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Fixed point results via O-F_ϱ contraction and applications to Fredholm and integro-differential equations

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Abstract. In this article, we introduce the new concept of an O-F_ϱ contraction on O-controlled O-b-Branciari metric type spaces. We furnish the validity of our findings with appropriate examples. This approach is completely new and will be beneficial for the future aspects of the related study. We provide some applications of Fredholm integral equation and an integro-differential equation to illustrate the usability of our theory.

1. Introduction

Banach [1] introduced one of the most essential Banach contraction principle (see several applications, e.g. [2–4] and reference therein). In 1993, Bakhtin [5] and Czerwik [6] introduced the notion of b-MS (b-metric space) by changing the triangle inequality as a development of metric space with a constant $t > 1$, readers can refer to numbers [7–12]. Recently, Kamran et al. [13] introduced the concept of extended b-metric space, in which the constant t was replaced by a non-negative function $\theta(\wp, \ell)$, where the variables $\wp$ and ℓ depend on the triangle inequality's left-hand side. He also helped to extend the b-MS and develop fixed point theorems for different types of contractions. More information on extended b-MS and EBb-DS (extended Branciari b-distance space) can be found in [14–16].

Mlaiki et al. [17] introduced a controlled metric type space (CMS), which is an expansion of b-MS, in 2018. Abdeljawad et al. [18], established the concept of double CMS by modifying CMS through two control functions, $\aleph(\wp, \ell)$ and $\mu(\wp, \ell)$, the parameters of which depend on the equation's right side, see [14–16, 19–21].

The orthogonal set (O_{set}) and orthogonal metric space notions were provided in 2017 by Gordji et al. [22]. Several authors have explored the orthogonal contractive type maps, and interesting results have been found in [23–30]. The novel idea of orthogonal Branciari metric space with the orthogonal L-contraction

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mapping was introduced in 2022 by Mukheimer et al. [31]. The study of novel generalized orthogonal Branciari metric spaces has recently attracted a lot of interest in fixed point theory (see [32–35]).

In this article, we present an O-F_{ϱ}-contraction and prove the unique fixed point theorems on O-controlled-b-Branciari metric spaces. Moreover, some examples and an applications to Fredholm integral equation and integro-differential equations are provided to illustrate the usability of the obtained results.

2. Preliminaries

We start with some fundamental definitions that will be used in the sequel. Mlaiki et al. [17] recently imitated a new type of CMS, which is as follows:

Definition 2.1. ([17]) *Let $\eth \neq \emptyset$ and $\aleph : \eth \times \eth \to [1, \infty)$. A function $\eth_{\aleph} : \eth \times \eth \to [0, \infty)$ is said to be a controlled metric type if*

1. $\eth_{\aleph}(\varsigma, \iota) = 0$ *iff* $\varsigma = \iota$;
2. $\eth_{\aleph}(\varsigma, \iota) = \eth_{\aleph}(\iota, \varsigma)$;
3. $\eth_{\aleph}(\varsigma, \iota) \leq \aleph(\varsigma, \mathfrak{w})\eth_{\aleph}(\varsigma, \mathfrak{w}) + \aleph(\mathfrak{w}, \iota)\eth_{\aleph}(\mathfrak{w}, \iota)$

for all $\varsigma, \iota, \mathfrak{w} \in \eth$. The pair $(\eth, \eth_{\aleph})$ is called a CMS.

Abdeljawad et al. [12] revealed the concept of an EBb-D as follows:

Definition 2.2. ([12]) *Let $\eth \neq \emptyset$ and a map $\mathfrak{z} : \eth \times \eth \to [1, \infty)$. we say that a mapping $\eth_{\mathfrak{z}} : \eth \times \eth \to [0, \infty)$ is said to be an EBb-D if*

1. $\eth_{\mathfrak{z}}(\varsigma, \iota) = 0$ *iff* $\varsigma = \iota$;
2. $\eth_{\mathfrak{z}}(\varsigma, \iota) = \eth_{\mathfrak{z}}(\iota, \varsigma)$;
3. $\eth_{\mathfrak{z}}(\varsigma, \iota) \leq \mathfrak{z}(\varsigma, \iota)[\eth_{\mathfrak{z}}(\varsigma, \mathfrak{r}) + \eth_{\mathfrak{z}}(\mathfrak{r}, \mathfrak{t}) + \eth_{\mathfrak{z}}(\mathfrak{t}, \iota)]$

for all $\varsigma, \iota \in \eth$ and all distinct $\mathfrak{r}, \mathfrak{t} \in \eth$.

In 2012, Wardkowski [36] initiated by the concept of F-contraction as below:

Definition 2.3. ([36]) *Let $(\eth, \eth)$ be a metric space. A function $\mathfrak{I} : \eth \to \eth$ is called a F-contraction if $\exists\ \tau > 0$ s.t. $\forall\ \varsigma, \iota \in \eth$,*

$$\eth(\mathfrak{I}\varsigma, \mathfrak{I}\iota) > 0 \Rightarrow \tau + F(\eth(\mathfrak{I}\varsigma, \mathfrak{I}\iota)) \leq F(\eth(\varsigma, \iota))$$

where a mapping $F : [0, \infty) \to (-\infty, +\infty)$ are satisfies the following condition:

1. *F is strictly increasing,that is, for all $\mathcal{Z}, \pounds \in [0, \infty)$ s.t. $\mathcal{Z} < \pounds$ implies $F(\mathcal{Z}) < F(\pounds)$;*
2. *For each sequence $\{\mathcal{Z}_n\}_{\varpi \in \mathbb{N}}$ of positive numbers* $\lim_{\varpi \to \infty} \mathcal{Z}_{\varpi} = 0$ *iff* $\lim_{\varpi \to \infty} F(\mathcal{Z}_{\varpi}) = -\infty$;
3. *There exists $\mathfrak{k} \in (0, 1)$ s.t.* $\lim_{\mathcal{Z} \to 0^+} \mathcal{Z}^{\mathfrak{k}} F(\mathcal{Z}) = 0$.

The new family of functions was defined by Hussain et al. [10].

Definition 2.4. ([10]) *Let $\triangle_{\pounds}$ be the non-empty set and a mapping $\pounds : [0, \infty) \to [0, \infty)$ fulfill the following:*
($\pounds_1$) $\lim_{\mathrm{i} \to \infty} \inf \pounds(\varsigma_{\mathrm{i}}) > 0,\ \forall\ (\varsigma_{\mathrm{i}})$ *be a sequence with* $(\varsigma_{\mathrm{i}}) > 0$;
It is worth nothing that $\pounds_1$ indicates:
($\pounds_2$) $\sum_{\mathrm{i}=0}^{\infty} \pounds(\varsigma_{\mathrm{i}}) = +\infty$, *for every sequence* (ς_{i}) *with* $\varsigma_{\mathrm{i}} > 0$.

Gordji et al. [22] presented the basic definitions of orthogonality as follows:

Definition 2.5. ([22]) *Let $\eth$ be non-void and $\perp \subseteq \eth \times \eth$ be an binary relation. If $\perp$ fulfills the following condition:*

$$\exists\, \wp_0 : (\forall\, \ell, \ell \perp \wp_0) \ \ or \ \ (\forall\, \ell, \wp_0 \perp \ell),$$

then $(\eth, \perp)$ is called an O_{set}.

Definition 2.6. ([35]) *Let $(\eth, \perp, \eth)$ be an O-B_bMS, if $(\eth, \perp)$ is an O_{set} and $(\eth, \eth)$ is a b-metric space.*

Definition 2.7. ([22]) *Let $(\eth, \perp, \eth)$ be an O-B_bMS.*

(1) Then $\varrho \colon \eth \to \eth$ is said to be orthogonally continuous in $\natural \in \eth$ if for each O_{seq} (orthogonal sequence) $\{\natural_\varpi\}_{\varpi\in\mathbb{N}}$ in $\eth$ with $\natural_\varpi \to \natural$, we have $\varrho(\natural_\varpi) \to \varrho(\natural)$. Also, ϱ is said to be orthogonal continuous on $\eth$ if ϱ is orthogonal continuous in each $\natural \in \eth$.

(2) Then $\eth$ is said to be orthogonally complete if every Cauchy O_{seq} is convergent.

(3) A function $\varrho \colon \eth \to \eth$ is said to be orthogonal preserving if $\varrho(\wp) \perp \varrho(\ell)$ if $\wp \perp \ell$.

We modify the concept of F_ϱ contraction to orthogonal sets in this article. To illustrate our results, we also give some examples and application.

3. Main results

Inspired by the F_ϱ contraction mappings defined by Zubair et al. [37], we implement a new orthogonally F_ϱ-contraction mapping and demonstrate some fixed point theorems in an O-complete O-controlled b-Branciari metric space for this contraction mapping.

Definition 3.1. *Let $\eth \neq \emptyset$ set and $\varrho : \eth \times \eth \to [1, \infty)$. A mapping $\eth_\varrho : \eth \times \eth \to [0, \infty)$ is called an O-C_bBM if it satisfies:*

1. $\eth_\varrho(\varsigma, \iota) = 0 \iff \varsigma = \iota$;
2. $\eth_\varrho(\varsigma, \iota) = \eth_\varrho(\iota, \varsigma)$;
3. $\eth_\varrho(\varsigma, \iota) \leq \varrho(\varsigma, \mathfrak{r})\eth_\varrho(\varsigma, \mathfrak{r}) + \varrho(\mathfrak{r}, \mathfrak{t})\eth_\varrho(\mathfrak{r}, \mathfrak{t}) + \varrho(\mathfrak{t}, \iota)\eth_\varrho(\mathfrak{t}, \iota)$,

for all $\varsigma, \iota, \mathfrak{r}, \mathfrak{t} \in \eth$ with $\varsigma \perp \iota, \iota \perp \mathfrak{r}, \mathfrak{r} \perp \mathfrak{t}$ and all distinct points $\mathfrak{r}, \mathfrak{t} \in \eth$, each distinct from ς and ι respectively. The pair $(\eth, \perp, \eth_\varrho)$ is called an O-C_bBMS.

Now in the sense of *O-C_bBMS*, we use the potentially generated definitions:

Definition 3.2. *Let $(\eth, \perp, \eth_\varrho)$ be an O-C_bBMS. Let $\{\varsigma_\varpi\}$ be an O_{seq} in $\eth$. We say that*

1. $\{\varsigma_\varpi\}$ *is a convergent, if* $\lim_{\varpi\to\infty} \eth_\varrho(\varsigma_\varpi, \varsigma) = 0$ *for some* $\varsigma \in \eth$.
2. $\{\varsigma_\varpi\}$ *is an orthogonal Cauchy, if* $\lim_{\varpi,\varpi\to\infty} \eth_\varrho(\varsigma_\mathsf{T}, \varsigma_\varpi) = 0$.
3. $(\eth, \eth_\varrho)$ *is an O-complete O-C_bBMS if every Cauchy O_{seq} is convergent in $\eth$.*

Definition 3.3. *Let $(\eth, \perp, \eth_\varrho)$ be an O-C_bBMS. A mapping $\mathfrak{I} : \eth \to \eth$ is called an O- F_ϱ-contraction if $\exists$ £ $\in \triangle_£$ be a function s.t.*

$$\eth_\varrho(\mathfrak{I}\varsigma, \mathfrak{I}\iota) > 0 \Rightarrow £(\eth_\varrho(\varsigma, \iota)) + F_\varrho(\eth_\varrho(\mathfrak{I}\varsigma, \mathfrak{I}\iota)) \leq F_\varrho(\eth_\varrho(\varsigma, \iota)), \tag{1}$$

with $\varsigma \perp \iota$, $\forall\, \varsigma, \iota \in \eth$ s.t. for each

$$\varsigma_0 \in \eth, \sup_{\mathsf{T}\geq 1} \lim_{i\to\infty} \varrho(\varsigma_{i+1}, \varsigma_{i+2})\varrho(\varsigma_{i+1}, \varsigma_\mathsf{T}) < \frac{1}{\lambda},$$

where $\varsigma_\varpi = \mathfrak{I}^\varpi \varsigma_0, \varpi = 0, 1, ..., \lambda \in (0, 1)$ and $F_\varrho : [0, \infty) \to (-\infty, +\infty)$ is a mapping satisfying:

(F_1) *F_ϱ is strictly increasing,that is, for all $\mathcal{Z}, £ \in [0,\infty)$ s.t. $\mathcal{Z} < £$ implies $F_\varrho(\mathcal{Z}) < F_\varrho(£)$;*
(F_2) *for each sequence $\{\mathcal{Z}_n\}_{\varpi\in\mathbb{N}}$ of positive numbers*

$$\lim_{\varpi\to\infty} \mathcal{Z}_\varpi = 0 \iff \lim_{\varpi\to\infty} F_\varrho(\mathcal{Z}_\varpi) = -\infty;$$

(F_3) *there exists $\lambda \in (0,1)$ s.t. $\lim_{\mathcal{Z}\to 0^+} \mathcal{Z}^\lambda F_\varrho(\mathcal{Z}) = 0$.*

We denote by F_ϱ, the set of all functions satisfying $(F_1) - (F_3)$.

