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A note on graphs containing spanning trees with bounded leaf degree

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Abstract. For a tree T, the leaf degree of $v \in V(T)$ is the number of leaves adjacent to v, and the leaf degree of T is the maximum leaf degree among all vertices of T. In this note, we study the existence of a spanning tree with bounded leaf degree in a graph G with minimum degree δ, and provide the sufficient conditions involving the size and the distance spectral radius, to guarantee a graph G with minimum degree δ to have a spanning tree with bounded leaf degree, which generalizes a result of Zhou *et al.* [*Discrete Math.*, **347** (2024), 113927].

1. Introduction

In this paper, we only consider finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, we use $|V(G)| = n$ and $|E(G)| = m$ to denote its order and size, respectively. For $v \in V(G)$, let $d_G(v)$ be the degree of v in G. A *leaf* is a vertex of degree one. We use $\delta(G)$ (or δ for short) to denote the minimum degree of a graph G. For a subset $S \subseteq V(G)$, we use $G[S]$ and $G - S = G[V(G)\backslash S]$ to denote the subgraphs of G induced by S and $V(G)\backslash S$, respectively. For $S \subseteq V(G)$, let $I(G - S)$ denote the set of isolated vertices in $G - S$ and $i(G - S)$ denote the number of isolated vertices in $G - S$. For two disjoint graphs G_1 and G_2, we use $G_1 \cup G_2$ and $G_1 \nabla G_2$ to denote respectively the *union* and *join* of G_1 and G_2.

The study of spanning trees has been shown to be very important to graphs and has many applications in fault-tolerance networks as well as network reliability [6, 9]. Thus it is quite interesting to explore a graph G to have some special kinds of spanning trees. In [13], Ozeki and Yamashita had dealt with spanning trees having some particular properties concerning hamiltonian properties, for example, spanning trees with bounded degree, with bounded number of leaves, or with bounded number of branch vertices. Moreover, they also studied spanning trees with some other properties. It is well known that finding some special kinds of spanning trees in G is NP-complete. Therefore, it is widely believed that it is impossible to find a good necessary and sufficient condition for a graph G to have some special kinds of spanning trees. Thus we mainly deal with sufficient conditions for a graph to have some special kinds of spanning trees. There exist several sufficient conditions, including the independence number conditions and the degree sum conditions, for a graph G to possess a spanning tree with bounded degree or a bounded number of leaves. For more results can be founded in [14, 15].

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In particular, in this note, we mainly focus on spanning trees with bounded degree. In [11], Kaneko introduced the concept of leaf degree of a spanning tree. Let T be a spanning tree of a connected graph G. The *leaf degree* of a vertex $v \in V(T)$ is defined as the number of leaves adjacent to vertex v in T. Furthermore, the *leaf degree of* T is the maximum leaf degree among all the vertices of T.

The following fundamental theorem of Kaneko [11] provided a sufficient and necessary condition for the existence of a spanning tree with bounded leaf degree in a graph.

Theorem 1.1 (Kaneko [11]). *Let G be a connected graph and $k \geq 1$ be an integer. Then G has a spanning tree with leaf degree at most k if and only if $i(G-S) < (k+1)|S|$ for all $\emptyset \neq S \subseteq V(G)$.*

Note that for any graph G of order $n \leq k+1$ and any $\emptyset \neq S \subseteq V(G)$, we have

$$i(G-S) \leq \begin{cases} n-1 \leq k < k+1 = (k+1)|S|, & \text{if } |S| = 1; \\ n-2 \leq k-1 < (k+1)|S|, & \text{if } |S| \geq 2. \end{cases}$$

It follows that $i(G-S) < (k+1)|S|$ for any $S \subseteq V(G)$. Thus Theorem 1.1 implies that for any graph G of order $n \leq k+1$, G has a spanning tree with leaf degree at most k. That is, when the number of vertices in a graph G is small ($n \leq k+1$), the existence of a spanning tree with bounded leaf degree in G can be guaranteed. Therefore, a natural and interesting question is *"whether it can also be guaranteed when there are enough edges. If so, what is the number of edges?"* Inspired by Theorem 1.1, we establish the following edge number condition to ensure that a graph G with minimum degree δ to have a spanning tree with bounded leaf degree.

