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Geometric invariants under a SMRC-transformation group on manifolds and an application to asset pricing

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Abstract. We investigate and confirm the geometric invariants under a SMRC-transformation group on manifolds and propose an interesting asset pricing model via the geometric invariant and martingale idea in a financial market. In this case we achieve, for the first time, an interesting example for the category of invariant geometries with respect to semi-symmetric connections. By virtue of the projective conformal semi-symmetric metric recurrent connection and the corresponding curvature tensors, the celebrated Schur's theorem, which is used to characterize the geometric properties of spaces, is also obtained.

1. Introduction

It is well known that A. Fridman and A. Schouten in [10] introduced and studied the metric connection with torsion. A. Hayden in [15] posed for the first time the concept of a semi-symmetric connection in a Riemannian manifold and investigated the basic geometric properties. Furthermore, K. Yano in [25], T. Tmai in [24] and K. Yano and J. Imai in [26] respectively introduced and studied deeply the geometries of a manifold associated with a semi-symmetric metric connection. In particular, U. C. De and B. C. Biswas in [6] obtained the geometrical and physical properties of its curvature tensor of a manifold with this connection. Afterwards N. S. Agache and M. R. Chafle in [1] and S. K. Chaubey and R. H. Ojha in [2] et al introduced and confirmed some kinds of a semi-symmetric non-metric connections and the geometrical characteristics. An interesting fact that K. A. Dunn in [8] introduced a basic physical model via the semi-symmetric non-metric connection, in particular, L. Csillag in [4] investigated systematically the basci charactistics of the Weyl-Schröinger, Yano-Schröinger and Friedmann-Weyl-Schröinger via Schröinger connections and semi-symmetry of connections, and then Fu, Yang and Zhao in [11, 12] and I. Suhendro in [20] and Zhao, De, Unal and De in [31] et al investigated the physics of this physical model based on the semi-symmetric non-metric connection. On the other hand, Zhao, De, Mandal and Han in [30] and De, Zhao, Mandal and

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Han in [7] also introduced a semi(quarter)-symmetric connection being projective equivalent to the Levi-Civita connection and studied some of its geometric and physical properties. At the same time, a projective conformal semi-symmetric connection was introduced and studied systematically in [5, 13, 14, 16, 27].

S. B. Edgar in [9] considered the curvature copy problem of a semi-symmetric connection and E. S. Stepanova in [19] introduced and studied the conjugate symmetry condition of Amari-Chentsor connection in the statistical manifold. S. S. Chern, W. H. Chen and K. S. Lam in [3] summarized systematically the Schur's theorem w.r.t. the Levi-Civita connection and Han, Ho and Zhao in [13], Ho, Jen and Piao in [17] posed and proved the Schur's theorem of a semi-symmetric non-metric connection. Tang, Ho, Fu and Zhao in [21, 22] introduced and studied respectively some invariant properties and geometric properties of semi(quarter)-symmetric metric recurrent connections, and Zhao, Ho and An in [27], Zhao, Ho, Wwak and Jon in [28], Zhao, Jen and Ho in [29] also investigated respectively some invariant properties of a manifold associated with semi-symmetric metric recurrent connections.

Recently, V. G. Ivancevic, T. T. Ivancevic in [18] studied deeply the connection homotopy based on geodesic of the Levi-Civita connection. In fact there were few results, about the projective invariant, the conformal invariant and the Schur's theorem of asymmetric non-metric connection and projective conformal asymmetric non-metric connection because of its formal complex and computational difficulty.

In this paper we newly defined a semi-symmetric metric recurrent connection homotopy which it is the connection homotopy from a semi-symmetric metric connection to a semi-symmetric metric recurrent connection family and its projective invariant and the conformal invariant. And it is extended as a projective conformal semi-symmetric metric recurrent connection homotopy and studied its properties of curvature tensor, conjugate symmetry condition and studied the Schur's theorem of this connection homotopy. Furthermore, we also give an application to this class of homotopy as a product posed by Tang and Zhao in [23] with respect to no-arbitrage principle in a financial market, and confirm some interesting models associated with this connection.

The present paper is organized as follows. Section 2 introduces one type of a semi-symmetric metric recurrent connection homotopy and proposes the projective invariants and conformal invariants. In section 3 the semi-symmetric metric recurrent connection homotopy is extended as the projective conformal semi-symmetric metric recurrent connection homotopy. Finally, the Schur's theorem of the projective conformal semi-symmetric metric recurrent connection homotopy is investigated. As an application, this article shows us a no-arbitrage principle by virtue of the invariant related with some option model.

2. Semi-symmetric metric recurrent connection homotopy

Let (M, g) be a Riemannian manifold ($\dim M \geq 3$), g be the Riemannian metric on M and $\overset{\circ}{\nabla}$ be the Levi-Civita connection with respect to g. Let $T(M)$ denote the collection of all vector fields on M.

Definition 2.1. *In a Riemannian manifold (M, g), a connection $\overset{t}{\nabla}$ is called a semi-symmetric metric recurrent connection homotopy, if it satisfies the relation*

$$\begin{aligned}&\overset{t}{\nabla}_z g(X,Y) = -2(\alpha-2)t\omega(Z)g(X,Y) - (\alpha-1)t\omega(X)g(Y,Z) - (\alpha-1)t\omega(Y)g(X,Z),\\&T(X,Y) = \pi(Y)X - \pi(X)Y\end{aligned} \tag{2.1}$$

for any $X, Y, Z \in T(M)$ and 1-form ω, π.

A semi-symmetric metric recurrent connection $\overset{t}{\nabla}$ is a non-metric connection and it is expressed as

$$\overset{t}{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + 2(\alpha-2)t\omega(X)Y + [(\alpha-2)t\omega(Y) + \pi(Y)]X + [t\omega(Z) - \pi(Z)]g(X,Y), \tag{2.2}$$

for $X, Y \in T(M)$, where $\overset{\circ}{\nabla}$ is the Levi-Civita connection.

The local expressions of the relations (2.1) and (2.2) are

$$\begin{cases} \overset{t}{\nabla}_k g_{ij} = -2(\alpha-2)t\omega_k g_{ij} - (\alpha-1)t\omega_i g_{jk} - (\alpha-1)t\omega_j g_{ik}, \\ T_{ij}^{\ k} = \pi_j\delta_i^k - \pi_i\delta_j^k \end{cases} \tag{2.3}$$

and

$$\overset{t}{\Gamma}{}_{ij}^{k} = \{{}_{ij}^{k}\} + 2(\alpha-2)t\omega_i\delta_j^k + [(\alpha-2)t\omega_j + \pi_j]\delta_i^k + (t\omega^k - \pi^k)g_{ij} \tag{2.4}$$

respectively, where $\overset{t}{\Gamma}{}_{ij}^{k}$ is the connection coefficient of $\overset{t}{\nabla}$ and $\{{}_{ij}^{k}\}$ is the connection coefficient of $\overset{\circ}{\nabla}$ and ω_i is a component of 1-form ω(this is called a metric recurrent component of $\overset{t}{\nabla}$) and π_i is a component of 1-form π(this is called a semi-symmetric component of $\overset{t}{\nabla}$) and $\omega^k = g^{kl}\pi_l$, $\pi^k = g^{kl}\pi_l$ and $\alpha = 1, 2, 3$, $t \in [0, 1]$.

If $t = 0$, then $\overset{t}{\nabla}$ is a semi-symmetric metric connection and if $t = 1$, then $\overset{t}{\nabla}$ is a semi- symmetric metric recurrent connection family. So $\overset{t}{\nabla}$ is a connection homotopy from a semi-symmetric metric connection to a semi-symmetric metric recurrent connection family.

Remark 2.1. *If $t = 1$ and $\alpha = 1$, then $\overset{t}{\nabla}$ is the first semi-symmetric metric recurrent connection. If $t = 1$ and $\alpha = 2$, then $\overset{t}{\nabla}$ is the second semi-symmetric metric recurrent connection. And if $t = 1$ and $\alpha = 3$, then $\overset{t}{\nabla}$ is third semi-symmetric metric connection.([13, 27])*

Using the expression (2.4), the curvature tensor of $\overset{t}{\nabla}$ is

$$\overset{t}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + \delta_j^l a_{ik} - \delta_i^l a_{jk} + g_{jk}\overset{t}{b}_i{}^{l} - g_{ik}\overset{t}{b}_j{}^{l} + \delta_k^l(\alpha-2)t\omega_{ij} \tag{2.5}$$

where $K_{ijk}{}^{l}$ is the curvature tensor of the Levi-Civita connection $\overset{\circ}{\nabla}$ and

$$\begin{cases} \overset{t}{a}_{ik} &= \overset{\circ}{\nabla}_i[(\alpha-2)t\omega_k + \pi_k] - [(\alpha-2)t\omega_i + \pi_i][(\alpha-2)t\omega_k + \pi_k] - g_{ik}[(\alpha-2)t\omega_p + \pi_p](t\omega^p - \pi^p) \\ \overset{t}{b}_{ik} &= \overset{\circ}{\nabla}_i(t\omega_k - \pi_k) + (t\omega_i - \pi_i)(t\omega_k - \pi_k) \\ \omega_{ik} &= \overset{\circ}{\nabla}_i\omega_k - \overset{\circ}{\nabla}_i\omega_k \end{cases}$$

And from the expression (2.4), the connection coefficient of the mutual connection $\overset{tm}{\nabla}$ of $\overset{t}{\nabla}$

$$\overset{tm}{\Gamma}{}_{ij}^{k} = \{{}_{ij}^{k}\} + [(\alpha-2)t\omega_i + \pi_i]\delta_j^k + (\alpha-2)t\omega_j\delta_i^k + (t\omega^k - \pi^k)g_{ij} \tag{2.6}$$

and from this expression, the curvature tensor of $\overset{tm}{\nabla}$ is

$$\overset{tm}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + \delta_j^l\overset{tm}{a}_{ik} - \delta_i^l\overset{tm}{a}_{jk} + g_{jk}\overset{t}{b}_i{}^{l} - g_{ik}\overset{t}{b}_j{}^{l} + \delta_k^l[(\alpha-2)t\omega_{ij} + \pi_{ij}] \tag{2.7}$$

where

$$\begin{cases} \overset{tm}{a}_{ik} &= \overset{\circ}{\nabla}_i[(\alpha-2)t\omega_k] - (\alpha-2)^2t^2\omega_i\omega_k - g_{ik}(\alpha-2)t\omega_p(t\omega^p - \pi^p) \\ \pi_{ij} &= \overset{\circ}{\nabla}_i\pi_j - \overset{\circ}{\nabla}_j\pi_i. \end{cases}$$

Definition 2.2. *In a Riemannian manifold (M, g), a connection $\overset{p}{\nabla}$ is called a projective semi-symmetric metric recurrent connection homotopy if $\overset{p}{\nabla}$ is projective equivalent to $\overset{t}{\nabla}$.*

From the expression (2.4) the connection coefficient of $\overset{p}{\nabla}$ is

$$\overset{p}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + [(\alpha-2)t\omega_i + \psi_i]\delta^k_j + [(\alpha-2)t\omega_j + \pi_j + \psi_j]\delta^k_i + (t\omega^k - \pi^k)g_{ij} \tag{2.8}$$

where ψ_i is a projective component of $\overset{p}{\nabla}$.

