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A note on pseudo Ricci symmetric space-times admitting a conformal vector field

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Abstract. In this paper, we characterize a pseudo Ricci symmetric space-time admitting a proper conformal vector field. At first, it is shown that under certain restriction on the associated scalar σ, such a space-time is of vanishing scalar curvature. Also, we prove that such a space-time with harmonic Weyl tensor is vacuum. Consequently, it is concluded that this space-time is of Petrov type N and represents a plane-fronted gravitational wave with parallel rays. Finally, it is shown that a pseudo Ricci symmetric space-time admitting a homothetic vector field represents a stiff matter fluid.

1. Introduction

Let M^4 be a 4−dimensional Lorentzian manifold admitting a globally time-like vector field with metric g of signature $(+, +, +, -)$. Such a Lorentzian manifold is physically known as space-time. A Lorentzian manifold is called Ricci symmetric if its Ricci tensor R_{hk} fulfills the condition [5]

$$\nabla_l R_{hk} = 0,$$

where ∇ denotes the covariant derivative with respect to the metric tensor g_{ij}. A Lorentzian manifold is said to be pseudo Ricci symmetric if its Ricci tensor satisfies the condition [4]

$$\nabla_l R_{hk} = 2\zeta_l R_{hk} + \zeta_h R_{lk} + \zeta_k R_{lh}, \tag{1}$$

with ζ_l being a non-zero covariant vector. Such a manifold is denoted by $(PRS)_4$ and the vector ζ_l is named the associated vector of $(PRS)_4$. If we put $\zeta_l = 0$ in the previous equation, then Ricci symmetric manifolds are obtained. Several authors have investigated pseudo Ricci symmetric manifolds and pseudo Ricci symmetric space-times (see e.g. [1, 2, 8, 18, 20–23]).

A vector field v is called conformal Killing if the following condition holds:

$$\mathcal{L}_v g_{hk} = 2\sigma g_{hk},$$

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for a smooth function σ on M^4. A conformal Killing vector field (CKV) is called conformal vector field or a conformal motion, and determines a conformal symmetry of the manifold (M^4, g). If σ =constant, then v is a homothetic vector field and for $\sigma = 0$, it becomes Killing. The set of all CKV forms a Lie algebra.

We have the following integrability condition for a CKV [10]:

$$\mathcal{L}_v\{^h_{ij}\} = (\nabla_i\sigma)\,\delta^h_j + \left(\nabla_j\sigma\right)\delta^h_i - g_{ij}\nabla^h\sigma, \tag{2}$$

$$\mathcal{L}_v R_{ij} = -2\nabla_i\nabla_j\sigma - \nabla_h\sigma^h g_{ij}, \tag{3}$$

$$\mathcal{L}_v r = -6\nabla_h\sigma^h, \tag{4}$$

$$\mathcal{L}_v C^h_{ijk} = 0, \tag{5}$$

where $\{^h_{ij}\}$, R_{ij}, r, and C^h_{ijk} denote respectively the Christoffel symbols of 2nd kind, Ricci tensor, scalar curvature, and Weyl tensor. A CKV V is said to be proper if v is neither homothetic nor Killing. CKV vector field in various space-times have been studied by many researchers, for example see [7, 12, 13, 15, 19].

In a perfect fluid space-time, the energy-momentum tensor T_{ij} has the subsequent expression

$$T_{ij} = pg_{ij} + (p + \mu)\,u_iu_j,$$

where p symbolizes the isotopic pressure, μ refers to the energy density, and u_i is a unit time-like vector field such that $u_iu^j = -1$. The perfect fluid space-time is named stiff matter fluid if $p = \mu$ [3].

The present paper is devoted to study proper CKV in a $(\text{PRS})_4$ space-times. Precisely, we prove the following theorems.

