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On compactifications of an orbit space

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Abstract. One of the important research topics in the theory of topological transformation groups is to examine the relationships between the topological properties of a G-space X and the orbit space X/G. In [7], Antonyan and Smirnov studied the relationship between the Stone-Čech compactification of a completely regular space and the Stone-Čech compactification of the orbit space. A similar study was done for the Hewitt realcompactification in [8, 14].

In this paper, we obtain that the c-realcompactification of the orbit space X/G is homeomorfic to the orbit space of the c -realcompactification of X, when G is a finite discrete group and X is a completely regular G-space. We will also show a similar result for almost realcompactification.

1. Introduction

All topological spaces are assumed to be completely regular Hausdorff spaces and all mapping are continuous. The letter G will always denote a Hausdorff (and hence, completely regular) topological group. We use the usual terminology and notation of the rings of continuous functions. For unexplained definitions and notations we refer to [9, 16]. For the convenience of the reader we recall some more special definitions and facts below.

A topological transformation group (G, X, θ) is a topological group G together with a Hausdorff topological space X and a continuous mapping $\theta : G \times X \longrightarrow X$ such that

a. $\theta(e, x) = x$ for $x \in X$ and

b. $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $x \in X$ and $g, h \in G$, where e is the identity of G.

By a G-space, we shall mean a space X, together with a given action θ of G on X. In general we will not specify the action θ explicitly and simply write gx for $\theta(g, x)$. For each $g \in G$, the mapping $\theta_g : X \longrightarrow X$ defined by $x \to gx$ is a homeomorphism. For a subgroup $H \subset G$ and a subset $Y \subset X$, then $H(Y)$ will denote the set $\{gy : g \in H, y \in Y\}$. In particular, $G(x)$ denotes the orbit $G(x) = \{gx : g \in G\}$ of x. If $H(Y) = Y$, then Y is called invariant under H. In other words, Y is under H if $\theta_h(Y) \subset Y$ for each $h \in H$. Let X/G denote the set whose elements are the subsets $G(x)$ of X. Let's denote the natural mapping $x \to G(x)$ of X onto X/G by π_X. We give X/G with the identification space topology, namely the strongest topology making π_X continuous. It is well-known that X/G is Hausdorff if G is a compact group acting on a Hausdorff space X. Moreover

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π_X is an open, closed and perfect mapping [9, 24]. Since θ is continuous and $Id \times \pi_X$ is open, it follows that G acts continuously and trivially on X/G. Moreover, if G is a compact group which acts on a completely regular X, then the orbit space X/G is also completely regular [24]. More generally, Chaber showed [10] that the image of a completely regular space under an open and closed mapping is also completely regular.

A surjective mapping $f : X \longrightarrow Y$ is called perfect if it is a closed and the fibers $f^{-1}(y)$ is compact for each $y \in Y$.

A mapping of G-spaces $f : X \longrightarrow Y$ is called equivariant if $f(gx) = gf(x)$ for every $(g, x) \in G \times X$.

For a completely regular space X, υX will denote the Hewitt realcompactification of X and βX will denote the Stone-Čech compactification of X.

Let X be a topological space and A be a subspace of X. Let θ be an action of G on A. An action Φ of G on X is said to be extension of θ if the restriction of Φ on $G \times A$ is θ, i.e. $\Phi(g, a) = \theta(g, a)$ for all $g \in G$ and $a \in A$.

Suppose that $A \subset X$ and $\mathcal{A}$ is a family of subsets of X. Let's denote the set $\theta_g(A) = \{g \cdot a : a \in A\}$ by $g \cdot A$ and the family $\theta_g(\mathcal{A}) = \{g \cdot A : A \in \mathcal{A}\}$ by $g \cdot \mathcal{A}$.

For any function $f : X \longrightarrow \mathbb{R}$, the set $Z(f) = f^{-1}(0) = \{x \in X : f(x) = 0\}$ is called a zero-set of X. Denote the family of all zero-sets in X by $Z(X)$.

