Filomat 38:29 (2024), 10351–10372
https://doi.org/10.2298/FIL2429351C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On inertial subgradient extragradient algorithms for equilibria systems and hierarchical variational inequalities for countable nonexpansive mappings

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Abstract. In a real Hilbert space, let the SGEP, VIP, HVI and CFPP represent a system of generalized equilibrium problems, a pseudomonotone variational inequality problem, a hierarchical variational inequality and a common fixed-point problem of countable nonexpansive mappings and an asymptotically nonexpansive mapping, respectively. We propose two relaxed inertial subgradient extragradient implicit rules with line-search process for finding a solution of the HVI with the SGEP, VIP and CFPP constraints. The designed algorithms are on the basis of the subgradient extragradient rule with line-search process, inertial iteration approach, hybrid deepest-descent method and Mann implicit iteration technique. Under suitable conditions, we prove the strong convergence of the designed algorithms to a solution of the HVI with the SGEP, VIP and CFPP constraints.

1. Introduction

Let H be a real Hilbert space and $\emptyset \neq C \subset H$ be a closed and convex set. For a nonlinear operator $S : C \to H$, we use the $\mathrm{Fix}(S)$ to indicate the fixed-point set of S. S is said to be asymptotically nonexpansive if $\exists\{\theta_l\}_{l=1}^{\infty} \subset [0, +\infty)$ s.t. $\theta_l \to 0$ $(l \to \infty)$ and $\theta_l\|v-w\|+\|v-w\| \geq \|S^l v - S^l w\|$, $\forall v, w \in C, l \geq 1$. If $\theta_l = 0$, $\forall l \geq 1$, S is called nonexpansive. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction. Consider the equilibrium problem (EP) of determining its equilibrium points, i.e., the set $\mathrm{EP}(\Theta) = \{u \in C : \Theta(u, v) \geq 0, \ \forall v \in C\}$. In the framework of the equilibrium problems theory, ones can take advantage of a unified approach for treating numerous problems arising in the physics, optimization, structural analysis, transportation, finance and economics. Throughout, we assume that

(H1) $\Theta(v, v) = 0$, $\forall v \in C$; (H2) Θ is monotone, i.e., $\Theta(v, w) + \Theta(w, v) \leq 0$, $\forall v, w \in C$; (H3) $\lim_{\lambda\to 0^+} \Theta((1-\lambda)v + \lambda u, w) \leq \Theta(v, w)$, $\forall u, v, w \in C$; and (H4) $w \mapsto \Theta(v, w)$ is convex and lower semicontinuous (l.s.c.) for every $v \in C$.

In 1994, Blum and Oettli [1] presented the following lemma, which had performed an important role in settling the equilibrium problems.

2020 *Mathematics Subject Classification*. Primary 47H09; Secondary 47H10, 47J20, 47J25.
Keywords. relaxed inertial subgradient extragradient implicit rule, variational inequality problem, system of generalized equilibria, countable nonexpansive mappings, common fixed point problem.
Received: 05 March 2024; Revised: 16 July 2024; Accepted: 14 August 2024
Communicated by Adrian Petrusel
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Lemma 1.1. *([1]) Suppose that $\Theta : C \times C \to \mathbf{R}$ suits for the hypotheses (H1)-(H4). For any $\tau > 0$, let the mapping $T_\tau^\Theta : H \to C$ be formulated as:*

$$T_\tau^\Theta(u) := \{v \in C : \Theta(v,w) + \frac{1}{\tau}\langle w - v, v - u\rangle \geq 0, \ \forall w \in C\}, \quad \forall u \in H.$$

Then T_τ^Θ is well defined and the following hold: (i) T_τ^Θ is single-valued, and firmly nonexpansive, i.e., $\|T_\tau^\Theta v - T_\tau^\Theta w\|^2 \leq \langle T_\tau^\Theta v - T_\tau^\Theta w, v - w\rangle, \ \forall v, w \in H$; and (ii) $\mathrm{Fix}(T_\tau^\Theta) = \mathrm{EP}(\Theta)$, and $\mathrm{EP}(\Theta)$ is convex and closed.

The EP reduces to the classical variational inequality problem (VIP) of finding $x \in C$ s.t. $\langle Ax, y - x\rangle \geq 0, \ \forall y \in C$, where A is a self-mapping on H. The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$. It is well known that, one of the most popular techniques for settling the VIP is the extragradient method with weak convergence, which was put forward by Korpelevich [21] in 1976. Up to now, the vast literature on the Korpelevich extragradient approach reveals that numerous authors have paid extensive attention to it and ameliorated it in different manners; see e.g., [2–5, 9–15, 17, 18, 20, 22, 24–34, 36–46] and references therein.

Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions and suppose that $B_1, B_2 : H \to H$ are two nonlinear mappings. In 2010, Ceng and Yao [7] considered the following problem of finding $(x^*, y^*) \in C \times C$ s.t.

$$\begin{cases} \Theta_1(x^*, x) + \langle B_1 y^*, x - x^*\rangle + \frac{1}{\mu_1}\langle x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \Theta_2(y^*, y) + \langle B_2 x^*, y - y^*\rangle + \frac{1}{\mu_2}\langle y^* - x^*, y - y^*\rangle \geq 0, & \forall y \in C, \end{cases} \tag{1}$$

which is referred to as a system of generalized equilibrium problems (SGEP). In particular, in case $\Theta_1 = \Theta_2 = 0$, then the SGEP reduces to the following general system of variational inequalities (GSVI) investigated in [5]: Find $(x^*, y^*) \in C \times C$ s.t.

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, y - y^*\rangle \geq 0 & \forall y \in C. \end{cases}$$

The SGEP (1) can be transformed into the fixed-point problem (FPP).

Lemma 1.2. *([7]) Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions s.t. the hypotheses (H1)-(H4) hold, and suppose that the mappings $B_1, B_2 : H \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$, respectively. Then, for given $x^*, y^* \in C$, (x^*, y^*) is a solution of SGEP (1) if and only if $x^* \in \mathrm{Fix}(G)$, where $G := T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1) T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)$ and $y^* = T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)x^*$.*

On the other hand, assume that the mappings $B_1, B_2 : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $f : C \to C$ be contractive with constant $\delta \in [0,1)$ and $F : C \to H$ be κ-Lipschitzian and η-strongly monotone with constants $\kappa, \eta > 0$ such that $\delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$ for $\mu \in (0, \frac{2\eta}{\kappa^2})$. Let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_l\}$ such that $\Omega := \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \neq \emptyset$ with $G := P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2)$ for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. In 2018, Cai, Shehu and Iyiola [2] proposed the modified viscosity implicit rule for finding an element of Ω, i.e., for any $x_1 \in C$, let the sequence $\{x_l\}$ be formulated below

$$\begin{cases} p_l = \beta_l x_l + (1 - \beta_l) y_l, \\ v_l = P_C(p_l - \mu_2 B_2 p_l), \\ y_l = P_C(v_l - \mu_1 B_1 v_l), \\ x_{l+1} = P_C[\alpha_l f(x_l) + (I - \alpha_l \mu F) S^l y_l], \quad \forall l \geq 1, \end{cases}$$

where $\{\alpha_l\}, \{\beta_l\} \subset (0,1]$ s.t. (i) $\sum_{l=1}^\infty |\alpha_{l+1} - \alpha_l| < \infty$, $\sum_{l=1}^\infty \alpha_l = \infty$; (ii) $\lim_{l\to\infty} \alpha_l = 0$, $\lim_{l\to\infty} \frac{\theta_l}{\alpha_l} = 0$; (iii) $0 < \varepsilon \leq \beta_l \leq 1$, $\sum_{l=1}^\infty |\beta_{l+1} - \beta_l| < \infty$; and (iv) $\sum_{l=1}^\infty \|S^{l+1} y_l - S^l y_l\| < \infty$. It was shown in [2] that the sequence $\{x_l\}$ converges strongly to an element $x^* \in \Omega$, which is a unique solution of the hierarchical variational inequality (HVI): $\langle (\mu F - f)x^*, p - x^*\rangle \geq 0$, $\forall p \in \Omega$. Very recently, Ceng and Shang [4] put forth the hybrid inertial subgradient extragradient method with line-search process for solving the pseudomonotone VIP with Lipschitz continuous A and the common fixed-point problem (CFPP) of finite nonexpansive mappings $\{S_l\}_{l=1}^N$ and an asymptotically nonexpansive mapping S in a real Hilbert space H. Assume that $\Omega := \bigcap_{l=0}^N \mathrm{Fix}(S_l) \cap \mathrm{VI}(C, A) \neq \emptyset$ with $S_0 := S$. Given a contraction $f : H \to H$ with constant $\delta \in [0, 1)$, and an η-strongly monotone and κ-Lipschitzian mapping $F : H \to H$ with $\delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$ for $\mu \in (0, \frac{2\eta}{\kappa^2})$. Let $\{\varepsilon_l\} \subset [0, 1]$ and $\{\alpha_l\}, \{\beta_l\} \subset (0, 1)$ with $\alpha_l + \beta_l < 1$, $\forall l \geq 1$. Moreover, one writes $S_l := S_{l \mathrm{mod} N}$ for integer $l \geq 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$, i.e., whenever $l = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, one has that $S_l = S_N$ if $q = 0$ and $S_l = S_q$ if $0 < q < N$. Given $\gamma > 0$, $\ell \in (0, 1)$, $\nu \in (0, 1)$ and $x_1, x_0 \in H$ arbitrarily, one calculates x_{k+1} $(k \geq 1)$ below:

Step 1. Set $p_l = S_l x_l + \varepsilon_l(S_l x_l - S_l x_{l-1})$ and calculate $y_l = P_C(p_l - \zeta_l A p_l)$, where ζ_l is chosen to be the largest $\zeta \in \{\gamma, \gamma\ell, \gamma\ell^2, ...\}$ satisfying $\zeta\|Ap_l - Ay_l\| \leq \nu\|p_l - y_l\|$.

Step 2. Calculate $z_l = P_{C_l}(p_l - \zeta_l A y_l)$ with $C_l := \{y \in H : \langle p_l - \zeta_l A p_l - y_l, y_l - y\rangle \geq 0\}$.

Step 3. Calculate $x_{l+1} = \alpha_l f(x_l) + \beta_l x_l + ((1 - \beta_l)I - \alpha_l \mu F)S^l z_l$.

Again set $l := l + 1$ and go to Step 1.