Using this expression, the curvature tensor of $\overset{p}{\nabla}$ is

$$\overset{p}{R}_{ijk}{}^l = K_{ijk}{}^l + \delta^l_j \overset{p}{a}_{ik} - \delta^l_i \overset{p}{a}_{jk} + g_{jk}\overset{t}{b}_i{}^l - g_{ik}\overset{t}{b}_j{}^l + \delta^l_k[(\alpha-2)t\omega_{ij} + \pi_{ij}] \tag{2.9}$$

where

$$\begin{cases} \overset{p}{a}_{ik} &= \overset{\circ}{\nabla}_i[(\alpha-2)t\omega_k + \pi_k + \psi_k] - [(\alpha-2)t\omega_i + \pi_i + \psi_i][(\alpha-2)t\omega_k + \pi_k + \psi_k] \\ & \quad -g_{ik}[(\alpha-2)t\omega_p + \pi_p + \psi_p](t\omega^p - \pi^p) \\ \psi_{ij} &= \overset{\circ}{\nabla}_i\psi_j - \overset{\circ}{\nabla}_j\psi_i. \end{cases}$$

From the expressions (2.5) and (2.9), we obtain

$$\overset{p}{R}{}^l_{ijk} = \overset{t}{R}{}^l_{ijk} + \delta^l_j \overset{p}{\alpha}_{ik} - \delta^l_i \overset{p}{\alpha}_{jk} + \delta^l_k \psi_{ij} \tag{2.10}$$

where $\overset{p}{\alpha}_{ik} = \overset{p}{a}_{ik} - \overset{t}{a}_{ik}$.

Using the expression (2.8), the connection coefficient of the mutual connection $\overset{pm}{\nabla}$ of $\overset{p}{\nabla}$ is

$$\overset{pm}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + [(\alpha-2)t\omega_i + \pi_i + \psi_i]\delta^k_j + [(\alpha-2)t\omega_j + \pi_j]\delta^k_i + (t\omega^k - \pi^k)g_{ij} \tag{2.11}$$

Using this expression the curvature tensor of $\overset{pm}{\nabla}$ is

$$\overset{pm}{R}_{ijk}{}^l = K_{ijk}{}^l + \delta^l_j \overset{pm}{a}_{ik} - \delta^l_i \overset{pm}{a}_{jk} + g_{jk}\overset{t}{b}_i{}^l - g_{ik}\overset{t}{b}_j{}^l + \delta^l_k[(\alpha-2)t\omega_{ij} + \pi_{ij} + \psi_{ij}] \tag{2.12}$$

where

$$\overset{pm}{a}_{ik} = \overset{\circ}{\nabla}_i[(\alpha-2)t\omega_k + \pi_k] - [(\alpha-2)t\omega_i + \psi_i][(\alpha-2)t\omega_k + \psi_k] - g_{ik}[(\alpha-2)t\omega_p + \psi_p](t\omega^p - \pi^p)$$

From the expressions (2.7) and (2.12), we obtain

$$\overset{pm}{R}{}^l_{ijk} = \overset{tm}{R}{}^l_{ijk} + \delta^l_j \overset{pm}{\alpha}_{ik} - \delta^l_i \overset{pm}{\alpha}_{jk} + \delta^l_k \psi_{ij} \tag{2.13}$$

where $\overset{pm}{\alpha}_{ik} = \overset{pm}{a}_{ik} - \overset{tm}{a}_{ik}$.

Theorem 2.1. *In a Riemannian manifold (M, g), if 1-form ψ is a closed form, then the Weyl projective curvature tensor for* $\overset{t}{\nabla}$

$$\overset{t}{W}{}^l_{ijk} = \overset{t}{R}{}^l_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik}) \tag{2.14}$$

is an invariant under the connection transformation $\overset{t}{\nabla} \to \overset{p}{\nabla}$.

Proof. If 1-form ψ is a closed form, then $\psi_{ij} = 0$. In this case the expression (2.10) becomes

$$\overset{p}{R}{}^l_{ijk} = \overset{t}{R}{}^l_{ijk} + \delta^l_j \overset{p}{\alpha}_{ik} - \delta^l_i \overset{p}{\alpha}_{jk} \tag{2.15}$$

Contracting the indices i and l of this expression, then we get

$$\overset{p}{R}_{jk} = \overset{t}{R}_{jk} - (n-1)\overset{p}{\alpha}_{jk}.$$

From this expression, we find

$$\overset{p}{\alpha}_{jk} = \frac{1}{n-1}(\overset{t}{R}_{jk} - \overset{p}{R}_{jk}).$$

Substituting this expression into (2.15) and by a direct computation, we have

$$\overset{p}{W}{}^{l}_{ijk} = \overset{t}{W}{}^{l}_{ijk}.$$

where the tensor $\overset{p}{W}{}^{l}_{ijk} \hat{=} \overset{p}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{p}{R}_{jk} - \delta^l_j \overset{p}{R}_{ik})$ is a Weyl projective curvature tensor for $\overset{p}{\nabla}$. □

Corollary 2.1. *In a Riemannian manifold $(M, g)(dimM \geq 3)$, if 1-form ψ is closed form, then the Weyl projective curvature tensor of* $\overset{tm}{\nabla}$

$$\overset{tm}{W}{}^{l}_{ijk} = \overset{tm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{tm}{R}_{jk} - \delta^l_j \overset{tm}{R}_{ik}) \tag{2.16}$$

is an invariant under the connection transformation $\overset{tm}{\nabla} \to \overset{pm}{\nabla}$.

Theorem 2.2. *In a Riemannian manifold $(M, g)(dimM \geq 3)$, the tensor*

$$\overline{\overset{t}{W}}{}^{l}_{ijk} = \overset{t}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik}) - \frac{1}{n^2-1}[\delta^l_i(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) - \delta^l_j(\overset{t}{R}_{ik} - \overset{t}{R}_{ki}) - (n-1)\delta^l_k(\overset{t}{R}_{ij} - \overset{t}{R}_{ji})] \tag{2.17}$$

for the connection $\overset{t}{\nabla}$ *is an invariant under the connection transformation* $\overset{t}{\nabla} \to \overset{p}{\nabla}$.

Proof. Contracting the indices i and l of the expression (2.10) then we find

$$\overset{p}{R}_{jk} = \overset{t}{R}_{jk} - (n-1)\overset{p}{\alpha}_{jk} - \psi_{jk}. \tag{2.18}$$

Alternating the indices j and k of this expression, using $\overset{p}{\alpha}_{jk} - \overset{p}{\alpha}_{kj} = \psi_{jk}$, we find

$$\psi_{jk} = \frac{1}{n+1}[(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) - (\overset{p}{R}_{jk} - \overset{p}{R}_{kj})].$$

Substituting this expression into (2.18), we have

$$\overset{p}{\alpha}_{jk} = \frac{1}{n-1}\Big\{\overset{t}{R}_{jk} - \overset{p}{R}_{jk} - \frac{1}{n+1}[(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) - (\overset{p}{R}_{jk} - \overset{p}{R}_{kj})]\Big\}$$

Substituting the above results into (2.10) and by a direct computation, we have

$$\overline{\overset{p}{W}}{}^{l}_{ijk} = \overline{\overset{t}{W}}{}^{l}_{ijk}.$$

where $\overline{\overset{p}{W}}{}^{l}_{ijk} = \overset{p}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{p}{R}_{jk} - \delta^l_j \overset{p}{R}_{ik}) - \frac{1}{n^2-1}[\delta^l_i(\overset{p}{R}_{jk} - \overset{p}{R}_{kj}) - \delta^l_j(\overset{p}{R}_{ik} - \overset{p}{R}_{ki}) - (n-1)\delta^l_k(\overset{p}{R}_{ij} - \overset{p}{R}_{ji})]$. □

Corollary 2.2. *In a Riemannian manifold (M, g), the tensor*

$$\overline{\overset{pm}{W}}{}^{l}_{ijk} = \overset{tm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{tm}{R}_{jk} - \delta^l_j \overset{tm}{R}_{ik}) - \frac{1}{n^2-1}[\delta^l_i(\overset{tm}{R}_{jk} - \overset{tm}{R}_{kj}) - \delta^l_j(\overset{tm}{R}_{ik} - \overset{tm}{R}_{ki}) - (n-1)\delta^l_k(\overset{tm}{R}_{ij} - \overset{tm}{R}_{ji})]$$

for the connection $\overset{tm}{\nabla}$ *is an invariant under the connection transformation* $\overset{tm}{\nabla} \to \overset{pm}{\nabla}$.

Definition 2.3. *In a Riemannian manifold $(M, g)(dimM \geq 3)$, a connection $\overset{c}{\nabla}$ is called a conformal semi-symmetric metric recurrent connection homotopy, if $\overset{c}{\nabla}$ is conformal equivalent to $\overset{t}{\nabla}$.*

From the expression (2.4), the connection coefficient of $\overset{c}{\nabla}$ is

$$\overset{c}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + [(\alpha-2)t\omega_i + \sigma_i]\delta^k_j + [(\alpha-2)t\omega_j + \sigma_j + \pi_j]\delta^k_i + (t\omega^k - \pi^k - \sigma^k)g_{ij} \tag{2.19}$$

where σ_i is a conformal component of $\overset{t}{\nabla}$.