A nonempty subset $\mathcal{A}$ of $Z(X)$ satisfying the following conditions is called z-filter;

i. $\emptyset \notin \mathcal{A}$

ii. If $A \in \mathcal{A}$ and $A \subset B \in Z(X)$, then $B \in \mathcal{A}$.

iii. If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

A maximal z-filter in $Z(X)$ is called z-ultrafilter.

The Stone-Čech compactification may be constructed out z-ultrafilters. We note that every point p of βX is the limit of a unique z-ultrafilter $\mathcal{A}^p$ on X, where $\mathcal{A}^p$ represents the (fixed) z-ultrafilter with limit p, i.e. $\mathcal{A}^p = \{A \in Z(X) : p \in A\}$.

It is well-known [28, p. 57] or [16, 6.5, 8.7] that a mapping $f : X \longrightarrow Y$ induces the Stone extension $\beta f : \beta X \longrightarrow \beta Y$ and the real extension $\upsilon f : \upsilon X \longrightarrow \upsilon Y$, where υf is the restriction of βf on υX. The Stone extension βf is defined as follows: For each $p \in \beta X$, there exists a unique ultrafilter $\mathcal{A}^p$ on X that converging to the point p. Consider $f^{\#}\mathcal{A}^p = \left\{E \in Z(Y) : f^{-1}(E) \in \mathcal{A}^p\right\}$. Since $\mathcal{A}^p$ is a prime z-filter on X, so is $f^{\#}\mathcal{A}^p$. Therefore $f^{\#}\mathcal{A}^p$ has a limit in βY. Then $\beta f(p)$ is defined by this limit: $\{\beta f(p)\} = \bigcap f^{\#}\mathcal{A}^p$. It can easily be seen that the Stone extension of an equivariant mapping is also equivariant mapping.

Let G be any Hausdorff topological group and X be a G-space. A compact Hausdorff G-space Y is called a G-compactification of X if there exists an equivariant dense embedding map $f : X \longrightarrow Y$. A G-compactification K of a G-space X will be called a maximal G-compactification (or Čech G-compactification) of X if every equivariant map of X into a compact G-space Y admits an (automatically unique) extension to an equivariant map of K into Y. We will denote the maximal G-compactification of a completely regular G-space by $\beta_G X$. When G is a topological group, the question of whether a complete regular Hausdorff space has a G-compactification was posed by de Vries in [31]. de Vries showed that every completely regular Hausdorff G-space admits a G-compactification if G is a locally compact group (see [32, Theorem 2.10] or [33]) (see also [24] for compact Lie groups and [5] for compact groups). However, this is not true in general, as first shown by Megrelishvili [23]. Also, an example of a countably compact G-space which fails to have a G-compactification was constructed by Sokolovskaya in [26].

Now let X be a G-space. For every $g \in G$, $\theta_g : X \longrightarrow X$, $x \mapsto gx$ is a homeomorphism, and so it uniquely extends to a homeomorphism $\beta\theta_g : \beta X \longrightarrow \beta X$. It is easy to see that the map which takes $g \in G$ to $\beta\theta_g$ is a homomorphism of G into the group of all autohomeomorphisms of βX. Therefore, the natural map $G \times \beta X \longrightarrow \beta X$, $(g, a) \mapsto \beta\theta_g(a)$ satisfies the two algebraic conditions a. and b. of an action. Thus, we obtain an algebraic action $G \times \beta X \longrightarrow \beta X$ which extends the given action $G \times X \longrightarrow X$. But, unfortunately this algebraic action is generally not continuous. An counterexample, given by M. Jerison, is the multiplication action of the circle group on complex plane [24, p. 23]. There are some important cases when the extended algebraic action $G \times \beta X \longrightarrow \beta X$ is continuous, i.e., $\beta_G X = \beta X$. Another interpretation of the extending of an action using ultrafilters can be given as follows.

