Filomat 38:29 (2024), 10261–10277
https://doi.org/10.2298/FIL2429261L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Further study on induced (L, M)-fuzzy bornological spaces

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Abstract. In this paper, the relationships of induced (L, M)-fuzzy bornology with (L, M)-fuzzy pseudo-quasi-metric and (L, M)-fuzzy topology are discussed. Moreover, quotient (L, M)-fuzzy bornology is introduced, and it is shown that the induced (L, M)-fuzzy bornology by quotient M-fuzzifying bornology is the quotient (L, M)-fuzzy bornology of induced (L, M)-fuzzy bornology.

1. Introduction

In [11, 12], S.T. Hu first introduced the axiomatic definition of bornology to define the conception of boundedness in a general topological space. Each bornology is an ideal in the powerset and contains all singletons. From the theoretical aspect, the theory of bornology as well as some related theories, such as hyperspace topologies [2, 5, 16], optimization theory [4], topologies in function spaces [3, 19], and so on [6, 7, 10, 22, 30], have received wide attention in recent years.

With the development of fuzzy set theory, bornology structures have been generalized to fuzzy case. Abel and Šostak [1] first introduced L-bornology. After that, Paseka et al. [20] introduced L-bornological vector spaces and systems. The categorical properties of them were studied, and it is proved that the category of L-bornological vector spaces is isomorphic to a full reflective subcategory of the category of L-bornological vector systems. In [32], Zhang and Zhang discussed the induced I-bornological vector spaces by general bornological vector spaces, and I-bornological linear mappings. Recently, Jin and Yan [14] introduced separation and L-Mackey convergence in L-bornological vector spaces, and discussed the equivalent characterization of separation in terms of L-Mackey convergence.

In an L-bornology, the bounded sets are fuzzy, but the bornology comprising those bounded sets is a crisp subset. In a different way, Šostak and Uljane [28] introduced $(L, *)$-valued bornology, which is considered as an L-subset of 2^X. In the setting of $(L, *)$-valued bornology, they proposed induced L-valued bornologies by fuzzy metrics and relative compactness-type L-valued bornologies in Chang-Goguen L-topological spaces. Adopting the terminology of fuzzy topology, we call this bornology an L-fuzzifying bornology. Shen and Yan [23] discussed fuzzifying bornologies induced by fuzzy pseudo-norms. They proved that the degree of bornological convergence is equivalent to the degree of topological convergence.

In 2017, Šostak and Uljane [29] introduced M-valued bornology on the L-powerset of an L-valued set (X, E) (where $E : X \times X \to L$ is an L-valued equality on X), which is called LM-fuzzy bornology for

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2020 *Mathematics Subject Classification*. Primary 52A01; Secondary 03E72
Keywords. (L, M)-fuzzy bornology, (L, M)-fuzzy pseudo-quasi-metric, (L, M)-fuzzy topology, base, quotient (L, M)-fuzzy bornology
Received: 07 March 2024; Revised: 22 August 2024; Accepted: 09 September 2024
Communicated by Ljubiša D. R. Kočinac
The work is supported by the National Natural Science Foundation of China (12101016, 12471432), and the Yuxiu Innovation Project of NCUT (No. 2024NCUTYXCX104).
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short. Recently, Liang et al. [18] introduced a new kind of M-valued L-fuzzy bornological vector space, namely (L, M)-fuzzy bornological vector space. The categorical properties were studied. In the classical case, bornology, topology and metric are closely related. In this paper, we will use L-fuzzy topology and (L, M)-fuzzy metric to characterize (L, M)-fuzzy bornology.

The structure of this paper is organized as follows. In Section 2, we review some preliminaries that are needed in the subsequent sections. In Section 3, we propose the relationships of induced (L, M)-fuzzy bornology with (L, M)-fuzzy pseudo-quasi-metric and (L, M)-fuzzy topology. In Section 4, we introduce quotient (L, M)-fuzzy bornology and discuss the relationships between quotient (L, M)-fuzzy bornological vector space and quotient L-bornological vector space. In Section 5, we discuss the relationships between (L, M)-fuzzy bornological space induced by M-fuzzifying bornological space with quotient space and product space.

2. Preliminaries

Throughout this paper, L (resp. M) is a frame with order-reversing involution. The smallest element and the largest element in L (resp. M) are denoted by $\bot_L$ and $\top_L$ (resp. $\bot_M$ and $\top_M$). An element a in L is called a prime element [9] if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. An element a in L is called co-prime element if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

For a nonempty set X, 2^X denotes the powerset of X. For any nonempty subset $A \subseteq 2^X$, let χ_A denote the characteristic function of A. L^X is the set of all L-subsets on X. For all $a \in L$ and $U \in L^X$[25], $U_{[a]} = \{x \in X \mid U(x) \geqslant a\}$, $U^{(a)} = \{x \in X \mid U(x) \nleqslant a\}$. The set of non-zero co-prime elements in L^X is denoted by $J(L^X)$. Each member in $J(L^X)$ is also called a point. It is easy to see that $J(L^X)$ is the set of all fuzzy points $x_\lambda(\lambda \in J(L))$. For each $a \in L$, $\underline{a}$ denotes the constant mapping $X \longrightarrow L$, $x \mapsto a$, which is called constant L-subset.

An M-fuzzy non-negative real number[13] is an equivalence class $[\varphi]$ of antitone mappings $\varphi : \mathbb{R} \longrightarrow M$ satisfying

$$\varphi(0-) = \bigwedge_{t<0} \varphi(t) = \top_M, \varphi(+\infty) = \bigwedge_{t\in R} \varphi(t) = \bot_M,$$

where the equivalence identifies two such mappings φ, ψ if and only if $\forall t > 0$, $\varphi(t-) = \psi(t-)$. We shall not distinguish an M-fuzzy real number $[\varphi]$ and its representative function φ being left continuous. The set of all non-negative M-fuzzy real numbers is denoted by $[0, +\infty)\,(M)$.

Let $f : X \longrightarrow Y$ be a mapping between two nonempty sets X, Y. The forward L-power operator $f^{\rightarrow} : L^X \longrightarrow L^Y$ and the backward L-powerset operator $f^{\leftarrow} : L^Y \longrightarrow L^X$ induced by f [21] are defined as follows:

(1) $\forall\, A \in L^X, y \in Y, f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$;

(2) $\forall\, B \in L^Y, f^{\leftarrow}(B) = B \circ f$.

Let X be a vector space over the field of real or complex numbers $\mathbb{K}$, θ be the zero vector in X. Using Zadeh's extension principle, the addition and scalar multiplication operator in L^X are defined as follows, respectively. For all $A, B \in L^X$, $x \in X$, and $k \in \mathbb{K}$,

(1) $(A + B)(x) = \bigvee_{y+z=x}(A(y) \wedge B(z))$;

(2) $(kA)(x) = A\left(\frac{x}{k}\right), k \neq 0$;

(3) $(0A)(x) = \begin{cases} \bigvee_{y\in X} A(y), x = \theta, \\ \bot_L, x \neq \theta. \end{cases}$

Lemma 2.1 ([25]). *For each $A \in L^X$ and $a \in L$, we have:*

(1) $f\left(A^{(a)}\right) = (f^{\rightarrow}(A))^{(a)}$;

(2) *If* $\{A_t\}_{t\in T} \subseteq L^X$, $\left(\bigvee_{t\in T} A_t\right)^{(a)} = \bigcup_{t\in T} (A_t)^{(a)}$;

(3) *For each* $B \in L^Y$, $(f^{\leftarrow}(B))^{(a)} = f^{-1}(B^{(a)})$.

Definition 2.2 ([20]). Let $\{X_i\}_{i\in I}$ be a family of sets. For all $A_i \in L^{X_i}$, define a mapping $\prod_{i\in I} A_i : \prod_{i\in I} X_i \longrightarrow L$ as follows $\prod_{i\in I} A_i(x) = \bigwedge_{i\in I} A_i(P_i(x))$, where $P_i : \prod_{i\in I} X_i \longrightarrow X_i$ be the projection.