Definition 3.4. *Let $(\eth, \perp, \eth_\varrho)$ be an O-C_bBMS. A mapping $\Im : \eth \to \eth$ is said to be an extended O-F_ϱ-contraction if $\exists\, £ \in \Delta_£$ s.t.*

$$\begin{aligned} &\eth_\varrho(\Im\varsigma, \Im\iota) > 0 \implies \\ &\quad £(\eth_\varrho(\varsigma,\iota)) + F_\varrho(\eth_\varrho(\Im\varsigma, \Im\iota)) \le F_\varrho\Big(\gamma_1 \eth_\varrho(\varsigma,\iota) + \gamma_2 \frac{\eth_\varrho(\varsigma, \Im\varsigma)}{1+\eth_\varrho(\varsigma,\Im\varsigma)} \\ &\qquad + \gamma_3 \frac{\eth_\varrho(\iota,\Im\iota)}{1+\eth_\varrho(\iota,\Im\iota)} + \gamma_4 \frac{\eth_\varrho(\varsigma,\Im\varsigma)\eth_\varrho(\iota,\Im\iota)}{\eth_\varrho(\varsigma,\iota)+\eth_\varrho(\varsigma,\Im\iota)+\eth_\varrho(\iota,\Im\varsigma)}\Big), \ \forall\, \varsigma,\iota \in \eth. \end{aligned} \tag{2}$$

where $F_\varrho \in \mathbb{F}_\varrho, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \ge 0$ satisfying $\gamma = \gamma_1+\gamma_2+\gamma_3+\gamma_4 < 1$. In addition, for each $\varsigma_0 \in \eth$, we have

$$\sup_{\tau\ge 1} \lim_{i\to\infty} \varrho(\varsigma_{i+1}, \varsigma_{i+2}) \varrho(\varsigma_{i+1}, \varsigma_\tau) < \frac{1}{\gamma},$$

here $\varsigma_\varpi = \Im^\varpi \varsigma_0, \varpi = 0, 1, ...$

Theorem 3.5. *Let $(\eth, \perp, \eth_\varrho)$ be an O-complete O-C_bBMS s.t. $\eth_\varrho$ is an orthogonal continuous function and $\Im : \eth \to \eth$ is an O-F_ϱ-contraction, $\Im$-continuous and $\Im$ orthogonal preserving. Moreover, if*

$$\lim_{\varpi\to\infty} \varrho(\varsigma_\varpi, \varsigma) \text{ and } \lim_{\varpi\to\infty} \varrho(\varsigma, \varsigma_\varpi), \tag{3}$$

exist and are finite, for every $\varsigma \in \eth$.Then, $\Im$ has a ufp (unique fixed point) in $\eth$.

Proof. By the definition of orthogonality, there exists an orthogonal element $\varsigma_0 \in \eth$ s.t.

$$\forall\, \ell \in \eth, \varsigma_0 \perp \ell \text{ or } \ell \perp \varsigma_0.$$

It follows that $\varsigma_0 \perp \Im(\varsigma_0)$ or $\Im(\varsigma_0) \perp \varsigma_0$. Let

$$\Im\varsigma_0 = \varsigma_1, \Im\varsigma_1 = \varsigma_2 \Rightarrow \varsigma_2 = \Im^2\varsigma_0, ..., \varsigma_{\varpi+1} = \Im^{\varpi+1}\varsigma_0,$$

for all $\varpi \in \mathbb{N}$. Since $\Im$ is an orthogonal preserving. $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}}$ is an O_{seq}.

If $\exists$ ‘$\mathfrak{k}_0 \in \mathbb{N}$ s.t. $\varsigma_{\mathfrak{k}_0} = \varsigma_{\mathfrak{k}_{0+1}}$, then $\varsigma_{\mathfrak{k}_0}$ is a fixed point of $\Im$. We now presume that $\varsigma_\varpi \ne \varsigma_{\varpi+1}\ \forall\, \varpi \ge 0$. This yields $\eth_\varrho(\varsigma_\varpi, \varsigma_{\mathfrak{n}+g}) > 0$, that is, $\eth_\varrho(\Im\varsigma_{\varpi-1}, \Im\varsigma_\varpi) > 0$. The evidence will now be broken down into four parts.

Step 1: The first step is prove

$$\lim_{\varpi\to\infty} \eth_\varrho(\varsigma_\varpi, \varsigma_{\mathfrak{n}+g}) = 0 \text{ and } \lim_{\varpi\to\infty} \eth_\varrho(\varsigma_\varpi, \varsigma_{\mathfrak{n}+}) = 0.$$

Taking $\varsigma = \varsigma_{\varpi-1}$ and $\iota = \varsigma_\varpi$ in (1), we get

$$£(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)) + F_\varrho(\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})) \le F_\varrho(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)). \tag{4}$$

Consequently, we have

$$F_\varrho(\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})) \le F_\varrho(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)) - £(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi))$$

$$\leq F_\varrho(\eth_\varrho(\varsigma_{\varpi-2},\varsigma_{\varpi-1})) - \pounds(\eth_\varrho(\varsigma_{\varpi-2},\varsigma_{\varpi-1})) - \pounds(\eth_\varrho(\varsigma_{\varpi-1},\varsigma_{\varpi})) \tag{5}$$
$$\vdots$$
$$\leq F_\varrho(\eth_\varrho(\varsigma,\varsigma_g)) - \sum_{i=1}^{\varpi} \pounds(\eth_\varrho(\varsigma_{i-1},\varsigma_i)).$$

By using ($\pounds_2$), we get

$$\lim_{\varpi\to\infty} F_\varrho(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1})) = -\infty, \tag{6}$$

which implies that

$$\lim_{\varpi\to\infty} \eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}) = 0. \tag{7}$$

From (F_3), there exists $\lambda \in (0,1)$ s.t.

$$\lim_{\varpi\to\infty} (\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda F_\varrho(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1})) = 0. \tag{8}$$

By (5), we have

$$\begin{aligned}(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda F_\varrho(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1})) - (\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda F_\varrho(\eth_\varrho(\varsigma,\varsigma_g))\\ \leq -(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda \sum_{i=1}^{\varpi} \pounds(\eth_\varrho(\varsigma_{i-1},\varsigma_i)).\end{aligned} \tag{9}$$

By ($\pounds_1$), there exists $\complement > 0$ s.t.

$$\pounds(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1})) > \complement, \ \forall\, \varpi > \varpi_0.$$

Subsequently, we get

$$\begin{aligned}&(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda F_\varrho(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1})) - (\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda F_\varrho(\eth_\varrho(\varsigma,\varsigma_g))\\ &\leq -(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda \sum_{i=1}^{\varpi} \pounds(\eth_\varrho(\varsigma_{i-1},\varsigma_i)) = (\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda\\ &\Big(-[\pounds(\eth_\varrho(\varsigma,\varsigma_g)) + \pounds(\eth_\varrho(\varsigma_g,\varsigma)) + ... + \pounds(\eth_\varrho(\varsigma_{\varpi-g},\varsigma_\varpi))]\\ &- [\pounds(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+g})) + ... + \pounds(\eth_\varrho(\varsigma_{\varpi-g},\varsigma_\varpi))]\Big)\\ &\leq -(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda(\varpi-\varpi_0)\complement.\end{aligned} \tag{10}$$

Letting $\varpi \to \infty$ in (10), we obtain

$$\lim_{\varpi\to\infty} \varpi(\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}))^\lambda = 0. \tag{11}$$

Then there exists $\varpi_1 \in \mathbb{N}$ s.t.

$$\varpi[\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1})]^\lambda \leq 1 \ \ \forall\, \varpi \geq \varpi_1.$$

Thus, we acquire

$$\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1}) \leq \frac{1}{\varpi^{\frac{1}{\lambda}}}. \tag{12}$$

Again taking $\varsigma = \varsigma_{\varpi-1}$ and $\iota = \varsigma_{\varpi+1}$ in (1), we get

$$\pounds(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+1})) + F_\varrho(\eth_\varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2})) \leq F_\varrho(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+1})). \tag{13}$$

Accordingly, we have

$$F_\varrho(\eth_\varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2})) \leq F_\varrho(\eth_\varrho(\varsigma, \varsigma)) - \sum_{i=1}^{\varpi} \pounds(\eth_\varrho(\varsigma_{i-1}, \varsigma_{i+1})). \tag{14}$$

By using ($\pounds_2$), we get

$$\lim_{\varpi\to\infty} F_\varrho(\eth_\varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2})) = -\infty, \tag{15}$$

which implies

$$\lim_{\varpi\to\infty} \eth_\varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2}) = 0. \tag{16}$$

Step 2: Now, we will take $\varsigma_\varpi \neq \varsigma_\top$, for $\varpi \neq \top$. Suppose, we take $\varsigma_\varpi = \varsigma_\top$ for some $\varpi = \top + 1 > \top$, we have

$$\varsigma_{\varpi+1} = \Im\varsigma_\varpi = \Im\varsigma_\top = \varsigma_{\top+1}.$$

Inequality (1), therefore implies that

$$\begin{aligned}
F_\varrho(\eth_\varrho(\varsigma_\top, \varsigma_{\top+1})) = F_\varrho(\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})) &= F_\varrho(\eth_\varrho(\Im\varsigma_{\varpi-1}, \Im\varsigma_\varpi)) \\
&\leq F_\varrho(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)) - \pounds(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)) \\
&< F_\varrho(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)) \\
&= F_\varrho(\eth_\varrho(\Im\varsigma_{\varpi-2}, \Im\varsigma_{\varpi-1})) \\
&\leq F_\varrho(\eth_\varrho(\varsigma_{\varpi-2}, \varsigma_{\varpi-1})) - \pounds(\eth_\varrho(\varsigma_{\varpi-2}, \varsigma_{\varpi-1})) \\
&\ \vdots \\
&< F_\varrho(\eth_\varrho(\varsigma_\top, \varsigma_{\top+1}))
\end{aligned}$$

this is a contradiction. Hence, we get $\varsigma_\varpi \neq \varsigma_\top$, for all $\varpi \neq \top$.

Step 3: In this step, we prove that $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}}$ is a Cauchy O_{seq} that is,

$$\lim_{\varpi\to\infty} \eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+\mathfrak{a}}) = 0, \text{ for all } \mathfrak{a} \in \mathbb{N}.$$

We have previously proved for the cases $\mathfrak{a} = 1$ and $\mathfrak{a} = 2$, respectively. Let us choose $\mathfrak{a} \geq 1$ arbitrary. We split the two cases.