Under mild conditions, it was shown in [4] that if $S^l z_l - S^{l+1} z_l \to 0$, then $\{x_l\}$ converges strongly to $x^* \in \Omega$ if and only if $x_l - x_{l+1} \to 0$ and $x_l - y_l \to 0$ as $l \to \infty$. Next, let the HVI, VIP, SGEP, and CFPP represent a hierarchical variational inequality, a pseudomonotone variational inequality problem, a system of generalized equilibrium problems, and a common fixed-point problem of countable nonexpansive mappings and an asymptotically nonexpansive mapping, respectively. We propose two relaxed inertial subgradient extragradient implicit rules with line-search process for finding a solution of the HVI with the CFPP, SGEP and VIP constraints. The suggested algorithms are based on the subgradient extragradient rule with line-search process, inertial iteration approach, hybrid deepest-descent method and Mann implicit iteration technique. Under mild restrictions, we prove the strong convergence of the suggested algorithms to a solution of the HVI with the CFPP, SGEP and VIP constraints. In addition, an illustrated instance is used to illustrate the feasibility and implementability of our proposed rules.

The architecture of this article is arranged below: In Sect. 2, we release some concepts and basic tools for further use. Sect. 3 provides the convergence analysis of the suggested algorithms. Lastly, Sect. 4 applies our main results to settle the SGEP, VIP and CFPP in an illustrated example. Our results improve and extend the corresponding ones in [2, 4, 31].

2. Preliminaries

Let C be a nonempty convex and closed subset of a real Hilbert space H. For all $v, y \in C$, an operator $T : C \to H$ is called

(a) L-Lipschitzian if $\exists L > 0$ s.t. $\|Tv - Ty\| \leq L\|v - y\|$;

(b) pseudomonotone if $\langle Tv, y - v\rangle \geq 0 \Rightarrow \langle Ty, y - v\rangle \geq 0$;

(c) monotone if $\langle Tv - Ty, v - y\rangle \geq 0$;

(d) α-strongly monotone if $\exists \alpha > 0$ s.t. $\langle Tv - Ty, v - y\rangle \geq \alpha\|v - y\|^2$;

(e) β-inverse-strongly monotone if $\exists \beta > 0$ s.t. $\langle Tv - Ty, v - y\rangle \geq \beta\|Tv - Ty\|^2$;

(f) sequentially weakly continuous if $\forall \{y_l\} \subset C$, the relation holds: $y_l \rightharpoonup y \Rightarrow Ty_l \rightharpoonup Ty$.

$\forall v \in H$, $\exists$ | (nearest point) $P_C(v) \in C$ s.t. $\|v - P_C(v)\| \leq \|v - u\|$, $\forall u \in C$. P_C is called a nearest point (or metric) projection of H onto C. The following conclusions hold (see [16]):

(a) $\langle v - y, P_C(v) - P_C(y)\rangle \geq \|P_C(v) - P_C(y)\|^2$, $\forall v, y \in H$;

(b) $y = P_C(v) \Leftrightarrow \langle v - y, x - y\rangle \leq 0$, $\forall v \in H, x \in C$;

(c) $\|v - y\|^2 \geq \|v - P_C(v)\|^2 + \|y - P_C(v)\|^2$, $\forall v \in H, y \in C$;

(d) $\|v - y\|^2 = \|v\|^2 - \|y\|^2 - 2\langle v - y, y\rangle$, $\forall v, y \in H$;

(e) $\|sv + (1 - s)y\|^2 = s\|v\|^2 + (1 - s)\|y\|^2 - s(1 - s)\|v - y\|^2$, $\forall v, y \in H, s \in [0, 1]$.

The following inequality is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$:

$$\|v + y\|^2 \leq \|v\|^2 + 2\langle y, v + y\rangle, \quad \forall v, y \in H. \tag{2}$$

The following concept and two propositions can be found in [8].

Definition 2.1. *Let $\{\xi_k\}_{k=1}^{\infty} \subset [0,1]$ and suppose that $\{S_k\}_{k=1}^{\infty}$ is a sequence of nonexpansive self-mappings on C. For each $m \geq 1$, the self-mapping W_m on C is formulated below:*

$$\begin{cases} U_{m,m+1} = I, \\ U_{m,m} = \xi_m S_m U_{m,m+1} + (1-\xi_m)I, \\ U_{m,m-1} = \xi_{m-1} S_{m-1} U_{m,m} + (1-\xi_{m-1})I, \\ \cdots \\ U_{m,k} = \xi_k S_k U_{m,k+1} + (1-\xi_k)I, \\ \cdots \\ U_{m,2} = \xi_2 S_2 U_{m,3} + (1-\xi_2)I, \\ W_m = U_{m,1} = \xi_1 S_1 U_{m,2} + (1-\xi_2)I. \end{cases} \tag{3}$$

Then, W_m is refereed to as a W-mapping generated by $S_m, \cdots, S_2, S_1$ and $\xi_m, \cdots, \xi_2, \xi_1$.

Proposition 2.2. *Let $\{\xi_k\}_{k=1}^{\infty} \subset (0,1]$ and suppose that $\{S_k\}_{k=1}^{\infty}$ is a sequence of nonexpansive self-mappings on C, such that $\bigcap_{k=1}^{\infty} \mathrm{Fix}(S_k) \neq \emptyset$. Then,*

(a) W_m is of nonexpansivity and $\bigcap_{k=1}^{m} \mathrm{Fix}(S_k) = \mathrm{Fix}(W_m),\ \forall m \geq 1$;

(b) $\forall q \in C,\ k \geq 1$, $\lim_{m\to\infty} U_{m,k}q$ exists;

(c) the mapping W, formulated as $Wq := \lim_{m\to\infty} W_m q = \lim_{m\to\infty} U_{m,1}q,\ \forall q \in C$, is a nonexpansive mapping s.t. $\bigcap_{k=1}^{\infty} \mathrm{Fix}(S_k) = \mathrm{Fix}(W)$, and it is refereed to as the W-mapping generated by $S_1, S_2, \ldots$ and $\xi_1, \xi_2, \ldots$.

Proposition 2.3. *Let $\{\xi_k\}_{k=1}^{\infty} \subset (0,\epsilon]$ for certain $\epsilon \in (0,1)$ and suppose that $\{S_k\}_{k=1}^{\infty}$ is a sequence of nonexpansive self-mappings on C, such that $\bigcap_{k=1}^{\infty} \mathrm{Fix}(S_k) \neq \emptyset$. Then, $\lim_{m\to\infty} \sup_{p\in D} \|W_m p - Wp\| = 0$ for each bounded set $D \subset C$.*

Proposition 2.4. *([19]) Let H_1 and H_2 be two real Hilbert spaces. Suppose that $F : H_1 \to H_2$ is uniformly continuous on each bounded subset of H_1 and D is bounded in H_1. Then, $F(D)$ is bounded.*

The following lemmas will be useful for concluding our main results in this paper.

Lemma 2.5. *Let the mapping $B : H \to H$ be η-inverse-strongly monotone. Then, for a given $\mu \geq 0$,*

$$\|(I-\mu B)y - (I-\mu B)z\|^2 \leq \|y-z\|^2 - \mu(2\eta - \mu)\|By - Bz\|^2.$$

In particular, if $0 \leq \mu \leq 2\eta$, then $I - \mu B$ is nonexpansive.

Using Lemma 1.1 and Lemma 2.5, we immediately obtain the following lemma.

Lemma 2.6. *([7]) Assume that $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ are two bifunctions satisfying the hypotheses (H1)-(H4), and the mappings $B_1, B_2 : H \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : H \to C$ be defined as $G := T_{\mu_1}^{\Theta_1}(I - \mu_1 B_1) T_{\mu_2}^{\Theta_2}(I - \mu_2 B_2)$. Then $G : H \to C$ is a nonexpansive mapping provided $0 < \mu_1 \leq 2\alpha$ and $0 < \mu_2 \leq 2\beta$.*

In particular, if $\Theta_1 = \Theta_2 = 0$, using Lemma 1.1 we deduce that $T_{\mu_1}^{\Theta_1} = T_{\mu_2}^{\Theta_2} = P_C$. Thus, from Lemma 2.6 we obtain the corollary below.

Corollary 2.7. *([5]) Suppose that the mappings $B_1, B_2 : H \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : H \to C$ be defined as $G := P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2)$. If $0 < \mu_1 \leq 2\alpha$ and $0 < \mu_2 \leq 2\beta$, then $G : H \to C$ is nonexpansive.*

Lemma 2.8. *Assume that $A : C \to H$ is pseudomonotone and continuous. Then $v \in C$ is a solution to the VIP $\langle Av, w - v\rangle \geq 0,\ \forall w \in C$, if and only if $\langle Aw, w - v\rangle \geq 0,\ \forall w \in C$.*

Lemma 2.9. *([35]) Suppose that $\{a_l\}$ is a sequence in $[0,\infty)$ s.t. $a_{l+1} \le (1-\varsigma_l)a_l + \varsigma_l \nu_l$, $\forall l \ge 1$, where $\{\varsigma_l\}$ and $\{\nu_l\}$ are sequences of real numbers s.t. (i) $\{\varsigma_l\} \subset [0,1]$ and $\sum_{l=1}^{\infty} \varsigma_l = \infty$, and (ii) $\limsup_{l\to\infty} \nu_l \le 0$ or $\sum_{l=1}^{\infty} |\varsigma_l \nu_l| < \infty$. Then $\lim_{l\to\infty} a_l = 0$.*

Lemma 2.10. *([6]) Assume that X is a Banach space which admits a weakly continuous duality mapping. Let C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$. Then $I - T$ is demiclosed at zero, i.e., if $\{v_l\}$ is a sequence in C such that $v_l \rightharpoonup v \in C$ and $(I-T)v_l \to 0$, then $(I-T)v = 0$, where I is the identity mapping of X.*

Lemma 2.11. *([23]) Let $\{\mathbf{\Gamma}_m\}$ be a sequence of real numbers that does not decrease at infinity in the sense that, $\exists \{\mathbf{\Gamma}_{m_k}\} \subset \{\mathbf{\Gamma}_m\}$ s.t. $\mathbf{\Gamma}_{m_k} < \mathbf{\Gamma}_{m_k+1}$ $\forall k \ge 1$. Let the sequence $\{\phi(m)\}_{m\ge m_0}$ of integers be formulated below:*

$$\phi(m) = \max\{k \le m : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\},$$

with integer $m_0 \ge 1$ satisfying $\{k \le m_0 : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\} \ne \emptyset$. Then the following hold:
(i) $\phi(m_0) \le \phi(m_0+1) \le \cdots$ and $\phi(m) \to \infty$;
(ii) $\mathbf{\Gamma}_{\phi(m)} \le \mathbf{\Gamma}_{\phi(m)+1}$ and $\mathbf{\Gamma}_m \le \mathbf{\Gamma}_{\phi(m)+1}$ $\forall m \ge m_0$.

