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On the construction of optimal quadrature formulas with equally spaced nodes

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Abstract. In this paper in Hilbert space $S_2(P_2)$ the problem of construction of an optimal quadrature formula in the sense of Sard is considered and using Sobolev's method a new optimal quadrature formula is obtained. For the optimal coefficients explicit formulas are given. The constructed optimal quadrature formula is exact for the functions e^{-x} and xe^{-x}.

1. Introduction. Statement of the problem

We consider the following quadrature formula

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) \tag{1}$$

with the error functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_\beta \delta(x - x_\beta), \tag{2}$$

where C_β are the coefficients and x_β are the nodes of formula (1), $x_\beta \in [0,1]$, $\varepsilon_{[0,1]}(x)$ is the indicator of the interval $[0,1]$, $\delta(x)$ is Dirac's delta-function and function $\varphi(x)$ belongs to the Hilbert space $S_2(P_2)$ embedded with the norm

$$\left\|\varphi(x)|S_2(P_2)\right\| = \left\{ \int_0^1 \left(\varphi''(x) + 2\varphi'(x) + \varphi(x)\right)^2 dx \right\}^{1/2}. \tag{3}$$

The difference

$$(\ell(x), \varphi(x)) = \int_0^1 \varphi(x)dx - \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx \tag{4}$$

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is called *the error* of the quadrature formula (1). The error of the formula (1) defines the linear functional (2) in $S_2^*(P_2)$, where $S_2^*(P_2)$ is the conjugate space to $S_2(P_2)$ space.

By the Cauchy-Schwartz inequality

$$|(\ell(x), \varphi(x))| \leq \|\varphi(x)|S_2(P_2)\| \cdot \|\ell(x)|S_2^*(P_2)\|$$

the error (4) of the formula (1) is estimated with the help of the norm

$$\left\|\ell(x)|S_2^*(P_2)\right\| = \sup_{\|\varphi(x)|S_2(P_2)\|=1} \left|(\ell(x), \varphi(x))\right|$$

of the error functional (2). Consequently, estimation of the error of the quadrature formula (1) on functions of the space $S_2(P_2)$ is reduced to finding the norm of the error functional $\ell(x)$ in the conjugate space $S_2^*(P_2)$.

Obviously, that the norm of the error functional $\ell(x)$ depends on the coefficients C_β and the nodes x_β. The problem of finding the minimum of the norm of the error functional $\ell(x)$ by coefficients C_β and nodes x_β, is called by *Nikolskii problem*, and obtained formula is called the *optimal quadrature formula in the sense of Nikolskii*. This problem was first considered by S.M.Nikolskii [16]. Further this problem were investigated by many authors for various cases (see e.g. [3]-[6] and references therein). Minimization of the norm of the error functional $\ell(x)$ by coefficients C_β when the nodes are fixed is called *Sard's problem*. And the obtained formula is called the *optimal quadrature formula in the sense of Sard*. First Second this problem was studied by A.Sard [17].

There are several methods of construction of optimal quadrature formulas in the sense of Sard such as the spline method, the $\varphi-$ function method (see, e.g. [3], [29]) and Sobolev's method. Note, the Sobolev method is based on construction of discrete analogue of a linear differential operator (see, e.g. [29], [30]). In the space $L_2^{(m)}(a,b)$, based on these methods, Sard's problem were investigated by many authors (see, for example, [2],[7]-[15],[17]-[19],[32]-[27] and references therein). Here $L_2^{(m)}(a,b)$ is the Sobolev space of functions, which $m-$th generalized derivative is square integrable.

It should be noted, that in the work [7], using the $\varphi-$ function method, the problem of construction of optimal quadrature formulas in the sense of Sard which are exact for solutions of linear differential equations were investigated and several examples were given for some number of nodes $N+1$.

In [23] the optimal quadrature formula in the sense of Sard was constructed using Sobolev's method in the space $W_2^{(m,m-1)}(0,1)$, which is defined by the formula

$$\left\|\varphi|W_2^{(m,m-1)}(0,1)\right\| = \left\{\int_0^1 \left(\varphi^{(m)}(x) + \varphi^{(m-1)}(x)\right)^2 dx\right\}^{\frac{1}{2}}.$$

The main aim of the present paper is to solve Sard's problem in the space $S_2(P_2)$ using Sobolev's method for any number of the nodes $N+1$ [28]-[30], i.e. finding the coefficients C_β satisfying the following equality

$$\left\|\overset{\circ}{\ell}(x)|S_2^*(P_2)\right\| = \inf_{C_\beta}\left\|\ell(x)|S_2^*(P_2)\right\|. \tag{5}$$

Thus, in order to construct of an optimal quadrature formula in the sense of Sard in the space $S_2(P_2)$ we need consequently to solve the following problems.

Problem 1. *Find the norm of the error functional $\ell(x)$ of the quadrature formula* (1).

Problem 2. *Find the coefficients C_β which satisfy equality* (5) *when the nodes x_β are fixed.*

The structure of the present paper is: in the second section the extremal function which corresponds to the error functional $\ell(x)$ is found and with its help the representation of the norm of the error functional (2) is calculated; in the third section the quantity $\|\ell\|^2$ is minimized by coefficients C_β and the system of linear equations is obtained for the coefficients of the optimal quadrature formula in the sense of Sard in the space $S_2(P_2)$; Moreover, existence and uniqueness of the solution for this system is proved; in the forth section explicit formulas for the coefficients of the optimal quadrature formula of the form (1) are found; finally, in the fifth section the results of numerical experiments are given.

2. The extremal function and representation of the error functional $\ell(x)$

In order to solve Problem 1, i.e. for calculation the norm of the error functional (2) in the space $S_2^*(P_2)$, it is used the concept of an extremal function for the given functional. Function $\psi_\ell(x)$ is called *the extremal* for the functional $\ell(x)$ (see, [25]), if the following equality is fulfilled

$$(\ell(x), \psi_\ell(x)) = \left\|\ell(x)|S_2^*(P_2)\right\| \cdot \left\|\psi_\ell(x)|S_2(P_2)\right\|. \tag{6}$$

Since the space $S_2(P_2)$ is a Hilbert space, then the extremal function $\psi_\ell(x)$ in this space is found with the help of Riesz theorem about general form of a linear continuous functional on a Hilbert space. Then for the functional $\ell(x)$ and for any $\varphi(x) \in S_2(P_2)$ there exists such a function $\psi_\ell(x) \in S_2(P_2)$, for which the following equality holds

$$(\ell(x), \varphi(x)) = \langle \psi_\ell(x), \varphi(x) \rangle, \tag{7}$$

where

$$\langle \psi_\ell(x), \varphi(x) \rangle = \int_0^1 \left(\psi_\ell''(x) + 2\psi_\ell'(x) + \psi_\ell(x)\right)\left(\varphi''(x) + 2\varphi'(x) + \varphi(x)\right) dx \tag{8}$$

is the inner product defined in the space $S_2(P_2)$.

Further, we solve equation (7).