Theorem 1.2. *Let G be a connected graph of order $n \geq (k+6)\delta - k + 2$, where δ is the minimum degree of G and $k \geq 1$ is an integer. If $|E(G)| \geq |E(K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1))|$, then G has a spanning tree with leaf degree at most k, unless $G \cong K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$.*

In the past decades, the study of relationships between the related eigenvalues of a graph G and the existence of some special kinds of spanning trees in G has received more and more attention [1, 7, 11, 13, 16, 21]. We won't list them all here, but we'll focus primarily on those related to a spanning tree with constrained leaf degree. Ao, Liu and Yuan [2] established several tight spectral conditions, in terms of the eigenvalues of the adjacency and signless Laplacian matrices, for a graph to have a spanning tree with leaf degree at most k. Moreover, they [3] also provided a tight spectral radius condition for the existence of a spanning tree with leaf degree at most k in a connected graph with minimum degree δ.

The *distance matrix* $\mathcal{D}(G)$ is a real symmetric matrix whose (i, j)-entry is the distance d_{ij} between v_i and v_j, and the distance is defined as the length of a shortest path from v_i to v_j. The *distance spectral radius* of G, denoted by $\partial_1(G)$, is the largest eigenvalue of its distance matrix $\mathcal{D}(G)$. Very recently, the problem of finding distance spectral radius conditions for graphs having certain structural properties or containing some specified kinds of spanning trees has received considerable attention. Zhang and Lin [18] presented a distance spectral radius condition to guarantee the existence of a perfect matching in a graph. In [17], Zhang and Dam gave a sufficient condition in terms of the eigenvalues of the distance matrix for the k-extendability of a graph. Zhou and Wu [20] proved a upper bound for distance spectral radius in a connected graph G to guarantee the existence of a spanning k-tree in G. Zhou, Sun and Liu [19] established an upper bound on the distance spectral radius of G to guarantee that G admits a spanning tree with leaf degree at most k.

Inspired by the above mentioned results, we generalize a recent result of Zhou et al [19] to graphs with minimum degree δ and prove the following result.

Theorem 1.3. *Let G be a connected graph of order $n \geq 4k\delta + 12\delta + 2$, where δ is the minimum degree of G and $k \geq 1$ is an integer. If $\partial_1(G) \leq \partial_1(K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1))$, then G has a spanning tree with leaf degree at most k, unless $G \cong K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$.*

The remainder of the paper is organized as follows. In Section 2, we present some preliminary results, which will be used in the subsequent. The proof of Theorem 1.2 is given in Section 3. Section 4 provides the proof of Theorem 1.3.

2. Preliminary

In this section, we present some preliminary results and lemmas which will be used in the subsequent sections.

The eigenvalues of an $n \times n$ real symmetric matrix M are denoted by $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$, where we always assume the eigenvalues to be arranged in nonincreasing order. Given a partition $\pi = (X_1, X_2, \cdots, X_t)$ of the set $\{1, 2, \cdots, n\}$ and a matrix B whose rows and columns are labelled with elements in $\{1, 2, \cdots, n\}$, B can be expressed as the following partitioned matrix

$$B = \begin{pmatrix} B_{11} & \cdots & B_{1t} \\ \vdots & \ddots & \vdots \\ B_{t1} & \cdots & B_{tt} \end{pmatrix}$$

with respect to π. The *quotient matrix* B_π of B with respect to π is the t by t matrix (b_{ij}) such that each entry b_{ij} is the average row sum of B_{ij}. If the row sum of each block B_{ij} is a constant, then the partition is *equitable*.

