E,F)$, $\mathcal{R}(E,F)$ and $\mathfrak{G}(E,F)$, respectively. The inclusions that follow are widely recognized:

$$\begin{aligned}&\mathcal{K}(E,F)\subseteq\mathcal{W}(E,F)\subseteq\mathcal{R}(E,F),\\ &\mathcal{W}(E,F)\subseteq\mathfrak{G}(E,F)\\ &\mathcal{K}(E,F)\subseteq\mathcal{S}(E,F).\end{aligned}$$

The monograph [11] by Pietsch contains a complete study on these operator ideals.

In this section, we will give two examples of bounded-holomorphic left-hand quotient ideals generated by an operator ideal and a bounded-holomorphic ideal, namely, the spaces of holomorphic maps which have Grothendieck range or Rosenthal range.

We refer the reader to [6] for a study of the Grothendieck property. According to [6, p. 298], a set $K \subseteq E$ is called *Grothendieck* if for every operator $T \in \mathcal{L}(E, c_0)$, $T(K)$ is a relatively weakly compact subset of $c_0$. It is known that $\mathfrak{G}$ is a closed surjective operator ideal.

**Definition 3.1.** *We will say that a map $f \in \mathcal{H}^\infty(U,F)$ is Grothendieck if $f(U)$ is a Grothendieck subset of $F$. Let $\mathcal{H}^\infty_{\mathfrak{G}}(U,F)$ denote the space of all Grothendieck holomorphic maps from $U$ into $F$.*

Following [7], $\mathcal{H}^\infty_{\mathcal{W}}(U,F)$ and $\mathcal{H}^\infty_{\mathcal{K}}(U,F)$ stand for the spaces of all bounded holomorphic maps from $U$ into $F$ with relatively weakly compact range and relatively compact range, respectively. By [2, Proposition 3.2] and [10, Proposition 3.4], $\mathcal{H}^\infty_{\mathcal{K}}$ and $\mathcal{H}^\infty_{\mathcal{W}}$ are bounded-holomorphic ideals with the LP in $\mathcal{K}$ and $\mathcal{W}$, respectively. We now show that $\mathcal{H}^\infty_{\mathfrak{G}}$ has the LP in $\mathfrak{G}$.

**Theorem 3.2.** *For a map $f \in \mathcal{H}^\infty(U,F)$, the following are equivalent:*

*(i)* $f \in \mathcal{H}^\infty_{\mathfrak{G}}(U,F)$.

*(ii)* $T_f \in \mathfrak{G}(\mathcal{G}^\infty(U),F)$.

*(iii)* $f \in \mathfrak{G} \circ \mathcal{H}^\infty(U,F)$.

*In this case,* $\|f\|_\infty = \|T_f\| = \|f\|_{\mathfrak{G}\circ\mathcal{H}^\infty}$. *As a consequence, the map* $f \mapsto T_f$ *is an isometric isomorphism from* $(\mathcal{H}^\infty_{\mathfrak{G}}(U,F), \|\cdot\|_\infty)$ *onto* $(\mathfrak{G}(\mathcal{G}^\infty(U),F), \|\cdot\|)$, *and from* $(\mathfrak{G} \circ \mathcal{H}^\infty(U,F), \|\cdot\|_{\mathfrak{G}\circ\mathcal{H}^\infty})$ *onto* $(\mathfrak{G}(\mathcal{G}^\infty(U),F), \|\cdot\|)$.

*Proof.* (i) ⇒ (ii): If $f \in \mathcal{H}^\infty_{\mathfrak{G}}(U,F)$, then $T_f(g_U(U)) = f(U)$ is Grothendieck in $F$. Notice that the norm-closed absolutely convex hull of a Grothendieck set is itself Grothendieck due to the norm-closed absolutely convex hull of a relatively weakly compact set is relatively weakly compact. In this way, $\overline{\mathrm{abco}}(T_f(g_U(U))$ is Grothendieck in $F$. Since $T_f(B_{\mathcal{G}^\infty(U)}) = T_f(\overline{\mathrm{abco}}(g_U(U)) \subseteq \overline{\mathrm{abco}}(T_f(g_U(U))$, it follows that $T_f(B_{\mathcal{G}^\infty(U)})$ is a Grothendieck subset of $F$.