First, suppose that $\varphi(x) \in \overset{\circ}{C}{}^{(\infty)}(0,1)$, where $\overset{\circ}{C}{}^{(\infty)}(0,1)$ is the compact support i.e. the space of functions, which are infinite times differentiable and finite on the interval $(0,1)$. Then from (8), integrating by parts, we obtain

$$\langle \psi_\ell(x), \varphi(x) \rangle = \int_0^1 \left(\psi_\ell^{(4)}(x) - 2\psi_\ell''(x) + \psi_\ell(x)\right)\varphi(x) dx. \tag{9}$$

Keeping in mind (9) from (7) we get

$$\psi_\ell^{(4)}(x) - 2\psi_\ell''(x) + \psi_\ell(x) = \ell(x). \tag{10}$$

So, when $\varphi(x) \in \overset{\circ}{C}{}^{(\infty)}(0,1)$ the extremal function $\psi_\ell(x)$ is a solution of equation (10). But, we have to find the solution of equation (7) when $\varphi(x) \in S_2(P_2)$. Since the space $\overset{\circ}{C}{}^{(\infty)}(0,1)$ is dense in the space $S_2(P_2)$, then we can approximate arbitrarily exact functions of the space $S_2(P_2)$ by a sequence of functions of the space $\overset{\circ}{C}{}^{(\infty)}(0,1)$. Next for $\varphi(x) \in S_2(P_2)$ we consider the inner product $\langle \psi_\ell(x), \varphi(x) \rangle$ and, integrating by parts, we have

$$\langle \psi_\ell(x), \varphi(x) \rangle = \left(\psi_\ell''(x) + 2\psi_\ell'(x) + \psi_\ell(x)\right)\left(\varphi'(x) + 2\varphi\right)\Big|_0^1$$

$$-\left(\psi_\ell'''(x) + 2\psi_\ell''(x) + \psi_\ell'(x)\right)\varphi(x)\Big|_0^1 + \int_0^1 \left(\psi_\ell^{(4)}(x) - 2\psi_\ell''(x) + \psi_\ell(x)\right)\varphi(x) dx.$$

Hence, taking into account arbitrariness of $\varphi(x)$ and uniqueness of the function $\psi_\ell(x)$ (up to linear combinetion of functions e^{-x} and xe^{-x}), taking into account (10), it should be fulfilled the following equation

$$\psi_\ell^{(4)}(x) - 2\psi_\ell''(x) + \psi_\ell(x) = \ell(x), \tag{11}$$

with the boundary conditions

$$\left(\psi_\ell''(x) + 2\psi_\ell'(x) + \psi_\ell(x)\right)|_{x=0}^{x=1} = 0, \tag{12}$$

$$\left(\psi_\ell'''(x) + 2\psi_\ell''(x) + \psi_\ell'(x)\right)|_{x=0}^{x=1} = 0. \tag{13}$$

Thus, we conclude, that the extremal function $\psi_\ell(x)$ is a solution of the boundary value problem (11)-(13).

The following theorem is valid.

Theorem 2.1. *The solution of the boundary value problem* (11)-(13) *is the extremal function* $\psi_\ell(x)$ *of the error functional* (2) *and has the following form*

$$\psi_\ell(x) = G_2(x) * \ell(x) + d_1 \cdot e^{-x} + d_2 \cdot x \cdot e^{-x},$$

where d_1 *and* d_2 *are real numbers, and*

$$G_2(x) = \frac{sign(x)}{4}\left(-\sinh(x) + x \cdot \cosh(x)\right) \tag{14}$$

is a solution of the equation

$$\psi_\ell^{(4)}(x) - 2\psi_\ell''(x) + \psi_\ell(x) = \delta(x). \tag{15}$$

Proof. It is known, that general solution of nonhomogeneous differential equation consists on sum of a partial solution of nonhomogeneous differential equation and general solution of corresponding homogeneous differential equation.

The homogeneous equation for the differential equation (11), have the form

$$\psi_\ell^{(4)}(x) - 2\psi_\ell''(x) + \psi_\ell(x) = 0. \tag{16}$$

It is easy to show, that general solution of the homogeneous equation (16) will be

$$\psi_{\ell,0}(x) = \cdot e^{-x} + d_2 \cdot x \cdot e^{-x} + d_3 \cdot e^{x} + d_4 \cdot x \cdot e^{x}. \tag{17}$$

It is not difficult to verify, that a partial solution of the differential equation (11) is $\ell(x) * G_2(x)$, where $G_2(x)$ is a fundamental solution of equation (11) and is defined by equation (14) and is a solution of equation (15), where * – is the operation of convolution, i.e.

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

The rule for finding a fundamental solution of a linear differential operator

$$L \equiv \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \ldots + a_n,$$

where a_j are constants, is given in [31, p.88]. Using this rule, it is found the function $G_2(x)$, which is the fundamental solution of the operator $\frac{d^4}{dx^4} - 2\frac{d^2}{dx^2} + 1$ and it has the form (14).

Thus, we have the following general solution of equation (11)

$$\psi_\ell(x) = \ell(x) * G_2(x) + d_1 \cdot e^{-x} + d_2 \cdot x \cdot e^{-x} + d_3 \cdot e^{x} + d_4 \cdot xe^{x}. \tag{18}$$

In order that in the space $S_2(P_2)$ the function $\psi_\ell(x)$ will be unique (up to a linear combinetion of functions e^{-x} and xe^{-x}), it has to satisfy conditions (12), (13). Here derivative is in generalized sense. In computations we need first three derivatives of the function $G_2(x)$:

$$\begin{aligned} G_2'(x) &= \frac{\text{signx}}{4}\cdot x\cdot\sinh(x),\\ G_2''(x) &= \frac{\text{signx}}{4}\cdot(\sinh(x)+x\cdot\cosh(x)),\\ G_2'''(x) &= \frac{\text{signx}}{4}\cdot(2\cdot\cosh(x)+x\cdot\sinh(x)), \end{aligned} \tag{19}$$

where the following formulas from the theory of generalized functions [31] are used

$$(\text{sign}x)' = 2\delta(x),\ \ \delta(x)f(x) = f(0).$$

Further, using the following formula [18]

$$(f(x)*g(x))' = f'(x)*g(x) = f(x)*g'(x)$$

from (17), taking account of (19), we get

$$\begin{aligned} \psi_\ell'(x) &= \ell(x)*G'(x) - d_1e^{-x} + d_2e^{-x} - d_2xe^{-x} + d_3e^x + d_4e^x + d_4xe^x,\\ \psi_\ell''(x) &= \ell(x)*G''(x) + d_1e^{-x} - 2d_2e^{-x} + d_2xe^{-x} + d_3e^x + 2d_4e^x + d_4xe^x,\\ \psi_\ell'''(x) &= \ell(x)*G'''(x) - d_1e^{-x} + 3d_2e^{-x} - d_2xe^{-x} + d_3e^x + 3d_4e^x + d_4xe^x. \end{aligned} \tag{20}$$

Then, using (18) and (20), keeping in mind (14) and (19), from (12) and (13) i.e. substituting instead of x respectively $x=0$, $x=1$ and equating to 0 we obtain the following system of linear equations

$$\begin{cases} (\ell(y),e^{-y}) - \frac{1}{2}\cdot(\ell(y),ye^{-y}) + 4d_3 + 12d_4 = 0,\\ -\frac{1}{2}\cdot(\ell(y),e^{-y}) + \frac{1}{2}\cdot(\ell(y),ye^{-y}) + 4d_3 + 8d_4 = 0,\\ \frac{1}{2}\cdot(\ell(y),e^{-y}) - \frac{1}{2}\cdot(\ell(y),ye^{-y}) + 4d_3 + 8d_4 = 0,\\ \frac{1}{2}\cdot(\ell(y),ye^{-y}) + 4d_3 + 4d_4 = 0. \end{cases}$$

Hence, we have

$$d_3 = 0,\ \ d_4 = 0, \tag{21}$$

$$(\ell(y),e^{-y}) = 0,\quad (\ell(y),ye^{-y}) = 0. \tag{22}$$

Thus, substituting (21) into equality (18) we get the assertion of the theorem. □

The equalities (22) mean that our quadrature formula is exact for functions e^{-x} and xe^{-x}.