Lemma 2.1 (Brouwer, Godsil [4, 8]). *Let M be a real symmetric matrix. If B_π is an equitable quotient matrix of M, then the eigenvalues of B_π are also eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then $\lambda_1(M) = \lambda_1(B_\pi)$.*

Lemma 2.2 (Berman, Horn [5, 10]). *Let $m < n$, A and B be $n\times n$ and $m\times m$ nonnegative matrices with the spectral radius $\lambda_1(A)$ and $\lambda_1(B)$, respectively. If B is a principal submatrix of A, then $\lambda_1(B) \leq \lambda_1(A)$. In addition, if A is irreducible, then $\lambda_1(B) < \lambda_1(A)$.*

Lemma 2.3 (Minc [12]). *Let G be a connected graph with two nonadjacent vertices $u, v \in V(G)$. Then $\partial_1(G+uv) < \partial_1(G)$.*

3. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2, which presents a sufficient edge number condition to ensure a graph with minimum degree δ to have a spanning tree with bounded leaf degree.

Proof of Theorem 1.2. Assume that G has no a spanning tree with leaf degree at most k. By Theorem 1.1, there exists a non-empty subset $S \subseteq V(G)$ satisfying $i(G-S) \geq (k+1)|S|$. Let $|S| = s$. Then G is a spanning subgraph of $G' = K_s\nabla(K_{n-(k+2)s} \cup ((k+1)s)K_1)$. Note that G has the minimum degree δ. Then $s \geq \delta$. It is obvious that

$$|E(G)| \leq |E(G')|, \tag{1}$$

where the equality holds if and only if $G \cong G'$. We proceed by the following two cases.

Case 1. $s = \delta$.

Then $G' \cong K_\delta\nabla(K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$. By (1), we have

$$|E(G)| \leq |E(K_\delta\nabla(K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1))|,$$

with equality holding if and only if $G \cong K_\delta\nabla(K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$. Observe that $K_\delta\nabla(K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$ has no a spanning tree with leaf degree at most k. Hence, $G \cong K_\delta\nabla(K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$.

Case 2. $s \geq \delta + 1$.

The vertex set of G' can be partitioned as $V(G') = V(K_s) \cup V(K_{n-(k+2)s}) \cup V(((k+1)s)K_1)$, where $V(K_s) = \{v_1, v_2, \cdots, v_s\}$, $V(K_{n-(k+2)s}) = \{w_1, w_2, \cdots, w_{n-(k+2)s}\}$ and $V(((k+1)s)K_1) = \{u_1, u_2, \cdots, u_{(k+1)s}\}$. Let

$$E_1 = \{u_iw_j \mid (k+1)\delta + 1 \leq i \leq (k+1)s,\ 1 \leq j \leq n-(k+2)s\},$$

$$E_2 = \{u_i u_j \mid (k+1)\delta + 1 \le i \le (k+1)s - 1,\ i+1 \le j \le (k+1)s\}$$

and

$$E_3 = \{v_i u_j \mid \delta + 1 \le i \le s,\ 1 \le j \le (k+1)\delta\}.$$

Assume that $G^* = G' + E_1 + E_2 - E_3$. It is obvious that $G^* \cong K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$. Since $n \ge (k+2)s$, we have

$$\begin{aligned}
|E(G^*)| - |E(G')| &= (k+1)(s-\delta)(n-(k+2)s) + \frac{1}{2}(k+1)(s-\delta)((k+1)(s-\delta)-1) \\
&\quad - (s-\delta)(k+1)\delta \\
&= \frac{1}{4}(k+1)(s-\delta)[4n - 2(k+3)s - 2(k+3)\delta - 2] \\
&\ge \frac{1}{4}(k+1)(s-\delta)[2n - 2s - 2(k+3)\delta - 2] \\
&\ge \frac{1}{4}(k+1)(s-\delta)(n + ks - 2(k+3)\delta - 2) \\
&\ge \frac{1}{4}(k+1)(s-\delta)(n + k(\delta+1) - 2(k+3)\delta - 2) \\
&> 0.
\end{aligned}$$