Theorem 1.1. *If a $(PRS)_4$ space-time admits a proper CKV with $\nabla_h\sigma^h = 0$, then the scalar curvature vanishes.*

A CKV is called special CKV if $\nabla_j\sigma_i = 0$. However $\nabla_j\sigma_i = 0 \Rightarrow \nabla_h\sigma^h = 0$. Thus, we have

Corollary 1.2. *If a $(PRS)_4$ space-time admits a special CKV, then the scalar curvature vanishes.*

In (M^4, g), the Weyl tensor C^l_{ijk} is demonstrated by [17]

$$C_{kijm} = R_{kijm} + \frac{1}{2}\left(g_{km}R_{ij} - g_{im}R_{kj} + g_{ij}R_{km} - g_{kj}R_{im}\right) - \frac{R}{6}\left(g_{km}g_{ij} - g_{im}g_{kj}\right).$$

in which R^l_{ijk} stands for the curvature tensor. Its divergence has the following structure [16]

$$\nabla_h C^h_{ijk} = \frac{1}{2}\left[\left(\nabla_k R_{ij} - \nabla_j R_{ik}\right) - \frac{1}{6}\left(g_{ij}\nabla_k R - g_{ik}\nabla_j R\right)\right]. \tag{6}$$

If $\nabla_l C^l_{ijk} = 0$, then the Weyl tensor is named harmonic. The Weyl tensor is a fundamental concept in differential geometry and general relativity. It measures the curvature of space-time in a way that is independent of the distribution of matter and energy. The Weyl tensor provides important information about the gravitational field and plays a crucial role in understanding the nature of gravitational waves and the overall geometry of the universe.

Thus, we prove that:

Theorem 1.3. *A $(PRS)_4$ space-time with harmonic Weyl tensor admitting a proper CKV is vacuum and such a space-time is of Petrov type N and represents plane-fronted gravitational waves with parallel rays, provided $\nabla_h\sigma^h = 0$.*

Remark 1.4. *The above theorem holds for special CKV.*

Considering homothetic vector field, we provide the following:

Theorem 1.5. *A $(PRS)_4$ space-time admitting a homothetic vector field represents a stiff matter fluid.*

2. Proof of Theorems

Proof. [Proof of Theorem 1] Let us consider the following commutation formula (see [24])

$$\mathcal{L}_v\left(\nabla_l R_{ij}\right) - \nabla_l\left(\mathcal{L}_v R_{ij}\right) = -R_{hj}\mathcal{L}_v\{^h_{il}\} - R_{ih}\mathcal{L}_v\{^h_{jl}\}. \tag{7}$$

We assume that the associated vector field ζ_l of a $(\mathrm{PRS})_4$ space-time is CKV. Therefore, using Eqs. (1), (2), and (3) in Eq. (7), we provide

$$\begin{aligned} & 2\zeta_l\mathcal{L}_v R_{ij} + \zeta_i\mathcal{L}_v R_{jl} + \zeta_j\mathcal{L}_v R_{il} + (n-2)\nabla_l\nabla_i\nabla_j\sigma \\ +g_{ij}\nabla_l\nabla_h\sigma^h \quad = \quad & -2\sigma_l R_{ij} - \sigma_i R_{jl} - \sigma_j R_{il} + R_{hj}g_{il}\sigma^h + R_{ih}g_{jl}\sigma^h, \end{aligned} \tag{8}$$

where $\sigma^h = \nabla^h\sigma$.

Again, utilizing Eq. (2) in Eq. (8), it can be inferred that

$$\begin{aligned} & 2\zeta_l\left[2\nabla_j\nabla_i\sigma + g_{ij}\nabla_h\sigma^h\right] + \zeta_i\left[2\nabla_j\nabla_l\sigma + g_{lj}\nabla_h\sigma^h\right] \\ & +\zeta_j\left[2\nabla_l\nabla_i\sigma + g_{il}\nabla_h\sigma^h\right] - \left[2\nabla_l\nabla_i\nabla_j\sigma + g_{ij}\nabla_l\nabla_h\sigma^h\right] \\ = \quad & 2\sigma_l R_{ij} + \sigma_i R_{jl} + \sigma_j R_{il} - R_{hj}g_{il}\sigma^h - R_{ih}g_{jl}\sigma^h. \end{aligned} \tag{9}$$

Multiplying the foregoing equation by g^{ij}, we deduce that

$$\begin{aligned} & 24\zeta_l\nabla_h\sigma^h + \left[2\nabla_j\nabla_l\sigma + g_{lj}\nabla_h\sigma^h\right]\zeta^j \\ & +\left[2\nabla_l\nabla_i\sigma + g_{li}\nabla_h\sigma^h\right]\zeta^i - 6\nabla_l\left(\nabla_h\sigma^h\right) \\ = \quad & 2\sigma_l R. \end{aligned} \tag{10}$$

By hypothesis $\nabla_h\sigma^h = 0$, the previous equation reduces to

$$2\left[\nabla_j\nabla_l\sigma + \nabla_l\nabla_j\sigma\right]\zeta^j = 2\sigma_l R. \tag{11}$$