Now, using Theorem 2.1, we obtain the following expression for the norm of the error functional $\ell(x)$:

$$\left\|\ell(x)|S_2^*(P_2)\right\|^2 = (\ell(x),\psi_\ell(x)) = \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma\, G_2(x_\beta - x_\gamma)$$

$$-2\sum_{\beta=0}^{N} C_\beta\int_0^1 G_2(x-x_\beta)dx + \int_0^1\int_0^1 G_2(x-y)dxdy. \tag{23}$$

Thus, Problem 1 is solved.

Further, in Sections 3 and 4, we solve Problem 2.

3. Existence and uniqueness of the optimal quadrature formula

Let the nodes x_β of the quadrature formula (1) be fixed and the error functional (2) be satisfied conditions (22). The norm of the error functional $\ell(x)$ is a multivariable function of the coefficients C_β ($\beta = \overline{0,N}$). For finding a conditional minimum of the norm square of the error functional (2) under the conditions (22) we apply the method of undetermined factors of Lagrange.

We consider the function

$$\Psi(C_0, C_1, ..., C_N, d_1, d_2) = \|\ell\|^2 - 2d_1\left(\ell(x), e^{-x}\right) - 2d_2\left(\ell(x), xe^{-x}\right).$$

Equating to 0 partial derivatives of $\Psi(C_0, C_1, ..., C_N, d_1, d_2)$ by C_β ($\beta = 0, ..., N$) and d_1, d_2, we get the following linear system

$$\sum_{\gamma=0}^{N} C_\gamma G_2(x_\beta - x_\gamma) + d_1 e^{-x_\beta} + d_2 x_\beta e^{-x_\beta} = f(x_\beta), \quad \beta = \overline{0,N}, \tag{24}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{-x_\gamma} = 1 - e^{-1}, \tag{25}$$

$$\sum_{\gamma=0}^{N} C_\gamma x_\gamma e^{-x_\gamma} = 1 - 2e^{-1}, \tag{26}$$

where $G_2(x)$ is defined by equality (14) and

$$f(x_\beta) = \int_0^1 G_2(x - x_\beta)dx. \tag{27}$$

The system (24)-(26) has a unique solution, and this solution gives minimum to $\|\ell\|^2$ under the conditions (25), (26).

The solution of system (24)-(26) we denote by $\mathbf{C} = (C_0, C_1, ..., C_N)$ and $\mathbf{d} = (d_1, d_2)$. It represents a stationary point of the function $\Psi(\mathbf{C}, \mathbf{d})$.

Now, in (23), we change variables as $C_\beta = \overline{C}_\beta + C_{1,\beta}$. Then (23) and the system (24)-(26) have the following forms:

$$\|\ell\|^2 = \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} \overline{C}_\beta \overline{C}_\gamma G_2(x_\beta - x_\gamma) - 2\sum_{\beta=0}^{N}(\overline{C}_\beta + C_{1,\beta})\int_0^1 G_2(x - x_\beta)dx$$

$$+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\left(2\overline{C}_\beta C_{1,\gamma} + C_{1,\beta}C_{1,\gamma}\right)G_2(x_\beta - x_\gamma) + \int_0^1\int_0^1 G_2(x - y)dxdy \tag{28}$$

and

$$\sum_{\gamma=0}^{N} \overline{C}_\gamma G_2(x_\beta - x_\gamma) + d_1 e^{-x_\beta} + d_2 x_\beta e^{-x_\beta} = F(x_\beta), \quad \beta = \overline{0,N}, \tag{29}$$

$$\sum_{\gamma=0}^{N} \overline{C}_\gamma e^{-x_\gamma} = 0, \tag{30}$$

$$\sum_{\gamma=0}^{N} \overline{C}_\gamma x_\gamma e^{-x_\gamma} = 0, \tag{31}$$

where $F(x_\beta) = f(x_\beta) - \sum_{\gamma=0}^{N} C_{1,\gamma} G_2(x_\beta - x_\gamma)$ and $C_{1,\beta}$ is a partial solution of the system (25), (26).

Hence, we directly get, that minimization of (23) under the conditions (22) by C_β is equivalent to minimization of the expression (28) by $\overline{C}_\beta$ under the conditions (30), (31). Therefore, it is sufficient to prove, that the system (29)-(31) has a unique solution with respect to unknowns $\overline{\mathbf{C}} = (\overline{C}_0, \overline{C}_1, ..., \overline{C}_N)$, $\mathbf{d} = (d_1, d_2)$ and this solution gives conditional minimum to the expression $\|\ell\|^2$.

From the theory of conditional extremum it is known the sufficient condition in which the solution of the system (29) - (31) gives conditional minimum to the expression $\|\ell\|^2$ on the manifold (30), (31). It consists on positiveness of the quadratic form

$$\Phi(\overline{\mathbf{C}}) = \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} \frac{\partial^2 \Psi}{\partial \overline{C}_\beta \partial \overline{C}_\gamma} \overline{C}_\beta \overline{C}_\gamma \tag{32}$$

on the set of vectors $\overline{\mathbf{C}} = (\overline{C}_0, \overline{C}_1, ..., \overline{C}_N)$, under the condition

$$S\overline{\mathbf{C}} = 0, \tag{33}$$

where S is the matrix of equations (30), (31):

$$S = \begin{pmatrix} e^{-x_0} & e^{-x_1} & \cdots & e^{-x_N} \\ x_0 e^{-x_0} & x_1 e^{-x_1} & \cdots & x_N e^{-x_N} \end{pmatrix}.$$

We show, that in our case this condition holds.

Theorem 3.1. *For any nonzero vector $\overline{\mathbf{C}} \in R^{N+1}$, lying in the subspace $S\overline{\mathbf{C}} = 0$, the function $\Phi(\overline{\mathbf{C}})$ is strictly positive.*

Proof. Using the definition of the function $\Psi(\mathbf{C}, \mathbf{d})$ and equations (28), (30), (31) from (32) we get

$$\Phi\left(\overline{\mathbf{C}}\right) = 2 \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} G_2(x_\beta - x_\gamma) \overline{C}_\beta \overline{C}_\gamma. \tag{34}$$

Consider the linear combination of delta functions

$$\delta_{\overline{\mathbf{C}}}(x) = \sqrt{2} \sum_{\beta=0}^{N} \overline{C}_\beta \delta(x - x_\beta). \tag{35}$$

By virtue of the condition (33), this functional belongs to the space $S_2^*(P_2)$. So, it has the extremal function $u_{\overline{\mathbf{C}}}(x) \in S_2(P_2)$, which is a solution to the equation

$$\left(\frac{d^4}{dx^4} - 2\frac{d^2}{dx^2} + 1\right) u_{\overline{\mathbf{C}}}(x) = \delta_{\overline{\mathbf{C}}}(x). \tag{36}$$

As $u_{\overline{\mathbf{C}}}(x)$ we can take a linear combination of shifts of the fundamental solution $G_2(x)$:

$$u_{\overline{\mathbf{C}}}(x) = \sqrt{2} \sum_{\beta=0}^{N} \overline{C}_\beta G_2(x - x_\beta).$$

Square of its norm in the space $S_2(P_2)$ coincide with $\Phi\left(\overline{\mathbf{C}}\right)$:

$$\left\| u_{\overline{\mathbf{C}}}(x) | S_2(P_2) \right\|^2 = \left(\delta_{\overline{\mathbf{C}}}(x), u_{\overline{\mathbf{C}}}(x)\right) = 2 \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} \overline{C}_\beta \overline{C}_\gamma G_2(x_\beta - x_\gamma).$$

Hence clearly, that for nonzero $\overline{\mathbf{C}}$ the function $\Phi\left(\overline{\mathbf{C}}\right)$ is strictly positive. □

If the nodes $x_0, x_1, ..., x_N$ are selected such that the matrix S has a right inverse matrix, then the system (29) - (31) has a unique solution. Then the system (24)-(26) also has a unique solution.