Lemma 2.12. *([35]) Let $\lambda \in (0,1]$ and Let $S : C \to C$ be a nonexpansive mapping. Let the mapping $S^\lambda : C \to H$ be formulated as $S^\lambda v := (I - \lambda\mu F)Sv$ $\forall v \in C$ with $F : C \to H$ being κ-Lipschitzian and η-strongly monotone. Then S^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$, i.e., $\|S^\lambda v - S^\lambda w\| \le (1-\lambda\tau)\|v-w\|$, $\forall v, w \in C$, where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1]$.*

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Assume always that the following conditions hold:

(r1): $\{S_k\}_{k=1}^{\infty}$ is a sequence of nonexpansive mappings on H and S is asymptotically nonexpansive mapping on H with a sequence $\{\theta_k\}$;

(r2): W_n is the W-mapping generated by $S_n, S_{n-1}, ..., S_1$ and $\xi_n, \xi_{n-1}, ..., \xi_1$, with $\{\xi_k\}_{k=1}^{\infty} \subset (0,\epsilon]$ for some $\epsilon \in (0,1)$;

(r3): A is pseudomonotone and L-Lipschitzian mapping on H, s.t. $\|Au\| \le \liminf_{n\to\infty} \|Au_n\|$ for each $\{u_n\} \subset C$ with $u_n \rightharpoonup u$;

(r4): $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ are two bifunctions satisfying the hypotheses (H1)-(H4), and $B_1, B_2 : H \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively;

(r5): $f : H \to H$ is a contractive map with constant $\delta \in [0,1)$, and $F : H \to H$ is η-strongly monotone and κ-Lipschitzian such that $\delta < \tau := 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)}$ for $\mu \in (0, \frac{2\eta}{\kappa^2})$;

(r6): $\Omega = \bigcap_{k=0}^{\infty} \mathrm{Fix}(S_k) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C,A) \ne \emptyset$ with $S_0 := S$ and $G := T_{\mu_1}^{\Theta_1}(I-\mu_1 B_1) T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)$ for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$.

(r7): $\{\varepsilon_n\} \subset [0,1]$, $\{\sigma_n\} \subset (0,1]$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ with $\alpha_n + \beta_n < 1$ $\forall n \ge 1$, s.t.
(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\sup_{n\ge1}(\varepsilon_n/\alpha_n) < \infty$ and $\lim_{n\to\infty}(\theta_n/\alpha_n) = 0$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$; (iv) $\limsup_{n\to\infty} \sigma_n < 1$.

Algorithm 3.1. Initial Step: *Given $\gamma > 0$, $\nu \in (0,1)$, $\ell \in (0,1)$. Let $x_1, x_0 \in H$ be arbitrary.*
Iterative Steps: *Compute x_{n+1} below:*
Step 1. Calculate $p_n = W_n x_n + \varepsilon_n(W_n x_n - W_n x_{n-1})$ and $q_n = \sigma_n p_n + (1-\sigma_n)u_n$, with $u_n = T_{\mu_1}^{\Theta_1}(v_n - \mu_1 B_1 v_n)$ and $v_n = T_{\mu_2}^{\Theta_2}(q_n - \mu_2 B_2 q_n)$.

Step 2. Calculate $y_n = P_C(q_n - \zeta_n A q_n)$*, with* ζ_n *being chosen to be the largest* $\zeta \in \{\gamma, \gamma\ell, \gamma\ell^2, ...\}$ *s.t.*

$$\zeta\|Aq_n - Ay_n\| \le \nu\|q_n - y_n\|. \tag{4}$$

Step 3. Calculate $t_n = \nu x_n + (1-\nu)z_n$ *with* $z_n = P_{C_n}(q_n - \zeta_n A y_n)$ *and* $C_n := \{y \in H : \langle q_n - \zeta_n A q_n - y_n, y_n - y\rangle \ge 0\}$.
Step 4. Calculate

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)S^n t_n. \tag{5}$$

Put $n := n + 1$ *and return to Step 1.*

Lemma 3.2. *The Armijo-like search rule* (4) *is well defined, and the relation holds:* $\min\{\gamma, \nu\ell/L\} \le \zeta_n \le \gamma$.

Proof. Thanks to $\|Aq_n - AP_C(q_n - \gamma\ell^j Aq_n)\| \le L\|q_n - P_C(q_n - \gamma\ell^j Aq_n)\|$, one obtains that (4) holds for all $\gamma\ell^j \le \frac{\nu}{L}$ and hence ζ_n is well defined. It is obvious that $\zeta_n \le \gamma$. In the case of $\zeta_n = \gamma$, the relation is true. In the case of $\zeta_n < \gamma$, from (4) one has $\|Aq_n - AP_C(q_n - \frac{\zeta_n}{\ell}Aq_n)\| > \frac{\nu}{(\zeta_n/\ell)}\|q_n - P_C(q_n - \frac{\zeta_n}{\ell}Aq_n)\|$. So this yields $\zeta_n > \frac{\nu\ell}{L}$. □

Lemma 3.3. *Suppose that* $\{q_n\}, \{p_n\}, \{y_n\}, \{z_n\}$ *are the sequences generated in Algorithm 3.1. Then for each* $p \in \Omega$*, one has*

$$\begin{aligned}\|z_n - p\|^2 \le \|p_n - p\|^2 &- (1-\sigma_n)[\|p_n - q_n\|^2 + \mu_2(2\beta - \mu_2)\|B_2 q_n - B_2 p\|^2 \\ &+ \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 q\|^2] - (1-\nu)[\|y_n - z_n\|^2 + \|y_n - q_n\|^2],\end{aligned}$$

where $q = T_{\mu_2}^{\Theta_2}(p - \mu_2 B_2 p)$ *and* $v_n = T_{\mu_2}^{\Theta_2}(q_n - \mu_2 B_2 q_n)$.

Proof. First, it is readily known from Lemma 2.6 that for each $n \ge 0$ there exists a unique element $q_n \in H$ such that

$$q_n = \sigma_n p_n + (1-\sigma_n)Gq_n.$$

Observe that for each $p \in \Omega \subset C \subset C_n$,

$$\begin{aligned}\|z_n - p\|^2 &= \|P_{C_n}(q_n - \zeta_n A y_n) - P_{C_n} p\|^2 \\ &\le \langle z_n - p, q_n - \zeta_n A y_n - p\rangle \\ &= \frac{1}{2}(\|z_n - p\|^2 + \|q_n - p\|^2 - \|z_n - q_n\|^2) - \zeta_n\langle z_n - p, Ay_n\rangle,\end{aligned}$$

which hence arrives at

$$\|z_n - p\|^2 \le \|q_n - p\|^2 - \|z_n - q_n\|^2 - 2\zeta_n\langle z_n - p, Ay_n\rangle.$$

Owing to $z_n = P_{C_n}(q_n - \zeta_n A y_n)$ with $C_n := \{y \in H : \langle q_n - \zeta_n A q_n - y_n, y_n - y\rangle \ge 0\}$, one gets $\langle q_n - \zeta_n A q_n - y_n, y_n - z_n\rangle \ge 0$. Combining (4) and the pseudomonotonicity of A, guarantees that

$$\begin{aligned}\|z_n - p\|^2 &\le \|q_n - p\|^2 - \|z_n - q_n\|^2 - 2\zeta_n\langle Ay_n, y_n - p + z_n - y_n\rangle \\ &\le \|q_n - p\|^2 - \|z_n - q_n\|^2 - 2\zeta_n\langle Ay_n, z_n - y_n\rangle \\ &= \|q_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - q_n\|^2 + 2\langle q_n - \zeta_n A y_n - y_n, z_n - y_n\rangle \\ &= \|q_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - q_n\|^2 + 2\langle q_n - \zeta_n A q_n - y_n, z_n - y_n\rangle + 2\zeta_n\langle Aq_n - Ay_n, z_n - y_n\rangle \\ &\le \|q_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - q_n\|^2 + 2\nu\|q_n - y_n\|\|z_n - y_n\| \\ &\le \|q_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - q_n\|^2 + \nu(\|q_n - y_n\|^2 + \|z_n - y_n\|^2) \\ &= \|q_n - p\|^2 - (1-\nu)[\|y_n - z_n\|^2 + \|y_n - q_n\|^2].\end{aligned} \tag{6}$$

Note that $q = T_{\mu_2}^{\Theta_2}(p - \mu_2 B_2 p)$, $v_n = T_{\mu_2}^{\Theta_2}(q_n - \mu_2 B_2 q_n)$ and $u_n = T_{\mu_1}^{\Theta_1}(v_n - \mu_1 B_1 v_n)$. Then $u_n = Gq_n$. By Lemma 2.5 we have

$$\|v_n - q\|^2 \le \|q_n - p\|^2 - \mu_2(2\beta - \mu_2)\|B_2 q_n - B_2 p\|^2$$

and

$$\|u_n - p\|^2 \leq \|v_n - q\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 q\|^2.$$

Combining the last two inequalities, we obtain

$$\|u_n - p\|^2 \leq \|q_n - p\|^2 - \mu_2(2\beta - \mu_2)\|B_2 q_n - B_2 p\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 q\|^2.$$

Thanks to $q_n = \sigma_n p_n + (1-\sigma_n)u_n$, we get $\|q_n - p\|^2 \leq \sigma_n \langle p_n - p, q_n - p\rangle + (1-\sigma_n)\|q_n - p\|^2$, which hence yields $\|q_n - p\|^2 \leq \langle p_n - p, q_n - p\rangle = \frac{1}{2}[\|p_n - p\|^2 + \|q_n - p\|^2 - \|p_n - q_n\|^2]$. This immediately implies that

$$\|q_n - p\|^2 \leq \|p_n - p\|^2 - \|p_n - q_n\|^2. \tag{7}$$

So it follows that

$$\begin{aligned}\|q_n - p\|^2 &\leq \sigma_n\|p_n - p\|^2 + (1-\sigma_n)\|u_n - p\|^2 \\ &\leq \sigma_n\|p_n - p\|^2 + (1-\sigma_n)[\|q_n - p\|^2 - \mu_2(2\beta - \mu_2)\|B_2 q_n - B_2 p\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 q\|^2] \\ &\leq \sigma_n\|p_n - p\|^2 + (1-\sigma_n)[\|p_n - p\|^2 - \|p_n - q_n\|^2 - \mu_2(2\beta - \mu_2)\|B_2 q_n - B_2 p\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 q\|^2] \\ &= \|p_n - p\|^2 - (1-\sigma_n)[\|p_n - q_n\|^2 + \mu_2(2\beta - \mu_2)\|B_2 q_n - B_2 p\|^2 + \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 q\|^2],\end{aligned}$$

which together with (6), yields

$$\begin{aligned}\|z_n - p\|^2 &\leq \|q_n - p\|^2 - (1-\nu)[\|y_n - z_n\|^2 + \|y_n - q_n\|^2] \\ &\leq \|p_n - p\|^2 - (1-\sigma_n)[\|p_n - q_n\|^2 + \mu_2(2\beta - \mu_2)\|B_2 q_n - B_2 p\|^2 \\ &\quad + \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 q\|^2] - (1-\nu)[\|y_n - z_n\|^2 + \|y_n - q_n\|^2].\end{aligned}$$