Case 1: Let $\mathfrak{a} = 2\top$, where $\top \geq 2$, we get

$$\begin{aligned}
\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+2\top}) &\leq \varrho(\varsigma_\varpi, \varsigma_{\varpi+2})\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+3})\eth_\varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+3}) \\
&\quad + \varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2\top})\eth_\varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2\top}) \\
&\leq \varrho(\varsigma_\varpi, \varsigma_{\varpi+2})\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+3})\eth_\varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+3}) + \varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2\top}) \\
&\quad [\varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+4})\eth_\varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+4}) + \varrho(\varsigma_{\varpi+4}, \varsigma_{\varpi+5})\eth_\varrho(\varsigma_{\varpi+4}, \varsigma_{\varpi+5}) \\
&\quad + \varrho(\varsigma_{\varpi+5}, \varsigma_{\varpi+2\top})\eth_\varrho(\varsigma_{\varpi+5}, \varsigma_{\varpi+2\top})] \\
&\quad \vdots
\end{aligned}$$

$$
\begin{aligned}
&\leq \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+3})\eth_{\varrho}(\varsigma_{\varpi+2}, \varsigma_{\varpi+3}) + \varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2_{\mathrm{T}}}) \\
&\quad [\varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+4})\eth_{\varrho}(\varsigma_{\varpi+3}, \varsigma_{\varpi+4}) + \varrho(\varsigma_{\varpi+4}, \varsigma_{\varpi+5})\eth_{\varrho}(\varsigma_{\varpi+4}, \varsigma_{\varpi+5})] + \\
&\quad \vdots \\
&\quad \varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2_{\mathrm{T}}})\varrho(\varsigma_{\varpi+5}, \varsigma_{\varpi+2_{\mathrm{T}}})\ldots\varrho(\varsigma_{\varpi+2_{\mathrm{T}}-3}, \varsigma_{\varpi+2_{\mathrm{T}}})[\varrho(\varsigma_{\varpi+2_{\mathrm{T}}-3}, \varsigma_{\varpi+2_{\mathrm{T}}-2}) \\
&\quad \eth_{\varrho}(\varsigma_{\varpi+2_{\mathrm{T}}-3}, \varsigma_{\varpi+2_{\mathrm{T}}-2}) + \varrho(\varsigma_{\varpi+2_{\mathrm{T}}-2}, \varsigma_{\varpi+2_{\mathrm{T}}-1})\eth_{\varrho}(\varsigma_{\varpi+2_{\mathrm{T}}-2}, \varsigma_{\varpi+2_{\mathrm{T}}-1})] \\
&\quad + \varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2_{\mathrm{T}}})\varrho(\varsigma_{\varpi+5}, \varsigma_{\varpi+2_{\mathrm{T}}})\ldots\varrho(\varsigma_{\varpi+2_{\mathrm{T}}-1}, \varsigma_{\varpi+2_{\mathrm{T}}})\eth_{\varrho}(\varsigma_{\varpi+2_{\mathrm{T}}-1}, \varsigma_{\varpi+2_{\mathrm{T}}}) \\
&\leq \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2}) + \sum_{i=\varpi+2}^{\varpi+2_{\mathrm{T}}-2} \eth_{\varrho}(\varsigma_i, \varsigma_{i+1}) \prod_{j=1}^{i} \varrho(\varsigma_j, \varsigma_{\varpi+2_{\mathrm{T}}})\varrho(\varsigma_i, \varsigma_{i+1}) \\
&\quad + \prod_{i=1}^{\varpi+2_{\mathrm{T}}-1} \varrho(\varsigma_i, \varsigma_{\varpi+2_{\mathrm{T}}})\eth_{\varrho}(\varsigma_{\varpi+2_{\mathrm{T}}-1}, \varsigma_{\varpi+2_{\mathrm{T}}}) \\
&\leq \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2}) + \sum_{i=\varpi+2}^{\varpi+2_{\mathrm{T}}-1} \eth_{\varrho}(\varsigma_i, \varsigma_{i+1}) \prod_{j=1}^{i} \varrho(\varsigma_j, \varsigma_{\varpi+2_{\mathrm{T}}})\varrho(\varsigma_i, \varsigma_{i+1}).
\end{aligned}
$$

We obtain the series

$$
\sum_{\varpi=1}^{\infty} \eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) \prod_{i=1}^{\varpi} \varrho(\varsigma_i, \varsigma_{\varpi+2_{\mathrm{T}}})\varrho(\varsigma_i, \varsigma_{i+1}),
$$

converges. Since,

$$
\begin{aligned}
\sum_{\varpi=1}^{\infty} \eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) \prod_{i=1}^{\varpi} \varrho(\varsigma_i, \varsigma_{\varpi+2_{\mathrm{T}}})\varrho(\varsigma_i, \varsigma_{i+1}) &\leq \sum_{\varpi=1}^{\infty} \frac{1}{\varpi^{\frac{1}{\lambda}}} \prod_{i=1}^{\varpi} \varrho(\varsigma_i, \varsigma_{\varpi+2_{\mathrm{T}}})\varrho(\varsigma_i, \varsigma_{i+1}) \\
&< \frac{1}{\lambda} \sum_{\varpi=1}^{\infty} \frac{1}{\varpi^{\frac{1}{\lambda}}},
\end{aligned}
$$

which is convergent. Let

$$
\begin{aligned}
\mathcal{Y} &= \sum_{\varpi=1}^{\infty} \eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) \prod_{i=1}^{\varpi} \varrho(\varsigma_i, \varsigma_{\varpi+2_{\mathrm{T}}})\varrho(\varsigma_{\varpi}, \varsigma_{\varpi+1}) \\
\mathcal{Y}_{\varpi} &= \sum_{j=1}^{\varpi} \eth_{\varrho}(\varsigma_j, \varsigma_{j+1}) \prod_{i=1}^{j} \varrho(\varsigma_i, \varsigma_{\varpi+2_{\mathrm{T}}})\varrho(\varsigma_j, \varsigma_{j+1}).
\end{aligned}
$$

Hence, we have

$$
\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2_{\mathrm{T}}}) \leq \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2}) + \mathcal{Y}_{\varpi+2_{\mathrm{T}}-1} - \mathcal{Y}_{\varpi+1}.
$$

Letting $\varpi \to \infty$ and using Equation (16), we simplify that

$$
\lim_{\varpi\to\infty} \eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2_{\mathrm{T}}}) = 0. \tag{17}
$$

Case 2: Let $\mathfrak{a} = 2\,\mathrm{T} + 1$, where $\mathrm{T} \geq 1$. Then, we find

$$
\begin{aligned}
&\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2_{\mathrm{T}}+1}) \\
&\quad \leq \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+1})\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) + \varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi+1}, \varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+2_{\mathrm{T}}+1})\eth_{\varrho}(\varsigma_{\varpi+2}, \varsigma_{\varpi+2_{\mathrm{T}}+1}) \\
&\quad \leq \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+1})\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) + \varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi+1}, \varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+2_{\mathrm{T}}+1})
\end{aligned}
$$

$$
\begin{aligned}
&[\varrho(\varsigma_{\varpi+2},\varsigma_{\varpi+3})\eth_{\varrho}(\varsigma_{\varpi+2},\varsigma_{\varpi+3})+\varrho(\varsigma_{\varpi+3},\varsigma_{\varpi+4})\eth_{\varrho}(\varsigma_{\varpi+3},\varsigma_{\varpi+4})+\varrho(\varsigma_{\varpi+4},\varsigma_{\varpi+2_{\mathsf{T}}+1})\\
&\eth_{\varrho}(\varsigma_{\varpi+4},\varsigma_{\varpi+2_{\mathsf{T}}+1})]\\
&\vdots\\
&\leq \varrho(\varsigma_{\varpi},\varsigma_{\varpi+1})\eth_{\varrho}(\varsigma_{\varpi},\varsigma_{\varpi+1})+\varrho(\varsigma_{\varpi+1},\varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi+1},\varsigma_{\varpi+2})+\varrho(\varsigma_{\varpi+2},\varsigma_{\varpi+2_{\mathsf{T}}+1})\\
&\quad[\varrho(\varsigma_{\varpi+2},\varsigma_{\varpi+3})\eth_{\varrho}(\varsigma_{\varpi+2},\varsigma_{\varpi+3})+\varrho(\varsigma_{\varpi+3},\varsigma_{\varpi+4})\eth_{\varrho}(\varsigma_{\varpi+3},\varsigma_{\varpi+4})]\\
&\quad+\varrho(\varsigma_{\varpi+2},\varsigma_{\varpi+2_{\mathsf{T}}+1})\varrho(\varsigma_{\varpi+4},\varsigma_{\varpi+2_{\mathsf{T}}+1})\ldots\varrho(\varsigma_{\varpi+2_{\mathsf{T}}-2},\varsigma_{\varpi+2_{\mathsf{T}}+1})[\varrho(\varsigma_{\varpi+2_{\mathsf{T}}-2},\varsigma_{\varpi+2_{\mathsf{T}}-1})\\
&\quad\eth_{\varrho}(\varsigma_{\varpi+2_{\mathsf{T}}-2},\varsigma_{\varpi+2_{\mathsf{T}}-1})+\varrho(\varsigma_{\varpi+2_{\mathsf{T}}-1},\varsigma_{\varpi+2_{\mathsf{T}}})\eth_{\varrho}(\varsigma_{\varpi+2_{\mathsf{T}}-1},\varsigma_{\varpi+2_{\mathsf{T}}})]\\
&\quad+\varrho(\varsigma_{\varpi+2_{\mathsf{T}}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\eth_{\varrho}(\varsigma_{\varpi+2_{\mathsf{T}}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\\
&\leq \sum_{\mathrm{i}=\varpi}^{\varpi+2_{\mathsf{T}}-1}\eth_{\varrho}(\varsigma_{\mathrm{i}},\varsigma_{\mathrm{i}+1})\prod_{\mathrm{j}=1}^{\mathrm{i}}\varrho(\varsigma_{\mathrm{j}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\varrho(\varsigma_{\mathrm{i}},\varsigma_{\mathrm{i}+1})+\prod_{\mathrm{i}=1}^{\varpi+2_{\mathsf{T}}}\varrho(\varsigma_{\mathrm{i}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\eth_{\varrho}(\varsigma_{\varpi+2_{\mathsf{T}}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\\
&\leq \sum_{\mathrm{i}=\varpi}^{\varpi+2_{\mathsf{T}}}\eth_{\varrho}(\varsigma_{\mathrm{i}},\varsigma_{\mathrm{i}+1})\prod_{\mathrm{j}=1}^{\mathrm{i}}\varrho(\varsigma_{\mathrm{j}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\varrho(\varsigma_{\mathrm{i}},\varsigma_{\mathrm{i}+1}).
\end{aligned}
$$

Note that the series

$$\sum_{\varpi=1}^{\infty}\eth_{\varrho}(\varsigma_{\varpi},\varsigma_{\varpi+1})\prod_{\mathrm{i}=1}^{\varpi}\varrho(\varsigma_{\mathrm{i}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\varrho(\varsigma_{\mathrm{i}},\varsigma_{\mathrm{i}+1}),$$

converges. Since

$$
\begin{aligned}
\sum_{\varpi=1}^{\infty}\eth_{\varrho}(\varsigma_{\varpi},\varsigma_{\varpi+1})\prod_{\mathrm{i}=1}^{\varpi}\varrho(\varsigma_{\mathrm{i}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\varrho(\varsigma_{\mathrm{i}},\varsigma_{\mathrm{i}+1}) &\leq \sum_{\varpi=1}^{\infty}\frac{1}{\varpi^{\frac{1}{\lambda}}}\prod_{\mathrm{i}=1}^{\varpi}\varrho(\varsigma_{\mathrm{i}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\varrho(\varsigma_{\mathrm{i}},\varsigma_{\mathrm{i}+1})\\
&\frac{1}{\lambda}\sum_{\varpi=1}^{\infty}\frac{1}{\varpi^{\frac{1}{\lambda}}},
\end{aligned}
$$

which is convergent. Let

$$
\begin{aligned}
\mathcal{Z} &= \sum_{\varpi=1}^{\infty}\eth_{\varrho}(\varsigma_{\varpi},\varsigma_{\varpi+1})\prod_{\mathrm{i}=1}^{\varpi}\varrho(\varsigma_{\mathrm{i}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\varrho(\varsigma_{\varpi},\varsigma_{\varpi+1})\\
\mathcal{Z}_{\varpi} &= \sum_{\mathrm{j}=1}^{\varpi}\eth_{\varrho}(\varsigma_{\mathrm{j}},\varsigma_{\mathrm{j}+1})\prod_{\mathrm{i}=1}^{\mathrm{j}}\varrho(\varsigma_{\mathrm{i}},\varsigma_{\varpi+2_{\mathsf{T}}+1})\varrho(\varsigma_{\mathrm{j}},\varsigma_{\mathrm{j}+1}).
\end{aligned}
$$

Thereby, the preceding inequality clearly indicates:

$$\eth_{\varrho}(\varsigma_{\varpi,\varsigma_{\varpi}+2_{\mathsf{T}}+1)}\leq\mathcal{Z}_{\varpi+2_{\mathsf{T}}}-\mathcal{Z}_{\varpi-1}.$$

Letting $\varpi\to\infty$ in the above inequality, we simplify that

$$\lim_{\varpi\to\infty}\eth_{\varrho}(\varsigma_{\varpi,\varsigma_{\varpi}+2_{\mathsf{T}}+1)}=0. \tag{18}$$

Consequentially, by Equations (16) and (17), we have

$$\lim_{\varpi\to\infty}\eth_{\varrho}(\varsigma_{\varpi,\varsigma_{\varpi}+\mathfrak{a})}=0,\ \text{for all } \mathfrak{a}\in\mathbb{N}. \tag{19}$$

Hence, we infer that $\{\varsigma_{\varpi}\}$ is a Cauchy O_{seq} that is, $\{\Im^{\varpi}\varsigma\}$ is a Cauchy O_{seq}. Since $(\Game,\eth_{\varrho})$ is a O-complete O-C_bBMS, let $\varsigma_{\varpi}\to\varsigma\in\Game$.