Now, suppose that a finite discrete group G acts on a completely regular space X. For any $g \in G$ and $p \in \beta X$, $g \cdot \mathcal{A}^p$ is a z-ultrafilter on X, which convergences to a point of βX, say gp. The mapping $\theta_\beta : G \times \beta X \longrightarrow \beta X$ defined by $\theta_\beta(g, p) = gp$ for $g \in G$, $x \in \beta X$, is an extension to βX of the action θ of G on X, that is the action θ_β keeps X invariant. For the continuity of θ_β, see [27]. This shows that $\beta_G X = \beta X$ if X is a completely regular space and G is a finite discrete group. The study of the equality $\beta_G X = \beta X$ was first tackled by Antonyan [4]. If G is a discrete group acting on a completely regular space X, then $\beta_G X = \beta X$ [35, 7.3.10 (ii)]. Later on, de Vries asked in [34] to find necessary and sufficient conditions on a G-space X for the equality $\beta_G X = \beta X$. In the same paper de Vries [34] proved that if G is a topological group whose underlying topological space is a k-space and X is a pseudocompact G-space, then $\beta_G X = \beta X$. Moreover, Antonyan [1] showed that if G is a pseudocompact group and X is a pseudocompact G-space, then again $\beta_G X = \beta X$. The same result can be found in [25, Theorem 2.4]. Recall that a completely regular space X is called pseudocompact if every real valued continuous function on X is bounded.

The structure of the orbit space in the theory of transformation groups has significant interest. Srivastava [27] proved that $\beta(X/G) = \beta X/G$ when G is a finite discrete group. But, six years ago, Antonyan and Smirnov proved [7] that for a completely regular G-space X, one has the formula $\beta_G X/G = \beta(X/G)$ for arbitrary compact group G. When G is a finite group then, clearly, $\beta_G X = \beta X$ and the above formula becomes into $\beta X/G = \beta(X/G)$. More generally, N. Antonyan and S. Antonyan [2] proved that if G is a compact group, H is a closed normal subgroup of G, and X is a G-space, then $(\beta_G X)/H = \beta_{G/H}(X/H)$. Note that for a compactification $B(X/G)$ of the orbit space X/G, the problem of existence of a compact G-extension $B_G X$ of X such that $B_G X/G = B(X/G)$ was studied by Antonyan in [6]. For more information on this topic we refer the reader to [3] which is an excellent survey.

Let us now recall the definition of some important subclasses of the Stone-Čech compactification. A subset A of a space X is called regular closed if $A = \mathrm{Cl}_X \mathrm{Int}_X A$. The family of all regular closed subsets of a space X will be denoted by $\mathcal{R}(X)$.

A family $\mathcal{F}$ of subsets of X is called the countable intersection property (CIP) if $\bigcap_n F_n \neq \emptyset$ for each sequence $(F_n)_{n\in\mathbb{N}}$ of sets drawn from $\mathcal{F}$. We will say that an open filter $\mathcal{U}$ has the closed countable intersection property (CCIP) if $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}\, U_n \neq \emptyset$ for any $U_n \in \mathcal{U}$. An ultrafilter $\mathcal{A}$ on $\mathcal{R}(X)$ is said to converges to a point $p \in \beta X$ if $\{p\} = \bigcap \left\{\mathrm{Cl}_{\beta X} A : A \in \mathcal{A}\right\}$.