This attains the desired conclusion. □

Lemma 3.4. *Assume that $\{q_n\}, \{p_n\}, \{y_n\}, \{z_n\}$ are bounded sequences generated in Algorithm 3.1. Suppose that $x_n - x_{n+1} \to 0$, $x_n - y_n \to 0$, $p_n - z_n \to 0$ and $S^n x_n - S^{n+1} x_n \to 0$. Then $\omega_w(\{x_n\}) \subset \Omega$, with $\omega_w(\{x_n\}) = \{z \in H : x_{n_k} \rightharpoonup z$ for some $\{x_{n_k}\} \subset \{x_n\}\}$.*

Proof. Take an arbitrary fixed $z \in \omega_w(\{x_n\})$. Then, $\exists \{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \rightharpoonup z \in H$. Thanks to $x_n - y_n \to 0$, we know that $\exists \{y_{n_k}\} \subset \{y_n\}$ s.t. $y_{n_k} \rightharpoonup z \in H$. In what follows, we claim that $z \in \Omega$. In fact, by Lemma 3.3 we obtain that for each $p \in \Omega$,

$$\begin{aligned}&(1-\sigma_n)[\|p_n - q_n\|^2 + \mu_2(2\beta - \mu_2)\|B_2 q_n - B_2 p\|^2 + \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 q\|^2] + (1-\nu)[\|y_n - z_n\|^2 + \|y_n - q_n\|^2] \\ &\leq \|p_n - p\|^2 - \|z_n - p\|^2 \leq \|p_n - z_n\|(\|p_n - p\| + \|z_n - p\|).\end{aligned}$$

Since $p_n - z_n \to 0$, $\nu \in (0,1)$, $\mu_1 \in (0, 2\alpha)$, $\mu_2 \in (0, 2\beta)$ and $0 < \liminf_{n\to\infty}(1-\sigma_n)$, from the boundedness of $\{p_n\}, \{z_n\}$ we deduce that

$$\lim_{n\to\infty} \|B_2 q_n - B_2 p\| = \lim_{n\to\infty} \|B_1 v_n - B_1 q\| = 0, \tag{8}$$

and

$$\lim_{n\to\infty} \|y_n - z_n\| = \lim_{n\to\infty} \|y_n - q_n\| = \lim_{n\to\infty} \|p_n - q_n\| = 0.$$

So it follows that

$$\|x_n - p_n\| \leq \|x_n - y_n\| + \|y_n - z_n\| + \|z_n - p_n\| \to 0 \quad (n \to \infty),$$

$$\|x_n - q_n\| \leq \|x_n - p_n\| + \|p_n - q_n\| \to 0 \quad (n \to \infty),$$

$$\|q_n - z_n\| \leq \|q_n - y_n\| + \|y_n - z_n\| \to 0 \quad (n \to \infty),$$

and

$$\|x_n - z_n\| \leq \|x_n - p_n\| + \|p_n - q_n\| + \|q_n - z_n\| \to 0 \quad (n \to \infty).$$

It is clear that

$$\|p_n - W_n x_n\| = \varepsilon_n \|W_n x_n - W_n x_{n-1}\| \leq \varepsilon_n \|x_n - x_{n-1}\| \to 0 \quad (n \to \infty).$$

Hence we have

$$\|x_n - W_n x_n\| \le \|x_n - p_n\| + \|p_n - W_n x_n\| \to 0 \quad (n \to \infty).$$

Also, from (5) we get $x_{n+1} - x_n = \alpha_n(f(x_n) - \mu F S^n t_n) + (1-\beta_n)(S^n t_n - x_n)$, which hence yields

$$\begin{aligned}(1-\beta_n)\|S^n t_n - x_n\| &= \|x_{n+1} - x_n - \alpha_n(f(x_n) - \mu F S^n t_n)\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n(\|f(x_n)\| + \|\mu F S^n t_n\|) \to 0 \quad (n \to \infty).\end{aligned}$$

This together with $0 < \liminf_{n\to\infty}(1-\beta_n)$, arrives at $\lim_{n\to\infty} \|S^n t_n - x_n\| = 0$. Noticing $\|t_n - x_n\| = (1-\nu)\|z_n - x_n\| \to 0$ $(n \to \infty)$ and using the asymptotical nonexpansivity of S, one deduces that

$$\|x_n - S^n x_n\| \le \|x_n - S^n t_n\| + \|S^n t_n - S^n x_n\| \le \|x_n - S^n t_n\| + (1+\theta_n)\|t_n - x_n\| \to 0 \quad (n \to \infty),$$

and hence

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - S^n x_n\| + \|S^n x_n - S^{n+1} x_n\| + \|S^{n+1} x_n - Sx_n\| \\ &\le \|x_n - S^n x_n\| + \|S^n x_n - S^{n+1} x_n\| + (1+\theta_1)\|S^n x_n - x_n\| \\ &= (2+\theta_1)\|x_n - S^n x_n\| + \|S^n x_n - S^{n+1} x_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{9}$$

On the other hand, by the firm nonexpansivity of $T^{\Theta_1}_{\mu_1}$ we obtain that

$$\begin{aligned}\|u_n - p\|^2 &\le \langle v_n - q, u_n - p\rangle + \mu_1\langle B_1 q - B_1 v_n, u_n - p\rangle \\ &\le \frac{1}{2}[\|v_n - q\|^2 + \|u_n - p\|^2 - \|v_n - u_n + p - q\|^2] + \mu_1\|B_1 q - B_1 v_n\|\|u_n - p\|,\end{aligned}$$

which hence arrives at

$$\|u_n - p\|^2 \le \|v_n - q\|^2 - \|v_n - u_n + p - q\|^2 + 2\mu_1\|B_1 q - B_1 v_n\|\|u_n - p\|.$$

In a similar way, one gets

$$\|v_n - q\|^2 \le \|q_n - p\|^2 - \|q_n - v_n + q - p\|^2 + 2\mu_2\|B_2 p - B_2 q_n\|\|v_n - q\|.$$

Combining the last two inequalities, one deduces that

$$\begin{aligned}\|u_n - p\|^2 &\le \|q_n - p\|^2 - \|q_n - v_n + q - p\|^2 - \|v_n - u_n + p - q\|^2 \\ &\quad + 2\mu_2\|B_2 p - B_2 q_n\|\|v_n - q\| + 2\mu_1\|B_1 q - B_1 v_n\|\|u_n - p\|,\end{aligned}$$

which together with (6) and (7), leads to

$$\begin{aligned}\|z_n - p\|^2 &\le \|q_n - p\|^2 \le \sigma_n\|p_n - p\|^2 + (1-\sigma_n)\|u_n - p\|^2 \\ &\le \sigma_n\|p_n - p\|^2 + (1-\sigma_n)[\|p_n - p\|^2 - \|q_n - v_n + q - p\|^2 - \|v_n - u_n + p - q\|^2 \\ &\quad + 2\mu_2\|B_2 p - B_2 q_n\|\|v_n - q\| + 2\mu_1\|B_1 q - B_1 v_n\|\|u_n - p\|] \\ &\le \|p_n - p\|^2 - (1-\sigma_n)[\|q_n - v_n + q - p\|^2 + \|v_n - u_n + p - q\|^2] + 2\mu_2\|B_2 p - B_2 q_n\| \\ &\quad \times \|v_n - q\| + 2\mu_1\|B_1 q - B_1 v_n\|\|u_n - p\|.\end{aligned}$$

This immediately implies that

$$\begin{aligned}&(1-\sigma_n)[\|q_n - v_n + q - p\|^2 + \|v_n - u_n + p - q\|^2] \\ &\le \|p_n - p\|^2 - \|z_n - p\|^2 + 2\mu_2\|B_2 p - B_2 q_n\|\|v_n - q\| + 2\mu_1\|B_1 q - B_1 v_n\|\|u_n - p\| \\ &\le \|p_n - z_n\|(\|p_n - p\| + \|z_n - p\|) + 2\mu_2\|B_2 p - B_2 q_n\|\|v_n - q\| \\ &\quad + 2\mu_1\|B_1 q - B_1 v_n\|\|u_n - p\|.\end{aligned}$$

Since $p_n - z_n \to 0$, and $0 < \liminf_{n\to\infty}(1-\sigma_n)$, from (8) and the boundedness of $\{u_n\}, \{v_n\}, \{p_n\}, \{z_n\}$ we obtain that

$$\lim_{n\to\infty} \|q_n - v_n + q - p\| = \lim_{n\to\infty} \|v_n - u_n + p - q\| = 0,$$

which hence yields

$$\|q_n - Gq_n\| = \|q_n - u_n\| \le \|q_n - v_n + q - p\| + \|v_n - u_n + p - q\| \to 0 \quad (n \to \infty).$$

This immediately implies that

$$\begin{aligned}\|x_n - Gx_n\| &\le \|x_n - q_n\| + \|q_n - Gq_n\| + \|Gq_n - Gx_n\| \\ &\le 2\|x_n - q_n\| + \|q_n - Gq_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{10}$$

We show that $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$. In fact, it is readily known that

$$\begin{aligned}\|x_n - Wx_n\| &\le \|x_n - W_nx_n\| + \|W_nx_n - Wx_n\| \\ &\le \|x_n - W_nx_n\| + \sup_{p\in D}\|W_np - Wp\|,\end{aligned}$$

where $D = \{x_n : n \ge 1\}$. Using Proposition 2.3, from $x_n - W_nx_n \to 0$ we get

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{11}$$

Next, let us show $z \in \mathrm{VI}(C, A)$. In fact, since C is of both convexity and closedness, from $\{y_n\} \subset C$ and $y_{n_k} \rightharpoonup z$ we get $z \in C$. In what follows, we consider two cases. In the case of $Az = 0$, it is easy to see that $z \in \mathrm{VI}(C, A)$ (due to $\langle Az, w - z\rangle \ge 0,\ \forall w \in C$). Let $Az \ne 0$. Since $y_{n_k} \rightharpoonup z$ as $k \to \infty$, using the condition imposed on A, instead of the sequentially weak continuity of A, one gets $0 < \|Az\| \le \liminf_{k\to\infty} \|Ay_{n_k}\|$. Thus, we might assume that $\|Ay_{n_k}\| \ne 0,\ \forall k \ge 1$. Meanwhile, from $y_n = P_C(q_n - \zeta_nAq_n)$, we have $\langle q_n - \zeta_nAq_n - y_n, w - y_n\rangle \le 0,\ \forall w \in C$, and hence

$$\frac{1}{\zeta_n}\langle q_n - y_n, w - y_n\rangle + \langle Aq_n, y_n - q_n\rangle \le \langle Aq_n, w - q_n\rangle, \quad \forall w \in C. \tag{12}$$

By the uniform continuity of A, one knows that $\{Aq_n\}$ is bounded. Note that $\{y_n\}$ is bounded as well. Using Lemma 3.2, from (12) we get $\liminf_{k\to\infty}\langle Aq_{n_k}, w - q_{n_k}\rangle \ge 0,\ \forall w \in C$. Moreover, note that $\langle Ay_n, w - y_n\rangle = \langle Ay_n - Aq_n, w - q_n\rangle + \langle Aq_n, w - q_n\rangle + \langle Ay_n, q_n - y_n\rangle$. Noticing $q_n - y_n \to 0$ and the uniform continuity of A, we have $Aq_n - Ay_n \to 0$, which hence yields $\liminf_{k\to\infty}\langle Ay_{n_k}, w - y_{n_k}\rangle \ge 0,\ \forall w \in C$.