We will now reveal that ς is a fixed point of $\Im$. Consider

$$\eth_\varrho(\varsigma, \varsigma_{\varpi+2}) \leq \varrho(\varsigma, \varsigma_\varpi)\eth_\varrho(\varsigma, \varsigma_\varpi) + \varrho(\varsigma_\varpi, \varsigma_{\varpi+1})\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) + \varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2})\eth_\varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2}).$$

Using (3) and (19), we obtain

$$\lim_{\varpi\to\infty} \eth_\varrho(\varsigma, \varsigma_{\varpi+2}) = 0. \tag{20}$$

Consider

$$\begin{aligned}\eth_\varrho(\varsigma, \Im\varsigma) &\leq \varrho(\varsigma, \varsigma_{\varpi+2})\eth_\varrho(\varsigma, \varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1})\eth_\varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1}) + \varrho(\varsigma_{\varpi+1}, \Im\varsigma)\eth_\varrho(\varsigma_{\varpi+1}, \Im\varsigma)\\ &= \varrho(\varsigma, \varsigma_{\varpi+2})\eth_\varrho(\varsigma, \varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1})\eth_\varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1}) + \varrho(\varsigma_{\varpi+1}, \Im\varsigma)\eth_\varrho(\Im^{\varpi+1}, \Im\varsigma).\end{aligned}$$

Letting $\varpi \to \infty$, we obtain $\eth_\varrho(\varsigma, \varsigma_{\varpi+2}) \to 0$ by (19). Since $\Im^\varpi\varsigma \to \varsigma$ and from the orthogonal continuity of $\Im$, $\lim_{\varpi\to\infty} \eth_\varrho(\Im^{\varpi+1}\varsigma, \Im\varsigma) = 0$. Thus,

$$\eth_\varrho(\varsigma, \Im\varsigma) = 0, \implies \varsigma = \Im\varsigma.$$

Hence ς is a fixed point of $\Im$.

Step 4: Now, we prove that ς is a ufp of $\Im$. Let ι be an another fixed point of $\Im$, then $\Im\iota = \iota \neq \varsigma = \Im\varsigma$. We have

$$[\varsigma_0 \perp \iota] \text{ or } [\iota \perp \varsigma_0].$$

Since $\Im$ is orthogonal preserving, we have

$$[\Im^\varpi(\varsigma_0) \perp \Im^\varpi(\iota)] \text{ or } [\Im^\varpi(\iota) \perp \Im^\varpi(\varsigma_0)],$$

for all $\varpi \in \mathbb{N}$. On the other hand $\Im$ is an F_ϱ-contraction. So, we get $\eth_\varrho(\varsigma, \iota) > 0$ that is, $\eth_\varrho(\Im\varsigma, \Im\iota) > 0$.
Now equation (1), implies

$$\pounds(\eth_\varrho(\varsigma, \iota)) + F_\varrho(\eth_\varrho(\Im\varsigma, \Im\iota)) \leq F_\varrho(\eth_\varrho(\varsigma, \iota)).$$

Therefore

$$\begin{aligned}\pounds(\eth_\varrho(\varsigma, \iota)) + F_\varrho(\eth_\varrho(\varsigma, \iota)) &\leq F_\varrho(\eth_\varrho(\varsigma, \iota))\\ \pounds(\eth_\varrho(\varsigma, \iota)) &\leq F_\varrho(\eth_\varrho(\varsigma, \iota)) - F_\varrho(\eth_\varrho(\varsigma, \iota))\\ &= 0\end{aligned}$$

which is a contradiction. Hence, $\Im$ has a ufp in Ͽ. □

Theorem 3.6. *Let* $(Ͽ, \eth_\varrho)$ *be a O-complete O-C_bBMS s.t.* $\eth_\varrho$ *is a continuous functional and* $\Im : Ͽ \to Ͽ$ *be an extended O-F_ϱ-contraction s.t. the following axioms are fulfill:*

1. $\Im$ *is orthogonal preserving;*
2. $\Im$ *is orthogonal continuous.*

Then, $\Im$ *has a ufp in* Ͽ.

Proof. By the definition of orthogonality, there exists an orthogonal element $\varsigma_0 \in Ͽ$ s.t.

$$\forall\, \ell \in Ͽ, \varsigma_0 \perp \ell \text{ or } \ell \perp \varsigma_0.$$

It follows that $\varsigma_0 \perp \Im(\varsigma_0)$ or $\Im(\varsigma_0) \perp \varsigma_0$. Let

$$\Im\varsigma_0 = \varsigma_1, \Im\varsigma_1 = \varsigma_2 \Rightarrow \varsigma_2 = \Im^2\varsigma_0, \ldots, \varsigma_{\varpi+1} = \Im^{\varpi+1}\varsigma_0,$$

for all $\varpi \in \mathbb{N}$. Since $\mathfrak{I}$ is an orthogonal preserving.

We define the sequence $\{\varsigma_\varpi\}$ by

$$\varsigma_0, \mathfrak{I}\varsigma_0 = \varsigma_1, \mathfrak{I}\varsigma_1 = \varsigma_2 \implies \varsigma_2 = \mathfrak{I}^2\varsigma_0, ..., \varsigma_{\varpi+1} = \mathfrak{I}^{\varpi+1}\varsigma_0.$$

If there is an $\varsigma_0 \in \mathbb{N}$ s.t. $\varsigma_{\mathrm{l}_0} = \varsigma_{\mathrm{l}_0+1}$, then ς_{l_0} is a fixed point of $\mho$. Therefore, assume that $\varsigma_\varpi \neq \varsigma_\varpi + 1 \; \forall \; \varpi \geq 0$. This yields $\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) > 0$, that is, $\eth_\varrho(\mathfrak{I}\varsigma_{\varpi-1}, \mathfrak{I}\varsigma_\varpi) > 0$.

Step 1: We will to prove

$$\lim_{\varpi\to\infty} \eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) = 0 \text{ and } \lim_{\varpi\to\infty} \eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+2}) = 0.$$

By using (20), for every $\varpi \in \mathbb{N}$, we have

$$\begin{aligned}
£(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)) + F_\varrho(\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})) &\leq F_\varrho\Big(\gamma_1\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + \gamma_2\frac{\eth_\varrho(\varsigma_{\varpi-1}, \mathfrak{I}\varsigma_{\varpi-1})}{1 + \eth_\varrho(\varsigma_{\varpi-1}, \mathfrak{I}\varsigma_{\varpi-1})} \\
&\quad + \gamma_3\frac{\eth_\varrho(\varsigma_\varpi, \mathfrak{I}\varsigma_\varpi)}{1 + \eth_\varrho(\varsigma_\varpi, \mathfrak{I}\varsigma_\varpi)} + \gamma_4\frac{\eth_\varrho(\varsigma_{\varpi-1}, \mathfrak{I}\varsigma_{\varpi-1})\eth_\varrho(\varsigma_\varpi, \mathfrak{I}\varsigma_\varpi)}{\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + \eth_\varrho(\varsigma_{\varpi-1}, \mathfrak{I}\varsigma_\varpi) + \eth_\varrho(\varsigma_\varpi, \mathfrak{I}\varsigma_{\varpi-1})}\Big) \\
&\leq F_\varrho\Big(\gamma_1\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + \gamma_2\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) \\
&\quad + \gamma_3\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) + \gamma_4\frac{\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})}{\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)}\Big) \\
&= F_\varrho\Big(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)(\gamma_1 + \gamma_2) + \eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})(\gamma_3 + \gamma_4)\Big).
\end{aligned} \tag{21}$$

This yields

$$\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) < \eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)(\gamma_1 + \gamma_2) + \eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})(\gamma_3 + \gamma_4)$$

that is,

$$(1 - \gamma_3 - \gamma_4)\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) \leq (\gamma_1 + \gamma_2)\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi).$$

As $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 < 1$, we have

$$\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) \leq \frac{\gamma_1 + \gamma_2}{1 - \gamma_3 - \gamma_4}\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) < \eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi).$$

From (21), we obtain

$$£(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)) + F_\varrho(\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})) \leq F(\eth_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)).$$

Reluctantly, we get

$$F_\varrho(\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})) \leq F_\varrho(\eth_\varrho(\varsigma_0, \varsigma_1)) - \sum_{\mathrm{i}=1}^{\varpi} £(\eth_\varrho(\varsigma_{\mathrm{i}-1}, \varsigma_\mathrm{i})).$$

By using ($£_2$), we get

$$\lim_{\varpi\to\infty} F_\varrho(\eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})) = -\infty, \tag{22}$$

which implies

$$\lim_{\varpi\to\infty} \eth_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) = 0. \tag{23}$$

Which implies in the proof of Theorem 3.5 that $\exists\, \varpi_1 \in \mathbb{N}$ and $\lambda \in (0,1)$ s.t.

$$ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) \leq \frac{1}{\varpi^{\frac{1}{\lambda}}}, \text{ for all } \varpi \geq \varpi_1.$$

Taking $\varsigma = \varsigma_{\varpi-1}$ and $\iota = \varsigma_{\varpi+1}$ in (20), we have

$$\begin{aligned}
&£(ð_\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+1})) + F_\varrho(ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+2}))\\
&\quad \leq F_\varrho\Big(\gamma_1 ð_\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+1}) + \gamma_2 \frac{ð_\varrho(\varsigma_{\varpi-1}, \Im\varsigma_{\varpi-1})}{1 + ð_\varrho(\varsigma_{\varpi-1}, \Im\varsigma_{\varpi-1})} + \gamma_3 \frac{ð_\varrho(\varsigma_{\varpi+1}, \Im\varsigma_{\varpi+1})}{1 + ð_\varrho(\varsigma_{\varpi+1}, \Im\varsigma_{\varpi+1})}\\
&\quad + \gamma_4 \frac{ð_\varrho(\varsigma_{\varpi-1}, \Im\varsigma_{\varpi-1})ð_\varrho(\varsigma_{\varpi+1}, \Im\varsigma_{\varpi+1})}{ð_\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+1}) + ð_\varrho(\varsigma_{\varpi-1}, \Im\varsigma_{\varpi+1}) + ð_\varrho(\varsigma_{\varpi+1}, \Im\varsigma_{\varpi-1})}\Big)\\
&\quad \leq F_\varrho\Big(\gamma_1 ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + \gamma_2 ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)\\
&\quad\quad + \gamma_3 ð_\varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2}) + \gamma_4 \frac{ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})}{ð_\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+1}) + ð_\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+2}) + ð_\varrho(\varsigma_{\varpi+1}, \varsigma_\varpi)}\Big)\\
&\quad \leq F_\varrho\Big(\gamma_1 ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + (\gamma_2 + \gamma_4)ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + \gamma_3 ð_\varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2})\Big).
\end{aligned} \tag{24}$$

This gives

$$\begin{aligned}
ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+2}) &\leq \gamma_1 ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + (\gamma_2 + \gamma_4)ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + \gamma_3 ð_\varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2})\\
&\leq \gamma_1[\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+3})ð_\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+3}) + \varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2})ð_\varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2})\\
&\quad + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1})ð_\varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1})] + (\gamma_2 + \gamma_4)ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + \gamma_3 ð_\varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2})\\
&\leq \gamma_1[\varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+3})[\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + \varrho(\varsigma_\varpi, \varsigma_{\varpi+2})ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+2})\\
&\quad + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1})ð_\varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1})] + \varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2})ð_\varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+2})\\
&\quad + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1})ð_\varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1})] + (\gamma_2 + \gamma_4)ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + \gamma_3 ð_\varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+2})[1 - \gamma_1 \varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+3})\varrho(\varsigma_\varpi, \varsigma_{\varpi+2})] &\leq [\gamma_2 + \gamma_4 + \gamma_1 \varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+3})\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)]ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)\\
&\quad + [\gamma_1 \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+1})(1 + \varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi-3}))]ð_\varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2})\\
&\quad + \gamma_1 \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+3})ð_\varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+3}).
\end{aligned}$$

Taking into account $\lim_{\varpi\to\infty} \varrho(\varsigma_{\varpi-1}, \varsigma_{\varpi+3})\varrho(\varsigma_\varpi, \varsigma_{\varpi+2}) < \frac{1}{\gamma} < \frac{1}{\gamma_1}$ and by employing equation (23), we obtain

$$\lim_{\varpi\to\infty} ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+2}) = 0. \tag{25}$$