Then we have $|E(G')| < |E(G^*)|$. Together with (1), we get $|E(G)| < |E(G^*)| = |E(K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1))|$, a contradiction to the condition. This completes the proof. □

4. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3, which presents a sufficient distance spectral radius condition to ensure a graph with minimum degree δ to have a spanning tree with bounded leaf degree, which generalizes a result of Zhou et al. [19, Theorem 1.1]

Proof of Theorem 1.3. Assume that G has no a spanning tree with leaf degree at most k. By Theorem 1.1, there exists a non-empty subset $S \subseteq V(G)$ satisfying $i(G-S) \ge (k+1)|S|$. Let $|S| = s$. Then G is a spanning subgraph of $G' = K_s \nabla (K_{n-(k+2)s} \cup ((k+1)s)K_1)$. Note that G has the minimum degree δ. Then $s \ge \delta$. By Lemma 2.3, we have

$$\partial_1(G) \ge \partial_1(G'), \tag{2}$$

with equality holding if and only if $G \cong G'$. Then we proceed by the following two cases.

Case 1. $s = \delta$.

Then $G' \cong K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$. By (2), we have

$$\partial_1(G) \ge \partial_1(K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)),$$

with equality holding if and only if $G \cong K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$. Observe that $K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$ has no a spanning tree with leaf degree at most k. Hence, $G \cong K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)$.

Case 2. $s \ge \delta + 1$.

Let B_s be an equitable quotient matrix of the distance matrix $\mathcal{D}(G')$ with respect to the partition $V(K_s) \cup V(K_{n-(k+2)s}) \cup V(((k+1)s)K_1)$. One can see that

$$B_s = \begin{pmatrix} s-1 & n-(k+2)s & (k+1)s \\ s & n-(k+2)s-1 & 2(k+1)s \\ s & 2n-2(k+2)s & 2((k+1)s-1) \end{pmatrix}.$$

By a simple calculation, the characteristic polynomial B_s is

$$\Phi(B_s, x) = x^3 - (n + ks + s - 4)x^2 + (2k^2s^2 + 7ks^2 + 5s^2 - 2kns - 2ns - ks - s - 3n + 5)x \\ - k^2s^3 - 3ks^3 - 2s^3 + kns^2 + ns^2 + 2k^2s^2 + 7ks^2 + 5s^2 - 2kns - 2ns - 2n + 2.$$

By Lemma 2.1, we know that $\partial_1(G') = \lambda_1(B_s)$ is the largest root of the equation $\Phi(B_s, x) = 0$. Note that $\mathcal{D}(K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1))$ has the equitable quotient matrix B_δ, which is obtained by replacing s with δ in B_s. Similarly, by Lemma 2.1, $\partial_1(K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k+1)\delta)K_1)) = \lambda_1(B_\delta)$ is the largest root of $\Phi(B_\delta, x) = 0$. Then

$$\Phi(B_s, x) - \Phi(B_\delta, x) = (s - \delta)(k + 1)[-x^2 + ((2k + 5)(s + \delta) - 2n - 1)x - ks^2 - 2s^2 + ns \\ - \delta ks + 2ks - 2\delta s + 5s + \delta n - 2n - \delta^2 k + 2\delta k - 2\delta^2 + 5\delta].$$

We claim that $\Phi(B_s, x) - \Phi(B_\delta, x) < 0$ for $x \in [n - (k + 1)\delta - 1, +\infty)$. In fact, let

$$f(x) = -x^2 + ((2k + 5)(s + \delta) - 2n - 1)x - ks^2 - 2s^2 + ns - \delta ks + 2ks - 2\delta s + 5s \\ + \delta n - 2n - \delta^2 k + 2\delta k - 2\delta^2 + 5\delta.$$