Using this expression, the curvature tensor of $\overset{t}{\nabla}$ is

$$\overset{c}{R}_{ijk}{}^l = K_{ijk}{}^l + \delta^l_j \overset{c}{a}_{ik} - \delta^l_i \overset{c}{a}_{jk} + g_{jk}\overset{c}{b}_i{}^l - g_{ik}\overset{c}{b}_j{}^l + \delta^l_k(\alpha-2)t\omega_{ij} \tag{2.20}$$

where

$$\begin{aligned}
\overset{c}{a}_{ik} &= \overset{\circ}{\nabla}_i[(\alpha-2)t\omega_k + \sigma_k + \pi_k] - [(\alpha-2)t\omega_i + \sigma_i + \pi_i][(\alpha-2)t\omega_k + \sigma_k + \pi_k] \\
&\quad - g_{ik}[(\alpha-2)t\omega_p + \sigma_p + \pi_p](t\omega^p - \sigma^p - \pi^p), \\
\overset{c}{b}_{ik} &= \overset{\circ}{\nabla}_i(t\omega_k - \sigma_k - \pi_k) + (t\omega_i - \sigma_i - \pi_i)(t\omega_k - \sigma_k - \pi_k).
\end{aligned}$$

On one hand, from the expression (2.19), the connection coefficient of dual connection $\overset{c*}{\nabla}$ of $\overset{c}{\nabla}$ is

$$\overset{c*}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - [(\alpha-2)t\omega_i + \sigma_i]\delta^k_j - [t\omega_j - \sigma_j - \pi_j]\delta^k_i - [(\alpha-2)t\omega^k + \pi^k + \sigma^k)g_{ij}$$

and the curvature tensor of $\overset{c*}{\nabla}$ is

$$\overset{c*}{R}_{ijk}{}^l = K_{ijk}{}^l - \delta^l_j \overset{c}{b}_{ik} + \delta^l_i \overset{c}{b}_{jk} + g_{jk}\overset{c}{a}_i{}^l - g_{ik}\overset{c}{a}_j{}^l - \delta^l_k(\alpha-2)t\omega_{ij} \tag{2.21}$$

From the expressions (2.20) and (2.21), we obtain

$$\overset{c}{R}_{ijk}{}^l + \overset{c*}{R}_{ijk}{}^l = 2K_{ijk}{}^l + \delta^l_j \overset{c}{\alpha}_{ik} - \delta^l_i \overset{c}{\alpha}_{jk} + g_{jk}\overset{c}{\alpha}_j{}^l - g_{ik}\overset{c}{\alpha}_i{}^l \tag{2.22}$$

where $\overset{c}{\alpha}_{ij} = \overset{c}{a}_{ij} - \overset{c}{b}_{ij}$.

On the other hand, from the expression (2.4), the connection coefficient of dual connection $\overset{t*}{\nabla}$ of $\overset{t}{\nabla}$ is

$$\overset{t*}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - (\alpha-2)t\omega_i\delta^k_j - (t\omega_j - \pi_j)\delta^k_i - [(\alpha-2)t\omega^k + \pi^k]g_{ij}$$

and the curvature tensor of $\overset{t*}{\nabla}$ is

$$\overset{t*}{R}_{ijk}{}^l = K_{ijk}{}^l - \delta^l_j \overset{c}{b}_{ik} + \delta^l_i \overset{c}{b}_{jk} - g_{jk}\overset{c}{a}_i{}^l + g_{ik}\overset{c}{a}_j{}^l - \delta^l_k(\alpha-2)t\omega_{ij} \tag{2.23}$$

From expressions (2.5) and (2.23), we obtain

$$\overset{t}{R}_{ijk}{}^l + \overset{t*}{R}_{ijk}{}^l = 2K_{ijk}{}^l + \delta^l_j \overset{t}{\alpha}_{ik} - \delta^l_i \overset{t}{\alpha}_{jk} + g_{jk}\overset{t}{\alpha}_j{}^l - g_{ik}\overset{t}{\alpha}_i{}^l \tag{2.24}$$

where $\overset{t}{\alpha}_{ij} = \overset{t}{a}_{ij} - \overset{t}{b}_{ij}$.

Theorem 2.3. *In a Riemannian manifold $(M, g)(dimM \geq 3)$, the tensor*

$$\overset{t}{C}_{ijk}{}^{l} + \overset{t*}{C}_{ijk}{}^{l} \tag{2.25}$$

for the connection $\overset{t}{\nabla}$ and $\overset{t*}{\nabla}$ is an invariant under the connection transformation $\overset{t}{\nabla} \to \overset{c}{\nabla}$ and $\overset{t*}{\nabla} \to \overset{c*}{\nabla}$, where $\overset{t}{C}_{ijk}{}^{l}$ and $\overset{t*}{C}_{ijk}{}^{l}$ are the Weyl conformal curvature tensors with respect to connection $\overset{t}{\nabla}$ and $\overset{t*}{\nabla}$ respectively, namely

$$\begin{cases} \overset{t}{C}{}^{l}_{ijk} &= \overset{t}{R}{}^{l}_{ijk} - \frac{1}{n-2}(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik} + g_{jk}\overset{t}{R}_i{}^l - g_{ik}\overset{t}{R}_j{}^l) - \frac{\overset{t}{R}}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_j g_{jk}), \\ \overset{t*}{C}{}^{l}_{ijk} &= \overset{t*}{R}{}^{l}_{ijk} - \frac{1}{n-2}(\delta^l_i \overset{t*}{R}_{jk} - \delta^l_j \overset{t*}{R}_{ik} + g_{jk}\overset{t*}{R}_i{}^l - g_{ik}\overset{t*}{R}_j{}^l) - \frac{\overset{t*}{R}}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_j g_{jk}). \end{cases} \tag{2.26}$$

Proof. Contracting the indices i and l of (2.24), we get

$$\overset{t}{R}_{jk} + \overset{t*}{R}_{jk} = 2K_{jk} - (n-2)\overset{t}{\alpha}_{jk} - g_{jk}\overset{t}{\alpha}{}^{i}_{i} \tag{2.27}$$

Multiplying both sides of this expression by g^{jk}, then we arrive at

$$\overset{t}{R} + \overset{t*}{R} = 2K - 2(n-1)\overset{t}{\alpha}{}^{i}_{i}$$

Thus we get

$$\overset{t}{\alpha}_{jk} = \frac{1}{n-2}[2K_{jk} - (\overset{t}{R}_{jk} + \overset{t*}{R}_{jk}) - \frac{g_{jk}}{2(n-1)}(2K - \overset{t}{R} - \overset{t*}{R})]$$

Substituting this expression into (2.24), then we arrive at

$$\overset{t}{C}_{ijk}{}^{l} + \overset{t*}{C}_{ijk}{}^{l} = 2\overset{\circ}{C}_{ijk}{}^{l} \tag{2.28}$$

where $\overset{\circ}{C}{}^{l}_{ijk} = K^{l}_{ijk} - \frac{1}{n-2}(\delta^l_i K_{jk} - \delta^l_j K_{ik} + g_{jk}K_i{}^l - g_{ik}K_j{}^l) - \frac{K}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_j g_{jk})$.

On the one hand, using the expression (2.22) and by using the same method above we have

$$\overset{c}{C}_{ijk}{}^{l} + \overset{c*}{C}_{ijk}{}^{l} = 2\overset{\circ}{C}_{ijk}{}^{l} \tag{2.29}$$

where

$$\begin{cases} \overset{c}{C}{}^{l}_{ijk} &= \overset{c}{R}{}^{l}_{ijk} - \frac{1}{n-2}(\delta^l_i \overset{c}{R}_{jk} - \delta^l_j \overset{c}{R}_{ik} + g_{jk}\overset{c}{R}_i{}^l - g_{ik}\overset{c}{R}_j{}^l) - \frac{\overset{c}{R}}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_j g_{jk}), \\ \overset{c*}{C}{}^{l}_{ijk} &= \overset{c*}{R}{}^{l}_{ijk} - \frac{1}{n-2}(\delta^l_i \overset{c*}{R}_{jk} - \delta^l_j \overset{c*}{R}_{ik} + g_{jk}\overset{c*}{R}_i{}^l - g_{ik}\overset{c*}{R}_j{}^l) - \frac{\overset{c*}{R}}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_j g_{jk}). \end{cases} \tag{2.30}$$

($\overset{c}{C}{}^{l}_{ijk}$ and $\overset{c*}{C}{}^{l}_{ijk}$ are the Weyl conformal curvature tensor with respect to $\overset{c}{\nabla}$ and $\overset{c*}{\nabla}$ respectively). From the expressions (2.28) and (2.29), we have

$$\overset{t}{C}_{ijk}{}^{l} + \overset{t*}{C}_{ijk}{}^{l} = \overset{c}{C}{}^{l}_{ijk} + \overset{c*}{C}{}^{l}_{ijk} \tag{2.31}$$

This expression shows that $\overset{t}{C}_{ijk}{}^{l} + \overset{t*}{C}_{ijk}{}^{l}$ is an invariant under the connection transformation $\overset{t}{\nabla} \to \overset{c}{\nabla}$ and $\overset{t*}{\nabla} \to \overset{c*}{\nabla}$. □

On the one hand from the expression (2.19), the connection coefficient of the mutual connection $\overset{cm}{\nabla}$ of $\overset{c}{\nabla}$

$$\overset{cm}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + [(\alpha - 2)t\omega_i + \sigma_i + \pi_i]\delta^k_j + [(\alpha - 2)t\omega_j + \sigma_j]\delta^k_i + [t\omega^k - \pi^k - \sigma^k]g_{ij} \tag{2.32}$$

and the curvature tensor of $\overset{cm}{\nabla}$ is

$$\overset{cm}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + \delta_j^l \overset{c}{a}_{ik} - \delta_i^l \overset{c}{a}_{jk} + g_{jk}\overset{c}{b}_i{}^{l} - g_{ik}\overset{c}{b}_j{}^{l} + \delta_k^l[(\alpha-2)t\omega_{ij} + \pi_{ij}] \tag{2.33}$$

where

$$\begin{aligned} \overset{cm}{a}_{ik} &= \overset{\circ}{\nabla}_i[(\alpha-2)t\omega_k + \sigma_k] - [(\alpha-2)t\omega_i + \sigma_i][(\alpha-2)t\omega_k + \sigma_k] \\ &\quad - g_{ik}[(\alpha-2)t\omega_p + \sigma_p](t\omega^p - \sigma^p - \pi^p) \end{aligned}$$

And from the expression (2.32), the connection coefficient of the dual connection $\overset{\overset{*}{cm}}{\nabla}$ of $\overset{cm}{\nabla}$ is

$$\overset{\overset{*}{cm}}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} - [(\alpha-2)t\omega_i + \sigma_i + \pi_i]\delta_j^k - [t\omega_j - \sigma_j - \pi_j]\delta_i^k - [(\alpha-2)t\omega^k + \sigma^k]g_{ij}$$