Theorem 3.2. *If the matrix S has a right inverse matrix, then the main matrix Q of the system* (29) - (31) *is nonsingular.*

Proof. We denote by M the matrix of the quadratic form $\Phi\left(\overline{\mathbf{C}}\right)/2$, where $\Phi\left(\overline{\mathbf{C}}\right)$ is defined by equality (34). As it is known, if homogenous system of a linear equations has only trivial solution, then corresponding nonhomogeneous system has a unique solution. Consider the homogeneous system, corresponding to the system (29) -(31), in the following matrix form:

$$Q\begin{pmatrix} \overline{\mathbf{C}} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} M & S^* \\ S & 0 \end{pmatrix}\begin{pmatrix} \overline{\mathbf{C}} \\ \mathbf{d} \end{pmatrix} = 0. \quad (37)$$

We verify, that a unique solution of (37) is identical 0. Let $\overline{\mathbf{C}}$ and $\mathbf{d}$ be a solution of (37).

Consider the function $\delta_{\overline{\mathbf{C}}}(x)$, which is determined by equality (35). As an a extremal function for $\delta_{\overline{\mathbf{C}}}(x)$ we can take the following function:

$$u_{\overline{\mathbf{C}}}(x) = \sqrt{2}\sum_{\beta=0}^{N} \overline{C}_\beta G_2(x - x_\beta) + d_1 e^{-x} + d_2 x e^{-x}.$$

This is possible, because $u_{\overline{\mathbf{C}}}(x)$ belongs to the space $S_2(P_2)$ and is a solution to equation (36). First $N + 1$ equations of the system (37) mean that $u_{\overline{\mathbf{C}}}(x)$ takes 0 values in all the nodes x_β. Then, for the norm of the functional $\delta_{\overline{\mathbf{C}}}(x)$ in $S_2^*(P_2)$, we have

$$\left\|\delta_{\overline{\mathbf{C}}}(x)|S_2^*(P_2)\right\|^2 = \left(\delta_{\overline{\mathbf{C}}}(x), u_{\overline{\mathbf{C}}}(x)\right) = \sqrt{2}\sum_{\beta=0}^{N} \overline{C}_\beta\, u_{\overline{\mathbf{C}}}(x_\beta) = 0,$$

that possible only when $\overline{\mathbf{C}} = 0$. Taking into account this, from the first $N + 1$ equation of the system (37) we obtain

$$S^*\mathbf{d} = 0. \quad (38)$$

By assertion of the theorem, the matrix S has a right inverse, then S^* has a left inverse matrix. Then from (38) we conclude, that the solution $\mathbf{d}$ is also equal to 0. □

From (23) and T

Theorems 3.1 and 3.2 it follows that in fixed values of the nodes x_β square of the norm of the error functional $\ell(x)$, being quadratic functions of the coefficients C_β has a unique minimum in certain value of $C_\beta = \overset{\circ}{C}_\beta$.

As it is said in the first section, the quadrature formula with the coefficients $\overset{\circ}{C}_\beta$ $(\beta = \overline{0, N})$, corresponding to this minimum in the fixed nodes x_β is called the *optimal quadrature formula in the ense of Sard* and $\overset{\circ}{C}_\beta$ $(\beta = \overline{0, N})$ are called *by the optimal coefficients*.

Below for convenience the optimal coefficients $\overset{\circ}{C}_\beta$ we remain as C_β.

4. Coefficients of the optimal quadrature formula in the sense of Sard

In the present section we solve the system (24)-(26) and find the explicit formulas for the coefficients C_β. Here we use similar method, offered by S.L.Sobolev [28] for finding optimal coefficients in the space $L_2^{(m)}(0, 1)$. Here mainly is used the concept of functions of discrete argument and operations on them (see. [29], [30]). For completeness we give some definitions about functions of discrete argument.

Suppose the nodes x_β are equal spaced, i.e. $x_\beta = h\beta$, $h = \frac{1}{N}$, $N = 1, 2, ...$

Assume that $\varphi(x)$ and $\psi(x)$ are real-valued functions of real variable and are defined on the real line $\mathbb{R}$.

Definition 4.1. Function $\varphi(h\beta)$ is called a *function of discrete argument*, if it is given on some set of integer values of β.

Definition 4.2. By inner product of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is called the number

$$[\varphi,\psi]=\sum_{\beta=-\infty}^{\infty}\varphi(h\beta)\cdot\psi(h\beta),$$

if the series on the right-hand side converges absolutely.

Definition 4.3. By convolution of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is called the inner product

$$\varphi(h\beta)*\psi(h\beta)=[\varphi(h\gamma),\psi(h\beta-h\gamma)]=\sum_{\gamma=-\infty}^{\infty}\varphi(h\gamma)\cdot\psi(h\beta-h\gamma).$$

Suppose that $C_\beta=0$ when $\beta<0$ and $\beta>N$. Using above mentioned definitions, the we rewrite system (24)-(26) in the convolution form:

$$G_2(h\beta)*C_\beta+d_1e^{-h\beta}+d_2(h\beta)e^{-h\beta}=f(h\beta),\quad \beta=0,1,...,N, \tag{39}$$

$$C_\beta=0,\ \text{ where }\quad \beta<0 \text{ and } \beta>N, \tag{40}$$

$$\sum_{\beta=0}^{N}C_\beta e^{-h\beta}=1-e^{-1}, \tag{41}$$

$$\sum_{\beta=0}^{N}C_\beta(h\beta)e^{-h\beta}=1-2e^{-1}, \tag{42}$$

where

$$f(h\beta)=1+(-\frac{1}{4}-\frac{3}{8}e^{-1})e^{h\beta}+(-\frac{1}{4}-\frac{1}{8}e^{1})e^{-h\beta}+(\frac{1}{8}+\frac{1}{8}e^{-1})(h\beta)e^{h\beta}+(-\frac{1}{8}-\frac{1}{8}e^{1})(h\beta)e^{-h\beta}. \tag{43}$$

Consider the following problem.