Definition 2.3 ([28]). An M-fuzzifying bornology on a set X is a mapping $\mathcal{B} : 2^X \longrightarrow M$ satisfying the following conditions:

(MB1) $\mathcal{B}(\{x\}) = \top_M, \forall x \in X$;
(MB2) For all $A, B \in 2^X with A \subseteq B, \mathcal{B}(A) \geqslant \mathcal{B}(B)$;
(MB3) $\mathcal{B}(A \cup B) \geqslant \mathcal{B}(A) \wedge \mathcal{B}(B), \forall A, B \in 2^X$.

The pair $(X, \mathcal{B})$ is called an M-fuzzifying bornological space. The value $\mathcal{B}(A)$ is interpreted as the degree of boundedness of a set A in the space $(X, \mathcal{B})$.

Definition 2.4 ([1, 20]). An L-bornology on a set X is a subfamily $\mathcal{B} \subseteq L^X$ such that:

(LB1) $\bigvee_{B\in\mathcal{B}} B(x) = \top_L$, $\forall x \in X$;
(LB2) $\forall B \in \mathcal{B}$, $D \in L^X$ with $D \leqslant B \Rightarrow D \in \mathcal{B}$;
(LB3) $A, B \in \mathcal{B} \Rightarrow A \vee B \in \mathcal{B}$.

The pair $(X, \mathcal{B})$ is called an L-bornological space.

Definition 2.5 ([18, 29]). An M-valued L-fuzzy bornology, or an (L, M)-fuzzy bornology for short on a set X is a mapping $\mathscr{B} : L^X \longrightarrow M$ which satisfies:

(LMB1) $\mathscr{B}(x_{\top_L}) = \top_M$;
(LMB2) For each $A, B \in L^X$, $A \leqslant B \Rightarrow \mathscr{B}(A) \geqslant \mathscr{B}(B)$;
(LMB3) $\mathscr{B}(A \vee B) \geqslant \mathscr{B}(A) \wedge \mathscr{B}(B), \forall A, B \in L^X$.

The pair $(X, \mathscr{B})$ is called an (L, M)-fuzzy bornological space. $\mathscr{B}(A)$ can be interpreted as the degree of boundedness of A.

Definition 2.6 ([18, 29]). Let $(X, \mathscr{B}_X)$ and $(Y, \mathscr{B}_Y)$ be two (L, M)-fuzzy bornological spaces. A mapping $f : X \longrightarrow Y$ is (L, M)-fuzzy bounded provided that $\mathscr{B}_X(A) \leqslant \mathscr{B}_Y(f^{\rightarrow}(A))$ for all $A \in L^X$.

Theorem 2.7 ([18]). *Supposed that* $\{(X_i, \mathscr{B}_i)\}_{i\in I}$ *is a family of* (L, M)*-fuzzy bornological spaces,* $X = \prod_{i\in I} X_i$ *and* $P_i : X \longrightarrow X_i$ *is the projection. Define* $\mathscr{B} : L^X \longrightarrow M$ *by* $\mathscr{B}(A) = \bigvee_{A\leqslant\prod_{i\in I} A_i} \bigwedge_{i\in I} \mathscr{B}_i(A_i)$, $\forall A \in L^X$. *Then* $(X, \mathscr{B})$ *is an* (L, M)*-fuzzy bornological space, which is called the product space of* $\{(X_i, \mathscr{B}_i)\}_{i\in I}$*, denoted by* $(X, \prod_{i\in I} \mathscr{B}_i)$.

Remark 2.8. The product space of (L, M)-fuzzy bornological spaces can degenerate to the product space of M-fuzzifying bornological spaces by restricting $L = \{0, 1\}$.

Remark 2.9. Let $(X, \mathscr{B})$ be a crisp bornological space. Then $\mathscr{B}$ can be regarded as a mapping $\chi_{\mathscr{B}} : L^X \longrightarrow M$ defined by

$$\chi_{\mathscr{B}}(A) = \begin{cases} \top_M, A \in \mathscr{B}, \\ \bot_M, A \notin \mathscr{B}. \end{cases}$$

Obviously, $(X, \chi_{\mathscr{B}})$ is a special (L, M)-fuzzy bornological space. In this way, $(X, \mathscr{B})$ can be regarded as an (L, M)-fuzzy bornological space determined by the crisp bornology.

Definition 2.10 ([18]). An (L,M)-fuzzy bornological vector space is a triple $(X,\mathbb{K},\mathscr{B})$, where X is a vector space over the field of real or complex numbers $\mathbb{K}$, and $(X,\mathscr{B})$ is an (L,M)-fuzzy bornological space such that :

(BV1) $f : X\times X \longrightarrow X$, $(x,y)\mapsto x+y$ is (L,M)-fuzzy bounded;
(BV2) $g : \mathbb{K}\times X \longrightarrow X$, $(k,x)\mapsto kx$ is (L,M)-fuzzy bounded,

where $X\times X$ and $\mathbb{K}\times X$ are equipped with the corresponding product (L,M)-fuzzy bornologies $\mathscr{B}\times\mathscr{B}$ and $\mathscr{B}_{\mathbb{K}}\times\mathscr{B}$ ($\mathscr{B}_{\mathbb{K}}$ is the (L,M)-fuzzy bornology determined by the crisp bornology on $\mathbb{K}$), respectively.

Theorem 2.11 ([18]). *Let X be a vector space over $\mathbb{K}$, and $(X,\mathscr{B})$ be an (L,M)-fuzzy bornological space. Then $(X,\mathbb{K},\mathscr{B})$ is an (L,M)-fuzzy bornological vector space if and only if $\mathscr{B}$ satisfies the following conditions :*
(BV3) $\mathscr{B}(A)\wedge\mathscr{B}(B)\leqslant\mathscr{B}(A+B)$, $\forall\, A,B\in L^X$;
(BV4) $\mathscr{B}(A)\leqslant\mathscr{B}(\lambda A)$, $\forall\, A\in L^X, \lambda\in\mathbb{K}$;
(BV5) $\mathscr{B}(A)\leqslant\mathscr{B}\left(\bigvee_{|\lambda|\leqslant 1}\lambda A\right)$, $\forall\, A\in L^X$.

3. Induced (L,M)-fuzzy bornologies by (L,M)-fuzzy pseudo-quasi-metric and (L,M)-fuzzy topology

In this section, we construct (L,M)-fuzzy bornologies in (L,M)-fuzzy pseudo-quasi-metric space and (L,M)-fuzzy topological space. In [24], the author introduced (L,M)-fuzzy pseudo-quasi-metric as follows.

Definition 3.1 ([24]). An (L,M)-fuzzy pseudo-quasi-metric on X is a mapping $d : J(L^X)\times J(L^X)\longrightarrow [0,+\infty)\,(M)$ satisfying: $\forall\, a,b,c\in J(L^X)$,

(LMd1) $a\leqslant b\Rightarrow d(a,b)(0+)=\perp_M$;
(LMd2) $\forall\, r,s>0,\ d(a,c)(r+s)\leqslant d(a,b)(r)\vee d(b,c)(s)$;
(LMd3) $d(a,b)=\bigwedge_{c\prec b} d(a,c)$.

Such a d is said to be an (L,M)-fuzzy pseudo-metric if d satisfies

(LMd4) $\forall\, u,v\in J(L^X),\ \bigwedge_{a\nleqslant u'} d(a,v)=\bigwedge_{b\nleqslant v'} d(b,u)$.

An (L,M)-fuzzy pseudo-metric d is said to be an (L,M)-fuzzy metric if d satisfies

(LMd5) $d(a,b)(0+)=\perp_M\Rightarrow a\leqslant b$.