(ii) ⇒ (i): If $T_f \in \mathfrak{G}(\mathcal{G}^\infty(U),F)$, then $T_f(B_{\mathcal{G}^\infty(U)})$ is a Grothendieck subset of $F$. Since $g_U(U) \subseteq B_{\mathcal{G}^\infty(U)}$, it follows that $f(U) = T_f(g_U(U))$ is Grothendieck in $F$.

(ii) ⇔ (iii): It is an application of [1, Theorem 3.2] (see also [2, Theorem 2.4]).

For the consequence, the first part follows easily applying Theorem 1.2 and (ii) ⇒ (i), and the second part from [2, Theorem 2.4]. □

Let us recall (see [2, Definition 2.1]) that a bounded-holomorphic ideal $\mathcal{J}^{\mathcal{H}^\infty}$ is said to be *closed* if every component $\mathcal{J}^{\mathcal{H}^\infty}(U,F)$ is a closed subspace of $\mathcal{H}^\infty(U,F)$ endowed with the topology of the supremum norm. Since $\mathfrak{G}$ is a closed operator ideal (the norm-limit of a convergent sequence of Grothendieck operators is Grothendieck) and $\mathcal{H}^\infty_{\mathfrak{G}} = \mathfrak{G} \circ \mathcal{H}^\infty$ by Theorem 3.2, then Corollary 2.5 in [2] yields the following.

**Corollary 3.3.** $[\mathcal{H}^\infty_{\mathfrak{G}}, \|\cdot\|_\infty]$ *is a closed bounded-holomorphic ideal.* □

We are now ready to describe the space of all Grothendieck holomorphic mappings in terms of a bounded-holomorphic left-hand quotient ideal.

**Theorem 3.4.** $[\mathcal{H}^\infty_{\mathfrak{G}}, \|\cdot\|_\infty] = [\mathcal{S}^{-1} \circ \mathcal{H}^\infty_{\mathcal{W}}, \|\cdot\|_{\mathcal{S}^{-1}\circ\mathcal{H}^\infty_{\mathcal{W}}}]$.

*Proof.* Let $f \in \mathcal{H}^\infty(U,F)$. Taking into account Theorem 3.2, [11, 3.2.6], Definition 1.1, and Theorem 2.6 joint to [10, Proposition 3.4], respectively, we have

$$\begin{aligned} f \in \mathcal{H}^\infty_{\mathfrak{G}}(U,F) &\Leftrightarrow T_f \in \mathfrak{G}(\mathcal{G}^\infty(U),F) \\ &\Leftrightarrow T \circ T_f \in \mathcal{W}(\mathcal{G}^\infty(U),G), \quad \forall\, T \in \mathcal{S}(F,G) \\ &\Leftrightarrow T_f \in \mathcal{S}^{-1} \circ \mathcal{W}(\mathcal{G}^\infty(U),F) \\ &\Leftrightarrow f \in \mathcal{S}^{-1} \circ \mathcal{H}^\infty_{\mathcal{W}}(U,F), \end{aligned}$$

and, in this case, $\left\|f\right\|_\infty = \left\|T_f\right\| = \left\|T_f\right\|_{\mathcal{S}^{-1}\circ\mathcal{W}} = \left\|f\right\|_{\mathcal{S}^{-1}\circ\mathcal{H}^\infty_{\mathcal{W}}}$. □

Let us look back on and recall that a subset $A$ of $E$ is said to be *conditionally weakly compact (or Rosenthal)* if every sequence in $A$ has a weak Cauchy subsequence.

On the other hand, an operator $T \in \mathcal{L}(E,F)$ is called *completely continuous* if every weakly convergent sequence $(x_n)$ is mapped into a norm convergent sequence $(T(x_n))$. Let $\mathcal{V}(E,F)$ be the space of all completely continuous operators from $E$ into $F$. By [11, 1.6.2 and 4.2.5], $\mathcal{V}$ is a closed operator ideal.

Next, we characterise the subclass of bounded holomorphic mappings which have Rosenthal range, denoted by $\mathcal{H}^\infty_{\mathcal{R}}$, as a bounded-holomorphic left-hand quotient ideal generated by the operator ideal $\mathcal{V}$ and the bounded-holomorphic ideal $\mathcal{H}^\infty_{\mathcal{K}}$.