Step 2: Let us $\varsigma_\varpi \neq \varsigma_\top$, for $\varpi \neq \top$. Suppose that, $\varsigma_\varpi = \varsigma_\top$ for any $\varpi = \top + ŧ > \top$, we have $\varsigma_{\varpi+1} = \Im\varsigma_\varpi = \Im\varsigma_\top = \varsigma_{\top+1}$. Inequality (25), signifies that

$$\begin{aligned}
F_\varrho(ð_\varrho(\varsigma_\top, \varsigma_{\top+1})) = F_\varrho(ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})) &= F_\varrho(ð_\varrho(\Im\varsigma_{\varpi-1}, \Im\varsigma_\varpi))\\
&\leq F_\varrho((\gamma_1 + \gamma_2)ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + (\gamma_3 + \gamma_4)ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}))\\
&\quad - £(ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi))\\
&< F_\varrho((\gamma_1 + \gamma_2)ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi) + (\gamma_3 + \gamma_4)ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1})).
\end{aligned}$$

By the property of F_ϱ, the above equation modified as

$$ð_\varrho(\varsigma_\top, \varsigma_{\top+1}) = ð_\varrho(\varsigma_\varpi, \varsigma_{\varpi+1}) \leq \frac{\gamma_1 + \gamma_2}{1 - \gamma_3 - \gamma_4} ð_\varrho(\varsigma_{\varpi-1}, \varsigma_\varpi)$$

$$\leq \Big(\frac{\gamma_1+\gamma_2}{1-\gamma_3-\gamma_4}\Big)^2 \eth_\varrho(\varsigma_{\varpi-2},\varsigma_{\varpi-1})$$
$$\vdots$$
$$\leq \Big(\frac{\gamma_1+\gamma_2}{1-\gamma_3-\gamma_4}\Big)^\varpi \eth_\varrho(\varsigma_\intercal,\varsigma_{\intercal+1}) < \eth_\varrho(\varsigma_\intercal,\varsigma_{\intercal+1})$$

which is contraction. Thus, we conclude that $\varsigma_\varpi \neq \varsigma_\intercal$, $\forall\ \varpi \neq \intercal$.

Step 3: Now, we will prove $\{\varsigma_\varpi\}_{\varpi\in\mathbb{N}}$ is a Cauchy O_{seq} that is,

$$\lim_{\varpi\to\infty} \eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+\mathfrak{a}}) = 0, \text{ for } \mathfrak{a}\in\mathbb{N}.$$

We have previously proved for the cases $\mathfrak{a}=1$ and $\mathfrak{a}=2$, respectively. Now, choose $\mathfrak{a}\geq 1$ arbitrary. We split into two cases.
Case 1: Let $\mathfrak{a}=2\intercal$, where $\intercal\geq 2$. Thereafter, we get

$$\begin{aligned}
\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+2\intercal}) &\leq \varrho(\varsigma_\varpi,\varsigma_{\varpi+2})\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2},\varsigma_{\varpi+3})\eth_\varrho(\varsigma_{\varpi+2},\varsigma_{\varpi+3}) + \varrho(\varsigma_{\varpi+3},\varsigma_{\varpi+2\intercal})\\
&\quad \eth_\varrho(\varsigma_{\varpi+3},\varsigma_{\varpi+2\intercal})\\
&\quad \vdots\\
&\leq \varrho(\varsigma_\varpi,\varsigma_{\varpi+2})\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2},\varsigma_{\varpi+3})\eth_\varrho(\varsigma_{\varpi+2},\varsigma_{\varpi+3})\\
&\quad + \varrho(\varsigma_{\varpi+3},\varsigma_{\varpi+2\intercal})[\varrho(\varsigma_{\varpi+3},\varsigma_{\varpi+4})\eth_\varrho(\varsigma_{\varpi+3},\varsigma_{\varpi+4}) + \varrho(\varsigma_{\varpi+4},\varsigma_{\varpi+5})\eth_\varrho(\varsigma_{\varpi+4},\varsigma_{\varpi+5})]+\\
&\quad \vdots\\
&\quad + \varrho(\varsigma_{\varpi+3},\varsigma_{\varpi+2\intercal})\varrho(\varsigma_{\varpi+5},\varsigma_{\varpi+2\intercal})\ldots\varrho(\varsigma_{\varpi+2\intercal-3},\varsigma_{\varpi+2\intercal})[\varrho(\varsigma_{\varpi+2\intercal-3},\varsigma_{\varpi+2\intercal-2})\\
&\quad \eth_\varrho(\varsigma_{\varpi+2\intercal-3},\varsigma_{\varpi+2\intercal-2}) + \varrho(\varsigma_{\varpi+2\intercal-2},\varsigma_{\varpi+2\intercal-1})\eth_\varrho(\varsigma_{\varpi+2\intercal-2},\varsigma_{\varpi+2\intercal-1})]\\
&\quad + \varrho(\varsigma_{\varpi+3},\varsigma_{\varpi+2\intercal})\varrho(\varsigma_{\varpi+5},\varsigma_{\varpi+2\intercal})\ldots\varrho(\varsigma_{\varpi+2\intercal-1},\varsigma_{\varpi+2\intercal})\eth_\varrho(\varsigma_{\varpi+2\intercal-1},\varsigma_{\varpi+2\intercal})\\
&\leq \varrho(\varsigma_\varpi,\varsigma_{\varpi+2})\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+2}) + \sum_{i=\varpi+2}^{\varpi+2\intercal-2}\eth_\varrho(\varsigma_i,\varsigma_{i+1})\prod_{j=1}^{i}\varrho(\varsigma_j,\varsigma_{\varpi+2\intercal})\varrho(\varsigma_i,\varsigma_{i+1})\\
&\quad + \prod_{i=1}^{\varpi+2\intercal-1}\varrho(\varsigma_i,\varsigma_{\varpi+2\intercal})\eth_\varrho(\varsigma_{\varpi+2\intercal-1},\varsigma_{\varpi+2\intercal})\\
&\leq \varrho(\varsigma_\varpi,\varsigma_{\varpi+2})\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+2}) + \sum_{i=\varpi+2}^{\varpi+2\intercal-1}\eth_\varrho(\varsigma_i,\varsigma_{i+1})\prod_{j=1}^{i}\varrho(\varsigma_j,\varsigma_{\varpi+2\intercal})\varrho(\varsigma_i,\varsigma_{i+1}).
\end{aligned}$$

Notice that the series

$$\sum_{\varpi=1}^{\infty}\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1})\prod_{i=1}^{\varpi}\varrho(\varsigma_i,\varsigma_{\varpi+2\intercal})\varrho(\varsigma_i,\varsigma_{i+1})$$

which is converges. Since

$$\begin{aligned}
\sum_{\varpi=1}^{\infty}\eth_\varrho(\varsigma_\varpi,\varsigma_{\varpi+1})\prod_{i=1}^{\varpi}\varrho(\varsigma_i,\varsigma_{\varpi+2\intercal})\varrho(\varsigma_i,\varsigma_{i+1}) &\leq \sum_{\varpi=1}^{\infty}\frac{1}{\varpi^{\frac{1}{\lambda}}}\prod_{i=1}^{\varpi}\varrho(\varsigma_i,\varsigma_{\varpi+2\intercal})\varrho(\varsigma_i,\varsigma_{i+1})\\
&< \frac{1}{\gamma_1}\sum_{\varpi=1}^{\infty}\frac{1}{\varpi^{\frac{1}{\lambda}}},
\end{aligned}$$

which is convergent. Let

$$\mathcal{Y} = \sum_{\varpi=1}^{\infty} \eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) \prod_{i=1}^{\varpi} \varrho(\varsigma_i, \varsigma_{\varpi+2\intercal})\varrho(\varsigma_{\varpi}, \varsigma_{\varpi+1})$$

$$\mathcal{Y}_{\varpi} = \sum_{j=1}^{\varpi} \eth_{\varrho}(\varsigma_j, \varsigma_{j+1}) \prod_{i=1}^{j} \varrho(\varsigma_i, \varsigma_{\varpi+2\intercal})\varrho(\varsigma_j, \varsigma_{j+1}).$$

From the above inequality, it follows that

$$\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2\intercal}) \leq \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2}) + \mathcal{Y}_{\varpi+2\intercal-1} - \mathcal{Y}_{\varpi+1}.$$

Letting $\varpi \to \infty$ and using (25), we have

$$\lim_{\varpi\to\infty} \eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2\intercal}) = 0. \tag{26}$$

Case 2: Let $\mathfrak{a} = 2\intercal + 1$, where $\intercal \geq 1$. Then, we find

$$\begin{aligned}
\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2\intercal+1}) &\leq \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+1})\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) + \varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi+1}, \varsigma_{\varpi+2}) + \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+2\intercal+1}) \\
&\quad \eth_{\varrho}(\varsigma_{\varpi+2}, \varsigma_{\varpi+2\intercal+1}) \\
&\quad \vdots \\
&\leq \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+1})\eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) + \varrho(\varsigma_{\varpi+1}, \varsigma_{\varpi+2})\eth_{\varrho}(\varsigma_{\varpi+1}, \varsigma_{\varpi+2}) + \\
&\quad \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+2\intercal+1})[\varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+3})\eth_{\varrho}(\varsigma_{\varpi+2}, \varsigma_{\varpi+3}) + \varrho(\varsigma_{\varpi+3}, \varsigma_{\varpi+4})\eth_{\varrho}(\varsigma_{\varpi+3}, \varsigma_{\varpi+4})] \\
&\quad \vdots \\
&\quad \varrho(\varsigma_{\varpi+2}, \varsigma_{\varpi+2\intercal+1})\varrho(\varsigma_{\varpi+4}, \varsigma_{\varpi+2\intercal+1})\ldots\varrho(\varsigma_{\varpi+2\intercal-2}, \varsigma_{\varpi+2\intercal+1})[\varrho(\varsigma_{\varpi+2\intercal-2}, \varsigma_{\varpi+2\intercal-1}) \\
&\quad \eth_{\varrho}(\varsigma_{\varpi+2\intercal-2}, \varsigma_{\varpi+2\intercal-1}) + \varrho(\varsigma_{\varpi+2\intercal-1}, \varsigma_{\varpi+2\intercal})\eth_{\varrho}(\varsigma_{\varpi+2\intercal-1}, \varsigma_{\varpi+2\intercal}) + \\
&\quad \varrho(\varsigma_{\varpi}, \varsigma_{\varpi+2\intercal+1})\eth_{\varrho}(\varsigma_{\varpi+2\intercal}, \varsigma_{\varpi+2\intercal+1})] \\
&\leq \sum_{i=\varpi}^{\varpi+2\intercal-1} \eth_{\varrho}(\varsigma_i, \varsigma_{i+1}) \prod_{j=1}^{i} \varrho(\varsigma_j, \varsigma_{\varpi+2\intercal+1})\varrho(\varsigma_i, \varsigma_{i+1}) + \prod_{i=1}^{\varpi+2\intercal} \varrho(\varsigma_i, \varsigma_{\varpi+2\intercal+1})\eth_{\varrho}(\varsigma_{\varpi+2\intercal}, \varsigma_{\varpi+2\intercal+1}) \\
&\leq \sum_{i=\varpi}^{\varpi+2\intercal} \eth_{\varrho}(\varsigma_i, \varsigma_{i+1}) \prod_{j=1}^{i} \varrho(\varsigma_j, \varsigma_{\varpi+2\intercal+1})\varrho(\varsigma_i, \varsigma_{i+1}).
\end{aligned}$$

We observe that the series $\sum_{\varpi=1}^{\infty} \eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) \prod_{i=1}^{\varpi} \varrho(\varsigma_i, \varsigma_{\varpi+2\intercal+1})\varrho(\varsigma_i, \varsigma_{i+1})$ converges. Since,

$$\begin{aligned}
\sum_{\varpi=1}^{\infty} \eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) \prod_{i=1}^{\varpi} \varrho(\varsigma_i, \varsigma_{\varpi+2\intercal+1})\varrho(\varsigma_i, \varsigma_{i+1}) &\leq \sum_{\varpi=1}^{\infty} \frac{1}{\varpi^{\frac{1}{\lambda}}} \prod_{i=1}^{\varpi} \varrho(\varsigma_i, \varsigma_{\varpi+2\intercal+1})\varrho(\varsigma_i, \varsigma_{i+1}) \\
&< \frac{1}{\gamma_1} \sum_{\varpi=1}^{\infty} \frac{1}{\varpi^{\frac{1}{\lambda}}}, \text{ which is convergent.}
\end{aligned}$$