Proposition 3.2. *Let d be an (L,M)-fuzzy pseudo-quasi-metric on X. For all $x_\lambda\in J(L^X)$, define a mapping $N^d_{x_\lambda} : L^X\longrightarrow M$ by*

$$N^d_{x_\lambda}(U)=\bigvee_{x_\lambda\leqslant V\leqslant U}\ \bigwedge_{y_\mu\nleqslant V'} d(y_\mu,V')(0+),$$

where $d(y_\mu,V')(0+)=\bigvee_{r>0}\bigwedge_{z_v\leqslant V'} d(y_\mu,z_v)(r)$. Then the following statements hold:

(1) $N^d_{x_\lambda}(\top_{L^X})=\top_M$;
(2) *For any $U,V\in L^X$ with $U\leqslant V$, $N^d_{x_\lambda}(U)\leqslant N^d_{x_\lambda}(V)$;*
(3) *For any $x\in X$ and $U\in L^X$ with $x_{\top_L}\nleqslant U$, $N^d_{x_{\top_L}}(U)=\perp_M$.*

Proof. (1) Since $(\top_{L^X})' = \bot_{L^X}$, we have

$$\begin{aligned} N^d_{x_\lambda}(\top_{L^X}) &= \bigvee_{x_\lambda \leqslant V \leqslant \top_{L^X}} \bigwedge_{y_\mu \nleqslant V'} d(y_\mu, V')(0+) \\ &= \bigvee_{x_\lambda \leqslant V \leqslant \top_{L^X}} \bigwedge_{y_\mu \nleqslant V'} \bigvee_{r>0} \bigwedge_{z_v \leqslant V'} d(y_\mu, z_v)(r) \\ &\geqslant \bigwedge_{y_\mu \nleqslant (\top_{L^X})'} \bigvee_{r>0} \bigwedge_{z_v \leqslant (\top_{L^X})'} d(y_\mu, z_v)(r) \\ &= \bigwedge_{y_\mu \nleqslant \bot_{L^X}} \bigvee_{r>0} \bigwedge_{z_v \leqslant \bot_{L^X}} d(y_\mu, z_v)(r) \\ &= \top_M. \end{aligned}$$

This implies $N^d_{x_\lambda}(\top_{L^X}) = \top_M$.

(2) Take any U, V with $U \leqslant V$. Then

$$\begin{aligned} N^d_{x_\lambda}(U) &= \bigvee_{x_\lambda \leqslant A \leqslant U} \bigwedge_{y_\mu \nleqslant A'} d(y_\mu, A')(0+) \\ &\leqslant \bigvee_{x_\lambda \leqslant B \leqslant V} \bigwedge_{y_\mu \nleqslant B'} d(y_\mu, B')(0+) \\ &= N^d_{x_\lambda}(V). \end{aligned}$$

This implies that $N^d_{x_\lambda}(U) \leqslant N^d_{x_\lambda}(V)$.
(3) Since $x_{\top_L} \nleqslant U$, it follows that

$$N^d_{x_{\top_L}}(U) = \bigvee_{x_{\top_L} \leqslant V \leqslant U} \bigwedge_{y_\mu \nleqslant V'} d(y_\mu, V')(0+) = \bot_M,$$

as desired. □

Theorem 3.3. *Let d be an (L, M)-fuzzy pseudo-quasi-metric on X. Define a mapping $\mathscr{B}_d : L^X \longrightarrow M$ by*

$$\mathscr{B}_d(A) = \bigvee_{A \leqslant U} \bigvee_{y \in X} N^d_{y_{\top_L}}(U).$$

Then $\mathscr{B}_d$ is an (L, M)-fuzzy bornology induced by d.

Proof. It suffices to show that $\mathscr{B}_d$ satisfies (LMB1)-(LMB3).

(LMB1) By Proposition 3.2, we have for all $y \in Y$, $N^d_{y_{\top_L}}(\top_{L^X}) = \top_M$. Therefore,

$$\mathscr{B}_d(x_{\top_L}) = \bigvee_{x_{\top_L} \leqslant U} \bigvee_{y \in X} N^d_{y_{\top_L}}(U) \geqslant \bigvee_{y \in X} N^d_{y_{\top_L}}(\top_{L^X}) = \top_M.$$

Then we have $\mathscr{B}_d(x_{\top_L}) = \top_M$.

(LMB2) For all $A, B \in L^X$ with $A \leqslant B$, we have

$$\mathscr{B}_d(A) = \bigvee_{A \leqslant U} \bigvee_{y \in X} N^d_{y_{\top_L}}(U) \geqslant \bigvee_{B \leqslant U} \bigvee_{y \in X} N^d_{y_{\top_L}}(U) = \mathscr{B}_d(B).$$

(LMB3) Take any $A, B, U_1, U_2 \in L^X$ with $A \leqslant U_1, B \leqslant U_2$. Then $A \vee B \leqslant U_1 \vee U_2$. By Proposition 3.2, we have

$$\bigvee_{y\in X} N^d_{y_{\top_L}}(U_1) \wedge \bigvee_{y\in X} N^d_{y_{\top_L}}(U_2) \leqslant \bigvee_{y\in X} N^d_{y_{\top_L}}(U_1 \vee U_2).$$

This implies

$$\mathscr{B}_d(A) \wedge \mathscr{B}_d(B) = \bigvee_{A\leqslant U_1} \bigvee_{y\in X} N^d_{y_{\top_L}}(U_1) \wedge \bigvee_{B\leqslant U_2} \bigvee_{y\in X} N^d_{y_{\top_L}}(U_2) \leqslant \bigvee_{A\vee B\leqslant U_1\vee U_2} \bigvee_{y\in X} N^d_{y_{\top_L}}(U_1 \vee U_2) = \mathscr{B}_d(A \vee B).$$

This shows that $\mathscr{B}_d$ satisfies (LMB1)-(LMB3). Therefore, $\mathscr{B}_d$ is an (L, M)-fuzzy bornology on X. □

Remark 3.4. In Theorem 3.3, let $L = M = \{0, 1\}$, and $B_d(A) = 1$. Then there exist $U \in L^X$ and $y \in X$ such that $A \leqslant U$, and $N^d_{y_{\top_L}}(U) = 1$(which means U is a neighborhood of $y_{\top_L}$).

Next we discuss the (L, M)-fuzzy bornology induced by fuzzy compactness. In [17], the authors introduced degrees of fuzzy compactness in L-fuzzy topological spaces. Definition 3.6 and Lemma 3.7 are presented in L-valued L-fuzzy topology. They can easily be transformed to M-valued L-fuzzy topology as follows.

Definition 3.5 ([15, 26, 27])**.** An M-valued L-fuzzy topology, or an (L, M)-fuzzy topology for short on a set X is a mapping $\tau : L^X \longrightarrow M$ which satisfies:

(LMT1) $\tau(\top_{L^X}) = \tau(\perp_{L^X}) = \top_M$;
(LMT2) $\forall A, B \in L^X, \tau(A \wedge B) \geqslant \tau(A) \wedge \tau(B)$;
(LMT3) $\forall \left\{A_j \mid j \in J\right\} \subseteq L^X, \tau(\bigvee_{j\in J} A_j) \geqslant \bigwedge_{j\in J} \tau(A_j)$.

The pair (X, τ) is called an (L, M)-fuzzy topological space. Given two (L, M)-fuzzy topological spaces (X, τ_X) and (Y, τ_Y), a mapping $f : X \longrightarrow Y$ is called continuous if $\tau_X(f^{\leftarrow}(B)) \geqslant \tau_Y(B)$ for each $B \in L^Y$.

Definition 3.6 ([17])**.** Let (X, τ) be an (L, M)-fuzzy topological space. For all $B \in L^X$, the degree of fuzzy compactness of B is defined as follows:

$$DFC_\tau(B) = \bigwedge_{\mathscr{U}\in \$(B)} \bigvee_{D\in\mathscr{U}} \tau'(D),$$

where $\$(B) = \left\{\mathscr{U} \subseteq L^X \mid \bigwedge_{x\in X}\left(B'(x) \vee \bigvee_{D\in\mathscr{U}} D(x)\right) \nleqslant \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\left(B'(x) \vee \bigvee_{D\in\mathscr{V}} D(x)\right)\right\}$.

Lemma 3.7 ([17])**.** *Let (X, τ) be an (L, M)-fuzzy topological spaces. Then for all $A, B \in L^X$, $DFC_\tau(A) \wedge DFC_\tau(B) \leqslant DFC_\tau(A \vee B)$.*

Theorem 3.8. *Let (X, τ) be an (L, M)-fuzzy topological space. Define a mapping $\mathscr{B}_c : L^X \longrightarrow M$ by*

$$\mathscr{B}_c(A) = \bigvee_{A\leqslant B} DFC_\tau(B).$$

Then $\mathscr{B}_c$ is an (L, M)-fuzzy bornology induced by the degree of fuzzy compactness.