To obtain $z \in \mathrm{VI}(C, A)$, we now pick a sequence $\{\delta_k\} \subset (0, 1)$ s.t. $\delta_k \downarrow 0$ as $k \to \infty$. $\forall k \ge 1$, one denotes by l_k the smallest positive integer s.t.

$$\langle Ay_{n_j}, w - y_{n_j}\rangle + \delta_k \ge 0, \quad \forall j \ge l_k. \tag{13}$$

Since $\{\delta_k\}$ is decreasing, it is clear that $\{l_k\}$ is increasing. Noticing $Ay_{l_k} \ne 0,\ \forall k \ge 1$ (due to $\{Ay_{l_k}\} \subset \{Ay_{n_k}\}$), we put $v_{l_k} = \frac{Ay_{l_k}}{\|Ay_{l_k}\|^2}$, and hence get $\langle Ay_{l_k}, v_{l_k}\rangle = 1,\ \forall k \ge 1$. So, from (13) we have $\langle Ay_{l_k}, w + \delta_kv_{l_k} - y_{l_k}\rangle \ge 0,\ \forall k \ge 1$. Also, by the pseudomonotonicity of A one gets $\langle A(w + \delta_kv_{l_k}), w + \delta_kv_{l_k} - y_{l_k}\rangle \ge 0,\ \forall k \ge 1$. This immediately arrives at

$$\langle Aw, w - y_{l_k}\rangle \ge \langle Aw - A(w + \delta_kv_{l_k}), w + \delta_kv_{l_k} - y_{l_k}\rangle - \delta_k\langle Aw, v_{l_k}\rangle, \quad \forall k \ge 1. \tag{14}$$

Let us show that $\lim_{k\to\infty} \delta_kv_{l_k} = 0$. In fact, noticing $\{y_{l_k}\} \subset \{y_{n_k}\}$ and $\delta_k \downarrow 0$ $(k \to \infty)$, one concludes that $0 \le \limsup_{k\to\infty} \|\delta_kv_{l_k}\| = \limsup_{k\to\infty} \frac{\delta_k}{\|Ay_{l_k}\|} \le \frac{\limsup_{k\to\infty}\delta_k}{\liminf_{k\to\infty}\|Ay_{l_k}\|} = 0$. Accordingly, one has $\delta_kv_{l_k} \to 0$ as $k \to \infty$. So, letting $k \to \infty$, one deduces that the right-hand side of (14) tends to zero by the uniform continuity of A, the boundedness of $\{y_{l_k}\}, \{v_{l_k}\}$ and the limit $\lim_{k\to\infty} \delta_kv_{l_k} = 0$. Consequently, one obtains $\langle Aw, w - z\rangle = \liminf_{k\to\infty}\langle Aw, w - y_{l_k}\rangle \ge 0,\ \forall w \in C$. By Lemma 2.8 one has $z \in \mathrm{VI}(C, A)$.

Next we show that $z \in \Omega$. In fact, noticing $x_{n_k} \rightharpoonup z$ and $x_{n_k} - Wx_{n_k} \to 0$ (due to (11)), from Proposition 2.2 and Lemma 2.10 we get the demiclosedness of $I - W$ at zero, which hence ensures $z \in \mathrm{Fix}(W) = \bigcap_{k=1}^{\infty} \mathrm{Fix}(S_k)$. Also, using (9) and (10) we have $x_{n_k} - Sx_{n_k} \to 0$ and $x_{n_k} - Gx_{n_k} \to 0$, respectively. In addition, from Lemma 2.10 it follows that $I - S$ and $I - G$ both are demiclosed at zero, and hence we get $(I - S)z = 0$ and $(I - G)z = 0$, i.e., $z \in \mathrm{Fix}(S) \cap \mathrm{Fix}(G)$. Therefore, $z \in \bigcap_{k=0}^{\infty} \mathrm{Fix}(S_k) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) = \Omega$ with $S_0 := S$. This completes the proof. □

Theorem 3.5. *Suppose that $\{x_n\}$ is the sequence generated in Algorithm 3.1. Then*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} S^n x_n - S^{n+1}x_n \to 0, \\ \sup_{n\geq 1} \|x_n - x_{n-1}\| < \infty, \end{cases}$$

where $x^* \in \Omega$ is only a solution of the HVI: $\langle(\mu F - f)x^*, p - x^*\rangle \geq 0, \ \forall p \in \Omega$.

Proof. Thanks to $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, we might assume that $\{\beta_n\} \subset [a,b] \subset (0,1)$ and $\theta_n \leq \frac{\alpha_n(\tau-\delta)}{2}$, $\forall n \geq 1$. We first show that $P_\Omega(I - \mu F + f) : H \to H$ is a contractive map. In fact, by Lemma 2.12 one has that $\forall x, y \in H$,

$$\|P_\Omega(I - \mu F + f)(x) - P_\Omega(I - \mu F + f)(y)\| \leq [1 - (\tau - \delta)]\|x - y\|,$$

which ensures that $P_\Omega(I - \mu F + f)$ is contractive. Banach's Contraction Mapping Principle guarantees that there exists a unique fixed point of $P_\Omega(I-\mu F+f)$. Say $x^* \in H$, i.e., $x^* = P_\Omega(I-\mu F+f)(x^*)$. Thus, $\exists\, |$ (solution) $x^* \in \Omega = \bigcap_{k=0}^{\infty} \mathrm{Fix}(S_k) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$ of the HVI:

$$\langle(\mu F - f)x^*, p - x^*\rangle \geq 0, \quad \forall p \in \Omega. \tag{15}$$

It is now easy to check that the necessity of the theorem is true. In fact, if $x_n \to x^* \in \Omega$, then we know that $x^* = Sx^*$ and

$$\|S^n x_n - S^{n+1}x_n\| \leq \|S^n x_n - x^*\| + \|x^* - S^{n+1}x_n\| \leq (2 + \theta_n + \theta_{n+1})\|x_n - x^*\| \to 0 \ \ (n \to \infty).$$

In addition, it is clear that

$$\|x_n - x_{n+1}\| \leq \|x_n - x^*\| + \|x^* - x_{n+1}\| \to 0 \quad (n \to \infty).$$

Next we show the sufficiency of the theorem. To the goal, under the condition $S^n x_n - S^{n+1}x_n \to 0$ with $\sup_{n\geq 1} \|x_n - x_{n-1}\| < \infty$, we divide the proof of the sufficiency into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, take an arbitrary $p \in \Omega = \bigcap_{k=0}^{\infty} \mathrm{Fix}(S_k) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$. Then $Gp = p$, $p = P_C(p - \zeta_n Ap)$, $S_n p = p$, $\forall n \geq 0$, and the inequalities (6) and (7) hold, i.e.,

$$\|z_n - p\|^2 \leq \|q_n - p\|^2 - (1 - \nu)\|y_n - z_n\|^2 - (1 - \nu)\|y_n - q_n\|^2, \tag{16}$$

and

$$\|q_n - p\|^2 \leq \|p_n - p\|^2 - \|p_n - q_n\|^2. \tag{17}$$

Combining (16) and (17), guarantees that

$$\|z_n - p\| \leq \|q_n - p\| \leq \|p_n - p\|, \quad \forall n \geq 1. \tag{18}$$

By the definition of p_n, one gets

$$\begin{aligned} \|p_n - p\| &\leq \|W_n x_n - p\| + \varepsilon_n\|W_n x_n - W_n x_{n-1}\| \\ &\leq \|x_n - p\| + \varepsilon_n\|x_n - x_{n-1}\| = \|x_n - p\| + \alpha_n \cdot \frac{\varepsilon_n}{\alpha_n}\|x_n - x_{n-1}\|. \end{aligned} \tag{19}$$

Since $\sup_{n\geq 1}(\varepsilon_n/\alpha_n) < \infty$ and $\sup_{n\geq 1} \|x_{n-1} - x_n\| < \infty$, we know that $\sup_{n\geq 1} \frac{\varepsilon_n}{\alpha_n}\|x_{n-1} - x_n\| < \infty$, which hence ensures that $\exists M_1 > 0$ s.t.

$$\frac{\varepsilon_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1, \quad \forall n \geq 1. \tag{20}$$

Combining (18)-(20), one obtains

$$\|z_n - p\| \leq \|q_n - p\| \leq \|p_n - p\| \leq \|x_n - p\| + \alpha_n M_1 \quad \forall n \geq 1. \tag{21}$$