**Theorem 3.5.** $[\mathcal{H}^\infty_{\mathcal{R}}, \|\cdot\|_\infty] = [\mathcal{V}^{-1} \circ \mathcal{H}^\infty_{\mathcal{K}}, \|\cdot\|_{\mathcal{V}^{-1}\circ\mathcal{H}^\infty_{\mathcal{K}}}]$.

*Proof.* Given $f \in \mathcal{H}^\infty(U,F)$, we obtain:

$$\begin{aligned} f \in \mathcal{H}^\infty_{\mathcal{R}}(U,F) &\Leftrightarrow T_f \in \mathcal{R}(\mathcal{G}^\infty(U),F) \\ &\Leftrightarrow T \circ T_f \in \mathcal{K}(\mathcal{G}^\infty(U),G), \quad \forall\, T \in \mathcal{V}(F,G) \\ &\Leftrightarrow T_f \in \mathcal{V}^{-1} \circ \mathcal{K}(\mathcal{G}^\infty(U),F) \\ &\Leftrightarrow f \in \mathcal{V}^{-1} \circ \mathcal{H}^\infty_{\mathcal{K}}(U,F), \end{aligned}$$

in whose case, $\left\|f\right\|_\infty = \left\|T_f\right\| = \left\|T_f\right\|_{\mathcal{V}^{-1}\circ\mathcal{K}} = \left\|f\right\|_{\mathcal{V}^{-1}\circ\mathcal{H}^\infty_{\mathcal{K}}}$, by using [7, Theorem 2.9], [11, 3.2.4], Definition 1.1, and Theorem 2.6 with [10, Proposition 3.4], respectively. □

We do not know what happens with Proposition 2.7 for the case in which the ideal of bounded holomorphic mappings $\mathcal{J}^{\mathcal{H}^\infty}$ does not have the LP in an operator ideal $\mathcal{A}$. The natural idea is to think that a holomorphic quotient $\mathcal{I}^{-1} \circ \mathcal{J}^{\mathcal{H}^\infty}$ may not coincide with a composition ideal of the form $(\mathcal{I}^{-1} \circ \mathcal{A}) \circ \mathcal{H}^\infty$.

Let us recall that a map $g : U \to F$ is called *locally weakly compact* if for each point $x \in U$, we can find a neighborhood $U_x \subseteq U$ for which $g(U_x)$ is relatively weakly compact in $F$. Let $\mathcal{H}^\infty_w(U,F)$ be the linear subspace of $\mathcal{H}^\infty(U,F)$ formed by all locally weakly compact maps. If $\Delta$ denotes the open unit disc of $\mathbb{C}$, let $g \in \mathcal{H}^\infty(\Delta, c_0)$ be given as

$$g(w) = (w^n)_{n=1}^\infty \qquad (w \in \Delta).$$

By [10, Example 3.2], $g \in \mathcal{H}^\infty_w(\Delta, c_0)$ but $g \notin \mathcal{H}^\infty_{\mathcal{W}}(\Delta, c_0)$. By [10, Proposition 3.4], it follows that $T_g \notin \mathcal{W}(\mathcal{G}^\infty(\Delta), c_0)$. Thus, $\mathcal{H}^\infty_w \neq \mathcal{W} \circ \mathcal{H}^\infty$ by [1, Theorem 3.2]. Hence $\mathcal{H}^\infty_w$ does not have the LP in $\mathcal{W}$, and since $\mathcal{I}^{-1} \circ \mathcal{H}^\infty_{\mathcal{W}} = (\mathcal{I}^{-1} \circ \mathcal{W}) \circ \mathcal{H}^\infty$ for any operator ideal $\mathcal{I}$ by Proposition 2.7 and [10, Proposition 3.4], we can assure that $\mathcal{I}^{-1} \circ \mathcal{H}^\infty_w \neq (\mathcal{I}^{-1} \circ \mathcal{W}) \circ \mathcal{H}^\infty$ for some operator ideal $\mathcal{I}$. Thus $\mathcal{I}^{-1} \circ \mathcal{H}^\infty_w$ can not be a composition ideal generated by its associated operator ideal $\mathcal{W}$ but it is a bounded-holomorphic left-hand quotient ideal according to Theorem 2.5 since $\mathcal{H}^\infty_w$ is a bounded-holomorphic ideal by [2, Proposition 3.1].

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