Using this expression, the curvature tensor of $\overset{\overset{*}{cm}}{\nabla}$ is

$$\overset{\overset{*}{cm}}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} - \delta_j^l \overset{c}{b}_{ik} + \delta_i^l \overset{c}{b}_{jk} - g_{jk}\overset{c}{a}_i{}^{l} + g_{ik}\overset{c}{a}_j{}^{l} - \delta_k^l[(\alpha-2)t\omega_{ij} + \pi_{ij}] \tag{2.34}$$

From the expressions (2.33) and (2.34), we obtain

$$\overset{cm}{R}_{ijk}{}^{l} + \overset{\overset{*}{cm}}{R}_{ijk}{}^{l} = 2K_{ijk}{}^{l} + \delta_j^l \overset{cm}{\alpha}_{ik} - \delta_i^l \overset{cm}{\alpha}_{jk} + g_{jk}\overset{cm}{\alpha}_i{}^{l} - g_{ik}\overset{cm}{\alpha}_j{}^{l} \tag{2.35}$$

where $\overset{cm}{\alpha}_{ik} = \overset{cm}{a}_{ik} - \overset{c}{b}_{ik}$. And from the expression (2.6) the connection coefficient of the dual connection $\overset{\overset{*}{tm}}{\nabla}$ of is

$$\overset{\overset{*}{tm}}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} - [(\alpha-2)t\omega_i + \pi_i]\delta_j^k - [t\omega_j - \pi_j]\delta_i^k - [(\alpha-2)t\omega^k]g_{ij}$$

and using this expression the curvature tensor of $\overset{\overset{*}{tm}}{\nabla}$ is

$$\overset{\overset{*}{tm}}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} - \delta_j^l \overset{t}{b}_{ik} + \delta_i^l \overset{t}{b}_{jk} - g_{jk}\overset{t}{a}_i{}^{l} + g_{ik}\overset{t}{a}_j{}^{l} - \delta_k^l[(\alpha-2)t\omega_{ij} + \pi_{ij}] \tag{2.36}$$

From the expressions (2.7) and (2.36), we obtain

$$\overset{tm}{R}_{ijk}{}^{l} + \overset{\overset{*}{tm}}{R}_{ijk}{}^{l} = 2K_{ijk}{}^{l} + \delta_j^l \overset{tm}{\alpha}_{ik} - \delta_i^l \overset{tm}{\alpha}_{jk} + g_{jk}\overset{tm}{\alpha}_i{}^{l} - g_{ik}\overset{tm}{\alpha}_j{}^{l} \tag{2.37}$$

where $\overset{tm}{\alpha}_{ik} = \overset{tm}{a}_{ik} - \overset{t}{b}_{ik}$.

Using the expressions (2.35) and (2.37), the following corollary is tenable.

Corollary 2.3. *In a Riemannian manifold $(m, g)(dimM \geq 3)$, the tensor*

$$\overset{tm}{C}_{ijk}{}^{l} + \overset{\overset{*}{tm}}{C}_{ijk}{}^{l} \tag{2.38}$$

for the connections $\overset{tm}{\nabla}$ and $\overset{\overset{*}{tm}}{\nabla}$ is an invariant under the connection transformation $\overset{tm}{\nabla} \to \overset{cm}{\nabla}$ and $\overset{\overset{*}{tm}}{\nabla} \to \overset{\overset{*}{cm}}{\nabla}$, where $\overset{tm}{C}_{ijk}{}^{l}$ and $\overset{\overset{*}{tm}}{C}_{ijk}{}^{l}$ are the Weyl conformal curvature tensor with respect to $\overset{tm}{\nabla}$ and $\overset{\overset{*}{tm}}{\nabla}$ respectively, namely

$$\begin{cases} \overset{tm}{C}{}^{l}_{ijk} &= \overset{tm}{R}{}^{l}_{ijk} - \frac{1}{n-2}(\delta_i^l \overset{tm}{R}_{jk} - \delta_j^l \overset{tm}{R}_{ik} + g_{jk}\overset{tm}{R}_i{}^{l} - g_{ik}\overset{tm}{R}_j{}^{l}) - \frac{\overset{tm}{R}}{(n-1)(n-2)}(\delta_j^l g_{ik} - \delta_j^l g_{jk}), \\ \overset{\overset{*}{tm}}{C}{}^{l}_{ijk} &= \overset{\overset{*}{tm}}{R}{}^{l}_{ijk} - \frac{1}{n-2}(\delta_i^l \overset{\overset{*}{tm}}{R}_{jk} - \delta_j^l \overset{\overset{*}{tm}}{R}_{ik} + g_{jk}\overset{\overset{*}{tm}}{R}_i{}^{l} - g_{ik}\overset{\overset{*}{tm}}{R}_j{}^{l}) - \frac{\overset{\overset{*}{tm}}{R}}{(n-1)(n-2)}(\delta_j^l g_{ik} - \delta_j^l g_{jk}). \end{cases} \tag{2.39}$$

3. Projective conformal semi-symmetric metric recurrent connection homotopy

Definition 3.1. *In a Riemannian manifold (M, g), a connection ∇ is called a projective conformal semi-symmetric metric recurrent connection homotopy, if ∇ is a conformal equivalent to $\overset{p}{\nabla}$.*

The projective conformal semi-symmetric metric recurrent connection homotopy ∇ satisfies the relation.

$$\begin{cases} \nabla_k g_{ij} &= -2[(\alpha-2)t\omega_k + \psi_k + \sigma_k]g_{ij} - [(\alpha-1)t\omega_i + \psi_i]g_{jk} - [(\alpha-1)t\omega_j + \psi + j]g_{ik}, \\ T^k_{ij} &= \pi_j\delta^k_i - \pi_i\delta^k_j, \end{cases} \tag{3.1}$$

and its connection coefficient is

$$\Gamma^{\ k}_{ij} = \{^{\,k}_{ij}\} + [(\alpha-2)t\omega_i + \sigma_i + \psi_i]\delta^k_j + [(\alpha-2)t\omega_j + \sigma_j + \psi_j]\delta^k_i + [t\omega^k - \pi^k - \sigma^k]g_{ij} \tag{3.2}$$

and the curvature tensor of ∇ is

$$R_{ijk}^{\ \ \ l} = K_{ijk}^{\ \ \ l} + \delta^l_j a_{ik} - \delta^l_i a_{jk} + g_{jk}b_i^{\ l} - g_{ik}b_j^{\ l} + \delta^l_k[(\alpha-2)t\omega_{ij} + \psi_{ij}] \tag{3.3}$$

where

$$\begin{cases} a_{ik} &= \overset{\circ}{\nabla}_i[(\alpha-2)t\omega_k + \sigma_k + \psi_k] - [(\alpha-2)t\omega_i + \sigma_i + \pi_i][(\alpha-2)t\omega_k + \sigma_k + \psi_k + \pi_k] \\ & -g_{ik}[(\alpha-2)t\omega_p + \sigma_p + \psi_p + \pi_p](t\omega^p - \sigma^p - \pi^p) \\ b_{ik} &= \overset{\circ}{\nabla}_i[t\omega_k - \sigma_k - \psi_k] + (t\omega_i - \sigma_i - \pi_i)(t\omega_k - \sigma_k - \pi_k) \end{cases} \tag{3.4}$$

The coefficient of dual connection $\overset{*}{\nabla}$ of ∇ is

$$\overset{*}{\Gamma}{}^{\ k}_{ij} = \{^{\,k}_{ij}\} - [(\alpha-2)t\omega_i + \sigma_i + \psi_i]\delta^k_j - [t\omega_j - \sigma_j - \pi_j]\delta^k_i - [(\alpha-2)t\omega^k + \psi^k + \sigma^k + \pi^k]g_{ij}$$

and from this expression the curvature tensor of $\overset{*}{\nabla}$ is

$$\overset{*}{R}_{ijk}{}^{l} = K_{ijk}^{\ \ \ l} - \delta^l_j a_{ik} + \delta^l_i a_{jk} - g_{jk}b_i^{\ l} + g_{ik}b_j^{\ l} - \delta^l_k[(\alpha-2)t\omega_{ij} + \psi_{ij}] \tag{3.5}$$

The mutual connection $\overset{m}{\nabla}$ of ∇ satisfies the relation

$$\begin{cases} \overset{m}{\nabla}_k g_{ij} &= -2[(\alpha-2)t\omega_k + \psi_k + \sigma_k + \pi_k]g_{ij} - [(\alpha-1)t\omega_i + \psi_i - \pi_i]g_{jk} - [(\alpha-1)t\omega_j + \psi_j - \pi_j]g_{ik}, \\ \overset{m}{T}{}^k_{ij} &= \pi_i\delta^k_j - \pi_j\delta^k_i, \end{cases} \tag{3.6}$$

and its connection coefficient is

$$\overset{m}{\Gamma}{}^{\ k}_{ij} = \{^{\,k}_{ij}\} + [(\alpha-2)t\omega_i + \sigma_i + \psi_i + \pi_i]\delta^k_j + [(\alpha-2)t\omega_j + \sigma_j + \psi_j]\delta^k_i + [t\omega^k - \pi^k - \sigma^k]g_{ij} \tag{3.7}$$

and from this expression, the curvature tensor of $\overset{m}{\nabla}$ is

$$\overset{m}{R}_{ijk}{}^{l} = K_{ijk}^{\ \ \ l} + \delta^l_j \overset{m}{a}_{ik} - \delta^l_i \overset{m}{a}_{jk} + g_{jk}b_i^{\ l} - g_{ik}b_j^{\ l} + \delta^l_k[(\alpha-2)t\omega_{ij} + \psi_{ij} + \pi_{ij}] \tag{3.8}$$

where

$$\begin{aligned} \overset{m}{a}_{ik} \ &= \ \overset{\circ}{\nabla}_i[(\alpha-2)t\omega_k + \sigma_k + \psi_k] - [(\alpha-2)t\omega_i + \sigma_i + \psi_i][(\alpha-2)t\omega_k + \sigma_k + \psi_k] \\ & \quad -g_{ik}[(\alpha-2)t\omega_p + \sigma_p + \psi_p](t\omega^p - \sigma^p - \pi^p) \end{aligned}$$