Note that $s \geq \delta + 1$. Then we only need to prove $f(x) < 0$ on $x \in [n - (k + 1)\delta - 1, +\infty)$. Note that $n \geq (k + 2)s$. Then $\delta + 1 \leq s \leq \frac{n}{k+2}$ and the symmetry axis of $f(x)$ is $x = \frac{(2k+5)(s+\delta)-2n-1}{2} < n - (k + 1)\delta - 1$ by $n \geq 4\delta k + 12\delta + 2$, we obtain

$$\begin{aligned} f(x) \leq\, & f(n - (k + 1)\delta - 1) \\ =\, & -(k + 2)s^2 + (2kn + 6n - 2\delta k^2 - 8\delta k - 7\delta)s - 3n^2 + (6\delta k + 10\delta + 1)n - 3\delta^2 k^2 \\ & - 10\delta^2 k - \delta k - 8\delta^2 - \delta \\ \leq\, & -(k + 2)\left(\frac{n}{k + 2}\right)^2 + (2kn + 6n - 2\delta k^2 - 8\delta k - 7\delta)\left(\frac{n}{k + 2}\right) - 3n^2 + (6\delta k + 10\delta + 1)n \\ & - 3\delta^2 k^2 - 10\delta^2 k - \delta k - 8\delta^2 - \delta \\ =\, & -\frac{1}{k + 2}[(k + 1)n^2 - (4\delta k^2 + 14\delta k + k + 13\delta + 2)n + 3\delta^2 k^3 + 16\delta^2 k^2 + \delta k^2 + 28\delta^2 k \\ & + 3\delta k + 16\delta^2 + 2\delta] \\ \leq\, & -\frac{1}{k + 2}[(k + 1)(4\delta k + 12\delta + 2)^2 - (4\delta k^2 + 14\delta k + k + 13\delta + 2)(4\delta k + 12\delta + 2) + 3\delta^2 k^3 \\ & + 16\delta^2 k^2 + \delta k^2 + 28\delta^2 k + 3\delta k + 16\delta^2 + 2\delta] \\ =\, & -\frac{1}{k + 2}[3\delta^2 k^3 + (24\delta^2 + 5\delta)k^2 + (48\delta^2 + 19\delta + 2)k + 4\delta^2] \\ <\, & 0. \end{aligned}$$

It follows that $\Phi(B_s, x) < \Phi(B_\delta, x)$ for $x \in [n - (k + 1)\delta - 1, +\infty)$. Note that $K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k + 1)\delta)K_1)$ contains $K_{n-(k+1)\delta}$ as a proper subgraph. By Lemma 2.2, we have

$$\partial_1(K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k + 1)\delta)K_1)) > \partial_1(K_{n-(k+1)\delta}) = n - (k + 1)\delta - 1,$$

and so $\lambda_1(B_s) > \lambda_1(B_\delta)$. This means that

$$\partial_1(G') > \partial_1(K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k + 1)\delta)K_1)).$$

Therefore,

$$\partial_1(G) \geq \partial_1(G') > \partial_1(K_\delta \nabla (K_{n-(k+2)\delta} \cup ((k + 1)\delta)K_1)),$$

a contradiction. This completes the proof. □

5. Concluding remarks

In this paper, we first provide an edge number condition to ensure a graph with minimum degree δ has a spanning tree with leaf degree at most k. Furthermore, we also provide a tight distance spectral radius condition for the existence of a spanning tree with leaf degree at most k in a graph with minimum degree δ. Inspired by the work above, it is natural to ask whether there are other types of spanning trees can be considered using the eigenvalues of the distance matrix? For further progress, we propose the following problem.

Problem 5.1. *Let G be a connected graph of order n with minimum degree δ which has no spanning k-ended tree. What is the minimum distance spectral radius of G? Moreover, characterize all the extremal graphs.*

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Declarations

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