Frolík [15] called a Hausdorff topological space X as almost realcompact if any open ultrafilter $\mathcal{U}$ with CCIP is fixed, that is $\bigcap \mathrm{Cl}_X U_n \neq \emptyset$. Unlike realcompact spaces, an almost realcompact space need not satisfy comletely regular separation property. However, we will assume that it has completely regular property. Every completely regular realcompact space is almost realcompact. A completely regular space X is almost realcompact if and only if each ultrafilter on $\mathcal{R}(X)$ with the countable intersection property has nonempty intersection [30]. Dykes [11, p. 576] introduced the concept of c-realcompact space. A well-known characterization of c-realcompact spaces is as follows. X is a c-realcompact space if and only if for each point $p \in \beta X \setminus X$, there exists a decreasing sequence $(A_n)_n$ of regular closed subsets of X such that $p \in \bigcap_n \mathrm{Cl}_{\beta X} A_n$ and $\bigcap_n A_n = \emptyset$ [17]. By using the characterization of realcompact spaces [13, p. 215] (or [16, p. 119]), it is clear that every realcompact space is c-realcompact. Every almost realcompact space is c-realcompact [11, Corollary 3.3]. See [12] for an example of an almost realcompact that is not realcompact. It well known that $X \subset uX \subset aX \subset \upsilon X \subset \beta X$ if X is a completely regular space [30].

Theorem 1.1. ([17]) *Let X be a completely regular space. Suppose that*

$$uX = \{p \in \beta X : \text{each ultrafilter in } \mathcal{R}(X) \text{ converging to } p \text{ has CIP}\}$$

Then uX is the smallest c-realcompact space of βX that contains X.

The space uX is called the c-realcompactification of X. Then X is c-realcompact if and only if $X = uX$.

For each completely regular Hausdorff space X, there is an almost realcompact space aX satisfying the followings:

1. $X \subset aX \subset \beta X$

2. If f is a mapping from X to any almost realcompact space Y, then f can be extended to a continuous mapping $af : aX \longrightarrow Y$. Further, af is the restriction to aX of the Stone extension $\beta f : \beta X \longrightarrow \beta Y$.

In [14], we proved that $\upsilon(X/G) = \upsilon X/G$ when G is a finite discrete group. But we later learned that this result had already been proven by Azad et al. [8] using different techniques.

In this article, we will prove that the c-realcompactification $u(X/G)$ of the orbit space X/G of a finite discrete group action on a completely regular space X is homeomorphic to the orbit space uX/G of the c-realcompactification uX of the space X. We also will show that $a(X/G) = aX/G$.

2. Main results

We will denote the restrictions of Stone extension βf to subspace uX by uf. First, we will examine whether the mappings uf and af are open and perfect.

Lemma 2.1. ([21, Theorem 4.4]) *If f is a closed mapping from X onto Y, then f is open if and only if the Stone extension βf is open.*

If $f : X \longrightarrow Y$ is perfect and open map, then υf is an open and perfect mapping of υX onto υY [18, 19]. Frolík [15] proved that if $f : X \longrightarrow Y$ is perfect and X is an almost realcompact space, then Y is also almost realcompact. Conversely, if Y is almost realcompact and X is regular, then X is almost realcompact space. Further, the image of a c-realcompact under an open and perfect map is also c-realcompact [22, Theorem 5.5(2)].

Let $\mathcal{U}$ ($\mathcal{F}$) denote a free open (closed) ultrafilter on X and $\mathcal{U}^p$ ($\mathcal{F}^p$) denote a free open (closed) ultrafilter converging to $p \in \beta X \setminus X = X^*$.

Let us denote a regular closed ultrafilter of X by $\mathcal{R}$ and the family of all regular closed ultrafilters $\mathcal{R}$ of X by $\mathfrak{R}$. Let's denote the family of all open ultrafilters $\mathcal{U}$ of X by $\mathfrak{U}$. Define $\mathrm{Cl}\,\mathcal{U} = \{\mathrm{Cl}\,U : U \in \mathcal{U}\}$. The following is well-known ([17, p. 649] or [20]).