Thus, using (21) and $\alpha_n + \beta_n < 1\ \forall n \geq 1$, from Lemma 2.12, we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n\mu F)S^n t_n - p\| \\
&= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + (1-\alpha_n-\beta_n)\{\frac{1-\beta_n}{1-\alpha_n-\beta_n}[(I - \frac{\alpha_n}{1-\beta_n}\mu F)S^n t_n \\
&\quad - (I - \frac{\alpha_n}{1-\beta_n}\mu F)p] + \frac{\alpha_n}{1-\alpha_n-\beta_n}(I-\mu F)p\}\| \\
&= \|\alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) + (1-\beta_n)[(I - \frac{\alpha_n}{1-\beta_n}\mu F)S^n t_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)p] \\
&\quad + \alpha_n(f - \mu F)p\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + \beta_n\|x_n - p\| + (1-\beta_n)\|(I - \frac{\alpha_n}{1-\beta_n}\mu F)S^n t_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)p\| \\
&\quad + \alpha_n\|(f - \mu F)p\| \\
&\leq \alpha_n\delta\|x_n - p\| + \beta_n\|x_n - p\| + (1-\beta_n)(1 - \frac{\alpha_n}{1-\beta_n}\tau)(1+\theta_n)\|t_n - p\| + \alpha_n\|(f-\mu F)p\| \\
&\leq \alpha_n\delta\|x_n - p\| + \beta_n(\|x_n - p\| + \alpha_n M_1) + (1-\beta_n-\alpha_n\tau)(\|x_n - p\| + \alpha_n M_1) + \theta_n\|t_n - p\| \\
&\quad + \alpha_n\|(f-\mu F)p\| \\
&\leq [\alpha_n\delta + \beta_n + (1-\beta_n-\alpha_n\tau)]\|x_n - p\| + \alpha_n M_1 + \frac{\alpha_n(\tau-\delta)(\|x_n - p\| + \alpha_n M_1)}{2} + \alpha_n\|(f-\mu F)p\| \\
&\leq [1 - \frac{\alpha_n(\tau-\delta)}{2}]\|x_n - p\| + \alpha_n(2M_1 + \|(f-\mu F)p\|) \\
&= [1 - \frac{\alpha_n(\tau-\delta)}{2}]\|x_n - p\| + \frac{\alpha_n(\tau-\delta)}{2}\cdot\frac{2(2M_1 + \|(f-\mu F)p\|)}{\tau-\delta} \\
&\leq \max\{\|x_n - p\|, \frac{2(2M_1 + \|(f-\mu F)p\|)}{\tau-\delta}\}.
\end{aligned}$$

By induction, we obtain

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{2(2M_1 + \|(f-\mu F)p\|)}{\tau-\delta}\}, \quad \forall n \geq 1. \tag{22}$$

Therefore, $\{x_n\}$ is bounded, and so are the sequences $\{q_n\}, \{p_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}, \{Ay_n\}, \{W_n x_n\}, \{S^n t_n\}$.

Step 2. We show that

$$\begin{aligned}
&(1-\beta_n-\alpha_n\tau)(1-\nu)[(1-\sigma_n)\|p_n - q_n\|^2 + (1-\nu)(\|y_n - z_n\|^2 \\
&\quad + \|y_n - q_n\|^2) + \nu\|x_n - z_n\|^2] \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\alpha_n + \theta_n)M_4,
\end{aligned}$$

for some $M_4 > 0$. In fact, observe that

$$\begin{aligned}
x_{n+1} - p &= \alpha_n(f(x_n) - p) + \beta_n(x_n - p) + (1-\alpha_n-\beta_n)\{\frac{1-\beta_n}{1-\alpha_n-\beta_n} \\
&\quad \times [(I - \frac{\alpha_n}{1-\beta_n}\mu F)S^n t_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)p] + \frac{\alpha_n}{1-\alpha_n-\beta_n}(I-\mu F)p\} \\
&= \alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) + (1-\beta_n) \\
&\quad \times [(I - \frac{\alpha_n}{1-\beta_n}\mu F)S^n t_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)p] + \alpha_n(f-\mu F)p.
\end{aligned}$$

Then by inequality (2), Lemma 2.12 and the convexity of the function $\phi(s) = s^2,\ \forall s \in \mathbf{R}$, we get

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|\alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) + (1-\beta_n)[(I - \frac{\alpha_n}{1-\beta_n}\mu F)S^n t_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)p]\|^2 \\
&\quad + 2\alpha_n\langle (f - \mu F)p, x_{n+1} - p\rangle \\
&\le [\alpha_n\delta\|x_n - p\| + \beta_n\|x_n - p\| + (1-\beta_n)(1 - \frac{\alpha_n}{1-\beta_n}\tau)(1+\theta_n)\|t_n - p\|]^2 \\
&\quad + 2\alpha_n\langle (f - \mu F)p, x_{n+1} - p\rangle \\
&= [\alpha_n\delta\|x_n - p\| + \beta_n\|x_n - p\| + (1-\beta_n - \alpha_n\tau)(1+\theta_n)\|t_n - p\|]^2 \\
&\quad + 2\alpha_n\langle (f - \mu F)p, x_{n+1} - p\rangle \\
&\le \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n - \alpha_n\tau)(1+\theta_n)\|t_n - p\|^2 + \alpha_n M_2
\end{aligned}
\tag{23}
$$

(due to $\alpha_n\delta + \beta_n + (1-\beta_n-\alpha_n\tau)(1+\theta_n) \le 1 - \alpha_n(\tau-\delta) + \theta_n \le 1 - \frac{\alpha_n(\tau-\delta)}{2}$), with $\sup_{n\ge1} 2\|(f-\mu F)p\|\|x_n - p\| \le M_2$ for some $M_2 > 0$. Using Lemma 3.3, from (23) we get

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n - \alpha_n\tau + \theta_n)\|t_n - p\|^2 + \alpha_n M_2 \\
&= \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n - \alpha_n\tau)[\nu\|x_n - p\|^2 + (1-\nu)\|z_n - p\|^2 \\
&\quad - \nu(1-\nu)\|x_n - z_n\|^2] + \theta_n\|t_n - p\|^2 + \alpha_n M_2 \\
&\le \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n - \alpha_n\tau)\{\nu\|x_n - p\|^2 + (1-\nu)[\|p_n - p\|^2 \\
&\quad - (1-\sigma_n)\|p_n - q_n\|^2 - (1-\nu)(\|y_n - z_n\|^2 + \|y_n - q_n\|^2)] - \nu(1-\nu)\|x_n - z_n\|^2\} \\
&\quad + \theta_n\|t_n - p\|^2 + \alpha_n M_2.
\end{aligned}
\tag{24}
$$

Also, from (21) we have

$$
\begin{aligned}
\|p_n - p\|^2 &\le (\|x_n - p\| + \alpha_n M_1)^2 \\
&= \|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2) \\
&\le \|x_n - p\|^2 + \alpha_n M_3,
\end{aligned}
\tag{25}
$$

where $\sup_{n\ge1}(2M_1\|x_n - p\| + \alpha_n M_1^2) \le M_3$ for some $M_3 > 0$. Combining (24) and (25), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n - \alpha_n\tau)\{\nu\|x_n - p\|^2 + (1-\nu) \\
&\quad \times [\|x_n - p\|^2 + \alpha_n M_3 - (1-\sigma_n)\|p_n - q_n\|^2 - (1-\nu)(\|y_n - z_n\|^2 + \|y_n - q_n\|^2)] \\
&\quad - \nu(1-\nu)\|x_n - z_n\|^2\} + \theta_n\|t_n - p\|^2 + \alpha_n M_2 \\
&\le [1 - \alpha_n(\tau-\delta)]\|x_n - p\|^2 - (1-\beta_n - \alpha_n\tau)(1-\nu)\{(1-\sigma_n)\|p_n - q_n\|^2 + (1-\nu)[\|y_n - z_n\|^2 \\
&\quad + \|y_n - q_n\|^2] + \nu\|x_n - z_n\|^2\} + \theta_n\|t_n - p\|^2 + \alpha_n M_2 + \alpha_n M_3 \\
&\le \|x_n - p\|^2 - (1-\beta_n - \alpha_n\tau)(1-\nu)[(1-\sigma_n)\|p_n - q_n\|^2 + (1-\nu)(\|y_n - z_n\|^2 \\
&\quad + \|y_n - q_n\|^2) + \nu\|x_n - z_n\|^2] + (\alpha_n + \theta_n)M_4,
\end{aligned}
$$

where $\sup_{n\ge1}(\|t_n - p\|^2 + M_3 + M_2) \le M_4$ for some $M_4 > 0$. This immediately implies that

$$
\begin{aligned}
&(1-\beta_n - \alpha_n\tau)(1-\nu)[(1-\sigma_n)\|p_n - q_n\|^2 + (1-\nu)(\|y_n - z_n\|^2 + \|y_n - q_n\|^2) + \nu\|x_n - z_n\|^2] \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\alpha_n + \theta_n)M_4.
\end{aligned}
\tag{26}
$$

Step 3. We show that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le [1 - \alpha_n(\tau-\delta)]\|x_n - p\|^2 + \alpha_n(\tau-\delta)\{\frac{2}{\tau-\delta}\langle (f-\mu F)p, x_{n+1} - p\rangle \\
&\quad + \frac{M}{\tau-\delta}(\frac{\varepsilon_n}{\alpha_n}3\|x_n - x_{n-1}\| + \frac{\theta_n}{\alpha_n})\}
\end{aligned}
$$

for some $M > 0$. In fact, we have

$$\begin{aligned}\|p_n - p\|^2 &\le (\|x_n - p\| + \varepsilon_n\|x_n - x_{n-1}\|)^2 \\ &= \|x_n - p\|^2 + \varepsilon_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \varepsilon_n\|x_n - x_{n-1}\|).\end{aligned} \tag{27}$$

Combining (21), (23) and (27), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\tau)(1 + \theta_n)\|t_n - p\|^2 + 2\alpha_n\langle(f - \mu F)p, x_{n+1} - p\rangle \\ &\le \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\tau)\|t_n - p\|^2 + \theta_n\|t_n - p\|^2 + 2\alpha_n\langle(f - \mu F)p, x_{n+1} - p\rangle \\ &\le \alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)[\|x_n - p\|^2 + \varepsilon_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \varepsilon_n\|x_n - x_{n-1}\|)] \\ &\quad + \theta_n\|t_n - p\|^2 + 2\alpha_n\langle(f - \mu F)p, x_{n+1} - p\rangle \\ &\le [1 - \alpha_n(\tau - \delta)]\|x_n - p\|^2 + \varepsilon_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \varepsilon_n\|x_n - x_{n-1}\|) \\ &\quad + \theta_n\|t_n - p\|^2 + 2\alpha_n\langle(f - \mu F)p, x_{n+1} - p\rangle \\ &\le [1 - \alpha_n(\tau - \delta)]\|x_n - p\|^2 + (\varepsilon_n\|x_n - x_{n-1}\|3 + \theta_n)M + 2\alpha_n\langle(f - \mu F)p, x_{n+1} - p\rangle \\ &= [1 - \alpha_n(\tau - \delta)]\|x_n - p\|^2 + \alpha_n(\tau - \delta)[\frac{2\langle(f - \mu F)p, x_{n+1} - p\rangle}{\tau - \delta} + \frac{M}{\tau - \delta}(\frac{\varepsilon_n}{\alpha_n}3\|x_n - x_{n-1}\| + \frac{\theta_n}{\alpha_n})],\end{aligned} \tag{28}$$

where $\sup_{n\ge1}\{\|x_n - p\|, \varepsilon_n\|x_n - x_{n-1}\|, \|t_n - p\|^2\} \le M$ for some $M > 0$.