Since $\nabla_j\nabla_l\sigma = \nabla_l\nabla_j\sigma$, Eq. (11) becomes

$$2\zeta^j\nabla_j\nabla_l\sigma = \sigma_l R. \tag{12}$$

In [20], it is proved that in a $(\mathrm{PRS})_4$ the following relation hold

$$\zeta^i R_{ij} = 0. \tag{13}$$

Consequently,

$$R_{ij}\left(\mathcal{L}_\zeta\zeta^i\right) + \zeta^i\left(\mathcal{L}_\zeta R_{ij}\right) = 0. \tag{14}$$

Inserting Eq. (3) in the aforementioned equation, we find that

$$2\zeta^i\nabla_i\nabla_j\sigma + \zeta^i g_{ij}\nabla_h\sigma^h = 0. \tag{15}$$

With the help of the hypothesis $\nabla_h\sigma^h = 0$, the foregoing equation becomes

$$\zeta^i\nabla_i\nabla_j\sigma = 0. \tag{16}$$

Therefore, Eq. (12) implies that

$$\sigma_l R = 0.$$

Assume that a $(\mathrm{PRS})_4$ space-time admits proper CKV, it arises

$$R = 0.$$

This completes the proof. □

Proof. [Proof of Theorem 2] From Theorem 1, we get

$$R = 0. \tag{17}$$

Consider that the divergence of the Weyl tensor vanishes, then Eq. (6) becomes

$$\left(\nabla_k R_{ij} - \nabla_j R_{ik}\right) - \frac{1}{6}\left(g_{ij}\nabla_k R - g_{ik}\nabla_j R\right) = 0.$$

Employing Eq. (17) in the previous equation, we get

$$\nabla_k R_{ij} = \nabla_j R_{ik}.$$

The use of Eq. (1) in the previous equation implies that

$$\zeta_k R_{ij} = \zeta_j R_{ik}.$$

Contracting with ζ^k and using $\zeta^k R_{ik} = 0$, we find that

$$R_{ij} = 0.$$

This means that the space-time under consideration is vacuum.

As stated in [6], a non-flat 4−dimensional vacuum space-time with non-homothetic conformal vector field is of Petrov type N and represents a plane-fronted gravitational waves with parallel rays [6].

Thus, our theorem is proved. □

Proof. [Proof of Theorem 3] For a homothetic vector field v, we have

$$\mathcal{L}_v\{^h_{ij}\} = 0, \qquad \mathcal{L}_v R_{ij} = 0.$$

From Eq. (7), we reveal that

$$\mathcal{L}_v\left(2\zeta_l R_{ij} + \zeta_i R_{jl} + \zeta_j R_{il}\right) = 0.$$

Since $\mathcal{L}_v R_{ij} = 0$, the previous equation becomes

$$2R_{ij}\left(\mathcal{L}_v \zeta_l\right) + R_{jl}\left(\mathcal{L}_v \zeta_i\right) + R_{il}\left(\mathcal{L}_v \zeta_j\right) = 0.$$

Let us assume the $\mathcal{L}_v \zeta_l = \lambda_l$ and λ_l is a unit time-like vector, that is, $\lambda_l \lambda^l = -1$. As a result, the previous equation has the following form

$$2\lambda_l R_{ij} + \lambda_i R_{jl} + \lambda_j R_{il} = 0.$$

Interchanging the indices l and i in the foregoing equation, we have

$$2\lambda_i R_{lj} + \lambda_l R_{ji} + \lambda_j R_{il} = 0.$$

Subtracting the last two equations, the result is

$$\lambda_l R_{ij} = \lambda_i R_{lj}. \tag{18}$$

Contracting the aforementioned equation with λ^l, we obtain

$$R_{ij} = -\lambda^l \lambda_i R_{lj}. \tag{19}$$

Again, transvecting Eq. (18) with g^{ij}, we find that

$$\lambda_l R = \lambda^j R_{lj}. \tag{20}$$

Utilizing Eq. (20) in Eq. (19), we realize that

$$R_{ij} = -\lambda_i \lambda_i R,$$

which implies that the considered space-time is Ricci simple [9].

In [14], the physical interpretation of the Ricci simple space-times is investigated. The authors proved that a Ricci simple space-time represents a stiff matter fluid.

Now, the proof is completed. □

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