Problem A. *Find the discrete function C_β and unknown coefficients d_1, d_2, which satisfy the system* (39)-(42) *for given $f(h\beta)$.*

Further, instead of C_β we introduce the functions

$$v(h\beta)=G_2(h\beta)*C_\beta \tag{44}$$

and

$$u(h\beta)=v\,(h\beta)+d_1e^{-h\beta}+d_2(h\beta)e^{-h\beta}. \tag{45}$$

In such statement it is necessary to express C_β by function $u(h\beta)$. For this we have to construct such operator $D_2(h\beta)$, which satisfies the equation

$$D_2(h\beta)*G_2(h\beta)=\delta(h\beta), \tag{46}$$

where $\delta(h\beta)$ is equal to 0 when $\beta\neq 0$ and is equal to 1 when $\beta=0$, i.e. $\delta(h\beta)$ is the discrete delta-function.

In connection with this a discrete analogue $D_2(h\beta)$ of the differential operator $\frac{d^4}{dx^4}-2\frac{d^2}{dx^2}+1$, which satisfies the equation (46) is constructed and its some properties are studied.

The following theorems are proved in [8].

Theorem 4.4. *A discrete analogue of the differential operator* $\frac{d^4}{dx^4} - 2\frac{d^2}{dx^2} + 1$ *satisfying equation* (46) *has the form*

$$D_2(h\beta) = K\begin{cases} A_1 \cdot \lambda_1^{|\beta|-1}, & |\beta| \geq 2, \\ 1 + A_1, & |\beta| = 1, \\ -t - 2(e^h + e^{-h}) + \dfrac{A_1}{\lambda_1}, & \beta = 0, \end{cases} \tag{47}$$

where $K = \frac{4}{h(e^h+e^{-h})+e^{-h}-e^h}$, $t = -\frac{4h-e^{2h}+e^{-2h}}{h(e^h+e^{-h})+e^{-h}-e^h}$,

$$A_1 = \frac{\lambda_1^4 - 2(e^h + e^{-h}) \cdot \lambda_1^3 + (e^{2h} + 4 + e^{-2h}) \cdot \lambda_1^2 - 2(e^h + e^{-h}) \cdot \lambda_1 + 1}{\lambda_1^2 - 1}, \tag{48}$$

$$\lambda_1 = \frac{4he^{2h} - e^{4h} + 1 + (e^{2h} - 1)\sqrt{(1 - e^{2h})^2 - 4h^2e^{2h}}}{2e^{2h}(h(e^h + e^{-h}) + e^{-h} - e^h)} \tag{49}$$

is the root of the polynomial

$$Q_2(\lambda) = \lambda^2 - \frac{4h - e^{2h} + e^{-2h}}{h(e^h + e^{-h}) + e^{-h} - e^h} \cdot \lambda + 1, \tag{50}$$

and $|\lambda_1| < 1$, h *is a small parameter.*

Theorem 4.5. *The discrete analogue* $D_2(h\beta)$ *to the differential operator* $\frac{d^4}{dx^4} - 2\frac{d^2}{dx^2} + 1$ *satisfies the following equalities*

1) $D_2(h\beta) * e^{-h\beta} = 0$,
2) $D_2(h\beta) * e^{h\beta} = 0$,
3) $D_2(h\beta) * (h\beta)e^{-h\beta} = 0$,
4) $D_2(h\beta) * (h\beta)e^{h\beta} = 0$,
5) $D_2(h\beta) * G_2(h\beta) = \delta(h\beta)$.

Here $G_2(h\beta)$ *is the function of discrete argument, corresponding to the function* $G_2(x)$ *defined by equality* (14), *and* $\delta(h\beta)$ *is the discrete delta-function.*

Then, taking into account (44)-(46) and Theorems 4.4 and 4.5, for the optimal coefficients we have

$$C_\beta = D_2(h\beta) * u(h\beta). \tag{51}$$

Thus, if we find the function $u(h\beta)$, then the optimal coefficients will be found from equality (51).

In order to calculate the convolution (51) it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From equality (39) we get, that $u(h\beta) = f(h\beta)$ when $h\beta \in [0, 1]$. Now we need to find the representation of the function $u(h\beta)$ for $\beta < 0$ and $\beta > N$.

Since $C_\beta = 0$ when $h\beta \notin [0, 1]$, then

$$C_\beta = D_2(h\beta) * u(h\beta) = 0, \qquad h\beta \notin [0, 1].$$

Now we calculate the convolution $v(h\beta) = G_2(h\beta) * C_\beta$ when $h\beta \notin [0, 1]$.

Assume $\beta < 0$, then, taking into account equalities (14), (40), (41), (42), we have

$$v(h\beta) = G_2(h\beta) * C_\beta = \sum_{\gamma=-\infty}^{\infty} C_\gamma \, G_2(h\beta - h\gamma)$$

$$= \sum_{\gamma=0}^{N} C_\gamma \frac{\text{sign}(h\beta - h\gamma)}{4} (-\sinh(h\beta - h\gamma) + (h\beta - h\gamma)\cosh(h\beta - h\gamma)) = -\frac{3e^{-1} - 2}{8} e^{h\beta}$$

$$-\frac{1-e^{-1}}{8}(h\beta)e^{h\beta}-\frac{1}{8}\left(\sum_{\gamma=0}^{N}C_\gamma e^{h\gamma}-\sum_{\gamma=0}^{N}C_\gamma(h\gamma)e^{h\gamma}\right)e^{-h\beta}-\frac{1}{8}\sum_{\gamma=0}^{N}C_\gamma e^{h\gamma}(h\beta)e^{-h\beta}.$$

We denote

$$q_1=\frac{1}{8}\left(\sum_{\gamma=0}^{N}C_\gamma e^{h\gamma}-\sum_{\gamma=0}^{N}C_\gamma(h\gamma)e^{h\gamma}\right),\quad q_2=\frac{1}{8}\sum_{\gamma=0}^{N}C_\gamma e^{h\gamma}.$$

Then for $\beta<0$ we get

$$v(h\beta)=-\frac{3e^{-1}-2}{8}e^{h\beta}-\frac{1-e^{-1}}{8}(h\beta)e^{h\beta}-q_1\cdot e^{-h\beta}-q_2\cdot(h\beta)e^{-h\beta}. \tag{52}$$

Similarly for $\beta>N$ we have

$$v(h\beta)=\frac{3e^{-1}-2}{8}e^{h\beta}+\frac{1-e^{-1}}{8}(h\beta)e^{h\beta}+q_1\cdot e^{-h\beta}+q_2\cdot(h\beta)e^{-h\beta}. \tag{53}$$

Denoting

$$d_1^-=-q_1+d_1,\quad d_2^-=-q_2+d_2, \tag{54}$$

$$d_1^+=q_1+d_1,\quad d_2^+=q_2+d_2 \tag{55}$$

and taking into account (52), (53), (45) we come to the following problem.