Proof. We need to prove that $\mathscr{B}_c$ satisfies (LMB1)-(LMB3).

(LMB1) For each $x \in X$ and $\mathscr{U} \subseteq L^X$, we have

$$\begin{aligned} \bigwedge_{y\in X}(x'_{\top_L}(y) \vee \bigvee_{D\in\mathscr{U}} D(y)) &= \bigvee_{D\in\mathscr{U}} D(x) \\ &= \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigvee_{D\in\mathscr{V}} D(x) \\ &= \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{y\in X}(x'_{\top_L}(y) \vee \bigvee_{D\in\mathscr{V}} D(y)). \end{aligned}$$

This implies $\mathbb{S}(x_{\top_L}) = \phi$. Thus

$$\begin{aligned}\mathscr{B}_c(x_{\top_L}) &= \bigvee_{x_{\top_L}\leqslant B} DFC_\tau(B)\\ &= \bigvee_{x_{\top_L}\leqslant B}\bigwedge_{\mathscr{U}\in\mathbb{S}(B)}\bigvee_{D\in\mathscr{U}}\tau'(D)\\ &\geqslant \bigwedge_{\mathscr{U}\in\mathbb{S}(x_{\top_L})}\bigvee_{D\in\mathscr{U}}\tau'(D)\\ &= \top_M.\end{aligned}$$

This shows $\mathscr{B}_c(x_{\top_L}) = \top_M$.

(LMB2) For all $A, B \in L^X$ with $A \leqslant B$, we can obtain

$$\mathscr{B}_c(A) = \bigvee_{A\leqslant D} DFC_\tau(D) \geqslant \bigvee_{B\leqslant D} DFC_\tau(D) = \mathscr{B}_c(B).$$

(LMB3) By Lemma 3.7, we know that

$$\begin{aligned}\mathscr{B}_c(A)\wedge\mathscr{B}_c(B) &= \bigvee_{A\leqslant D_1} DFC_\tau(D_1)\wedge\bigvee_{B\leqslant D_2} DFC_\tau(D_2)\\ &= \bigvee_{A\leqslant D_1}\bigvee_{B\leqslant D_2}(DFC_\tau(D_1)\wedge DFC_\tau(D_2))\\ &\leqslant \bigvee_{A\vee B\leqslant D_1\vee D_2} DFC_\tau(D_1\vee D_2)\\ &= \mathscr{B}_c(A\vee B).\end{aligned}$$

Therefore, $\mathscr{B}_c$ satisfies (LMB1)-(LMB3). Thus $\mathscr{B}_c$ is an (L, M)-fuzzy bornology induced by the degree of fuzzy compactness. □

Lemma 3.9 ([17]). *If $f : (X, \tau) \longrightarrow (Y, \delta)$ is continuous with respect to (L, M)-fuzzy topologies τ and δ, then $DFC_\tau(A) \leqslant DFC_\delta(f^\rightarrow(A))$.*

Proposition 3.10. *Let (X, τ_X) and (Y, τ_Y) be two (L, M)-fuzzy topological spaces, the mapping $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be continuous, $(X, \mathscr{B}_{c_X})$ and $(Y, \mathscr{B}_{c_Y})$ be two (L, M)-fuzzy bornological spaces induced by the degree of fuzzy compactness. Then $f : (X, \mathscr{B}_{c_X}) \longrightarrow (Y, \mathscr{B}_{c_Y})$ is (L, M)-fuzzy bounded.*

Proof. By Lemma 3.9, we know that for all $A \in L^X$,

$$\begin{aligned}\mathscr{B}_{c_X}(A) &= \bigvee_{A\leqslant B} DFC_{\tau_X}(B)\\ &\leqslant \bigvee_{f^\rightarrow(A)\leqslant f^\rightarrow(B)} DFC_{\tau_Y}(f^\rightarrow(B))\\ &\leqslant \mathscr{B}_{c\_Y}(f^\rightarrow(A)).\end{aligned}$$

Thus f is (L, M)-fuzzy bounded. □

In the following, we introduce the (L, M)-fuzzy bornology induced by (L, M)-fuzzy topology.

Proposition 3.11. *Let (X, τ) be an (L, M)-fuzzy topological space. Define $\mathscr{B}_\tau : L^X \longrightarrow M$ as follows:*

$$\mathscr{B}_\tau(A) = \bigvee_{A \leqslant B} \bigvee_{x_\lambda \in J(L^X)} \bigvee_{x_\lambda \leqslant D \leqslant B} \tau(D).$$

Then $\mathscr{B}_\tau$ is an (L, M)-fuzzy bornology on X induced by τ.

Proof. It suffices to show that $\mathscr{B}_\tau$ satisfies (LMB1)-(LMB3).

(LMB1) For all $x \in X$, we have

$$\mathscr{B}_\tau(x_{\top_L}) = \bigvee_{x_{\top_L} \leqslant B} \bigvee_{y_\mu \in J(L^X)} \bigvee_{y_\mu \leqslant D \leqslant B} \tau(D) \geqslant \tau(\top_{L^X}) = \top_M.$$

This implies $\mathscr{B}_\tau(x_{\top_L}) = \top_M$.

(LMB2) For all $A, B \in L^X$ with $A \leqslant B$, we have

$$\mathscr{B}_\tau(A) = \bigvee_{A \leqslant D} \bigvee_{x_\lambda \in J(L^X)} \bigvee_{x_\lambda \leqslant C \leqslant D} \tau(C) \geqslant \bigvee_{B \leqslant D} \bigvee_{x_\lambda \in J(L^X)} \bigvee_{x_\lambda \leqslant C \leqslant D} \tau(C) = \mathscr{B}_\tau(B).$$

(LMB3) Take any $A, B \in L^X$. Then we have

$$\begin{aligned}
\mathscr{B}_\tau(A) \wedge \mathscr{B}_\tau(B) &= \left[\bigvee_{A \leqslant C_1} \bigvee_{x_\lambda \in J(L^X)} \bigvee_{x_\lambda \leqslant C_2 \leqslant C_1} \tau(C_2)\right] \wedge \left[\bigvee_{B \leqslant D_1} \bigvee_{x_\lambda \in J(L^X)} \bigvee_{x_\lambda \leqslant D_2 \leqslant D_1} \tau(D_2)\right] \\
&\leqslant \bigvee_{A \vee B \leqslant C_1 \vee D_1} \bigvee_{x_\lambda \in J(L^X)} \bigvee_{x_\lambda \leqslant C_2 \vee D_2 \leqslant C_1 \vee D_1} \tau(C_2 \vee D_2) \\
&= \mathscr{B}_\tau(A \vee B).
\end{aligned}$$

This proves that $\mathscr{B}_\tau$ is an (L, M)-fuzzy bornology on X. □

Proposition 3.12. Let $\mathscr{B} : L^X \longrightarrow M$ be an (L, M)-fuzzy bornology on X. Define $\tau_{\mathscr{B}} : L^X \longrightarrow M$ as follows:

$$\tau_{\mathscr{B}}(A) = \bigwedge \{\mathcal{T}_X(A) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\}, \ \forall A \in L^X,$$

where $\mathfrak{T}_X$ denotes the family of all (L, M)-fuzzy topologies on X. Then $\tau_{\mathscr{B}}$ is an (L, M)-fuzzy topology on X.

Proof. It suffices to show that $\tau_{\mathscr{B}}$ satisfies (LMT1)-(LMT3).