From the expressions (3.6) and (3.7), the connection coefficient of the dual connection $\overset{m*}{\nabla}$ of $\overset{m}{\nabla}$ is

$$\overset{m*}{\Gamma}{}^{\ k}_{ij} = \{^{\,k}_{ij}\} - [(\alpha-2)t\omega_i + \sigma_i + \psi_i + \pi_i]\delta^k_j - [t\omega_j - \sigma_j - \pi_j]\delta^k_i - [(\alpha-2)t\omega^k + \psi^k + \sigma^k]g_{ij}$$

and from this expression, the curvature tensor $\overset{m*}{\nabla}$ is

$$\overset{m*}{R}_{ijk}{}^{l} = K_{ijk}^{\ \ \ l} - \delta^l_j b_{ik} + \delta^l_i b_{jk} - g_{jk}\overset{m}{a}_i{}^{l} + g_{ik}\overset{m}{a}_j{}^{l} - \delta^l_k[(\alpha-2)t\omega_{ij} + \psi_{ij} + \pi_{ij}] \tag{3.9}$$

Theorem 3.1. *In a manifold (M, g) if 1-form ψ, ω and π are closed form for the projective conformal semi-symmetric metric recurrent connection homotopy ∇, then we have*

$$R_{ijk}{}^{l} + R_{jki}{}^{l} + R_{kij}{}^{l} = 0, \textit{first Bianchi identity} \tag{3.10}$$

$$R_{jk} = R_{kj}, P_{ij} = 0 (R_{jk} = R^{i}_{ijk}, P_{ij} = R_{ijk}{}^{k}), \tag{3.11}$$

$$R_{ijkl} + \overset{*}{R}_{ijkl} = R_{klij} + \overset{*}{R}_{klij}. \tag{3.12}$$

Proof. If 1-form ψ, ω and π are closed form, then $\psi_{ij} = \omega_{ij} = \pi_{ij} = 0$. From these facts, the expression (3.3) is

$$R_{ijk}{}^{l} = K_{ijk}{}^{l} + \delta^{l}_{j} a_{ik} - \delta^{l}_{i} a_{jk} + g_{jk} b_{i}{}^{l} - g_{ik} b_{j}{}^{l} \tag{3.13}$$

Using the expression $K_{ijk}{}^{l} + K_{jki}{}^{l} + K_{kij}{}^{l} = 0$, a cyclic permutation of the indices of the expression (3.13) yields

$$R_{ijk}{}^{l} + R_{jki}{}^{l} + R_{kij}{}^{l} = \delta^{l}_{j}(a_{ik} - a_{ki}) + \delta^{l}_{i}(a_{jk} - a_{kj}) + \delta^{l}_{k}(a_{ji} - a_{ij}) \tag{3.14}$$

On the one hand, from the expression (3.4), we have

$$\begin{cases} a_{ij} - a_{ji} &= (\alpha - 2) t \omega_{ij} + \psi_{ij} + \pi_{ij} = 0, \\ b_{ij} - b_{ji} &= t \omega_{ij} - \pi_{ij} = 0. \end{cases} \tag{3.15}$$

Using these expressions, from the expression (3.14), the expression (3.10) is proved.

Contracting the indices k and l of the expression (3.13) we find

$$P_{ij} = R_{ijk}{}^{k} = \overset{\circ}{P}_{ij} + a_{ij} - a_{ji} + b_{ij} - b_{ji}$$

where $\overset{\circ}{P}_{ij} = K_{ijk}{}^{k} = 0$. Using the expression (3.15), from this expression we obtain $P_{ij} = 0$. And contracting the indices i and l of the expression (3.10), we have

$$R_{jk} + P_{jk} - R_{kj} = 0.$$

Using $P_{ij} = 0$, from this expression we obtain $R_{jk} = R_{kj}$.

Adding the expressions (3.3) and (3.5), we obtain

$$R_{ijkl} + \overset{*}{R}_{ijkl} = 2K_{ijkl} + g_{jl}\alpha_{ik} - g_{il}\alpha_{jk} - g_{jk}\alpha_{il} + g_{ik}\alpha_{jl} \tag{3.16}$$

where $\alpha_{ik} = a_{ik} - b_{ik}$.

Putting $A_{ijkl} = g_{jl}\alpha_{ik} - g_{il}\alpha_{jk} + g_{ik}\alpha_{jl} - g_{jk}\alpha_{il}$ and using the expression (3.15) we have $A_{ijkl} = A_{klij}$. Using this expression and $K_{ijkl} = K_{klij}$, we proved the expression (3.12). □

Remark 3.1. *If $R_{ijk}{}^{l}$ is a curvature tensor of ∇, then*

$$\nabla_h R_{ijkl} + \nabla_i R_{jhkl} + \nabla_j R_{hikl} = 2(\pi_h R_{ijkl} + \pi_i R_{jhkl} + \pi_j R_{hikl}), \textit{second Bianchi identity} \tag{3.17}$$

and

$$R_{jhkl} + \overset{*}{R}_{jhkl} = -R_{ijlk} - \overset{*}{R}_{jhlk}.$$

It is easy to see that the following Corollary is tenable.

Corollary 3.1. *In a Riemannian manifold (M, g), if 1-form ψ, ω and π are closed form, then there holds*

$$\overset{m}{R}_{ijk}{}^{l} + \overset{m}{R}_{jki}{}^{l} + \overset{m}{R}_{kij}{}^{l} = 0, \textit{first Bianchi identity}$$

$$\overset{m}{R}_{jk} = \overset{m}{R}_{kj}, \overset{m}{P}_{ij} = 0, (\overset{m}{R}_{jk} = \overset{m}{R}_{ijk}{}^{i}, \overset{m}{P}_{ij} = \overset{m}{R}_{ijk}{}^{k})$$

$$\overset{m}{R}_{ijkl} + \overset{m*}{R}_{ijkl} = \overset{m}{R}_{klij} + \overset{m*}{R}_{klij}$$

Remark 3.2. *If $\overset{m}{R}_{ijk}{}^{l}$ is a curvature tensor of $\overset{m}{\nabla}$ to ∇, then*

$$\overset{m}{\nabla}_h \overset{m}{R}_{ijkl} + \overset{m}{\nabla}_i \overset{m}{R}_{jhkl} + \overset{m}{\nabla}_j \overset{m}{R}_{hikl} = 2(\pi_h \overset{m}{R}_{ijkl} + \pi_i \overset{m}{R}_{jhkl} + \pi_j \overset{m}{R}_{hikl}), \textit{second Bianchi identity} \tag{3.18}$$

Second Bianchi identity and

$$\overset{m}{R}_{jhkl} + \overset{m*}{R}_{jhkl} = -\overset{m}{R}_{ijlk} - \overset{m*}{R}_{jhlk}.$$

Theorem 3.2. *If a Riemannian metric admits the projective conformal semi-symmetric recurrent connection homotopy ∇ with a vanishing curvature tensor in the Riemannian manifold $(M, g)(dimM \geq 3)$, then the Riemannian metric is conformal flat.*

Proof. Contracting the indices i and l of (3.16), then we find

$$R_{jk} + \overset{*}{R}_{jk} = 2K_{jk} - (n-2)\alpha_{jk} - g_{jk}\alpha^i_i, \tag{3.19}$$

Multiplying both sides of this expression by g^{jk}, we have

$$R + \overset{*}{R} = 2K - 2(n-1)\alpha^i_i.$$

Thus we arrive at

$$\alpha_{jk} = \frac{1}{n-2}[2K_{jk} - R_{jk} - \overset{*}{R}_{jk} - \frac{1}{2(n-1)} g_{jk}(2K - R - \overset{*}{R})].$$

Substituting this expression for the expression (3.16), and by a direct computation, we obtain

$$C_{ijk}{}^{l} + \overset{*}{C}_{ijk}{}^{l} = 2\overset{\circ}{C}_{ijk}{}^{l} \tag{3.20}$$

where

$$\begin{cases} C^{\,l}_{\,ijk} & = R^{\,l}_{\,ijk} - \frac{1}{n-2}(\delta^l_i \overset{tm}{R}_{jk} - \delta^l_j \overset{tm}{R}_{ik} + g_{jk}\overset{tm}{R}_i{}^{l} - g_{ik}\overset{tm}{R}_j{}^{l}) - \frac{\overset{tm}{R}}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_j g_{jk}), \\ \overset{*}{C}{}^{\,l}_{\,ijk} & = \overset{*}{R}{}^{\,l}_{\,ijk} - \frac{1}{n-2}(\delta^l_i \overset{*}{R}_{jk} - \delta^l_j \overset{*}{R}_{ik} + g_{jk}\overset{*}{R}_i{}^{l} - g_{ik}\overset{*}{R}_j{}^{l}) - \frac{\overset{*}{R}}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_j g_{jk}), \\ \overset{\circ}{C}{}^{\,l}_{\,ijk} & = K^{\,l}_{\,ijk} - \frac{1}{n-2}(\delta^l_i K_{jk} - \delta^l_j K_{ik} + g_{jk}K_i{}^{l} - g_{ik}K_j{}^{l}) - \frac{K}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_j g_{jk}) \end{cases} \tag{3.21}$$

From the expression (3.21) if $R^{\,l}_{\,ijk} = 0$, then $C^{\,l}_{\,ijk} = \overset{*}{C}{}^{\,l}_{\,ijk} = 0$, we have $\overset{\circ}{C}{}^{\,l}_{\,ijk} = 0$. This means that the Riemannian metric is conformal flat. □

Following Theorem 3.2, it is not hard to see that there holds

Corollary 3.2. *If a Riemannian metric admits the mutual connection $\overset{m}{\nabla}$ with a vanishing curvature tensor of ∇ in the Riemannian manifold (M, g), then Riemannian metric is conformal flat.*

Theorem 3.3. *In order that the projective conformal semi-symmetric metric recurrent connection homotopy ∇ is a conjugate symmetry in a Riemannian manifold $(M, g)(dimM \geq 3)$, it is necessary and sufficient that it is a conjugate Ricci symmetry and a conjugate volume symmetry.*

Proof. From the expressions (3.3) and (3.5), we have

$$\overset{*}{R}_{ijk}{}^{l} = R_{ijk}{}^{l} - \delta^l_j \beta_{ik} + \delta^l_i \beta_{jk} - g_{jk}\beta_i{}^l + g_{ik}\beta_j{}^l - 2\delta^l_k[(\alpha-2)t\omega_{ij} + \psi_{ij}] \tag{3.22}$$

where $\beta_{ik} = a_{ik} + b_{ik}$. Contracting the indices i and l of this expression, we have

$$\overset{*}{R}_{jk} = R_{jk} + n\beta_{jk} - g_{jk}\beta_i{}^i - 2[(\alpha-2)t\omega_{jk} + \psi_{jk}] \tag{3.23}$$