Lemma 2.2. i. $\mathfrak{U} \ni \mathcal{U} \Rightarrow \mathcal{R} = \mathrm{Cl}\,\mathcal{U} \in \mathfrak{R}$. *If $\mathcal{U}$ has CCIP, so has $\mathcal{R}$.*

ii. $\mathfrak{R} \ni \mathcal{R} \Rightarrow \mathcal{U}(\mathcal{R}) = \{U : \mathrm{Int}\,R \subset U$ *for some* $R \in \mathcal{R}$ *and* U *is open*$\} \in \mathfrak{U}$. *If $\mathcal{R}$ has CCIP, so has $\mathcal{U}(\mathcal{R})$.*

iii. $\mathfrak{U} \ni \mathcal{U} \Rightarrow \mathcal{U}(\mathrm{Cl}\,\mathcal{U}) = \mathcal{U}$.

iv. $\mathfrak{R} \ni \mathcal{R} \Rightarrow \mathrm{Cl}(\mathcal{U}(\mathcal{R})) = \mathcal{R}$.

Let us define

$$U(X;0) = \{p \in X^* : \text{any } \mathcal{U}^p \text{ has CCIP}\}$$

and

$$U(X;0,\Delta) = \left\{p \in X^* : \text{there are } \mathcal{U}_1^p \text{ with CCIP and } \mathcal{U}_2^p \text{ without CCIP}\right\}.$$

From Lemma 2.2 and the definitions, we have the followings (see [21, 22])

1. X is almost realcompact if and only if $U(X;0) \cup U(X;0,\Delta) = \emptyset$.
2. X is c-realcompact if and only if $U(X;0) = \emptyset$.
3. $aX = X \cup U(X;0) \cup U(X;0,\Delta)$ and $uX = X \cup U(X;0)$.

If $f : X \longrightarrow Y$ is perfect and open, then we have $(\beta f)^{-1}(Y \cup U(Y;0)) = X \cup U(X;0)$ [22, 4.6(6)], that is, $(\beta f)^{-1}(uY) = uX$.

If $f : X \longrightarrow Y$ is perfect and open, then we have $(\beta f)(U(X;0,\Delta)) \subset U(Y;0,\Delta)$ [22, 4.6(4)] and $(\beta f)^{-1}(U(Y;0,\Delta)) \subset U(X;0,\Delta)$ [22, 4.6(5)]. It follows that $(\beta f)^{-1}(aY) = aX$.

Now we are ready to prove the following similar lemma, which we will use to prove our main theorems.

Lemma 2.3. *If $f : X \longrightarrow Y$ is a perfect and open mapping, then $uf : uX \to uY$ is perfect and open mapping between the c-realcompactifications.*

Proof. Since f is an open and closed mapping, then βf is an open by Lemma 2.1. Since $(\beta f)^{-1}(uY) = uX$ and βf is a perfect, it is clear that the restriction $uf : (\beta f)^{-1}(uY) = uX \longrightarrow uY$ is also a perfect mapping [13, Proposition 3.7.6].

Moreover, since $\beta(uf) = \beta f$, βf is open and uf is closed, then uf is an open mapping by Lemma 2.1. □

Similarly we have

Lemma 2.4. *If $f : X \longrightarrow Y$ is a perfect and open mapping, then $af : aX \to aY$ is perfect and open mapping between the almost realcompactifications.*

Corollary 2.5. *If G is a compact group acting on a completely regular space X, then the orbit map $\pi : X \longrightarrow X/G$ induces the open and perfect map $u\pi$ from uX onto $u(X/G)$ and the open and perfect map $a\pi$ from aX onto $a(X/G)$. In particular, the orbit space of a c-realcompact space (almost realcompact) is also c-realcompact (almost realcompact).*

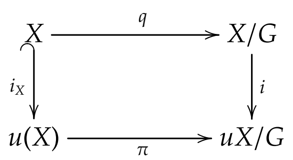
It follows by Lemma 2.3 and Lemma 2.4 that aX and uX are invariant subspaces of βX with respect to G, where G is a finite discrete group acting on a completely regular X.

We now have the necessary tools to prove our main theorem.

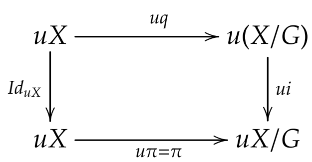
Theorem 2.6. *If G is a finite discrete group and X is a completely regular G-space, then uX/G is homeomorphic to $u(X/G)$.*

Proof. Recall that uX is invariant subspace of βX.