(LMT1) By the definition of $\tau_{\mathscr{B}}$, we have

$$\tau_{\mathscr{B}}(\top_{L^X}) = \bigwedge \{\mathcal{T}_X(\top_{L^X}) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} = \top_M,$$

and

$$\tau_{\mathscr{B}}(\bot_{L^X}) = \bigwedge \{\mathcal{T}_X(\bot_{L^X}) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} = \top_M.$$

(LMT2) For all $A, B \in L^X$, we have

$$\begin{aligned}
\tau_{\mathscr{B}}(A) \wedge \tau_{\mathscr{B}}(B) &= \bigwedge \{\mathcal{T}_X(A) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} \wedge \bigwedge \{\mathcal{T}_X(B) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} \\
&= \bigwedge \{\mathcal{T}_X(A) \wedge \mathcal{T}_X(B) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} \\
&\leqslant \bigwedge \{\mathcal{T}_X(A \wedge B) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} \\
&= \tau_{\mathscr{B}}(A \wedge B).
\end{aligned}$$

(LMT3) For each $\left\{A_j \mid j \in J\right\} \subseteq L^X$, we have

$$
\begin{aligned}
\bigwedge_{j\in J} \tau_{\mathscr{B}}(A_j) &= \bigwedge_{j\in J} \bigwedge\{\mathcal{T}_X(A_j) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} \\
&= \bigwedge \bigwedge_{j\in J}\{\mathcal{T}_X(A_j) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} \\
&\leqslant \bigwedge\{\mathcal{T}_X\left(\bigvee_{j\in J} A_j\right) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} \\
&= \tau_{\mathscr{B}}\left(\bigvee_{j\in J} A_j\right).
\end{aligned}
$$

This proves that $\tau_{\mathscr{B}}$ is an (L, M)-fuzzy topology on X. □

Remark 3.13. Let $\mathscr{B}$ be an (L, M)-fuzzy bornology on X. By Proposition 3.12, $\tau_{\mathscr{B}}$ is the (L, M)-fuzzy topology induced by $\mathscr{B}$. By Proposition 3.11, $\mathscr{B}_{\tau_{\mathscr{B}}}$ is the (L, M)-fuzzy bornology induced by $\tau_{\mathscr{B}}$, and

$$
\mathscr{B}_{\tau_{\mathscr{B}}}(A) = \bigvee_{A\leqslant B} \bigvee_{x_\lambda\in J(L^X)} \bigvee_{x_\lambda\leqslant D\leqslant B} \tau_{\mathscr{B}}(D).
$$

Proposition 3.14. *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological space. Then $\mathscr{B}_{\tau_{\mathscr{B}}} \geqslant \mathscr{B}$.*

Proof. For all $A \in L^X$, we have

$$
\begin{aligned}
\mathscr{B}_{\tau_{\mathscr{B}}}(A) &= \bigvee_{A\leqslant B} \bigvee_{x_\lambda\in J(L^X)} \bigvee_{x_\lambda\leqslant D\leqslant B} \tau_{\mathscr{B}}(D) \\
&= \bigvee_{A\leqslant B} \bigvee_{x_\lambda\in J(L^X)} \bigvee_{x_\lambda\leqslant D\leqslant B} \bigwedge\{\mathcal{T}_X(D) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} \\
&\geqslant \bigvee_{x_\lambda\in J(L^X)} \bigvee_{x_\lambda\leqslant D\leqslant A} \bigwedge\{\mathcal{T}_X(D) : \mathscr{B} \leqslant \mathcal{T}_X \in \mathfrak{T}_X\} \\
&\geqslant \bigvee_{x_\lambda\in J(L^X)} \bigvee_{x_\lambda\leqslant D\leqslant A} \mathscr{B}(D) \\
&\geqslant \mathscr{B}(A).
\end{aligned}
$$

Therefore we can obtain $\mathscr{B}_{\tau_{\mathscr{B}}} \geqslant \mathscr{B}$. □

4. Quotient (L, M)-fuzzy bornology

In this section, we introduce the quotient (L, M)-fuzzy bornology induced by surjective mapping, and discuss the relationships between quotient (L, M)-fuzzy bornological vector space and quotient L-bornological vector space.

Proposition 4.1. *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological space and $f : X \longrightarrow Y$ be a surjective mapping. Define a mapping $\mathscr{B}/f : L^Y \longrightarrow M$ by*

$$
(\mathscr{B}/f)(B) = \textstyle\bigvee_{B\leqslant f^{\rightarrow}(A)} \mathscr{B}(A), \ \forall\, B \in L^Y.
$$

Then $\mathscr{B}/f$ is the finest (L, M)-fuzzy bornology on Y such that f is (L, M)-fuzzy bounded. We call $\mathscr{B}/f$ a quotient (L, M)-fuzzy bornology on Y induced by f and $\mathscr{B}$.

Proof. Firstly, we verify that $\mathscr{B}/f$ is an (L, M)-fuzzy bornology on Y.

(LMB1) Take any $y \in Y$. Since f is a surjective mapping, we can obtain that there exists $x \in X$ such that $f(x) = y$. This implies $f^{\rightarrow}(x_{\top_L}) = f(x)_{\top_L} = y_{\top_L}$. Then

$$(\mathscr{B}/f)(y_{\top_L}) = \bigvee_{y_{\top_L} \leqslant f^{\rightarrow}(A)} \mathscr{B}(A) \geqslant \mathscr{B}(x_{\top_L}) = \top_M.$$

This shows $(\mathscr{B}/f)(y_{\top_L}) = \top_M$.

(LMB2) For all $A, B \in L^Y$ with $A \leqslant B$, we have

$$(\mathscr{B}/f)(B) = \bigvee_{B \leqslant f^{\rightarrow}(C)} \mathscr{B}(C) \leqslant \bigvee_{A \leqslant f^{\rightarrow}(C)} \mathscr{B}(C) = (\mathscr{B}/f)(A).$$

(LMB3) Take any $A, B \in L^Y$. Since for all $C_1, C_2 \in L^X$, $f^{\rightarrow}(C_1 \vee C_2) = f^{\rightarrow}(C_1) \vee f^{\rightarrow}(C_2)$, it follows that

$$\begin{aligned}
(\mathscr{B}/f)(A) \wedge (\mathscr{B}/f)(B) &= \bigvee_{A \leqslant f^{\rightarrow}(C_1)} \mathscr{B}(C_1) \wedge \bigvee_{B \leqslant f^{\rightarrow}(C_2)} \mathscr{B}(C_2) \\
&= \bigvee_{A \leqslant f^{\rightarrow}(C_1)} \bigvee_{B \leqslant f^{\rightarrow}(C_2)} \Big(\mathscr{B}(C_1) \wedge \mathscr{B}(C_2)\Big) \\
&\leqslant \bigvee_{A \leqslant f^{\rightarrow}(C_1)} \bigvee_{B \leqslant f^{\rightarrow}(C_2)} \mathscr{B}(C_1 \vee C_2) \\
&\leqslant \bigvee_{A \vee B \leqslant f^{\rightarrow}(C_1) \vee f^{\rightarrow}(C_2)} \mathscr{B}(C_1 \vee C_2) \\
&= \bigvee_{A \vee B \leqslant f^{\rightarrow}(C_1 \vee C_2)} \mathscr{B}(C_1 \vee C_2) \\
&= (\mathscr{B}/f)(A \vee B).
\end{aligned}$$

This shows $\mathscr{B}/f$ satisfies (LMB1)-(LMB3). Therefore, $\mathscr{B}/f$ is an (L, M)-fuzzy bornology on Y.

Secondly, we prove $f : (X, \mathscr{B}) \longrightarrow (Y, \mathscr{B}/f)$ is (L, M)-fuzzy bounded. For each $A \in L^X$, it follows that

$$(\mathscr{B}/f)(f^{\rightarrow}(A)) = \bigvee_{f^{\rightarrow}(A) \leqslant f^{\rightarrow}(C)} \mathscr{B}(C) \geqslant \mathscr{B}(A).$$

Hence $f : (X, \mathscr{B}) \longrightarrow (Y, \mathscr{B}/f)$ is (L, M)-fuzzy bounded.