Problem B. *Find the solution of the equation*

$$D_2(h\beta)*u(h\beta)=0,\quad h\beta\notin[0,1] \tag{56}$$

having the form:

$$u(h\beta)=\begin{cases}-\frac{3e^{-1}-2}{8}e^{h\beta}-\frac{1-e^{-1}}{8}(h\beta)e^{h\beta}+d_1^-\cdot e^{-h\beta}+d_2^-\cdot(h\beta)e^{-h\beta}, & \beta<0,\\ f(h\beta), & 0\le\beta\le N,\\ \frac{3e^{-1}-2}{8}e^{h\beta}+\frac{1-e^{-1}}{8}(h\beta)e^{h\beta}+d_1^+\cdot e^{-h\beta}+d_2^+\cdot(h\beta)e^{-h\beta}, & \beta>N,\end{cases} \tag{57}$$

where $d_1^-,\ d_2^-,\ d_1^+,\ d_2^+$ *are unknowns.*
If we will find $d_1^-,\ d_2^-,\ d_1^+,\ d_2^+$, then from (54), (55) we have

$$\begin{aligned}d_1&=\tfrac{1}{2}\left(d_1^-+d_1^+\right),\quad d_2=\tfrac{1}{2}\left(d_2^-+d_1^+\right),\\ q_1&=\tfrac{1}{2}\left(d_1^+-d_1^-\right),\quad q_2=\tfrac{1}{2}\left(d_2^+-d_2^-\right).\end{aligned} \tag{58}$$

Unknowns $d_1^-,\ d_2^-,\ d_1^+,\ d_2^+$ will be found from equation (56), using the function $D_2(h\beta)$. Then we will obtain explicit form of the function $u(h\beta)$ and we will find the optimal coefficients C_β. Thus, problem B and respectively problem A will be solved.

But here we will not find $d_1^-,\ d_2^-,\ d_1^+,\ d_2^+$. Instead of them, using $D_2(h\beta)$ and $u(h\beta)$, taking into account (51), we find expressions for the optimal coefficients C_β when $\beta=1,...,N-1$.

We introduce the following notations:

$$m=\frac{A_1K}{\lambda_1}\sum_{\gamma=1}^{\infty}\lambda_1^{\gamma}\Big[-\frac{3e^{-1}-2}{8}\cdot e^{-h\gamma}+\frac{1-e^{-1}}{8}\cdot(h\gamma)e^{-h\gamma}+d_1^-\cdot e^{h\gamma}-d_2^-\cdot(h\gamma)e^{h\gamma}-f(-h\gamma)\Big], \tag{59}$$

$$n=\frac{A_1K}{\lambda_1}\sum_{\gamma=1}^{\infty}\lambda_1^{\gamma}\Big[\frac{3e^{-1}-2}{8}e^{h(N+\gamma)}+\frac{1-e^{-1}}{8}(N+\gamma)he^{h(N+\gamma)}+d_1^+e^{-h(N+\gamma)}+d_2^+(N+\gamma)he^{-h(N+\gamma)}-f(1+h\gamma)\Big]. \tag{60}$$

Since $|\lambda_1|<1$, then the series in (59), (60) are convergent.

The following is true

Theorem 4.6. *The coefficients of optimal quadrature formulas in the sense of Sard of the form* (1) *in the space* $S_2(P_2)$ *have the following representation*

$$C_\beta = \frac{4(e^h + e^{-h} - 2)}{2h + e^h - e^{-h}} + m\lambda_1^\beta + n\lambda_1^{N-\beta}, \ \ \beta = \overline{1, N-1}, \tag{61}$$

where m *and* n *are defined by formulas* (59) *and* (60), *respectively,* λ_1 *is determined by equality* (49).

Proof. Let $\beta = \overline{1, N-1}$. Then from (51), using (47) and (57), we have

$$C_\beta = D_2(h\beta) * u(h\beta) = \sum_{\gamma=-\infty}^{\infty} D_2(h\beta - h\gamma)u(h\gamma)$$

$$= \sum_{\gamma=-\infty}^{-1} D_2(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=0}^{N} D_2(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_2(h\beta - h\gamma)u(h\gamma)$$

$$= D_2(h\beta) * f(h\beta) + \frac{A_1 K}{\lambda_1}\sum_{\gamma=1}^{\infty}\lambda_1^{\beta+\gamma}\Big[-\frac{3e^{-1}-2}{8}\cdot e^{-h\gamma} + \frac{1-e^{-1}}{8}\cdot(h\gamma)e^{-h\gamma} + d_1^-\cdot e^{h\gamma} - d_2^-\cdot(h\gamma)e^{h\gamma} - f(-h\gamma)\Big]$$

$$+\frac{A_1 K}{\lambda_1}\sum_{\gamma=1}^{\infty}\lambda_1^{N+\gamma-\beta}\Big[\frac{3e^{-1}-2}{8}e^{h(N+\gamma)} + \frac{1-e^{-1}}{8}(N+\gamma)he^{h(N+\gamma)} + d_1^+ e^{-h(N+\gamma)} + d_2^+(N+\gamma)he^{-h(N+\gamma)} - f(1+h\gamma)\Big].$$

Hence, taking into account notations (59) and (60), we get

$$C_\beta = D_2(h\beta) * f(h\beta) + m\lambda_1^\beta + n\lambda_1^{N-\beta}. \tag{62}$$

Now, using Theorems 4.4 and 4.5 and equality (43), we calculate the convolution $D_2(h\beta) * f(h\beta)$. Then

$$D_2(h\beta) * f(h\beta) = D_2(h\beta) * 1 = \sum_{\gamma=-\infty}^{\infty} D_2(h\gamma) = D_2(0) + 2D_2(h) + 2\sum_{\gamma=2}^{\infty} D_2(h\gamma) = \frac{4(e^h + e^{-h} - 2)}{2h + e^h - e^{-h}}. \tag{63}$$

Substituting (63) into (62) we obtain (61). □

From Theorem 4.6 it is clear, that in order to obtain explicit expressions of the optimal coefficients C_β it is sufficient to find m and n. But here we will not calculate series (59) and (60). Instead of them substituting equality (61) into (39) we obtain identity with respect to $(h\beta)$. Whence, equating the coefficients of similar terms of the left and the right-hand sides of equation (39) we will find m and n. And the coefficient C_0 and C_N will be found from (41), (42). Below we will do it.

Finally, the main result of the present paper is the following.

Theorem 4.7. *The coefficients of optimal quadrature formulas in the sense of Sard of the form* (1) *in the space* $S_2(P_2)$ *have the following form*

$$\begin{aligned}
C_0 = \ & e^{-1} + \frac{4(he^h - he^{h-1} - e^h + 1)}{e^{2h} + 2he^h - 1} + \frac{e^{2h} - 2he^h - 1}{he^h(e^{2h} + 2he^h - 1)(\lambda_1^N + 1)} \\
& \times\left(he^h - he^{h-1}\lambda_1^N - e^h + \lambda_1 + he^h\lambda_1^N - he^{h-1} - \lambda_1^N e^h + \lambda_1^{N-1}\right), \\
C_\beta = \ & \frac{4(e^h-1)^2}{e^{2h} + 2he^h - 1} + \frac{(e^{2h} - 2he^h - 1)((e^h - \lambda_1)^2\lambda_1^\beta + (\lambda_1 e^h - 1)^2\lambda_1^N)}{h\lambda_1 e^h(e^{2h} + 2he^h - 1)(\lambda_1^N + 1)}, \beta = \overline{1, N-1}, \\
C_N = \ & e^1 - 2 - \frac{4(he^{h+1} - he^h - e^{2h} + e^h)}{e^{2h} + 2he^h - 1} - \frac{e^{2h} - 2he^h - 1}{h(e^{2h} + 2he^h - 1)(\lambda_1^N + 1)} \\
& \times\left(he^1 - h\lambda_1^N - e^h\lambda_1^{N-1} + \lambda_1^N + he^1\lambda_1^N - h - \lambda_1 e^h + 1\right),
\end{aligned}$$

where λ_1 *is defined by equality* (49) *and* $|\lambda_1| < 1$.