Step 4. We show that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the HVI (15). In fact, putting $p = x^*$, we deduce from (28) that

$$\|x_{n+1} - x^*\|^2 \le [1 - \alpha_n(\tau - \delta)]\|x_n - x^*\|^2 + \alpha_n(\tau - \delta)[\frac{2\langle(f - \mu F)x^*, x_{n+1} - x^*\rangle}{\tau - \delta} + \frac{M}{\tau - \delta}(\frac{\varepsilon_n}{\alpha_n}3\|x_n - x_{n-1}\| + \frac{\theta_n}{\alpha_n})]. \tag{29}$$

Putting $\mathbf{\Gamma}_n = \|x_n - x^*\|^2$, we show the convergence of $\{\mathbf{\Gamma}_n\}$ to zero by the following two cases.

Case 1. Suppose that there exists an integer $n_0 \ge 1$ such that $\{\mathbf{\Gamma}_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty} \mathbf{\Gamma}_n = d < +\infty$ and $\lim_{n\to\infty}(\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1}) = 0$. Putting $p = x^*$, from (26) and $\{\beta_n\} \subset [a, b] \subset (0, 1)$ we obtain

$$\begin{aligned}&(1 - b - \alpha_n\tau)(1 - \nu)[(1 - \sigma_n)\|p_n - q_n\|^2 + (1 - \nu)(\|y_n - z_n\|^2 + \|y_n - q_n\|^2) \\ &\quad + \nu\|x_n - z_n\|^2] \le (1 - \beta_n - \alpha_n\tau)(1 - \nu)[(1 - \sigma_n)\|p_n - q_n\|^2 \\ &\quad + (1 - \nu)(\|y_n - z_n\|^2 + \|y_n - q_n\|^2) + \nu\|x_n - z_n\|^2] \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (\alpha_n + \theta_n)M_4 = \mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} + (\alpha_n + \theta_n)M_4.\end{aligned}$$

Noticing $0 < \nu < 1,\ 0 < \liminf_{n\to\infty}(1 - \sigma_n)$, $\alpha_n \to 0,\ \theta_n \to 0$ and $\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} \to 0$, one has

$$\lim_{n\to\infty} \|p_n - q_n\| = \lim_{n\to\infty} \|y_n - z_n\| = \lim_{n\to\infty} \|y_n - q_n\| = \lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{30}$$

Thus, we get

$$\|x_n - y_n\| \le \|x_n - z_n\| + \|z_n - y_n\| \to 0 \quad (n \to \infty), \tag{31}$$

and

$$\|p_n - z_n\| \le \|p_n - q_n\| + \|q_n - y_n\| + \|y_n - z_n\| \to 0 \quad (n \to \infty). \tag{32}$$

Moreover, noticing $x_{n+1} - x^* = \beta_n(x_n - x^*) + (1-\beta_n)(S^n t_n - x^*) + \alpha_n(f(x_n) - \mu FS^n t_n)$, we obtain from (21) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\beta_n(x_n - x^*) + (1-\beta_n)(S^n t_n - x^*) + \alpha_n(f(x_n) - \mu FS^n t_n)\|^2 \\
&\le \|\beta_n(x_n - x^*) + (1-\beta_n)(S^n t_n - x^*)\|^2 + 2\langle \alpha_n(f(x_n) - \mu FS^n t_n), x_{n+1} - x^*\rangle \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|S^n t_n - x^*\|^2 - \beta_n(1-\beta_n)\|x_n - S^n t_n\|^2 \\
&\quad + 2\|\alpha_n(f(x_n) - \mu FS^n t_n)\|\|x_{n+1} - x^*\| \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(1+\theta_n)^2\|t_n - x^*\|^2 - \beta_n(1-\beta_n)\|x_n - S^n t_n\|^2 \\
&\quad + 2\alpha_n(\|f(x_n)\| + \|\mu FS^n t_n\|)\|x_{n+1} - x^*\| \\
&\le \beta_n(1+\theta_n)^2(\|x_n - x^*\| + \alpha_n M_1)^2 + (1-\beta_n)(1+\theta_n)^2(\|x_n - x^*\| + \alpha_n M_1)^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - S^n t_n\|^2 + 2\alpha_n(\|f(x_n)\| + \|\mu FS^n t_n\|)\|x_{n+1} - x^*\| \\
&= (1+\theta_n)^2(\|x_n - x^*\| + \alpha_n M_1)^2 - \beta_n(1-\beta_n)\|x_n - S^n t_n\|^2 \\
&\quad + 2\alpha_n(\|f(x_n)\| + \|\mu FS^n t_n\|)\|x_{n+1} - x^*\| \\
&= (1+\theta_n)^2\|x_n - x^*\|^2 + (1+\theta_n)^2\alpha_n M_1[2\|x_n - x^*\| + \alpha_n M_1] \\
&\quad - \beta_n(1-\beta_n)\|x_n - S^n t_n\|^2 + 2\alpha_n(\|f(x_n)\| + \|\mu FS^n t_n\|)\|x_{n+1} - x^*\|,
\end{aligned}$$

which immediately arrives at

$$\begin{aligned}
&\beta_n(1-\beta_n)\|x_n - S^n t_n\|^2 \le (1+\theta_n)^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + (1+\theta_n)^2\alpha_n M_1[2\|x_n - x^*\| + \alpha_n M_1] + 2\alpha_n(\|f(x_n)\| + \|\mu FS^n t_n\|)\|x_{n+1} - x^*\| \\
&\le (1+\theta_n)^2\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} + (1+\theta_n)^2\alpha_n M_1[2\mathbf{\Gamma}_n^{\frac{1}{2}} + \alpha_n M_1] + 2\alpha_n(\|f(x_n)\| + \|\mu FS^n t_n\|)\mathbf{\Gamma}_{n+1}^{\frac{1}{2}}.
\end{aligned}$$

Since $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, $\theta_n \to 0$, $\alpha_n \to 0$, $\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} \to 0$ and $\lim_{n\to\infty}\mathbf{\Gamma}_n = d < +\infty$, from the boundedness of $\{f(x_n)\}$, $\{S^n t_n\}$, we infer that

$$\lim_{n\to\infty}\|x_n - S^n t_n\| = 0.$$

So it follows from Algorithm 3.1 that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\beta_n)(S^n t_n - x_n) - \alpha_n\mu FS^n t_n\| \\
&\le (1-\beta_n)\|S^n t_n - x_n\| + \alpha_n\|f(x_n) - \mu FS^n t_n\| \\
&\le \|S^n t_n - x_n\| + \alpha_n(\|f(x_n)\| + \|\mu FS^n t_n\|) \to 0 \quad (n\to\infty).
\end{aligned} \tag{33}$$

From the boundedness of $\{x_n\}$, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (f-\mu F)x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle (f-\mu F)x^*, x_{n_k} - x^*\rangle. \tag{34}$$

Since H is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_k} \rightharpoonup \widetilde{x}$. Thus, from (34) one gets

$$\begin{aligned}
\limsup_{n\to\infty}\langle (f-\mu F)x^*, x_n - x^*\rangle &= \lim_{k\to\infty}\langle (f-\mu F)x^*, x_{n_k} - x^*\rangle \\
&= \langle (f-\mu F)x^*, \widetilde{x} - x^*\rangle.
\end{aligned} \tag{35}$$

Since $x_n - x_{n+1} \to 0$, $x_n - y_n \to 0$, $p_n - z_n \to 0$ and $S^n x_n - S^{n+1}x_n \to 0$, by Lemma 3.4 we infer that $\widetilde{x} \in\in \omega_w(\{x_n\}) \subset \Omega$. Hence from (15) and (35) one gets

$$\limsup_{n\to\infty}\langle (f-\mu F)x^*, x_n - x^*\rangle = \langle (f-\mu F)x^*, \widetilde{x} - x^*\rangle \le 0, \tag{36}$$

which together with (33), leads to

$$\begin{aligned}\limsup_{n\to\infty}\langle(f-\mu F)x^*, x_{n+1}-x^*\rangle &= \limsup_{n\to\infty}[\langle(f-\mu F)x^*, x_{n+1}-x_n\rangle + \langle(f-\mu F)x^*, x_n-x^*\rangle] \\ &\le \limsup_{n\to\infty}[\|(f-\mu F)x^*\|\|x_{n+1}-x_n\| + \langle(f-\mu F)x^*, x_n-x^*\rangle] \le 0.\end{aligned} \tag{37}$$

Note that $\{\alpha_n(\tau-\delta)\}\subset[0,1]$, $\sum_{n=1}^{\infty}\alpha_n(\tau-\delta)=\infty$, and

$$\limsup_{n\to\infty}[\frac{2\langle(f-\mu F)x^*, x_{n+1}-x^*\rangle}{\tau-\delta} + \frac{M}{\tau-\delta}(\frac{\varepsilon_n}{\alpha_n}3\|x_n-x_{n-1}\| + \frac{\theta_n}{\alpha_n})] \le 0.$$

Consequently, applying Lemma 2.9 to (29), one has $\lim_{n\to\infty}\|x_n-x^*\|^2=0$.