Let

$$\mathcal{Z} = \sum_{\varpi=1}^{\infty} \eth_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+1}) \prod_{i=1}^{\varpi} \varrho(\varsigma_i, \varsigma_{\varpi+2\intercal+1})\varrho(\varsigma_{\varpi}, \varsigma_{\varpi+1})$$

$$\mathcal{Z}_{\varpi} = \sum_{j=1}^{\varpi} \eth_{\varrho}(\varsigma_j, \varsigma_{j+1}) \prod_{i=1}^{j} \varrho(\varsigma_i, \varsigma_{\varpi+2\intercal+1})\varrho(\varsigma_j, \varsigma_{j+1}).$$

Eventually, the above inequality yields:

$$ð_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2_{\mathsf{T}}+1}) \leq \mathcal{Z}_{\varpi+2_{\mathsf{T}}} - \mathcal{Z}_{\varpi-1}.$$

Letting $\varpi \to \infty$, we deduce that

$$\lim_{\varpi\to\infty} ð_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+2_{\mathsf{T}}+1}) = 0. \tag{27}$$

Consequently, we obtain by combining equations (26) and (27).

$$\lim_{\varpi\to\infty} ð_{\varrho}(\varsigma_{\varpi}, \varsigma_{\varpi+\mathfrak{a}}) = 0, \ \forall \ \mathfrak{a} \in \mathbb{N}. \tag{28}$$

Hence, we conclude that $\{\varsigma_{\varpi}\}$ is a Cauchy O_{seq} that is, $\{\mathfrak{I}^{\varpi}\varsigma\}$ is a Cauchy O_{seq}. Since $\eth$ is O-complete, let $\varsigma_{\varpi} \to \varsigma \in \eth$. By continuity of $\mathfrak{I}$, we have

$$\varsigma = \lim_{\varpi\to\infty} \varsigma_{\varpi+1} = \lim_{\varpi\to\infty} \mathfrak{I}\varsigma_{\varpi} = \mathfrak{I} \lim_{\varpi\to\infty} \varsigma_{\varpi} = \mathfrak{I}\varsigma.$$

that is, ς is a fixed point of $\mathfrak{I}$.

Step 4: Let $\iota \neq \varsigma$ be a another fixed point of $\mathfrak{I}$ that is, $\mathfrak{I}\iota = \iota$. We have

$$[\varsigma_0 \perp \iota] \text{ or } [\iota \perp \varsigma_0].$$

Since $\mathfrak{I}$ is orthogonal preserving, we have

$$[\mathfrak{I}^{\varpi}(\varsigma_0) \perp \mathfrak{I}^{\varpi}(\iota)] \text{ or } [\mathfrak{I}^{\varpi}(\iota) \perp \mathfrak{I}^{\varpi}(\varsigma_0)],$$

for all $\varpi \in \mathbb{N}$. On the other hand $\mathfrak{I}$ is an F_{ϱ}-contraction. From Equation (2), we that

$$\begin{aligned}
\pounds(ð_{\varrho}(\varsigma,\iota)) + F_{\varrho}(ð_{\varrho}(\varsigma,\iota)) &= \pounds(ð_{\varrho}(\varsigma,\iota)) + F_{\varrho}(ð_{\varrho}(\mathfrak{I}\varsigma,\mathfrak{I}\iota)) \\
&\leq F_{\varrho}\Big(\gamma_1 ð_{\varrho}(\varsigma,\iota) + \gamma_2 \frac{ð_{\varrho}(\varsigma,\mathfrak{I}\varsigma)}{1+ð_{\varrho}(\varsigma,\mathfrak{I}\varsigma)} + \gamma_3 \frac{ð_{\varrho}(\iota,\mathfrak{I}\iota)}{1+ð_{\varrho}(\iota,\mathfrak{I}\iota)} + \\
&\gamma_4 \frac{ð_{\varrho}(\varsigma,\mathfrak{I}\varsigma)ð_{\varrho}(\iota,\mathfrak{I}\iota)}{ð_{\varrho}(\varsigma,\iota)+ð_{\varrho}(\varsigma,\mathfrak{I}\iota)+ð_{\varrho}(\iota,\mathfrak{I}\varsigma)}\Big) \\
&\leq F_{\varrho}(\gamma_1 ð_{\varrho}(\varsigma,\iota)) < F_{\varrho}(ð_{\varrho}(\varsigma,\iota))
\end{aligned}$$

that is, $\pounds(ð_{\varrho}(\varsigma,\iota)) < 0$, which is a contradiction. Hence, $\mathfrak{I}$ has a ufp in $\eth$. □

Example 3.7. *Let* $\eth = \{0, 1, 2, 3\}$. *Define* $ð_{\varrho} : \eth \times \eth \to [0, \infty)$ *is orthogonal continuous as follows:*

- $ð_{\varrho}(\varsigma,\varsigma) = 0, \ \forall \ \varsigma \in \eth, ð_{\varrho}(0,1) = ð_{\varrho}(1,0) = 2,.$
- $ð_{\varrho}(0,2) = ð_{\varrho}(2,0) = ð_{\varrho}(0,3) = ð_{\varrho}(3,0) = 3,$
- $ð_{\varrho}(1,2) = ð_{\varrho}(2,1) = ð_{\varrho}(1,3) = ð_{\varrho}(3,1) = 5,$
- $ð_{\varrho}(2,3) = ð_{\varrho}(3,2) = 15.$

Let ϱ: $\eth \times \eth \to [0, \infty)$ *be symmetric and the binary relation* $\perp$ *on* $\eth$ *s.t.* $\varsigma \perp \iota$. *Then* $(\eth, \perp)$ *is an* O_{set} *and* $ð_{\varrho}$ *is a metric on* $\eth$. *We can be defined as follows:*

- $\varrho(\varsigma,\varsigma) = 1, \ \forall \ \varsigma \in \eth,$
- $\varrho(0,1) = \varrho(0,2) = \varrho(0,3) = \dfrac{3}{2},$

- $\varrho(1,2) = \varrho(1,3) = \varrho(2,3) = \dfrac{5}{4}$.

Then $(\eth, \perp, \eth_\varrho)$ *is an O-complete O-*C_b*MS.*
Note that

1. $(\eth, \eth_\varrho)$ *is not an extended O-*B_b*MS. Since*

$$\eth_\varrho(3,2) = 15 > \varrho(3,2)[\eth_\varrho(3,0) + \eth_\varrho(0,1) + \eth_\varrho(1,2)] = 12.5$$

2. $(\eth, \eth_\varrho)$ *is not an O-CMS. Since*

$$\eth_\varrho(3,2) = 15 > \varrho(3,0)\eth_\varrho(3,0) + \varrho(0,2)\eth_\varrho(0,2) = 9.$$

Let $\Im : \eth \to \eth$ *given by* $\Im 0 = \Im 1 = 0,\ \Im 2 = \Im 3 = 1$. *Define* $F_\varrho\colon [0,\infty) \to (-\infty, +\infty)$ *by* $F_\varrho(\varsigma) = \varsigma - \dfrac{1}{2}$, $\forall\, \varsigma \in [0,\infty)$ *and* $\pounds\colon [0,\infty) \to [0,\infty)$ *defined by* $\pounds(\varsigma) = \frac{\varsigma+1}{\varsigma+2}$, $\forall\, \varsigma \in [0,\infty)$.

Case A: *Let* $\varsigma = 0$. *Now* $\eth_\varrho(\Im 0, \Im 1) = \eth_\varrho(0,0) = 0$. *Therefore, we only need to assume* $\iota = 2,3$. *Consider*

$$\begin{aligned}\pounds(\eth_\varrho(0,2)) + F_\varrho(\eth_\varrho(\Im 0, \Im 2)) &= \frac{\eth_\varrho(0,1)+1}{\eth_\varrho(0,2)+2} + \eth_\varrho(\Im 0, \Im 2) - \frac{1}{2}\\ &= \frac{4}{5} + 2 - \frac{1}{2}\\ &= \frac{23}{10}.\end{aligned}$$

Hence

$$\pounds(\eth_\varrho(0,2)) + F_\varrho(\eth_\varrho(\Im 0, \Im 2)) < 2.5 = F_\varrho(\eth_\varrho(0,2)).$$

Similar arguments may be made for $\iota = 3$.

Case B: *Let* $\varsigma = 2$. *Now* $\eth_\varrho(\Im 2, \Im 3) = \eth_\varrho(1,1) = 0$. *Therefore, we only need to consider for* $\iota = 1$. *Assume*

$$\begin{aligned} eta(\eth_\varrho(2,1)) + F_\varrho(\eth_\eth(\Im 2, \Im 1)) &= \frac{\eth_\varrho(2,1)+1}{\eth_\varrho + 2} + \eth_\varrho(\Im 2, \Im 1) - \frac{1}{2}\\ &= \frac{6}{7} + 2 - \frac{1}{2}\\ &= \frac{33}{14}.\end{aligned}$$

Hence

$$\pounds(\eth_\varrho(2,1) + F_\varrho(\Im 2, \Im 1)) < 4.5 = F_\varrho(\eth_\varrho(2,1)).$$

The proof is the same as in the instances above for $\varsigma = 3$. *In addition, for every* $\varsigma \in \eth$, *we get*

$$\sup_{\mathrm{T}\geq 1} \lim_{\mathrm{i}\to\infty} \varrho(\varsigma_{\mathrm{i}+1}, \varsigma_{\mathrm{i}+2})\varrho(\varsigma_{\mathrm{i}+1}, \varsigma_{\mathrm{T}}) < \frac{1}{\lambda},$$

with $\lambda = \frac{1}{2}$. *Now, we verify that*

$$\lim_{\varpi\to\infty} \varrho(\varsigma_\varpi, \varsigma) \text{ and } \lim_{\varpi\to\infty} \varrho(\varsigma, \varsigma_\varpi),$$

exist and are finite, for all $\varsigma \in \eth$. *Thus,* $\Im$ *satisfies all the axioms of Theorem 3.5 and hence* $\varsigma = 0$ *is a ufp.*

4. Application to Fredholm integral equation

In this final section, we try to use Theorem 3.5 to demonstrate the existence and uniqueness of the provided Fredholm integral equation's solution.

$$\varsigma(\mathfrak{H}) = \int_{\Bbbk}^{\flat} \tau(\mathfrak{H}, \mathrm{t}, \varsigma(\mathrm{t}))dt + \mathfrak{f}(\mathfrak{H}), \ \forall\ \mathfrak{H}, \mathrm{t} \in [\Bbbk, \flat], \tag{29}$$

where $\tau, \mathfrak{f} \in \complement([\Bbbk, \flat], (-\infty, +\infty))$ (say that $\eth = \complement([\Bbbk, \flat], (-\infty, +\infty))$). Define $\eth_\varrho : \eth \times \eth \to [0, \infty)$ and $\varrho : \eth \times \eth \to [1, \infty)$ by:

$$\eth_\varrho(\varsigma, \iota) = \sup_{\mathfrak{H}\in[\Bbbk,\flat]} \left|\varsigma(\mathfrak{H}) - \iota(\mathfrak{H})\right|^2$$

and

$$\varrho(\varsigma, \iota) = \begin{cases} 1 + \sup_{\mathfrak{H}\in[\Bbbk,\flat]} |\varsigma(\mathfrak{H}) - \iota(\mathfrak{H})|, & \text{if } \varsigma(\mathfrak{H}) \neq \iota(\mathfrak{H}) \\ 1, & \text{if } \varsigma(\mathfrak{H}) = \iota(\mathfrak{H}). \end{cases}$$

It is clear that $(\eth, \eth_\varrho)$ is an O-complete O-C_bMS.