Further, for any (L, M)-fuzzy bornological space $(Y, \mathscr{B}_Y)$ such that $f : (X, \mathscr{B}) \longrightarrow (Y, \mathscr{B}_Y)$ is (L, M)-fuzzy bounded, and $A \in L^Y$, we have

$$(\mathscr{B}/f)(A) = \bigvee_{A \leqslant f^{\rightarrow}(C)} \mathscr{B}(C) \leqslant \bigvee_{A \leqslant f^{\rightarrow}(C)} \mathscr{B}_Y(f^{\rightarrow}(C)) \leqslant \mathscr{B}_Y(A).$$

This implies $\mathscr{B}/f$ is the finest (L, M)-fuzzy bornology on Y such that f is (L, M)-fuzzy bounded. □

Lemma 4.2. *Let $f : X \longrightarrow Y$ be a surjective linear mapping. Then for all $A, B \in L^X$, $f^{\rightarrow}(A) + f^{\rightarrow}(B) = f^{\rightarrow}(A + B)$.*

Proof. Take any $A, B \in L^X$ and $z \in Y$,

$$\begin{aligned}(f^{\rightarrow}(A) + f^{\rightarrow}(B))(z) &= \bigvee_{z_1+z_2=z} (f^{\rightarrow}(A)(z_1) \wedge f^{\rightarrow}(B)(z_2)) \\ &= \bigvee_{z_1+z_2=z} \left(\bigvee_{f(x_1)=z_1} A(x_1) \wedge \bigvee_{f(x_2)=z_2} B(x_2) \right) \\ &= \bigvee_{f(x)=z} \bigvee_{x_1+x_2=x} (A(x_1) \wedge B(x_2)) \\ &= \bigvee_{f(x)=z} (A + B)(x) \\ &= f^{\rightarrow}(A + B)(z).\end{aligned}$$

Therefore, $f^{\rightarrow}(A) + f^{\rightarrow}(B) = f^{\rightarrow}(A + B)$. □

Proposition 4.3. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space and $f : X \longrightarrow Y$ be a surjective linear mapping. Then $(Y, \mathbb{K}, \mathscr{B}/f)$ is an (L, M)-fuzzy bornological vector space.*

Proof. By Proposition 4.1 and Theorem 2.11, we only need to check $\mathscr{B}/f$ satisfies (BV3)-(BV5).

(BV3) Take any $A, B \in L^Y$. By Lemma 4.2, we have

$$\begin{aligned}(\mathscr{B}/f)(A) \wedge (\mathscr{B}/f)(B) &= \bigvee_{A \leqslant f^{\rightarrow}(C_1)} \mathscr{B}(C_1) \wedge \bigvee_{B \leqslant f^{\rightarrow}(C_2)} \mathscr{B}(C_2) \\ &= \bigvee_{A \leqslant f^{\rightarrow}(C_1)} \bigvee_{B \leqslant f^{\rightarrow}(C_2)} \Big(\mathscr{B}(C_1) \wedge \mathscr{B}(C_2) \Big) \\ &\leqslant \bigvee_{A \leqslant f^{\rightarrow}(C_1)} \bigvee_{B \leqslant f^{\rightarrow}(C_2)} \mathscr{B}(C_1 + C_2) \\ &\leqslant \bigvee_{A+B \leqslant f^{\rightarrow}(C_1)+f^{\rightarrow}(C_2)} \mathscr{B}(C_1 + C_2) \\ &= \bigvee_{A+B \leqslant f^{\rightarrow}(C_1+C_2)} \mathscr{B}(C_1 + C_2) \\ &\leqslant (\mathscr{B}/f)(A + B).\end{aligned}$$

(BV4) For all $A \in L^Y$ and $\lambda \in \mathbb{K}$, we have

$$\begin{aligned}(\mathscr{B}/f)(A) &= \bigvee_{A \leqslant f^{\rightarrow}(B)} \mathscr{B}(B) \\ &= \bigvee_{\lambda A \leqslant \lambda f^{\rightarrow}(B)} \mathscr{B}(B) \\ &\leqslant \bigvee_{\lambda A \leqslant f^{\rightarrow}(\lambda B)} \mathscr{B}(\lambda B) \\ &\leqslant (\mathscr{B}/f)(\lambda A).\end{aligned}$$

(BV5) For all $A \in L^Y$, we have

$$(\mathscr{B}/f)(A) = \bigvee_{A \leqslant f^{\rightarrow}(B)} \mathscr{B}(B) \leqslant \bigvee_{A \leqslant f^{\rightarrow}(B)} \mathscr{B}\left(\bigvee_{|\lambda| \leqslant 1} \lambda B \right).$$

For all $B \in L^X$ with $A \leqslant f^{\rightarrow}(B)$, we can obtain

$$\bigvee_{|\lambda|\leqslant 1} \lambda A \leqslant \bigvee_{|\lambda|\leqslant 1} \lambda f^{\rightarrow}(B) = \bigvee_{|\lambda|\leqslant 1} f^{\rightarrow}(\lambda B) = f^{\rightarrow}\left(\bigvee_{|\lambda|\leqslant 1} \lambda B\right).$$

This implies $\bigvee_{|\lambda|\leqslant 1} \lambda A \leqslant f^{\rightarrow}(\bigvee_{|\lambda|\leqslant 1} \lambda B)$. Then

$$\begin{aligned}(\mathscr{B}/f)(A) &\leqslant \bigvee_{\bigvee_{|\lambda|\leqslant 1} \lambda A \leqslant f^{\rightarrow}(\bigvee_{|\lambda|\leqslant 1} \lambda B)} \mathscr{B}\left(\bigvee_{|\lambda|\leqslant 1} \lambda B\right) \\ &\leqslant \bigvee_{\bigvee_{|\lambda|\leqslant 1} \lambda A \leqslant f^{\rightarrow}(C)} \mathscr{B}(C) \\ &= (\mathscr{B}/f)\left(\bigvee_{|\lambda|\leqslant 1} \lambda A\right).\end{aligned}$$

This shows that $\mathscr{B}/f$ satisfies (BV3)-(BV5). Thus $(Y, \mathbb{K}, \mathscr{B}/f)$ is an (L, M)-fuzzy bornological vector space. □

In the following, we discuss the relationships between quotient (L, M)-fuzzy bornological vector space and quotient L-bornological vector space.

Proposition 4.4 ([14]). *Let $(X, \mathcal{B})$ be an L-bornological space. Then $(X, \mathbb{K}, \mathcal{B})$ is an L-bornological vector space if and only if $\mathcal{B}$ satisfies the following conditions:*
(B3) $U, V \in \mathcal{B} \Rightarrow U + V \in \mathcal{B}$;
(B4) $\forall\, t \in \mathbb{K},\ U \in \mathcal{B} \Rightarrow tU \in \mathcal{B}$;
(B5) $U \in \mathcal{B} \Rightarrow \bigvee_{|t|\leqslant 1} tU \in \mathcal{B}$.

Proposition 4.5 ([18]). *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological space. Then*

(1) $\forall\, a \in M$, $\mathscr{B}_{[a]} = \{A \in L^X : \mathscr{B}(A) \geqslant a\}$ *is an L-bornology on X.*
(2) $\forall\, a \in P(M)$, $\mathscr{B}^{(a)} = \{A \in L^X : \mathscr{B}(A) \nleqslant a\}$ *is an L-bornology on X.*

Proposition 4.6. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space. Then*

(1) $\forall\, a \in M$, $(X, \mathbb{K}, \mathscr{B}_{[a]})$ *is an L-bornological vector space;*
(2) $\forall\, a \in P(M)$, $(X, \mathbb{K}, \mathscr{B}^{(a)})$ *is an L-bornological vector space.*

Proof. (1) By Proposition 4.5, it is enough to show that $\mathscr{B}_{[a]}$ satisfies (B3)-(B5).

(B3) Take any $A, B \in \mathscr{B}_{[a]}$. We have $\mathscr{B}(A) \geqslant a$ and $\mathscr{B}(B) \geqslant a$. Thus $a \leqslant \mathscr{B}(A) \wedge \mathscr{B}(B) \leqslant \mathscr{B}(A + B)$. This implies $A + B \in \mathscr{B}_{[a]}$.

(B4) For all $t \in \mathbb{K},\ A \in \mathscr{B}_{[a]}$, we know that $a \leqslant \mathscr{B}(A) \leqslant \mathscr{B}(tA)$. This shows that $tA \in \mathscr{B}_{[a]}$.

(B5) Take any $A \in \mathscr{B}_{[a]}$. $a \leqslant \mathscr{B}(A) \leqslant \mathscr{B}(\bigvee_{|t|\leqslant 1} tA)$. This implies $\bigvee_{|t|\leqslant 1} tA \in \mathscr{B}_{[a]}$.

Hence, $(X, \mathbb{K}, \mathscr{B}_{[a]})$ is an L-bornological vector space.

(2) For all $a \in P(M)$, it suffices to verify that $\mathscr{B}^{(a)}$ satisfies (B3)-(B5).