We show that the c-realcompact extension ui of the inclusion map $i : X/G \longrightarrow uX/G$ is a homeomorphism. Assume that $i_X : X \longrightarrow \beta X$ is the inclusion map, $q : X \longrightarrow X/G$ and $\pi : uX \longrightarrow uX/G$ are the orbit maps. Then we obtain the next commutative diagram, that is $i \circ q = \pi \circ i_X$.



Then we have $(ui) \circ (uq) = u(\pi) \circ u(i_X)$ since u is functorial (because β is functorial and u is restriction of β). Since uX/G is a c-realcompact space, then we obtain $u(\pi) = \pi$. Moreover, since $u(i_X) = Id_{uX}$ (since the Stone extension $\beta(i_X) : \beta X \longrightarrow \beta X$ is the identity map), we have the following commutative diagram, that is, $ui \circ uq = \pi \circ Id_{uX}$.



Now, let us show that ui is injective. Assume $ui(q_1) = ui(q_2)$ for $q_1, q_2 \in u(X/G)$. Since uq is surjective, we can choose $p_1 \in (uq)^{-1}(q_1)$ and $p_2 \in (uq)^{-1}(q_2)$. Then $\pi(p_1) = \pi(p_2)$ implies $G(p_1) = G(p_2)$. Hence $p_1 = g \cdot p_2$ for some $g \in G$. Since G acts trivially on $u(X/G)$ and uq is equivariant (because βq is equivariant), we have that $q_1 = uq(p_1) = uq(g \cdot p_2) = g \cdot uq(p_2) = uq(p_2) = q_2$.

The surjectivity of ui is clear from the commutative diagram above.

Since uq and π are continuous and open, it is obtained that ui is continuous and open. □

Notice that we do not use the equality $\beta(X/G) = \beta X/G$ in the proof of the Theorem 2.6. If we had used this equality then the result was obvious. In fact, since $X/G \subset uX/G \subset \beta X/G \approx \beta(X/G)$, we have $X/G \subset u(X/G) \subset uX/G \subset \beta(X/G)$ and so $u(X/G) = uX/G$, because uX/G is c-realcompact and $u(X/G)$ is the smallest c-realcompact space between X/G and $\beta(X/G)$. The reason why we do not prefer this kind of proof is this: If we can extend a given action of G on X to an action of G on uX (may not be on βX) then the functionality of our proof will become apparent.

Similarly the following can be proven.

Theorem 2.7. *Suppose that G is a finite discrete group acting on a completely regular X. Then aX/G is homeomorphic to $a(X/G)$.*

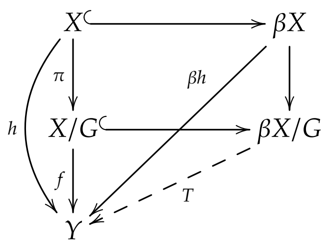
Now, we give a different proof of the equality $\beta X/G = \beta(X/G)$.

Theorem 2.8. *Suppose that a finite discrete group G acting on a completely regular X. Then $\beta X/G = \beta(X/G)$.*

Proof. Since X is dense in βX, so is X/G in $\beta X/G$. It is sufficient to show that every continuous mapping f from X/G into any compact Hausdorff space Y has an extension to a continuous map from $\beta X/G$ to Y [16, Theorem 6.4, Theorem 6.7]. For any mapping $f : X/G \longrightarrow Y$, the mapping $h := f \circ \pi : X \longrightarrow Y$ is continuous and it is constant map on each orbit (i.e. $h(gx) = h(x)$ for each $g \in G$ and $x \in X$). Therefore, we have that the Stone extension $\beta(h) : \beta X \longrightarrow Y$ and $\beta(h)$ is constant on each orbit. To show this, let's take any $g \in G$ and $p \in \beta X$. Then let $\mathcal{A}$ be a unique z-ultrafilter on X that converging to the point gp. Hence $g^{-1} \cdot \mathcal{A}$ is a z-ultrafilter on X that converges to the point p. Thus we obtain