Proof. First from equations (41) and (42) we express the coefficients C_0 and C_N by C_β $(\beta = \overline{1, N-1})$. Then we obtain

$$C_0 = e^{-1} - \sum_{\gamma=1}^{N-1} C_\gamma e^{-h\gamma} + \sum_{\gamma=1}^{N-1} C_\gamma (h\gamma) e^{-h\gamma},$$
$$C_N = e^1 - 2 - e^1 \cdot \sum_{\gamma=1}^{N-1} C_\gamma (h\gamma) e^{-h\gamma}.$$

Hence, using (61), after some simplifications we have

$$C_0 = e^{-1} + 4\frac{he^h(1-e^{-1}) + 1 - e^h}{e^{2h} + 2he^h - 1} + m \cdot \left(h\frac{e^h(\lambda_1 - \lambda_1^{N+1} e^{-1})}{(e^h - \lambda_1)^2} - \frac{\lambda_1}{e^h - \lambda_1} \right)$$
$$+ n \cdot \left(h\frac{e^h(\lambda_1^{N+1} - \lambda_1 e^{-1})}{(\lambda_1 e^h - 1)^2} - \frac{\lambda_1^N}{\lambda_1 e^h - 1} \right), \tag{64}$$
$$C_N = e^1 - 2 + 4\frac{he^h(e^1 - 1) - e^h(e^h - 1)}{e^{2h} + 2he^h - 1} + m \cdot \left(h\frac{e^h(\lambda_1 e^1 - \lambda_1^{N+1})}{(e^h - \lambda_1)^2} - \frac{\lambda_1^N e^h}{e^h - \lambda_1} \right)$$
$$+ n \cdot \left(h\frac{e^h(\lambda_1^{N+1} e^1 - \lambda_1)}{(\lambda_1 e^h - 1)^2} - \frac{\lambda_1 e^h}{\lambda_1 e^h - 1} \right). \tag{65}$$

Further, we consider the convolution $G_2(h\beta) * C_\beta$ in equation (39). Using the definition of convolution of two discrete functions and taking into account (40), we get

$$G_2(h\beta) * C_\beta = \sum_{\gamma=0}^{N} C_\gamma G_2(h\beta - h\gamma) = \sum_{\gamma=0}^{N} C_\gamma \frac{\mathrm{sign}(h\beta - h\gamma)}{4} \left(-\sinh(h\beta - h\gamma) + (h\beta - h\gamma)\cosh(h\beta - h\gamma) \right)$$
$$= \tfrac{1}{2} \sum_{\gamma=0}^{\beta} C_\gamma \left[\tfrac{-e^{h\beta - h\gamma} + e^{-h\beta + h\gamma}}{2} + (h\beta - h\gamma) \tfrac{e^{h\beta - h\gamma} + e^{-h\beta + h\gamma}}{2} \right] - \tfrac{1}{4} \sum_{\gamma=0}^{N} C_\gamma \left[\tfrac{-e^{h\beta - h\gamma} + e^{-h\beta + h\gamma}}{2} + (h\beta - h\gamma) \tfrac{e^{h\beta - h\gamma} + e^{-h\beta + h\gamma}}{2} \right]. \tag{66}$$

We denote

$$B_1 = \tfrac{1}{2} \sum_{\gamma=0}^{\beta} C_\gamma \left[\tfrac{-e^{h\beta - h\gamma} + e^{-h\beta + h\gamma}}{2} + (h\beta - h\gamma) \tfrac{e^{h\beta - h\gamma} + e^{-h\beta + h\gamma}}{2} \right], \tag{67}$$

$$B_2 = \tfrac{1}{4} \sum_{\gamma=0}^{N} C_\gamma \left[\tfrac{-e^{h\beta - h\gamma} + e^{-h\beta + h\gamma}}{2} + (h\beta - h\gamma) \tfrac{e^{h\beta - h\gamma} + e^{-h\beta + h\gamma}}{2} \right]. \tag{68}$$

First we consider B_1. For B_1, using formula (61), we obtain

$$B_1 = \frac{1}{4} C_0 \left[-e^{h\beta} + e^{-h\beta} + h\beta e^{h\beta} + h\beta e^{-h\beta} \right]$$
$$+ \frac{1}{4} \sum_{\gamma=1}^{\beta} (T + m\lambda_1^{\gamma} + n\lambda_1^{N-\gamma}) \left(-e^{h\beta - h\gamma} + e^{-h\beta + h\gamma} + (h\beta - h\gamma) \cdot (e^{h\beta - h\gamma} + e^{-h\beta + h\gamma}) \right)$$
$$= \frac{1}{4} C_0 \left[-e^{h\beta} + e^{-h\beta} + h\beta e^{h\beta} + h\beta e^{-h\beta} \right]$$
$$+ \frac{1}{4} \sum_{\gamma=0}^{\beta-1} (T + m\lambda_1^{\beta-\gamma} + n\lambda_1^{N+\gamma-\beta}) \left(-e^{h\gamma} + e^{-h\gamma} + (h\gamma) \cdot (e^{h\gamma} + e^{-h\gamma}) \right), \tag{69}$$

where $T = \frac{4(e^h+e^{-h}-2)}{2h+e^h-e^{-h}}$. After some calculations and simplifications B_1 has the form:

$$B_1 = \frac{1}{4}\left[-C_0 - T\cdot\left(\frac{1}{e^h-1} + h\frac{e^h}{(e^h-1)^2}\right) - m\cdot\left(\frac{\lambda_1}{e^h-\lambda_1} + h\frac{\lambda_1 e^h}{(\lambda_1-e^h)^2}\right) - n\cdot\left(\frac{\lambda_1^N}{\lambda_1 e^h-1} + h\frac{\lambda_1^{N+1}e^h}{(\lambda_1 e^h-1)^2}\right)\right]\cdot e^{h\beta}$$
$$+\frac{1}{4}\left[C_0 - T\cdot\left(\frac{e^h}{e^h-1} + h\frac{e^h}{(e^h-1)^2}\right) - m\cdot\left(\frac{\lambda_1 e^h}{\lambda_1 e^h-1} + h\frac{\lambda_1 e^h}{(\lambda_1 e^h-1)^2}\right) - n\cdot\left(\frac{\lambda_1^N e^h}{e^h-\lambda_1} + h\frac{\lambda_1^{N+1}e^h}{(e^h-\lambda_1)^2}\right)\right]\cdot e^{-h\beta}$$
$$+\frac{1}{4}\left[C_0 + T\cdot\frac{1}{e^h-1} + m\cdot\frac{\lambda_1}{e^h-\lambda_1} + n\cdot\frac{\lambda_1^N}{\lambda_1 e^h-1}\right]\cdot(h\beta)e^{h\beta}$$
$$+\frac{1}{4}\left[C_0 - T\cdot\frac{e^h}{e^h-1} - m\cdot\frac{\lambda_1 e^h}{\lambda_1 e^h-1} - n\cdot\frac{\lambda_1^N e^h}{e^h-\lambda_1}\right]\cdot(h\beta)e^{-h\beta}$$
$$+\frac{T}{4}\cdot\left(\frac{e^h+1}{e^h-1} + h\frac{2e^h}{(e^h-1)^2}\right). \tag{70}$$

It should be noted, that in simplifications of B_1 it is used that λ_1 is the root of the polynomial $Q_2(\lambda)$ which is defined by formula (50).