Alternating the indices j and k of this expression, we have

$$\overset{*}{R}_{jk} - \overset{*}{R}_{kj} = R_{jk} - R_{kj} + n(\beta_{jk} - \beta_{kj}) - 4[(\alpha-2)t\omega_{jk} + \psi_{jk}].$$

On the one hand, contracting the indices k and l of the expression (3.2) and changing the index i for j and index j for k, we have

$$\overset{*}{P}_{jk} = P_{jk} - 2(\beta_{jk} - \beta_{kj}) - 2n[(\alpha-2)t\omega_{jk} + \psi_{jk}].$$

From the above two expressions, we find

$$(\alpha-2)t\omega_{jk} + \psi_{jk} = \frac{1}{2(n^2-4)}\Big\{2[(R_{jk} - R_{kj}) - (\overset{*}{R}_{jk} - \overset{*}{R}_{kj})] - (P_{jk} - \overset{*}{P}_{kj})]\Big\} \tag{3.24}$$

Substituting this expression for the expression (3.23), we find

$$\beta_{jk} = \frac{1}{n}\Big\{\overset{*}{R}_{jk} - R_{jk} + g_{jk}\beta^i_i + \frac{1}{(n^2-4)}(2[(\overset{*}{R}_{jk} - \overset{*}{R}_{kj}) - (R_{jk} - R_{kj})] - (\overset{*}{P}_{kj} - P_{jk}))\Big\} \tag{3.25}$$

And substituting the expressions (3.24) and (3.25) for the expression (3.22) and by a direct computation we obtain

$$\begin{aligned}
&\overset{*}{R}_{ijk}{}^{l} - \frac{1}{n}(\delta^l_i \overset{*}{R}_{jk} - \delta^l_j \overset{*}{R}_{ik} + g_{ik}\overset{*}{R}{}^l_j - g_{jk}\overset{*}{R}{}^l_i) - \frac{2}{n(n^2-4)}[\delta^l_i(\overset{*}{R}_{jk} - \overset{*}{R}_{kj}) - \delta^l_j(\overset{*}{R}_{ik} - \overset{*}{R}_{ki}) + n\delta^l_k(\overset{*}{R}_{ij} - \overset{*}{R}_{ji})] \\
&- \frac{1}{n^2-4}(\delta^l_i \overset{*}{P}_{jk} - \delta^l_j \overset{*}{P}_{ik} + g_{ik}\overset{*}{P}{}^l_j - g_{jk}\overset{*}{P}{}^l_i + n\delta^l_k \overset{*}{P}_{ij})) \\
&= R_{ijk}{}^{l} - \frac{1}{n}(\delta^l_i R_{jk} - \delta^l_j R_{ik} + g_{ik}R^l_j - g_{jk}R^l_i) - \frac{2}{n(n^2-4)}[\delta^l_i(R_{jk} - R_{kj}) - \delta^l_j(R_{ik} - R_{ki}) + n\delta^l_k(R_{ij} - R_{ji})] \\
&- \frac{1}{n^2-4}(\delta^l_i P_{jk} - \delta^l_j P_{ik} + g_{ik}P^l_j - g_{jk}P^l_i + n\delta^l_k P_{ij}))
\end{aligned}$$

From this expression we arrive at $\overset{*}{R}_{ijk}{}^{l} = R_{ijk}{}^{l}$ if and only if $\overset{*}{R}_{jk} = R_{jk}$ and $\overset{*}{P}_{jk} = P_{jk}$. □

Theorem 3.4. *In a Riemannian manifold $(M, g)(dimM \geq 3)$, if 1-form ω, ψ and π are closed form for ∇ the in order that ∇ is a conjugate symmetry it is necessary and sufficient that it is a conjugate Ricci symmetry.*

Proof. If 1-form ω, ψ and π are closed form, then $\omega_{ij} = \psi_{ij} = \pi_{ij} = 0$. From this fact and formulas (3.8), (3.9), the expression (3.22), is

$$\overset{*}{R}_{ijk}{}^{l} = R_{ijk}{}^{l} + \delta^l_i \beta_{jk} - \delta^l_j \beta_{ik} + g_{ik}\beta^l_j - g_{jk}\beta^l_i \tag{3.26}$$

Contracting the indices i and l of this expression, we have

$$\overset{*}{R}_{jk} = R_{jk} + n\beta_{jk} - g_{jk}\beta^l_l$$

Thus we get

$$\beta_{jk} = \frac{1}{n}(\overset{*}{R}_{jk} - R_{jk} + g_{jk}\beta_l^l)$$

Substituting this expression for the expression (3.26) and by a direct computation we obtain

$$\overset{*}{R}_{ijk}{}^{l} - \frac{1}{n}(\overset{*}{R}_{jk} - \delta_j^l \overset{*}{R}_{ik} + g_{ik}\overset{*}{R}{}_j^l - g_{jk}\overset{*}{R}{}_i^l) = R_{ijk}{}^{l} - \frac{1}{n}(R_{jk} - \delta_j^l R_{ik} + g_{ik}R_j^l - g_{jk}R_i^l)$$

From this expression we arrive at $\overset{*}{R}_{ijk}{}^{l} = R_{ijk}{}^{l}$ if and only if $\overset{*}{R}_{jk} = R_{jk}$. □

4. Schur's theorem w.r.t. the projective conformal semi-symmetric metric recurrent connection homotopy

Theorem 4.1. *(Schur's theorem) Suppose there is a connected Riemannian manifold $(M, g)(dimM \geq 3)$ with the projective conformal semi-symmetric metric recurrent connection homotopy ∇ that is everywhere isotropic. If it satisfies the relation*

$$(\alpha - 3)t\omega_h + \psi_h + 2\sigma_h + 2\pi_h = 0 \tag{4.1}$$

then the Riemannian manifold (M, g, ∇) is a constant curvature manifold.

Proof. From the fact that (M, g) is isotropic at p, the curvature tensor of ∇ is

$$R_{ijk}{}^{l} = k(p)(\delta_j^l g_{ik} - \delta_i^l g_{jk}) \tag{4.2}$$

Substituting (4.2) into (3.17), we obtain

$$\begin{aligned}
&\nabla_h k(\delta_j^l g_{ik} - \delta_i^l g_{jk}) + \nabla_i k(\delta_h^l g_{jk} - \delta_j^l g_{hk}) + \nabla_j k(\delta_i^l g_{hk} - \delta_h^l g_{ik}) \\
&+k(\delta_j^l \nabla_h g_{ik} - \delta_i^l \nabla_h g_{jk} + \delta_h^l \nabla_i g_{jk} - \delta_j^l \nabla_i g_{ik} + \delta_i^l \nabla_j g_{hk} - \delta_h^l \nabla_j g_{ik}) \\
&= 2k[\pi_h(\delta_j^l g_{ik} - \delta_i^l g_{jk}) + \pi_i(\delta_h^l g_{jk} - \delta_j^l g_{hk}) + \pi_j(\delta_i^l g_{hk} - \delta_h^l g_{ik})]
\end{aligned}$$

Using the expression (3.1), we arrive at the following

$$\begin{aligned}
&\{\nabla_h k - k[(\alpha - 3)t\omega_h + \psi_h + 2\sigma_h + 2\pi_h]\}(\delta_j^l g_{ik} - \delta_i^l g_{jk}) + \{\nabla_i k - k[(\alpha - 3)t\omega_i + \psi_i + 2\sigma_i + 2\pi_i]\}(\delta_h^l g_{jk} - \delta_j^l g_{hk}) \\
&+\{\nabla_j k - k[(\alpha - 3)t\omega_j + \psi_j + 2\sigma_j + 2\pi_j]\}(\delta_i^l g_{hk} - \delta_h^l g_{ik}) = 0
\end{aligned}$$

Contracting the indices i and l of this expression, then we find

$$\{\nabla_j k - [(\alpha - 3)t\omega_j + \psi_j + 2\sigma_j + 2\pi_j]\}g_{hk} = \{\nabla_h k - [(\alpha - 3)t\omega_h + \psi_h + 2\sigma_h + 2\pi_h]\}g_{jk}$$

And multiplying both sides of this expression by g^{jk}, then we obtain

$$\{\nabla_h k - k[(\alpha - 3)t\omega_h + \psi_h + 2\sigma_h + 2\pi_h]\} = 0$$

Consequently, from $dimM \geq 3$, $k = const$ if and only if

$$(\alpha - 3)t\omega_h + \psi_h + 2\sigma_h + 2\pi_h = 0.$$

This ends the proof of Theorem 4.1. □

Using the expression (4.1), from (3.1) and (3.2), we have

$$\begin{cases} \nabla_k g_{ij} & = -2(t\omega_k - \sigma_k - 2\pi_k)g_{ij} - 2(t\omega_i - \sigma_i - 2\pi_i)g_{jk} - 2(t\omega_j - \sigma_j - 2\pi_j)g_{ik}, \\ T^{\,k}_{ij} & = \pi_j\delta^k_i - \pi_i\delta^k_j, \\ \Gamma^{\,k}_{ij} & = \{^{\,k}_{ij}\} + (t\omega_i - \sigma_i - 2\pi_i)\delta^k_j + (t\omega_j - \sigma_j - 2\pi_j)\delta^k_i + g_{ij}(t\omega^k - \sigma^k - 2\pi^k) \end{cases} \tag{4.3}$$

This expression is expresses a projective conformal semi-symmetric metric recurrent homotopy with constant curvature. The expression (4.3) is independent of 1-form ψ. This fact shows that $\overset{c}{\nabla}$ is different from ∇, but that $\overset{c}{\nabla}$ with constant curvature is not different from ∇ with constant curvature.

If $t = 0$, then from the expression (4.3) we have

$$\begin{cases} \nabla_k g_{ij} & = 2(\sigma_k + 2\pi_k)g_{ij} + 2(\sigma_i + 2\pi_i)g_{jk} + 2(\sigma_j + 2\pi_j)g_{ik}, \\ T^{\,k}_{ij} & = \pi_j\delta^k_i - \pi_i\delta^k_j, \\ \Gamma^{\,k}_{ij} & = \{^{\,k}_{ij}\} - (\sigma_i + 2\pi_i)\delta^k_j - (\sigma_j + \pi_j)\delta^k_i - g_{ij}(\sigma^k + \pi^k) \end{cases} \tag{4.4}$$

This connection is a projective conformal (or conformal) semi-symmetric metric connection with constant curvature.