(B3) Take any $A, B \in \mathscr{B}^{(a)}$, we have $\mathscr{B}(A) \nleqslant a$ and $\mathscr{B}(B) \nleqslant a$. Thus $\mathscr{B}(A) \wedge \mathscr{B}(B) \nleqslant a$. Since $\mathscr{B}(A + B) \geqslant \mathscr{B}(A) \wedge \mathscr{B}(B)$, this implies $A + B \in \mathscr{B}^{(a)}$.

(B4) For all $t \in \mathbb{K},\ A \in \mathscr{B}^{(a)}$, we know that $\mathscr{B}(A) \nleqslant a$. Since $\mathscr{B}(tA) \geqslant \mathscr{B}(A)$, this shows that $tA \in \mathscr{B}^{(a)}$.

(B5) Take any $A \in \mathscr{B}^{(a)}$. Since $\mathscr{B}(\bigvee_{|t|\leqslant 1} tA) \geqslant \mathscr{B}(A)$, this implies $\mathscr{B}(\bigvee_{|t|\leqslant 1} tA) \nleqslant a$. Therefore, $\bigvee_{|t|\leqslant 1} tA \in \mathscr{B}^{(a)}$. Hence, $(X, \mathbb{K}, \mathscr{B}^{(a)})$ is an L-bornological vector space. □

By restricting $M = \{0, 1\}$, we can obtain the following corollary.

Corollary 4.7. *Let $(X, \mathbb{K}, \mathcal{B})$ be an L-bornological vector space, and $f : X \longrightarrow Y$ be a surjective mapping. Define $\mathcal{B}/f = \left\{B \in L^Y \mid \exists A \in \mathcal{B},\ B \leqslant f^{\rightarrow}(A)\right\}$. Then $\mathcal{B}/f$ is the finest L-bornology on Y such that f is L-bounded. We call $\mathcal{B}/f$ a quotient L-bornology on Y induced by f and $\mathcal{B}$.*

Theorem 4.8. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space and $f : X \longrightarrow Y$ be a surjective mapping. Then*

(1) $\forall a \in P(M)$, $\mathscr{B}^{(a)}/f = (\mathscr{B}/f)^{(a)}$;

(2) $\forall a \in M$, $\mathscr{B}_{[a]}/f = (\mathscr{B}/f)_{[a]}$.

Proof. (1) Firstly, we verify that $\mathscr{B}^{(a)}/f \subseteq (\mathscr{B}/f)^{(a)}$. Take any $B \in \mathscr{B}^{(a)}/f$. Then there exists $A \in \mathscr{B}^{(a)}$ such that $B \leqslant f^{\rightarrow}(A)$. Thus

$$(\mathscr{B}/f)(B) = \bigvee_{B \leqslant f^{\rightarrow}(A)} \mathscr{B}(A) \nleqslant a.$$

This shows that $B \in (\mathscr{B}/f)^{(a)}$.

Conversely, for all $B \in (\mathscr{B}/f)^{(a)}$, we have

$$(\mathscr{B}/f)(B) = \bigvee_{B \leqslant f^{\rightarrow}(A)} \mathscr{B}(A) \nleqslant a.$$

Then there exists $A \in L^X$ with $B \leqslant f^{\rightarrow}(A)$ such that $\mathscr{B}(A) \nleqslant a$. Since $A \in \mathscr{B}^{(a)}$ it follows that $B \in \mathscr{B}^{(a)}/f$. This implies $(\mathscr{B}/f)^{(a)} \subseteq \mathscr{B}^{(a)}/f$. Therefore, $\mathscr{B}^{(a)}/f = (\mathscr{B}/f)^{(a)}$.

(2) Similar to the proof of (1), we can obtain $\mathscr{B}_{[a]}/f = (\mathscr{B}/f)_{[a]}$. □

5. Induced (L, M)-fuzzy bornological vector space by M-fuzzifying bornological vector space

In this section, we discuss the relationships of (L, M)-fuzzy bornological spaces induced by M-fuzzifying bornological spaces with quotient spaces and product spaces. In [18], the authors introduced the induced (L, M)-fuzzy bornological vector space by M-fuzzifying bornological vector space as follows.

Definition 5.1 ([18])**.** Let $(X, \mathbb{K}, \mathcal{B})$ be an M-fuzzifying bornological vector space. Define a mapping $\omega(\mathcal{B}) : L^X \longrightarrow M$ by

$$\omega(\mathcal{B})(A) = \bigwedge_{a \in L} \mathcal{B}\left(A^{(a)}\right), \forall A \in L^X.$$

Then $(X, \mathbb{K}, \omega(\mathcal{B}))$ is an (L, M)-fuzzy bornological vector space.

Next we discuss the relationships between M-fuzzifying bornology and (L, M)-fuzzy bornology induced by M-fuzzifying bornology.

Theorem 5.2. *Let $(X,\mathcal{B})$ be an M-fuzzifying bornological space and $f : X \longrightarrow Y$ be a surjective mapping. Then $\omega(\mathcal{B}/f) \leqslant \omega(\mathcal{B})/f$. In addition, if f is a bijective mapping, then $\omega(\mathcal{B}/f) = \omega(\mathcal{B})/f$.*

Proof. Since $f : (X, \mathcal{B}) \longrightarrow (Y, \mathcal{B}/f)$ is M-fuzzifying bounded, then for all $A \in L^X$ and $a \in L$,

$$\mathcal{B}(A^{(a)}) \leqslant (\mathcal{B}/f)(f(A^{(a)})).$$

Take any $B \in L^Y$. By Lemma 2.1, we have

$$\begin{aligned}
(\omega(\mathcal{B})/f)\,(B) &= \bigvee_{B \leqslant f^{\rightarrow}(A)} \omega(\mathcal{B})(A) \\
&= \bigvee_{B \leqslant f^{\rightarrow}(A)} \bigwedge_{a \in L} \mathcal{B}(A^{(a)}) \\
&\leqslant \bigvee_{B \leqslant f^{\rightarrow}(A)} \bigwedge_{a \in L} (\mathcal{B}/f)(f(A^{(a)})) \\
&\leqslant \bigwedge_{a \in L} (\mathcal{B}/f)(B^{(a)}) \\
&= \omega(\mathcal{B}/f)(B).
\end{aligned}$$

This shows that $\omega(\mathcal{B})/f \leqslant \omega(\mathcal{B}/f)$.

Additionally, if f is a bijective mapping, then for all $B \in L^Y$,

$$\begin{aligned}
\omega(\mathcal{B}/f)(B) &= \bigwedge_{a\in L}(\mathcal{B}/f)(B^{(a)}) \\
&= \bigwedge_{a\in L}\bigvee_{B^{(a)}\leqslant f^{\rightarrow}(D)}\mathcal{B}(D) \\
&= \bigwedge_{a\in L}\bigvee_{f^{\leftarrow}(B^{(a)})\leqslant D}\mathcal{B}(D) \\
&\leqslant \bigwedge_{a\in L}\mathcal{B}(f^{\leftarrow}(B)^{(a)}) \\
&= \omega(\mathcal{B})(f^{\leftarrow}(B)) \\
&\leqslant \bigvee_{B\leqslant f^{\rightarrow}(C)}\omega(\mathcal{B})(C) \\
&= (\omega(\mathcal{B})/f)(B).
\end{aligned}$$

This implies $\omega(\mathcal{B}/f) = \omega(\mathcal{B})/f$. □

Let f be a bijective mapping, $\mathcal{B}(X)$ be the set of all M-fuzzifying bornology on X, $\Delta(X)$ be the set of all (L,M)-fuzzy bornology on X. The previous theorem shows that the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{B}(X) & \xrightarrow{1/f} & \mathcal{B}(Y) \\
\omega\downarrow & & \omega\downarrow \\
\Delta(X) & \xrightarrow{1/f} & \Delta(Y)
\end{array}$$

Corollary 5.3. *Let $(X,\mathbb{K},\mathcal{B})$ be an M-fuzzifying bornological vector space and $f : X \longrightarrow Y$ be a surjective mapping. Then $\omega(\mathcal{B}/f) \leqslant \omega(\mathcal{B})/f$. Futher if f is a bijective mapping, then $\omega(\mathcal{B}/f) = \omega(\mathcal{B})/f$.*

Next we discuss the relationship between induced (L,M)-fuzzy bornological space and product space.