$$\begin{aligned}(\beta h)(p) &= \bigcap h^{\#}\left(g^{-1} \cdot \mathcal{A}\right) = \bigcap \left\{E \subset Z(Y) : h^{-1}(E) \in g^{-1} \cdot \mathcal{A}\right\} \\ &= \bigcap \left\{E \subset Z(Y) : h^{-1}(E) = g \cdot h^{-1}(E) \in \mathcal{A}\right\} \\ &= \bigcap h^{\#}(\mathcal{A}) = (\beta h)(gp)\end{aligned}$$

Thus we can define $T : \beta X/G \longrightarrow Y$, $G(x) \mapsto (\beta h)(x)$, that makes the next diagram commutative:



Hence T is an extension of f and $\beta(X/G) = \beta X/G$ is obtained. □

Let X be dense in T. Then $\upsilon X = \upsilon T$ if and only if every continuous mapping f from X into any realcompact space Y can be extended to a continuous mapping from T into Y [16, Theorem 8.6]. By taking any realcompact space instead of compact space Y and using the fact that X/G is dense in $\upsilon X/G$ and υX is invariant subspace of βX, similar to the proof above, we can prove the next theorem.

Theorem 2.9. ([8, 14]) *Let X be a completely regular G-space, where G is a finite discrete group. Then $\upsilon X/G = \upsilon(X/G)$.*

Note that if X is dense in T, then $aX = aT$ if and only if every continuous mapping f from X into any almost realcompact space Y can be extended to a continuous mapping from T into Y [30]. Therefore Theorem 2.7 can also be proven as above.

From the above explanations the following result is immediately obtained.

Theorem 2.10. *If G is a compact group acting on a pseudocompact X, then $\beta X/G = \beta(X/G)$.*

Notice that if X is pseudocompact G-space, Theorem 2.6 and 2.9 does not make sense, because in this case, $uX = \beta X$ [17, Theorem 1.13] and $\upsilon X = \beta X$ [16, 8.A] and $aX = \beta X$.

It is a natural question to investigate under what conditions a action of G on X can be extended to the action G on uX (υX or aX). In other words, under what conditions does $u(G \times X) = G \times uX$ ($\upsilon(G \times X) = G \times \upsilon X$ or $a(G \times X) = G \times aX$)?