If $t = 1$, then from the expression (4.3) we have

$$\begin{cases} \nabla_k g_{ij} & = -2(\omega_k - \sigma_k - \pi_k)g_{ij} - 2(\omega_i - \sigma_i - \pi_i)g_{jk} - 2(\omega_j - \sigma_j - \pi_j)g_{ik}, \\ T^{\,k}_{ij} & = \pi_j\delta^k_i - \pi_i\delta^k_j, \\ \Gamma^{\,k}_{ij} & = \{^{\,k}_{ij}\} + (\omega_i - \sigma_i - 2\pi_i)\delta^k_j + (\omega_j - \sigma_j - \pi_j)\delta^k_i + g_{ij}(\omega^k - \sigma^k - \pi^k) \end{cases} \tag{4.5}$$

This connection is the projective conformal (or conformal) semi-symmetric metric recurrent connection with constant curvature. So the expression (4.3) expresses a projective conformal (or conformal) semi-symmetric metric recurrent connection homotopy ∇ with constant.

If $\sigma = 0$, then $\nabla = \overset{p}{\nabla}$. In this case from the expression (4.3) we have

$$\begin{cases} \overset{p}{\nabla}_k g_{ij} & = -2(t\omega_k - 2\pi_k)g_{ij} - 2(t\omega_i - \pi_i)g_{jk} - 2(t\omega_j - \pi_j)g_{ik}, \\ T^{\,k}_{ij} & = \pi_j\delta^k_i - \pi_i\delta^k_j, \\ \overset{p}{\Gamma}{}^{\,k}_{ij} & = \{^{\,k}_{ij}\} + (t\omega_i - 2\pi_i)\delta^k_j + (t\omega_j - 2\pi_j)\delta^k_i + g_{ij}(t\omega^k - \pi^k) \end{cases} \tag{4.6}$$

This expression expresses the projective semi-symmetric metric recurrent connection homotopy $\overset{p}{\nabla}$ with constant curvature.

If $t = 0$, then from expression (4.6) we have

$$\begin{cases} \overset{p}{\nabla}_k g_{ij} & = 4\pi_k g_{ij} + 2\pi_i g_{jk} + 2\pi_j g_{ik}, \\ T^{\,k}_{ij} & = \pi_j\delta^k_i - \pi_i\delta^k_j, \\ \overset{p}{\Gamma}{}^{\,k}_{ij} & = \{^{\,k}_{ij}\} - 2\pi_i\delta^k_j - \pi_j\delta^k_i - g_{ij}\pi^k \end{cases} \tag{4.7}$$

This connection expresses the projective semi-symmetric metric connection with constant curvature.

If $t = 1$, then from expression (4.6) we have

$$\begin{cases} \overset{p}{\nabla}_k g_{ij} & = -2(\omega_k - 2\pi_k)g_{ij} - 2(\omega_i - \pi_i)g_{jk} - 2(\omega_j - \pi_j)g_{ik}, \\ T^{\,k}_{ij} & = \pi_j\delta^k_i - \pi_i\delta^k_j, \\ \overset{p}{\Gamma}{}^{\,k}_{ij} & = \{^{\,k}_{ij}\} + (\omega_i - 2\pi_i)\delta^k_j + (\omega_j - \pi_j)\delta^k_i + g_{ij}(\omega^k - \pi^k) \end{cases} \tag{4.8}$$

This connection expresses the projective semi-symmetric metric recurrent connection with constant curvature. So the expression (4.6) expresses the projective semi-symmetric metric recurrent connection homotopy $\overset{p}{\nabla}$ with constant curvature.

In particular, if $\omega_i = \pi_i$, then the expression (4.8) becomes

$$\begin{cases} \overset{p}{\nabla}_k g_{ij} & = 2\pi_k g_{ij}, \\ T_{ij}^{\ k} & = \pi_j\delta_i^k - \pi_i\delta_j^k, \\ \overset{p}{\Gamma}{}_{ij}^{\ k} & = \left\{ {k \atop ij} \right\} - \pi_i\delta_j^k \end{cases} \tag{4.9}$$

This connection is studied as the semi-symmetric non-metric connection with constant curvature in [8].

If $\omega_i = 2\pi_i$, the expression (4.8) becomes

$$\begin{cases} \overset{p}{\nabla}_k g_{ij} & = -2\pi_k g_{ij} - 2\pi_j g_{ik}, \\ T_{ij}^{\ k} & = \pi_j\delta_i^k - \pi_i\delta_j^k, \\ \overset{p}{\Gamma}{}_{ij}^{\ k} & = \left\{ {k \atop ij} \right\} + \pi_j\delta_+^k g_{ij}\pi^k \end{cases} \tag{4.10}$$

If $\psi = \sigma = 0$, then $\nabla = \overset{t}{\nabla}$. In this case from the expression (4.3) we have

$$\begin{cases} \overset{t}{\nabla}_k g_{ij} & = -2(t\omega_k - 2\pi_k)g_{ij} - 2(t\omega_i - \pi_i)g_{jk} - 2(t\omega_j - \pi_j)g_{ik}, \\ T_{ij}^{\ k} & = \pi_j\delta_i^k - \pi_i\delta_j^k, \\ \overset{t}{\Gamma}{}_{ij}^{\ k} & = \{ {k \atop ij} \} + (t\omega_i - 2\pi_i)\delta_j^k + (t\omega_j - \pi_j)\delta_i^k + g_{ij}(t\omega^k - \pi^k) \end{cases} \tag{4.11}$$

This expression coincides with expression (4.6) in form but this connection does not express the semi-symmetric metric recurrent connection homotopy $\overset{t}{\nabla}$, (namely the connection homotopy from the semi-symmetric metric connection to the semi-symmetric metric recurrent connection) with constant curvature.

In fact, in this case the expression (4.1) becomes

$$(\alpha - 3)t\omega_h + 2\pi_h = 0. \tag{4.12}$$

From this expression, if $t = 0$ or $\alpha = 3$, then $\pi_h = 0$. So if $t = 0$, then the expression (4.11) expressed as

$$\begin{cases} \overset{t}{\nabla}_k g_{ij} & = 0, \\ T_{ij}^{\ k} & = 0, \\ \overset{t}{\Gamma}{}_{ij}^{\ k} & = \left\{ {k \atop ij} \right\}. \end{cases} \tag{4.13}$$

This connection is the Levi-Civita connection. Consequently the expression (4.11) expresses the semi-symmetric metric recurrent connection family $\overset{t}{\nabla}$ with constant curvature together with connection homotopy from the Levi-Civita connection to the semi-symmetric metric recurrent connection with constant curvature.

Remark 4.1. *If $\alpha = 3$, then the expression (4.11) becomes*

$$\begin{cases} \overset{t}{\nabla}_k g_{ij} & = -2t\omega_k g_{ij} - 2t\omega_i g_{jk} - 2t\omega_j g_{ik}, \\ T_{ij}^{\ k} & = 0, \\ \overset{t}{\Gamma}{}_{ij}^{\ k} & = \{ {k \atop ij} \} + t\omega_i\delta_j^k + t\omega_j\delta_i^k + g_{ij}t\omega^k. \end{cases} \tag{4.14}$$

This expression is expresses the connection homotopy from the Levi-Civita connection to the Amari-Chentsor connection with metric recurrent connection satisfying the Schur's theorem.

Theorem 4.2. *(Schur's Lemma) Suppose a Riemannian manifold $(M, g)(dimM \geq 3)$ with the mutual connection $\overset{m}{\nabla}$ of ∇ is isotropic everywhere. If it satisfies the relation*

$$(\alpha - 3)t\omega_h + \psi_h + 2\sigma_h + \pi_h = 0. \tag{4.15}$$

then the Riemannian manifold $(M, g, \overset{m}{\nabla})$ is a constant curvature manifold.

Proof. From the fact that $(M, g)(dimM \geq 3)$ is isotropic at point p, the curvature tensor of $\overset{m}{\nabla}$ is

$$\overset{m}{R}_{ijk}{}^{l} = k(p)(\delta^l_j g_{ik} - \delta^l_i g_{jk})$$

Substituting this expression into (3.18), we obtain

$$\begin{aligned}
&\overset{m}{\nabla}_h k(\delta^l_j g_{ik} - \delta^l_i g_{jk}) + \overset{m}{\nabla}_i k(\delta^l_h g_{jk} - \delta^l_j g_{hk}) + \overset{m}{\nabla}_j k(\delta^l_i g_{hk} - \delta^l_h g_{ik}) \\
&+k(\delta^l_j \overset{m}{\nabla}_h g_{ik} - \delta^l_i \overset{m}{\nabla}_h g_{jk} + \delta^l_h \overset{m}{\nabla}_i g_{jk} - \delta^l_j \overset{m}{\nabla}_i g_{ik} + \delta^l_i \overset{m}{\nabla}_j g_{hk} - \delta^l_h \overset{m}{\nabla}_j g_{ik}) \\
&= -2k[\pi_h(\delta^l_j g_{ik} - \delta^l_i g_{jk}) + \pi_i(\delta^l_h g_{jk} - \delta^l_j g_{hk}) + \pi_j(\delta^l_i g_{hk} - \delta^l_h g_{ik})]
\end{aligned}$$

Using the expression (3.6), from this expression we obtain

$$\begin{aligned}
&\{\overset{m}{\nabla}_h k - k[(\alpha-3)t\omega_h + \psi_h + 2\sigma_h + 2\pi_h]\}(\delta^l_j g_{ik} - \delta^l_i g_{jk}) + \{\overset{m}{\nabla}_i k - k[(\alpha-3)t\omega_i + \psi_i + 2\sigma_i + 2\pi_i]\}(\delta^l_h g_{jk} - \delta^l_j g_{hk}) \\
&+\{\overset{m}{\nabla}_j k - k[(\alpha-3)t\omega_j + \psi_j + 2\sigma_j + 2\pi_j]\}(\delta^l_i g_{hk} - \delta^l_h g_{ik}) = 0
\end{aligned}$$

Contracting the indices i and l of this expression, then we find

$$\{\overset{m}{\nabla}_j k - [(\alpha-3)t\omega_j + \psi_j + 2\sigma_j + \pi_j]\}g_{hk} = \{\overset{m}{\nabla}_h k - [(\alpha-3)t\omega_h + \psi_h + 2\sigma_h + \pi_h]\}g_{jk}$$

And multiplying both sides of this expression by g^{jk}, the we obtain

$$\{\overset{m}{\nabla}_h k - k[(\alpha-3)t\omega_h + \psi_h + 2\sigma_h + \pi_h]\} = 0.$$

Consequently, from $dimM \geq 3$, $k = const$ if and only if

$$(\alpha - 3)t\omega_h + \psi_h + 2\sigma_h + \pi_h = 0.$$

This completes the proof of Theorem 4.2. □

Using the expression (4.15), from (3.6) and (3.7), we have

$$\begin{cases}
\overset{m}{\nabla}_k g_{ij} &= -2(t\omega_k - \sigma_k)g_{ij} - 2(t\omega_i - \sigma_i - \pi_i)g_{jk} - 2(t\omega_j - \sigma_j - \pi_j)g_{ik}, \\
\overset{m}{T}{}^{k}_{ij} &= \pi_j\delta^k_i - \pi_i\delta^k_j, \\
\overset{m}{\Gamma}{}^{k}_{ij} &= \{^{k}_{ij}\} + (t\omega_i - \sigma_i)\delta^k_j + (t\omega_j - \sigma_j - \pi_j)\delta^k_i + g_{ij}(t\omega^k - \sigma^k - \pi^k)
\end{cases} \tag{4.16}$$

This connection is the mutual connection $\overset{m}{\nabla}$ with constant curvature of ∇ and is independent of 1-form ψ. This fact shows that $\overset{cm}{\nabla}$ is different from $\overset{m}{\nabla}$ but that $\overset{cm}{\nabla}$ with constant curvature is not different from $\overset{m}{\nabla}$ with constant curvature.