Theorem 4.1. *Assume that for all* $\varsigma, \iota \in \complement([\Bbbk, \flat], (-\infty, +\infty))$

$$|\tau(\mathfrak{H}, \mathrm{t}, \varsigma(\mathrm{t})) - \tau(\mathfrak{H}, \mathrm{t}, \iota(\mathrm{t}))| \leq \frac{e^{-\frac{1}{|\varsigma(\mathrm{t})-\iota(\mathrm{t})|}}}{\flat - \Bbbk}|\varsigma(\mathrm{t}) - \iota(\mathrm{t})|, \tag{30}$$

with $\varsigma \perp \iota$*, for all* $\mathfrak{H}, \mathrm{t} \in [\Bbbk, \flat]$*. Then, the integral equation* (29) *has a solution.*

Proof. We consider orthogonal relation $\perp$ on $\eth$ as defined by

$$\varsigma \perp \iota \iff \begin{array}{l} \varsigma(\mathrm{t})\iota(\mathrm{t}) \geq \varsigma(\mathrm{t}) \ \text{ or} \\ \varsigma(\mathrm{t})\iota(\mathrm{t}) \geq \iota(\mathrm{t}), \end{array}$$

for all $\mathrm{t} \in [\Bbbk, \flat]$. Then $(\eth, \perp)$ is an O_{set}. Define $\mathfrak{I} : \eth \to \eth$ by

$$\varsigma(\mathfrak{H}) = \int_{\Bbbk}^{\flat} \tau(\mathfrak{H}, \mathrm{t}, \varsigma(\mathrm{t}))dt + \mathfrak{f}(\mathfrak{H}), \ \forall\ \mathfrak{H}, \mathrm{t} \in [\Bbbk, \flat].$$

The operator $\mathfrak{I}$ meets of Theorem 3.5, $\forall\ \varsigma, \iota \in \eth$, we get

$$\begin{aligned} |\mathfrak{I}\varsigma(\mathfrak{H}) - \mathfrak{I}\iota(\mathfrak{H})|^2 &\leq \Big(\int_{\Bbbk}^{\flat} |\tau(\mathfrak{H}, \mathrm{t}, \varsigma(\mathrm{t})) - \tau(\mathfrak{H}, \mathrm{t}, \iota(\mathrm{t}))|dt\Big)^2 \\ &\leq \Big(\int_{\Bbbk}^{\flat} \frac{e^{-\frac{1}{|\varsigma(\mathrm{t})-\iota(\mathrm{t})|}}}{\flat - \Bbbk}|\varsigma(\mathrm{t}) - \iota(\mathrm{t})|dt\Big)^2 \\ &\leq \frac{1}{(\flat - \Bbbk)^2} e^{-\frac{1}{\sup_{\mathrm{t}\in[\Bbbk,\flat]}|\varsigma(\mathfrak{H})-\iota(\mathfrak{H})|^2}} \sup_{\mathrm{t}\in[\Bbbk,\flat]} |\varsigma(\mathfrak{H}) - \iota(\mathfrak{H})|^2 \Big(\int_{\Bbbk}^{\flat} dt\Big)^2 \\ &= e^{\frac{-1}{\eth_\varrho(\varsigma,\iota)}} \eth_\varrho(\varsigma, \iota), \end{aligned}$$

which implies

$$\eth_\varrho(\mathfrak{I}\varsigma, \mathfrak{I}\iota) \leq e^{\frac{-1}{\eth_\varrho(\varsigma,\iota)}} \eth_\varrho(\varsigma, \iota).$$

Taking log on both sides, we have

$$\ln(\eth_{\varrho}(\Im\varsigma, \Im\iota)) \leq \frac{-1}{\eth_{\varrho}(\varsigma, \iota)} + \ln(\eth_{\varrho}(\varsigma, \iota)).$$

Resultant, we have

$$\frac{1}{\eth_{\varrho}(\varsigma, \iota)} + \ln(\eth_{\varrho}(\Im\varsigma, \Im\iota)) \leq \ln(\eth_{\varrho}(\varsigma, \iota)).$$

Let us define $F_{\varrho}\colon [0,\infty) \to (-\infty, +\infty)$ and $\pounds\colon [0,\infty) \to [0,\infty)$ by $F_{\varrho}(\mathfrak{s}) = \ln(\mathfrak{s}), \mathfrak{s} > 0$ and $\pounds(\varsigma) = \frac{1}{\varsigma}, \varsigma \in [0,\infty)$. Therefore, from the inequality above we get

$$\pounds(\eth_{\varrho}(\varsigma, \iota)) + F_{\varrho}(\eth_{\varrho}(\Im\varsigma, \Im\iota)) \leq F_{\varrho}(\eth_{\varrho}(\varsigma, \iota)).$$

Hence, all the requirements of Theorem 3.5 are satisfied. Operator $\Im$, therefore has a ufp, that is the Fredholm integral equation has a solution. □

4.1. Application to integro-differential equation

An application to integro-differential equation for two dimensional nonlinear partial Volterra equation with desired order:

$$\begin{cases} \frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}} = \varrho(\delta,\xi) + \int_0^{\xi}\int_0^{\delta} \mathcal{K}\left(\delta, \xi, \mathfrak{b}, \natural, \frac{\varsigma^{\mathfrak{b},\natural}\mathfrak{f}(\mathfrak{b},\natural)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right) d\mathfrak{b}\, d\natural, (\delta,\xi) \in [0,1]\times[0,1] \\ \text{Appropriate intial conditions,} \end{cases} \tag{31}$$

where the kernel function is a known nonlinear orthogonal continuous function in $\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}$ with $\delta \perp \xi$ and $\varrho(\delta,\xi)$ is a known function where $\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}$ is an unknown function.

It is intended to approximate the function $\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}$, with Haar wavelests. For this purpose, the mesh nodes on the square $0 \leq \delta, \xi \leq 1$ are obtained using the following collocation points:

$$\delta_{\tau} = \frac{\tau - 0.5}{2\mathcal{M}}, \quad \tau = 1, 2, ...2\mathcal{M}. \tag{32}$$

$$\xi_{\varpi} = \frac{\varpi - 0.5}{2\mathcal{N}}, \quad \varpi = 1, 2, ...2\mathcal{N}. \tag{33}$$

The two-dimensional function $\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}$ is approximated with two dimensional Haar wavelet on $0 \leq \delta, \xi \leq 1$ as follows:

$$\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}} = \sum_{i=1}^{2\mathcal{M}}\sum_{j=1}^{2\mathcal{N}} \mathfrak{b}_{i,j}\mathfrak{h}_i(\delta)\mathfrak{h}_j(\xi). \tag{34}$$

To calculate the coefficients of $\mathfrak{b}_{i,j}$ in Equation (34), we substitute the point defined in (31), and (32) in Equation (33) to the following linear system of $4\mathcal{MN} \times 4\mathcal{MN}$ with the coefficients of $\mathfrak{b}_{i,j}$.

$$\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\xi_{\varpi}}^{\delta=\delta_{\tau}} = \sum_{i=1}^{2\mathcal{M}}\sum_{j=1}^{2\mathcal{N}} \mathfrak{b}_{i,j}\mathfrak{h}_i(\delta_{\tau})\mathfrak{h}_j(\xi_{\varpi}), \quad \tau = 1, 2, ..., 2\mathcal{M}, \ \varpi = 1, 2, ..., 2\mathcal{N}.$$

Unknown coefficients of $\mathfrak{b}_{i,j}$ are achieved using Theorem 3.6.

Theorem 4.2. *Suppose a function $F(\delta,\xi)$ of two variables δ and ξ is approximated using Haar wavelet approximation given as*

$$F(\delta,\xi)=\sum_{i=1}^{2\mathcal{M}}\sum_{j=1}^{2\mathcal{N}}\mathfrak{b}_{i,j}\mathfrak{h}_i(\delta)\mathfrak{h}_j(\xi),\ \ \delta\perp\xi.$$

Suppose that $F(\delta,\xi)$ is known at collocation points $(\delta_\top,\xi_\varpi)$, $\top=1,2,.....,2\mathcal{M}$, $\varpi=1,2,.....,2\mathcal{N}$ and F is orthogonal continuous. Then, the approximate value of the function $F(\delta,\xi)$ at any other point of the domain can be calculated as follows:

$$\begin{aligned}F(\delta,\xi)&=\frac{1}{2\mathcal{M}\times 2\mathcal{N}}\sum_{\mathfrak{k}=1}^{2\mathcal{M}}\sum_{\mathfrak{p}=1}^{2\mathcal{N}}F(\delta_\top,\xi_\varpi)\mathfrak{h}_g(\delta)\mathfrak{h}_1(\xi)\\&+\sum_{i=1}^{2\mathcal{M}}\frac{1}{\rho_2\times 2\mathcal{N}}\left(\sum_{\mathfrak{k}=\aleph_1}^{\beta_1}\sum_{\mathfrak{p}=1}^{2\mathcal{N}}F(\delta_\top,\xi_\varpi)-\sum_{\mathfrak{k}=\beta_1+1}^{\gamma_1}\sum_{\mathfrak{p}=1}^{2\mathcal{N}}F(\delta_\top,\xi_[\varpi])\right)\mathfrak{h}_i(\delta)\mathfrak{h}_1\\&+\sum_{j=1}^{2\mathcal{N}}\frac{1}{2\mathcal{M}\times\rho_2}\left(\sum_{\mathfrak{k}=1}^{2\mathcal{M}}\sum_{\mathfrak{p}=\aleph_2}^{\beta_2}F(\delta_\top,\xi_\varpi)-\sum_{\mathfrak{k}=1}^{2\mathcal{M}}\sum_{\mathfrak{p}=\beta_2+1}^{\gamma_2}F(\delta_\top,\xi_\varpi)\right)\mathfrak{h}_1(\delta)\mathfrak{h}_j(\xi)\\&+\sum_{i=1}^{2\mathcal{M}}\sum_{j=1}^{2\mathcal{N}}\frac{1}{\rho_1\times\rho_2}\left(\sum_{\mathfrak{k}=\aleph_1}^{\beta_1}\sum_{\mathfrak{p}=\aleph_2}^{\beta_2}F(\delta_\top,\xi_\varpi)-\sum_{\mathfrak{k}=\aleph_1}^{\beta_1}\sum_{\mathfrak{p}=\beta_2+1}^{\gamma_2}F(\delta_\top,\xi_\varpi)\right.\\&\left.-\sum_{\mathfrak{k}=\beta_1+1}^{\gamma_1}\sum_{\mathfrak{p}=\aleph_2}^{\beta_2}F(\delta_\top,\xi_\varpi)+\sum_{\mathfrak{k}=\beta_1+1}^{\gamma_1}\sum_{\mathfrak{p}=\beta_2+1}^{\gamma_2}F(\delta_\top,\xi_\varpi)\right)\mathfrak{h}_i(\delta)\mathfrak{h}_j(\xi)\end{aligned}$$

where

$$\begin{aligned}\aleph_1&=\aleph_1(\sigma_1-1)+1,\\\beta_1&=\rho_1(\sigma_1-1)+\frac{\rho_1}{2},\\\gamma_1&=\rho_1\sigma_1,\\\rho_1&=\frac{2\mathcal{M}}{\nu_1},\\\sigma_1&=i-\nu_1,\\\nu_1&=2^{\log_2(i-1)}.\end{aligned}\tag{35}$$

And similarly

$$\begin{aligned}\aleph_2&=\rho_2(\sigma_2-1)+1,\\\beta_2&=\rho_2(\sigma_2-1)+\frac{\rho_2}{2},\\\gamma_2&=\rho_2\sigma_2,\\\rho_2&=\frac{2\mathcal{N}}{\nu_2},\\\sigma_2&=i-\nu_2,\\\nu_2&=2^{\log_2(j-1)}\end{aligned}\tag{36}$$

First, the Kernel of (31) *is orthogonal continuous and approximated by two dimensional Haar wavelet as follows:*

$$\mathcal{K}\left(\delta,\xi,\mathfrak{b},\natural,\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\mathfrak{b},\natural)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right)\approx\sum_{i=1}^{2\mathcal{M}}\sum_{j=1}^{2\mathcal{N}}\mathfrak{b}_{i,j}(\delta,\xi)\mathfrak{h}_i(\mathfrak{b})\mathfrak{h}_j(\natural).$$

Substituting the above approximation in Equation (31)*, the following equation is obtained. Thus, we have*

$$\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}} = \varrho(\delta,\xi) + \int_0^{\xi}\int_0^{\delta}\sum_{i=1}^{2\mathcal{M}}\sum_{j=1}^{2\mathcal{N}} \mathfrak{b}_{i,j}(\delta,\xi)\mathfrak{h}_i(\mathfrak{b})\mathfrak{h}_j(\natural)d\mathfrak{b}d\natural.$$

With the help of Haar wavelet properties, the following equation is obtained.

$$\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}} = \varrho(\delta,\xi) + \sum_{i=1}^{2\mathcal{M}}\sum_{j=1}^{2\mathcal{N}} \mathfrak{b}_{i,j}(\delta,\xi)\mathfrak{k}_{i,1}(\delta)\mathfrak{k}_{j,1}(\xi).$$

Now, collocation point $\delta_{\mathsf{T}}, \xi_{\varpi}$ *are inserted in Equation* (31) *to get the following system of equations.*

$$\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\xi_{\varpi}}^{\delta=\delta_{\mathsf{T}}} = \varrho(\delta_{\mathsf{T}},\xi_{\varpi}) + \sum_{i=1}^{2\mathcal{M}}\sum_{j=1}^{2\mathcal{N}} \mathfrak{b}_{i,j}(\delta_{\mathsf{T}},\xi_{\varpi})\mathfrak{k}_{i,1}(\delta_{\mathsf{T}})\mathfrak{k}_{j,1}(\xi_{\varpi}) \tag{37}$$

$\mathsf{T} = 1,2,...,2\mathcal{M}, \varpi = 1,2,...,2\mathcal{N}$.