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References

[1] N. Antonyan, *On the maximal G-compactification of products of two G-spaces*, Int. J. Math. Sci., 2006. https://doi.org/10.1155/IJMMS/2006/93218
[2] N. Antonyan, S. Antonyan, *Free G-spaces and maximal equivariant compactifications* Annali di Matematica **184** (2005), 407–420.
[3] N. Antonyan, S. Antonyan, M. Sanchis, *Pseudocompactness in the Realm of Topological Transformation Groups*, In: Hrušák, M., Tamariz-Mascarúa, Á., Tkachenko, M. (eds) Pseudocompact Topological Spaces. Developments in Mathematics, vol 55. Springer, 2018.
[4] S. A. Antonyan, *G-pseudocompact and G-Hewitt spaces*, Russian Math. Surv. **35**(6) (1980), 81–82.
[5] S. A. Antonyan, *New proof of existance of a bicompact G-extension*— Comment Math. Univ. Carolin. **22**(4),(1981), 761–772.
[6] S. A. Antonyan, *On a problem of L. G. Zambakhidze*(in Russian), Uspehi Mat. Nauk **41**(5) (1986), 159–161; English translat.: Russian Math. Surv. **41**(5) (1986), 125–126.
[7] S. A. Antonyan, Yu. M. Smirnov, *Universal objects and bicompact extensions for topological transformation groups*(in Russian), Doklady. Akad. Nauk S.S.S.R. **257**(3) (1981), 521–526; English translat.: Soviet Math. Doklady. **23**(2) (1981), 279–284.
[8] K. K. Azad, K. Srivastava, G. Agrawal, *Extending group actions to extensions and absolutes-A survey*, Far East J. Math. Sci. **1**(2),(1993), 223–245.
[9] G. E. Bredon, I*ntroduction to Compact Transformation Groups*, New York, NY, USA: Academic Press, 1972.
[10] J. Chaber, *Remarks on open-closed mappings*, Fund. Math. **74**,(1972), 197–208.
[11] N. Dykes, *Generalizatioms of realcompact spaces*, Pacific J. Math. **33**,(1970), 571–581.
[12] N. Dykes, *Mappings and realcompact spaces*, Pacific J. Math. **31** (1969), 347-358.
[13] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.
[14] S. Eyidoğan, M. Onat, *The Hewitt realcompactification of an orbit space*, Turkish J. Math. **44** (2020), 2330–2336.
[15] Z. Frolík, *A generalization of realcompact spaces*, Czechoslovak Math. J., **13**(1963), 127–138.
[16] L. Gillman, M. Jerison, *Rings of Continuous Functions*, Princeton, 1960.
[17] K. Hardy, R. G. Woods, *On c-realcompact spaces and locally bounded normal functions*, Pacific J. Math. **43** (1972), 647–656.
[18] T. Ishii, *On the completions of maps*, Proc. Japan Acad. **50** (1974), 39–43.
[19] T. Isiwata, *Clopen realcompactification of a mapping*, **119** (1985), 153-158.
[20] T. Isiwata, *Closed ultrafilters and realcompactness*, Pacific J. Math. **92** (1981), 68–78.
[21] T. Isiwata, *Mappings and spaces*, Pacific J. Math. **20** (1967), 455–480.
[22] T. Isiwata, *Ultrafilters and mappings*, Pacific J. Math. **104** (1983), 371–389.
[23] M. Megrelishvili, *A Tychonoff G-space not admitting a compact Hausdorff G-extension or a G-linearization*, Russian Math. Surveys **43** (1988), 177–178.
[24] R. Palais, *The classification of G-spaces*, Mem. Amer. Math. Soc. **36** (1960).
[25] E. A. Reznichenko, *Extension of functions defined on products of pseudocompact spaces and continuity of the inverse in pseudocompact groups*, Topology Appl. **59** (1994), 233–244.
[26] A. Sokolovskaya, *G-compactifications of pseudocompact G-spaces*, Topology Appl. **155** (2008), 342–346.
[27] K. Srivastava, *On the Stone-Čech compactification of an orbit space*, Bull. Austral. Math. Soc. **36** (1987), 435–439.
[28] R. C. Walker, *The Stone-Čech Compactification*, Berlin, Germany: Springer-Verlag, 1974.
[29] R. G. Woods, *-absolutes of remainders of Stone-Čech compactifications* Pacific J. Math. **37** (1971), 545–560.
[30] R. G. Woods, A Tychonoff almost real-compactification, Proc. Amer. Math. Soc. **43**, (1974), 200–208.
[31] J. de Vries, *Can every Tychonoff G-space equivariantly be embedded in a compact Hausdorff G-space?*, Math. Centrum, Amsterdam, afd. Zuivere Wisk. 36, 1975.
[32] J. de Vries, *Equivariant embeddings of G-spaces*, in: General Topology and its Rel. to Modern Analysis and Algebra IV (Proc. 4th Prague Top. Sym., 1976), Part B, 485–493.
[33] J. de Vries, *On the existence of G-compactifications*, Bull. Acad. Polon. Sci. ser. Math.**26** (1978), 275–280.
[34] J. de Vries, *On the G-compactification of products*, Pacific J. Math. **110** (1984), 447–470.
[35] J. de Vries, *Topological Transformation Groups* 1, Math. Centre Tracts no. 65, Mathematisch Centrum, Amsterdam, 1975.