So the expression (4.16) expresses the mutual connection $\overset{m}{\nabla}$ (or $\overset{cm}{\nabla}$) with constant curvature of ∇ (or $\overset{c}{\nabla}$).

If $\sigma = 0$, then $\overset{m}{\nabla} = \overset{pm}{\nabla}$. In this case from the expression (4.16), we have

$$\begin{cases} \overset{pm}{\nabla}_k g_{ij} &= -2t\omega_k g_{ij} - 2(t\omega_i - \pi_i)g_{jk} - 2(t\omega_j - \pi_j)g_{ik}, \\ \overset{pm}{T}{}^{k}_{ij} &= \pi_j \delta^k_i - \pi_i \delta^k_j, \\ \overset{pm}{\Gamma}{}^{k}_{ij} &= \left\{ {k \atop ij} \right\} + t\omega_i \delta^k_j + (t\omega_j - \pi_j)\delta^k_i + g_{ij}(t\omega^k - \pi^k) \end{cases} \tag{4.17}$$

This connection expression the mutual connection $\overset{pm}{\nabla}$ with constant curvature of $\overset{p}{\nabla}$.

If $t = 0$, then the expression (4.17) is

$$\begin{cases} \overset{pm}{\nabla}_k g_{ij} &= 2\pi_i g_{jk} + 2\pi_j g_{ik}, \\ \overset{pm}{T}{}^{k}_{ij} &= \pi_j \delta^k_i - \pi_i \delta^k_j, \\ \overset{pm}{\Gamma}{}^{k}_{ij} &= \left\{ {k \atop ij} \right\} - \pi_j \delta^k_i - g_{ij}\pi^k. \end{cases} \tag{4.18}$$

This connection is the same connection as (4.10) and the mutual connection $\overset{pm}{\nabla}$ with constant curvature of the projective semi-symmetric metric connection $\overset{p}{\nabla}$. If $t = 1$, then the expression (4.17) is

$$\begin{cases} \overset{pm}{\nabla}_k g_{ij} &= -2\omega_k g_{ij} - 2(\omega_i - \pi_i)g_{jk} - 2(\omega_j - \pi_j)g_{ik}, \\ \overset{pm}{T}{}^{k}_{ij} &= \pi_j \delta^k_i - \pi_i \delta^k_j, \\ \overset{pm}{\Gamma}{}^{k}_{ij} &= \left\{ {k \atop ij} \right\} + \omega_i \delta^k_j + (\omega_j - \pi_j)\delta^k_i + g_{ij}(\omega^k - \pi^k) \end{cases} \tag{4.19}$$

This connection is the mutual connection $\overset{pm}{\nabla}$ with constant curvature of the projective semi-symmetric metric recurrent connection $\overset{p}{\nabla}$. So the expression (4.17) expresses the mutual connection $\overset{pm}{\nabla}$ with constant curvature of the projective semi-symmetric metric recurrent connection homotopy $\overset{p}{\nabla}$. In particular, if $\omega_i = \pi_i$, then from (4.19) there holds

$$\begin{cases} \overset{pm}{\nabla}_k g_{ij} &= -2\pi_k g_{ij}, \\ \overset{pm}{T}{}^{k}_{ij} &= \pi_j \delta^k_i - \pi_i \delta^k_j, \\ \overset{pm}{\Gamma}{}^{k}_{ij} &= \left\{ {k \atop ij} \right\} + \pi_i \delta^k_j. \end{cases} \tag{4.20}$$

This is the same as the expression (4.9). If $\omega_i = 2\pi_i$, then from the expression (4.19), we have

$$\begin{cases} \overset{pm}{\nabla}_k g_{ij} &= -4\pi_k g_{ij} - 2\pi_i g_{jk} - 2\pi_j g_{ik}, \\ \overset{pm}{T}{}^{k}_{ij} &= \pi_j \delta^k_i - \pi_i \delta^k_j, \\ \overset{pm}{\Gamma}{}^{k}_{ij} &= \left\{ {k \atop ij} \right\} + 2\pi_i \delta^k_j + \pi_j \delta^k_i + g_{ij}\pi^k. \end{cases} \tag{4.21}$$

This connection is the same connection as the expression (4.7) and expresses the mutual connection $\overset{pm}{\nabla}$ with constant curvature of projective semi-symmetric metric current connection $\overset{p}{\nabla}$.

If $\psi = \sigma = 0$, then $\overset{m}{\nabla} = \overset{tm}{\nabla}$. In this case, from expression (4.16), we have

$$\begin{cases} \overset{tm}{\nabla}_k g_{ij} &= -2t\omega_k g_{ij} - 2(t\omega_i - \pi_i)g_{jk} - 2(t\omega_j - \pi_j)g_{ik}, \\ \overset{tm}{T}{}^{k}_{ij} &= \pi_j \delta^k_i - \pi_i \delta^k_j, \\ \overset{tm}{\Gamma}{}^{k}_{ij} &= \{ {k \atop ij} \} + t\omega_i \delta^k_j + (t\omega_j - \pi_j)\delta^k_i + g_{ij}(t\omega^k - \pi^k) \end{cases} \tag{4.22}$$

This expression coincides with the expression (4.17) in from but this connection does not express the mutual connection $\overset{tm}{\nabla}$ with constant curvature of the semi-symmetric metric recurrent connection homotopy $\overset{t}{\nabla}$. In fact, in this case from the expression (4.15), we have the expression

$$(\alpha - 3)t\omega_h + \pi_h = 0.$$

From this expression, if $t = 0$ or $\alpha = 3$ then $\pi_h = 0$. From this fact, if $t = 0$, then from the expression (4.22) we have

$$\overset{tm}{\nabla}_k g_{ij} = 0,\ \overset{tm}{T}{}^k_{ij} = 0,\ \overset{tm}{\Gamma}{}^k_{ij} = 0.$$

This expression expresses the Levi-Civita connection and from the expression (4.13), we have $\overset{t}{\nabla} = \overset{tm}{\nabla}$. And if $\alpha = 3$, then from the expression (4.22) we have

$$\begin{cases} \overset{tm}{\nabla}_k g_{ij} & = -2t\omega_k g_{ij} - 2t\omega_i g_{jk} - 2t\omega_j g_{ik}, \\ \overset{tm}{T}{}^k_{ij} & = 0, \\ \overset{tm}{\Gamma}{}^k_{ij} & = \{^{\,k}_{ij}\} + t\omega_i \delta^k_j + t\omega_j \delta^k_i + g_{ij} t\omega^k. \end{cases} \tag{4.23}$$

This connection is the same connection as the expression (4.14), namely $\overset{t}{\nabla} = \overset{tm}{\nabla}$. So the expression (4.22) expresses the mutual connection $\overset{t}{\nabla}$ with constant curvature of the connection homotopy from the Levi-Civitaconnection to the semi-symmetric metric recurrent connection with constant curvature.

5. An Application to Projective Conformal Semi-symmetric Metric Recurrent Connections

In this subsection we will investigate no-arbitrage properties of a manifold associated with projective conformal semi-symmetric metric recurrent connections as a financial market. This view is new and interesting for a manifold with some semi-symmetric connections to describe the financial information, in particular, no-arbitrage principle. This research is a good example of the application of manifolds in financial engineering. The present paper poses only one application of the manifold associated with a special connection. The other further researches we will state them the the following articles (we omit them here). All the related information about financial market represented by manifolds, one can refer to [23] for details.

Theorem 5.1. *In a financial market $\mathcal{M} = (M^n_S, \overset{t}{\nabla}, \{\phi_t\}; F)$, if a one-parameter transformation ϕ_s satisfying $\phi_s(\overset{t}{\nabla}) \hat{=} \overset{p}{\nabla}$ is a projective conformal semi-symmetric metric recurrent connection transformation, then there exists an invariant $\overset{t}{\overline{W}}{}^l_{ijk}$ satisfying the following*

$$\overset{t}{\overline{W}}{}^l_{ijk} = \overset{t}{R}{}^l_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik}) - \frac{1}{n^2-1}[\delta^l_i(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) - \delta^l_j(\overset{t}{R}_{ik} - \overset{t}{R}_{ki}) - (n-1)\delta^l_k(\overset{t}{R}_{ij} - \overset{t}{R}_{ji})] \tag{5.1}$$

such that $\mathcal{M} = (M^n_S, \overset{t}{\nabla}, \{\phi_t\}; F)$ is of geometric no-arbitrage (where F is the friction factor in the market, M^n is a manifold with dimension n and S is the asset prices process).

Proof. One can directly see [23] for the proof of Theorem 5.1. □

Remark 5.1. *Following [23], one gets that the invariant $\overset{t}{\overline{W}}{}^l_{ijk}$ can be regarded as an option up to $\{\phi_s\}_{s\geq 0}$ such that $\phi_s(\overset{t}{\overline{W}}{}^l_{ijk})$, which satisfies the corresponding option pricing equation, is exactly the pricing formula of this class of options.*

Following the work posed by L. Csillag in [4], we can consider Schröinger connections associated with projective conformal semi-symmetric metric recurrent connections (briefly, PCSMR connection) and arrive at similarly the generalized Friedmann Equations and obtain the De Sitter Solution, we will prove this fact in our next manuscript and omit it here.

6. Ackonowedement

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