The values $\mathfrak{b}_{i,j}(\delta,\xi)$ *are obtained from Theorem 3.6 and inserted in Equation* (37) *to reach the following system of equations.*

$$\begin{aligned}
\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\xi_{\varpi}}^{\delta=\delta_{\mathsf{T}}} = {}& \varrho(\delta_{\mathsf{T}},\xi_{\varpi}) + \frac{\mathfrak{k}_{1,1}(\delta_{\mathsf{T}})\mathfrak{k}_{1,1}(\xi_{\varpi})}{2\mathcal{M}\times 2\mathcal{N}}\sum_{\mathfrak{k}=1}^{2\mathcal{M}}\sum_{\mathfrak{p}=1}^{2\mathcal{N}}\mathcal{K}\Big(\delta_{\mathsf{T}},\xi_{\varpi},\mathfrak{b}_{\mathfrak{k}},\natural_{\mathfrak{p}},\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\natural_{\mathfrak{p}}}^{\delta=\mathfrak{b}_{\mathfrak{k}}}\Big)\\
&+\sum_{i=2}^{2\mathcal{M}}\frac{\mathfrak{k}_{i,1}(\delta_{\mathsf{T}})\mathfrak{k}_{1,1}(\xi_{\varpi})}{\rho_1\times 2\mathcal{N}}\Big(\sum_{\mathfrak{k}=\aleph_1}^{\beta_1}\sum_{\mathfrak{p}=1}^{2\mathcal{N}}\mathcal{K}\Big(\delta_{\mathsf{T}},\xi_{\varpi},\mathfrak{b}_{\mathfrak{k}},\natural_{\mathfrak{p}},\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\natural_{\mathfrak{p}}}^{\delta=\mathfrak{b}_{\mathfrak{k}}}\Big)\\
&-\sum_{\mathfrak{k}=\beta_1+1}^{\gamma_1}\sum_{\mathfrak{p}=1}^{2\mathcal{N}}\mathcal{K}\Big(\delta_{\mathsf{T}},\xi_{\varpi},\mathfrak{b}_{\mathfrak{k}},\natural_{\mathfrak{p}},\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\natural_{\mathfrak{p}}}^{\delta=\mathfrak{b}_{\mathfrak{k}}}\Big)\Big)\\
&+\sum_{j=2}^{2\mathcal{N}}\frac{\mathfrak{k}_{1,1}(\delta_{\mathsf{T}})\mathfrak{k}_{j,1}(\xi_{\varpi})}{2\mathcal{M}\times\rho_2}\Big(\sum_{\mathfrak{k}=1}^{2\mathcal{M}}\sum_{\mathfrak{p}=\aleph_2}^{\beta_2}\mathcal{K}\Big(\delta_{\mathsf{T}},\xi_{\varpi},\mathfrak{b}_{\mathfrak{k}},\natural_{\mathfrak{p}},\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\natural_{\mathfrak{p}}}^{\delta=\mathfrak{b}_{\mathfrak{k}}}\Big)\\
&-\sum_{\mathfrak{k}=1}^{2\mathcal{M}}\sum_{\mathfrak{p}=\beta_2+1}^{\gamma_2}\mathcal{K}\Big(\delta_{\mathsf{T}},\xi_{\varpi},\mathfrak{b}_{\mathfrak{k}},\natural_{\mathfrak{p}},\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\natural_{\mathfrak{p}}}^{\delta=\mathfrak{b}_{\mathfrak{k}}}\Big)\Big)\\
&+\sum_{i=2}^{2\mathcal{M}}\sum_{j=2}^{2\mathcal{N}}\frac{\mathfrak{k}_{i,1}(\delta_{\mathsf{T}})\mathfrak{k}_{j,1}(\xi_{\varpi})}{\rho_1\times\rho_2}\Big(\sum_{\mathfrak{k}=\aleph_1}^{\beta_1}\sum_{\mathfrak{p}=\aleph_2}^{\beta_2}\mathcal{K}\Big(\delta_{\mathsf{T}},\xi_{\varpi},\mathfrak{b}_{\mathfrak{k}},\natural_{\mathfrak{p}},\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\natural_{\mathfrak{p}}}^{\delta=\mathfrak{b}_{\mathfrak{k}}}\Big)\\
&-\sum_{\mathfrak{k}=\aleph_1}^{\beta_1}\sum_{\mathfrak{p}=\beta_2+1}^{\gamma_2}\mathcal{K}\Big(\delta_{\mathsf{T}},\xi_{\varpi},\mathfrak{b}_{\mathfrak{k}},\natural_{\mathfrak{p}},\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\natural_{\mathfrak{p}}}^{\delta=\mathfrak{b}_{\mathfrak{k}}}\Big)\\
&-\sum_{\mathfrak{k}=\beta_1+1}^{\gamma_1}\sum_{\mathfrak{p}=\aleph_2}^{\beta_2}\mathcal{K}\Big(\delta_{\mathsf{T}},\xi_{\varpi},\mathfrak{b}_{\mathfrak{k}},\natural_{\mathfrak{a}},\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\natural_{\mathfrak{a}}}^{\delta=\mathfrak{b}_{\mathfrak{k}}}\Big)\\
&+\sum_{\mathfrak{k}=\beta_1+1}^{\gamma_1}\sum_{\mathfrak{a}=\beta_2+1}^{\gamma_2}\mathcal{K}\Big(\delta_{\mathsf{T}},\xi_{\varpi},\mathfrak{b}_{\mathfrak{k}},\natural_{\mathfrak{a}},\left.\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}\right|_{\xi=\natural_{\mathfrak{a}}}^{\delta=\mathfrak{b}_{\mathfrak{k}}}\Big)\Big),
\end{aligned}$$

where $\mathsf{T} = 1,2,...,2\mathcal{M}, \varpi = 1,2,...,2\mathcal{N}$.

Equation (41) is a 4MN×4MN a nonlinear system which can be solved either by Broyden or Newton methods. The solution of this system gives values of $\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}$ *at the collocation points.*

The value of $\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}\varsigma\xi^{\mathfrak{w}}}$ *at points other than collocation points can be calculated using Theorem 4.2. The equation*

$$\frac{\varsigma^{\mathfrak{v}+\mathfrak{w}}\mathfrak{f}(\delta,\xi)}{\varsigma\delta^{\mathfrak{v}}_{\intercal}\varsigma\xi^{\mathfrak{w}}_{\varpi}} = \mathcal{A}(\delta,\xi),$$

can be solved using one of the method of partial differential equations.

Example 4.3. *Consider partial integro-differential equation as follows:*

$$\frac{\varsigma^2\mathfrak{f}(\delta,\xi)}{\varsigma\delta\varsigma\xi} = \varrho(\delta,\xi) + \int_0^{\xi}\int_0^{\delta}\left(\frac{\varsigma^2\mathfrak{f}(\mathfrak{b},\natural)}{\varsigma\mathfrak{b}\varsigma\natural} + 2\delta\xi\Big(\frac{\varsigma^2\mathfrak{f}(\mathfrak{b},\natural)}{\varsigma\mathfrak{b}\varsigma\natural}\Big)^3 + \xi^2\Big(\frac{\varsigma^2\mathfrak{f}(\mathfrak{b},\natural)}{\varsigma\mathfrak{b}\varsigma\natural}\Big)^5\right)d\mathfrak{b}d\natural,$$

where

$$\varrho(\delta,\xi) = e^{\xi} - (-1+e^{\xi})\delta - \frac{1}{5}(-1+e^{5\xi})\delta\xi^2 - \frac{2}{3}(-1+e^{3\xi})\delta^2\xi.$$

The exact solution of this problem is

$$\mathfrak{f}(\delta,\xi) = \delta e^{\xi}.$$

And $\mathfrak{f}(\delta,\xi)$ *is orthogonal continuous on*$[0,1]$, *and a supplementary conditions are*

$$\frac{\varsigma\mathfrak{f}(\delta,0)}{\varsigma\delta} = 1, \mathfrak{f}(0,0) = 0.$$

The approximation solution of this equation is

$$\frac{\varsigma^2\mathfrak{f}(\delta,\xi)}{\varsigma\delta\varsigma\xi} = e^{\xi}. \tag{38}$$

Error of the integral equation is

$$\mathfrak{f}(\delta,\xi) - \frac{\varsigma^2\mathfrak{f}(\delta,\xi)}{\varsigma\delta\varsigma\xi} = \delta e^{\xi} - e^{\xi} = e^{\xi}(\delta-1). \tag{39}$$

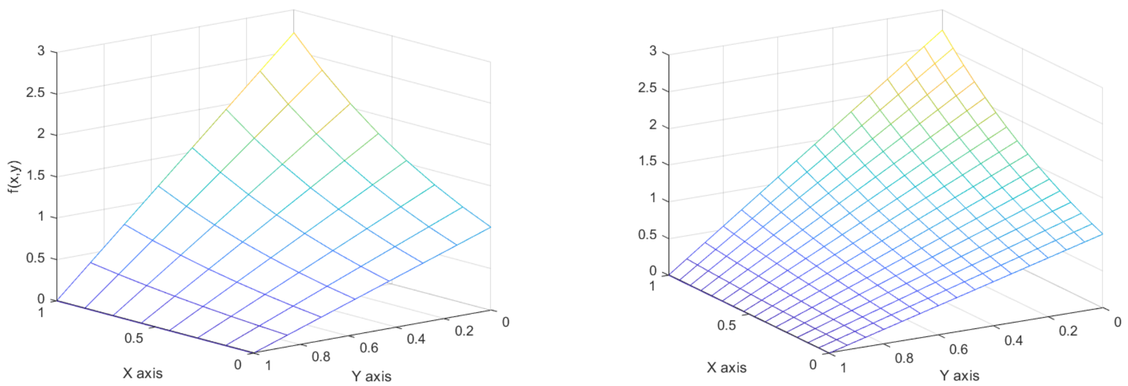

Figure 1: (a) Equation (39) with the interval difference h=0.1 and (b) Equation (39) with the interval difference h=0.0625

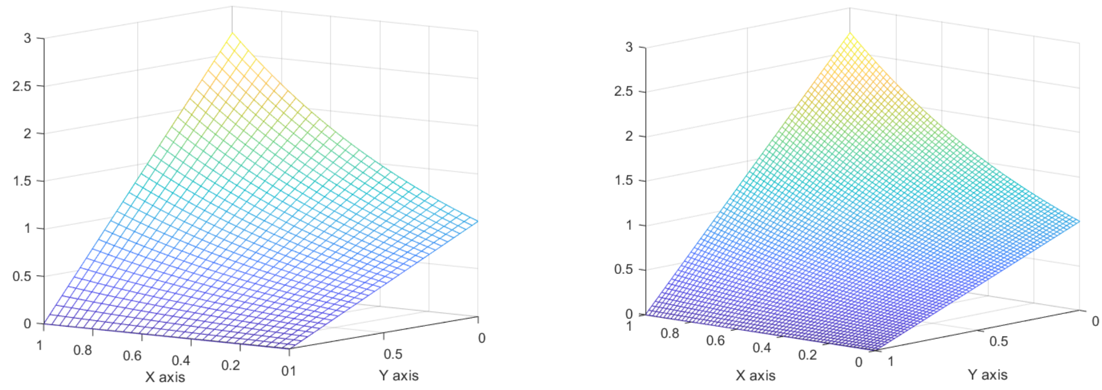

Figure 2: (c) Equation (39) with the interval difference h=0.03125 and (d) Equation (39) with the interval difference h=0.015625

The comparison between the exact and approximation solutions using different intervals, such as 0.1, 0.0625, 0.03125, and 0.015625, is presented in the aforementioned Figures 1 and 2.

5. Conclusion

In this article, we established fixed point theorems for an O-F_{ϱ} contraction mapping in O-complete O-controlled b-Branciari metric type spaces. Along with our main results, we provided appropriate examples. We have also provided an application to find the solution to the integro-differential equation. This concept can be applied for further investigations in studying fixed points for other structures in metric spaces.

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