Proposition 5.4. *Let $\{(X_i,\mathscr{B}_i)\}_{i\in I}$ be a family of (L,M)-fuzzy bornological spaces, $X = \prod_{i\in I} X_i$ and $(X,\mathscr{B})$ be the product space of $\{(X_i,\mathscr{B}_i)\}_{i\in I}$. Then the projection P_i is (L,M)-fuzzy bounded.*

Proof. For any $\prod_{i\in I} A_i \in L^X$, $i \in I$ and $x_i \in X_i$, we have

$$\begin{aligned}
P_i^{\rightarrow}\left(\prod_{i\in I}A_i\right)(x_i) &= \bigvee_{P_i(x)=x_i}\left(\prod_{i\in I}A_i\right)(x) \\
&= \bigvee_{P_i(x)=x_i}\bigwedge_{i\in I}A_i(P_i(x)) \\
&= \bigwedge_{i\in I}A_i(x_i) \\
&\leqslant A_i(x_i).
\end{aligned}$$

This implies $P_i^{\rightarrow}\left(\prod\limits_{i\in I} A_i\right) \leqslant A_i$. *For any* $A \in L^X$, *if* $A \leqslant \prod\limits_{i\in I} A_i$, *we have* $P_i^{\rightarrow}(A) \leqslant P_i^{\rightarrow}\left(\prod\limits_{i\in I} A_i\right) \leqslant A_i$. *Then for any* $A \in L^X$ *and* $i \in I$,

$$\begin{aligned}
\mathscr{B}(A) &= \bigvee_{A\leqslant \prod\limits_{i\in I} A_i} \bigwedge_{i\in I} \mathscr{B}_i(A_i) \\
&\leqslant \bigvee_{A\leqslant \prod\limits_{i\in I} A_i} \mathscr{B}_i(A_i) \\
&\leqslant \bigvee_{P_i^{\rightarrow}(A)\leqslant A_i} \mathscr{B}_i(A_i) \\
&\leqslant \mathscr{B}_i(P_i^{\rightarrow}(A)).
\end{aligned}$$

This shows that $P_i : X \longrightarrow X_i$ *is* (L, M)*-fuzzy bounded.* □

The above proposition can degenerate to M-fuzzifying bounded by restricting $L = \{0, 1\}$.

Theorem 5.5. *Supposed that* $\{(X_i, \mathcal{B}_i)\}_{i\in I}$ *is a family of M-fuzzifying bornological spaces,* $X = \prod_{i\in I} X_i$ *and* $(X, \prod_{i\in I} \mathcal{B}_i)$ *is the product space of* $\{(X_i, \mathcal{B}_i)\}_{i\in I}$. *Then* $\prod_{i\in I} \omega(\mathcal{B}_i) = \omega(\prod_{i\in I} \mathcal{B}_i)$.

Proof. First we verify that $\omega(\prod_{i\in I} \mathcal{B}_i) \leqslant \prod_{i\in I} \omega(\mathcal{B}_i)$. Since $P_i : (X, \prod_{i\in I} \mathcal{B}_i) \longrightarrow (X_i, \mathcal{B}_i)$ is M-fuzzifying bounded, then for all $B \in 2^X$ and $i \in I$, it follows that $(\prod_{i\in I} \mathcal{B}_i)(B) \leqslant \mathcal{B}_i(P_i(B))$. For all $A \in L^X$, let $A = \prod_{i\in I} A_i$. Then for each $i \in I$,

$$\begin{aligned}
\omega(\prod_{i\in I} \mathcal{B}_i)(A) &= \bigwedge_{a\in L} (\prod_{i\in I} \mathcal{B}_i)(A^{(a)}) \\
&\leqslant \bigwedge_{a\in L} \mathcal{B}_i(P_i(A^{(a)})) \\
&= \bigwedge_{a\in L} \mathcal{B}_i((P_i^{\rightarrow}(A))^{(a)}) \\
&= \omega(\mathcal{B}_i)(A_i).
\end{aligned}$$

This implies

$$\begin{aligned}
\omega(\prod_{i\in I} \mathcal{B}_i)(A) &\leqslant \bigwedge_{i\in I} \omega(\mathcal{B}_i)(A_i) \\
&\leqslant \bigvee_{\prod_{i\in I} A_i\leqslant \prod_{i\in I} B_i} \bigwedge_{i\in I} \omega(\mathcal{B}_i)(B_i) \\
&= \prod_{i\in I} \omega(\mathcal{B}_i)(\prod_{i\in I} A_i) \\
&= \prod_{i\in I} \omega(\mathcal{B}_i)(A).
\end{aligned}$$

Conversely, we shows that $\prod_{i\in I} \omega(\mathcal{B}_i) \leqslant \omega(\prod_{i\in I} \mathcal{B}_i)$. For all $A = \prod_{i\in I} A_i \in L^X$, we have

$$\begin{aligned}
\prod_{i\in I} \omega(\mathcal{B}_i)(A) &= \bigvee_{A\leqslant \prod_{i\in I} B_i} \bigwedge_{i\in I} \omega(\mathcal{B}_i)(B_i) \\
&= \bigvee_{A\leqslant \prod_{i\in I} B_i} \bigwedge_{i\in I} \bigwedge_{a\in L} \mathcal{B}_i(B_i^{(a)}).
\end{aligned}$$

Take any $\prod_{i\in I} B_i \in L^X$ with $A \leqslant \prod_{i\in I} B_i$, i.e., $\prod_{i\in I} A_i \leqslant \prod_{i\in I} B_i$. Since for all $a \in L$, $A^{(a)} = (\prod_{i\in I} A_i)^{(a)} \leqslant (\prod_{i\in I} B_i)^{(a)} \leqslant \prod_{i\in I} B_i^{(a)}$,

$$\begin{aligned}\bigwedge_{a\in L}\bigwedge_{i\in I}\mathcal{B}_i(B_i^{(a)}) &\leqslant \bigwedge_{a\in L}\bigvee_{A^{(a)}\leqslant\prod_{i\in I}C_i}\bigwedge_{i\in I}\mathcal{B}_i(C_i)\\ &= \bigwedge_{a\in L}(\prod_{i\in I}\mathcal{B}_i)(A^{(a)})\\ &= \omega(\prod_{i\in I}\mathcal{B}_i)(A).\end{aligned}$$

Then

$$\begin{aligned}\prod_{i\in I}\omega(\mathcal{B}_i)(A) &= \bigvee_{A\leqslant\prod_{i\in I}B_i}\bigwedge_{i\in I}\omega(\mathcal{B}_i)(B_i)\\ &= \bigvee_{A\leqslant\prod_{i\in I}B_i}\bigwedge_{i\in I}\bigwedge_{a\in L}\mathcal{B}_i(B_i^{(a)})\\ &\leqslant \omega(\prod_{i\in I}\mathcal{B}_i)(A).\end{aligned}$$

This implies $\prod_{i\in I}\omega(\mathcal{B}_i) \leqslant \omega(\prod_{i\in I}\mathcal{B}_i)$. Therefore, $\prod_{i\in I}\omega(\mathcal{B}_i) = \omega(\prod_{i\in I}\mathcal{B}_i)$. □

6. Conclusions

In this paper, we used (L, M)-fuzzy pseudo-quasi-metric, (L, M)-fuzzy topology, surjective mapping, and M-fuzzifying bornology to characterize (L, M)-fuzzy bornology, respectively. Moreover, we proposed some properties of them and discussed the relationships between induced (L, M)-fuzzy bornology and induced (L, M)-fuzzy topology, quotient (L, M)-fuzzy bornological vector space and quotient L-bornological vector space. Further, we discussed the relationships of (L, M)-fuzzy bornology induced by M-fuzzifying bornology with quotient space and product space. Following induced (L, M)-fuzzy bornology in this paper, we will consider how to generalize (L, M)-fuzzy norms and use (L, M)-fuzzy norms to characterize (L, M)-fuzzy bornology in the future.

Acknowledgments

The authors would like to express their sincere thanks to the Editors and anonymous reviewers for their most valuable comments and suggestions in improving this paper greatly.

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