Next we consider B_2. Keeping in mind (41) and (42) for B_2 we get the following expression

$$B_2 = \left(\frac{3}{8}e^{-1} - \frac{1}{4}\right)\cdot e^{h\beta} + \frac{1}{8}\left[C_0 - T\left(\frac{e^h}{e^h-1} + h\frac{e^h(1-e^1)}{(e^h-1)^2}\right) - m\left(\frac{\lambda_1 e^h}{\lambda_1 e^h-1}\right.\right.$$
$$\left.\left.+h\frac{\lambda_1 e^h(1-\lambda_1^N e)}{(\lambda_1 e^h-1)^2}\right) - n\left(\frac{\lambda_1^N e^h}{e^h-\lambda_1} + h\frac{\lambda_1 e^h(\lambda_1^N - e^1)}{(e^h-\lambda_1)^2}\right)\right]\cdot e^{-h\beta} - \frac{e^{-1}-1}{8}\cdot(h\beta)e^{h\beta}$$
$$+\frac{1}{8}\left(C_0 + e^1 C_N + T\frac{e^1-e^h}{e^h-1} + m\frac{\lambda_1^N e^1 - \lambda_1 e^h}{\lambda_1 e^h-1} + n\frac{\lambda_1 e^1 - \lambda_1^N e^h}{e^h-\lambda_1}\right)\cdot(h\beta)e^{-h\beta}. \tag{71}$$

Now, keeping in mind the notations (67), (68), substituting (66) into equation (39) we get the following identity with respect to $(h\beta)$

$$B_1 - B_2 + d_1 e^{-h\beta} + d_2(h\beta)e^{-h\beta} = f(h\beta), \tag{72}$$

where $f(h\beta)$ is defined by equality (43).

In (72) unknowns are $m,\ n,\ d_1$ and d_2. Equating the corresponding coefficients of $e^{h\beta}$ and $(h\beta)e^{h\beta}$ of both sides of the identity (72), using (70), (71), (73), for unknowns m and n we get the following system of linear equations

$$h\frac{\lambda_1 e^h}{(\lambda_1-e^h)^2}\cdot m + h\frac{\lambda_1^{N+1}e^h}{(\lambda_1 e^h-1)^2}\cdot n = \frac{e^{2h}-2he^h-1}{e^{2h}+2he^h-1},$$
$$h\frac{e^h(\lambda_1-\lambda_1^{N+1}e^{-1})}{(e^h-\lambda_1)^2}\cdot m + h\frac{e^h(\lambda_1^{N+1}-\lambda_1 e^{-1})}{(\lambda_1 e^h-1)^2}\cdot n = \frac{(1-e^{-1})(e^{2h}-2he^h-1)}{e^{2h}+2he^h-1}.$$

Solving this system we get

$$m = \frac{(e^{2h}-2he^h-1)(e^h-\lambda_1)^2}{h\lambda_1 e^h(e^{2h}+2he^h-1)(1+\lambda_1^N)} \ \ \textit{and}\ \ n = \frac{(e^{2h}-2he^h-1)(\lambda_1 e^h-1)^2}{h\lambda_1 e^h(e^{2h}+2he^h-1)(1+\lambda_1^N)}. \tag{73}$$

Unknowns d_1 and d_2 can be found from the identity (72) by equating the corresponding coefficients of $e^{-h\beta}$ and $(h\beta)e^{-h\beta}$. In this case is used equations (70), (71), (43), (73).

Thus, from (64), (65), (61), keeping in mind (73), we obtain the assertion of the theorem. □

So, proving Theorem 4.7 we have solved Problem 1, which is equivalent to Problem 2. Thus, Problem 2 is solved, i.e. the coefficients of optimal quadrature formula of the form (1) in the sense of Sard in the space $S_2(P_2)$ for equal spaced nodes are found.

5. Numerical results

As integrands we take, for example, the following functions

$$\varphi_1(x) = x^4 + e^{2x}, \ \ \varphi_2(x) = \tan(x) \ \ and \ \ \varphi_3(x) = \frac{1}{1+x^2}.$$

Exact values of these integrals we denote by

$$I_1 = \int_0^1 \left(x^4 + e^{2x}\right) dx, \ I_2 = \int_0^1 \tan(x) dx \ \ and \ \ I_3 = \int_0^1 \frac{1}{x^2+1} dx,$$

respectively.

We calculate these integrals approximately, with the help of optimal quadrature formula which is constructed above, for the cases $N = 10;\ 100;\ 1000$. Approximate values of integrals $I_1,\ I_2, I_3$ we denote by $AppI_1,\ AppI_2,\ AppI_3$, respectively.

The results of calculations are given in Tables 1, 2, 3 and 4.

Table 1: The norm of the error functional of the optimal quadrature formula for $N = 10;\ 100;\ 1000$.

N	**10**	**100**	**1000**
$\lvert \ell \rvert$	4.230640e-4	3.780180e-6	3.732155e-8

Table 1 shows the norm of the error functional of the constructed optimal quadrature formula for $N = 10$, 100, and 1000. From Table 1 it is clear that as the number of nodes N increases, the norm decreases.

Table 2: For the cases $N = 10; 100$ and 1000, the absolute values of the errors between the exact value of integral I_1 and the approximate values calculated by the quadrature formulas given by, where App_1, TTu_1 and Tr_1 are approximate values calculated for I_1 by the optimal quadrature formula in the space $S_2(P_2)$, the midpoint formula and the trapezoid formula, respectively.

N	$\lvert I_1 - AppI_1 \rvert$	$\lvert I_1 - TTu_1 \rvert$	$\lvert I_1 - Tr_1 \rvert$
10	2.208779e-3	6.981583e-3	1.397133e-2
100	2.309964e-6	6.990788e-5	1.398165e-4
1000	2.320312e-9	6.990879e-7	1.398175e-6

Table 3: For the cases $N = 10; 100$ and 1000, the absolute values of the errors between the exact value of integral I_2 and the approximate values calculated by the quadrature formulas given by, where App_2, TTu_2 and Tr_2 are approximate values calculated for I_2 by the optimal quadrature formula in the space $S_2(P_2)$, the midpoint formula and the trapezoid formula, respectively.

N	$\lvert I_2 - AppI_2 \rvert$	$\lvert I_2 - TTu_2 \rvert$	$\lvert I_2 - Tr_2 \rvert$
10	4.718291e-4	1.004092e-3	2.013778e-3
100	5.039290e-7	1.010566e-5	2.021187e-5
1000	5.067572e-10	1.010632e-7	2.021264e-7

Table 4: For the cases $N = 10; 100$ and 1000, the absolute values of the errors between the exact value of integral I_3 and the approximate values calculated by the quadrature formulas given by, where App_3, TTu_3 and Tr_3 are approximate values calculated for I_3 by the optimal quadrature formula in the space $S_2(P_2)$, the midpoint formula and the trapezoid formula, respectively.

N	$\|I_3 - AppI_3\|$	$\|I_3 - TTu_3\|$	$\|I_3 - Tr_3\|$
10	2.629061e-5	2.083328e-4	4.166661e-4
100	2.422557e-8	2.083333e-6	4.166666e-6
1000	2.407268e-11	2.083333e-8	4.166666e-8

Numerical experiments demonstrate that when calculating the integrals I_1, I_2, I_3 approximately, the optimal quadrature formulas give better approximations than the others.

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