Case 2. Suppose that $\exists\{\mathbf{\Gamma}_{n_k}\}\subset\{\mathbf{\Gamma}_n\}$ s.t. $\mathbf{\Gamma}_{n_k}<\mathbf{\Gamma}_{n_k+1}$, $\forall k\in\mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\psi:\mathcal{N}\to\mathcal{N}$ by

$$\psi(n) := \max\{k\le n : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\}.$$

By Lemma 2.11, we get

$$\mathbf{\Gamma}_{\psi(n)} \le \mathbf{\Gamma}_{\psi(n)+1} \quad \text{and} \quad \mathbf{\Gamma}_n \le \mathbf{\Gamma}_{\psi(n)+1}.$$

Putting $p=x^*$, from (26) we have

$$\begin{aligned}&(1-b-\alpha_{\psi(n)}\tau)(1-\nu)[(1-\sigma_{\psi(n)})\|p_{\psi(n)}-q_{\psi(n)}\|^2 + (1-\nu)(\|y_{\psi(n)}-z_{\psi(n)}\|^2 \\ &\quad + \|y_{\psi(n)}-q_{\psi(n)}\|^2) + \nu\|x_{\psi(n)}-z_{\psi(n)}\|^2] \\ &\le (1-\beta_{\psi(n)}-\alpha_{\psi(n)}\tau)(1-\nu)[(1-\sigma_{\psi(n)})\|p_{\psi(n)}-q_{\psi(n)}\|^2 + (1-\nu)(\|y_{\psi(n)}-z_{\psi(n)}\|^2 \\ &\quad + \|y_{\psi(n)}-q_{\psi(n)}\|^2) + \nu\|x_{\psi(n)}-z_{\psi(n)}\|^2] \\ &\le \|x_{\psi(n)}-x^*\|^2 - \|x_{\psi(n)+1}-x^*\|^2 + (\alpha_{\psi(n)}+\theta_{\psi(n)})M_4 \\ &= \mathbf{\Gamma}_{\psi(n)} - \mathbf{\Gamma}_{\psi(n)+1} + (\alpha_{\psi(n)}+\theta_{\psi(n)})M_4,\end{aligned} \tag{38}$$

which immediately yields

$$\lim_{n\to\infty}\|p_{\psi(n)}-q_{\psi(n)}\| = \lim_{n\to\infty}\|y_{\psi(n)}-z_{\psi(n)}\| = \lim_{n\to\infty}\|y_{\psi(n)}-q_{\psi(n)}\| = \lim_{n\to\infty}\|x_{\psi(n)}-z_{\psi(n)}\| = 0.$$

Utilizing the same inferences as in the proof of Case 1, we deduce that

$$\lim_{n\to\infty}\|x_{\psi(n)}-y_{\psi(n)}\| = \lim_{n\to\infty}\|p_{\psi(n)}-z_{\psi(n)}\| = \lim_{n\to\infty}\|x_{\psi(n)+1}-x_{\psi(n)}\| = 0,$$

and

$$\limsup_{n\to\infty}\langle(f-\mu F)x^*, x_{\psi(n)+1}-x^*\rangle \le 0. \tag{39}$$

On the other hand, from (29) we obtain

$$\begin{aligned}\alpha_{\psi(n)}(\tau-\delta)\mathbf{\Gamma}_{\psi(n)} &\le \mathbf{\Gamma}_{\psi(n)} - \mathbf{\Gamma}_{\psi(n)+1} + \alpha_{\psi(n)}(\tau-\delta)[\frac{2\langle(f-\mu F)x^*, x_{\psi(n)+1}-x^*\rangle}{\tau-\delta} + \frac{M}{\tau-\delta}(\frac{\varepsilon_{\psi(n)}}{\alpha_{\psi(n)}}3\|x_{\psi(n)}-x_{\psi(n)-1}\| + \frac{\theta_{\psi(n)}}{\alpha_{\psi(n)}})] \\ &\le \alpha_{\psi(n)}(\tau-\delta)[\frac{2\langle(f-\mu F)x^*, x_{\psi(n)+1}-x^*\rangle}{\tau-\delta} + \frac{M}{\tau-\delta}(\frac{\varepsilon_{\psi(n)}}{\alpha_{\psi(n)}}3\|x_{\psi(n)}-x_{\psi(n)-1}\| + \frac{\theta_{\psi(n)}}{\alpha_{\psi(n)}})],\end{aligned}$$

which hence arrives at

$$\limsup_{n\to\infty}\mathbf{\Gamma}_{\psi(n)} \le \limsup_{n\to\infty}[\frac{2\langle(f-\mu F)x^*, x_{\psi(n)+1}-x^*\rangle}{\tau-\delta} + \frac{M}{\tau-\delta}(\frac{\varepsilon_{\psi(n)}}{\alpha_{\psi(n)}}3\|x_{\psi(n)}-x_{\psi(n)-1}\| + \frac{\theta_{\psi(n)}}{\alpha_{\psi(n)}})] \le 0.$$

Thus, $\lim_{n\to\infty}\|x_{\psi(n)}-x^*\|^2=0$. Also, note that

$$\begin{aligned}\|x_{\psi(n)+1}-x^*\|^2-\|x_{\psi(n)}-x^*\|^2&=2\langle x_{\psi(n)+1}-x_{\psi(n)},x_{\psi(n)}-x^*\rangle+\|x_{\psi(n)+1}-x_{\psi(n)}\|^2\\&\le 2\|x_{\psi(n)+1}-x_{\psi(n)}\|\|x_{\psi(n)}-x^*\|+\|x_{\psi(n)+1}-x_{\psi(n)}\|^2.\end{aligned}\tag{40}$$

Owing to $\mathbf{\Gamma}_n\le\mathbf{\Gamma}_{\psi(n)+1}$, we get

$$\begin{aligned}\|x_n-x^*\|^2&\le\|x_{\psi(n)+1}-x^*\|^2\\&\le\|x_{\psi(n)}-x^*\|^2+2\|x_{\psi(n)+1}-x_{\psi(n)}\|\|x_{\psi(n)}-x^*\|+\|x_{\psi(n)+1}-x_{\psi(n)}\|^2\to 0\quad(n\to\infty).\end{aligned}$$

That is, $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

According to Theorem 3.5, we can obtain the following corollary.

Corollary 3.6. *Let $S:H\to H$ be nonexpansive and the sequence $\{x_n\}$ be constructed by the modified version of Algorithm 3.1, that is, for any initial $x_1,x_0\in H$,*

$$\begin{cases}p_n=W_nx_n+\varepsilon_n(W_nx_n-W_nx_{n-1}),\\q_n=\sigma_np_n+(1-\sigma_n)u_n,\\u_n=T_{\mu_1}^{\Theta_1}(v_n-\mu_1B_1v_n),\\v_n=T_{\mu_2}^{\Theta_2}(q_n-\mu_2B_2q_n),\\y_n=P_C(q_n-\zeta_nAq_n),\\z_n=P_{C_n}(q_n-\zeta_nAy_n),\\t_n=\nu x_n+(1-\nu)z_n,\\x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_n\mu F)St_n,\quad\forall n\ge 1,\end{cases}\tag{41}$$

where for each $n\ge1$, C_n and ζ_n are chosen as in Algorithm 3.1. Then $x_n\to x^*\in\Omega \Leftrightarrow \sup_{n\ge1}\|x_n-x_{n-1}\|<\infty$, where $x^*\in\Omega$ is the unique solution to the HVI: $\langle(\mu F-f)x^*,p-x^*\rangle\ge0,\ \forall p\in\Omega$.

Next, we put forth another modification of relaxed inertial subgradient extragradient implicit rule with line-search process.

Algorithm 3.7. Initial Step: *Given $\gamma>0,\ \nu\in(0,1),\ \ell\in(0,1)$. Let $x_1,x_0\in H$ be arbitrary.*

Iterative Steps: *Compute x_{n+1} below:*

Step 1. Calculate $p_n=W_nx_n+\varepsilon_n(W_nx_n-W_nx_{n-1})$ and $q_n=\sigma_np_n+(1-\sigma_n)u_n$, with $u_n=T_{\mu_1}^{\Theta_1}(v_n-\mu_1B_1v_n)$ and $v_n=T_{\mu_2}^{\Theta_2}(q_n-\mu_2B_2q_n)$.

Step 2. Calculate $y_n=P_C(q_n-\zeta_nAq_n)$, with ζ_n being chosen to be the largest $\zeta\in\{\gamma,\gamma\ell,\gamma\ell^2,...\}$ s.t.

$$\zeta\|Aq_n-Ay_n\|\le\nu\|q_n-y_n\|.\tag{42}$$

Step 3. Calculate $t_n=\nu z_n+(1-\nu)S^nt_n$ with $z_n=P_{C_n}(q_n-\zeta_nAy_n)$ and $C_n:=\{y\in H:\langle q_n-\zeta_nAq_n-y_n,y_n-y\rangle\ge0\}$.

Step 4. Calculate

$$x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_n\mu F)S^nt_n.\tag{43}$$

Set $n:=n+1$ and return to Step 1.

Theorem 3.8. *Suppose that $\{x_n\}$ is the sequence generated in Algorithm 3.7. Then*

$$x_n\to x^*\in\Omega\quad\Leftrightarrow\quad\begin{cases}S^nx_n-S^{n+1}x_n\to0,\\\sup_{n\ge1}\|x_n-x_{n-1}\|<\infty,\end{cases}$$

where $x^*\in\Omega$ is only a solution of the HVI: $\langle(\mu F-f)x^*,p-x^*\rangle\ge0,\ \forall p\in\Omega$.

Proof. Using the same arguments as in the proof of Theorem 3.5, we deduce that there exists the unique solution $x^* \in \Omega = \bigcap_{k=0}^{\infty} \mathrm{Fix}(S_k) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$ to the HVI (15), and that the necessity of the theorem is valid. Thanks to $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, we might assume that $\{\theta_n\} \subset [0, \nu)$ and $\tilde{\theta}_n = \theta_n(\frac{2}{\nu-\theta_n} + \frac{\theta_n}{(\nu-\theta_n)^2}) \leq \frac{\alpha_n(\tau-\delta)}{2}$, $\forall n \geq 1$. In what follows, we show the sufficiency of the theorem. To the goal, under the condition $S^n x_n - S^{n+1} x_n \to 0$ with $\sup_{n\geq 1} \|x_n - x_{n-1}\| < \infty$, we divide the proof of the sufficiency into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, noticing $t_n = \nu z_n + (1-\nu) S^n t_n$ and the asymptotical nonexpansivity of S, we get

$$\begin{aligned}\|t_n - p\| &\leq \nu\|z_n - p\| + (1-\nu)\|S^n t_n - p\| \\ &\leq \nu\|z_n - p\| + (1-\nu)(1+\theta_n)\|t_n - p\| \\ &\leq \nu\|z_n - p\| + (1-\nu+\theta_n)\|t_n - p\|.\end{aligned}$$

This together with (21), ensures that

$$\|t_n - p\| \leq (1 + \frac{\theta_n}{\nu - \theta_n})\|z_n - p\| \leq (1 + \frac{\theta_n}{\nu-\theta_n})(\|x_n - p\| + \alpha_n M_1), \quad \forall n \geq 1. \tag{44}$$

From $\alpha_n + \beta_n < 1$, Lemma 2.12 and (44) it follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) + (1-\beta_n)[(I - \frac{\alpha_n}{1-\beta_n}\mu F)S^n t_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)p + \frac{\alpha_n}{1-\beta_n}(f - \mu F)p]\| \\ &\leq \alpha_n\|f(x_n) - f(p)\| + \beta_n\|x_n - p\| + (1-\beta_n)\|(I - \frac{\alpha_n}{1-\beta_n}\mu F)S^n t_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)p + \frac{\alpha_n}{1-\beta_n}(f-\mu F)p\| \\ &\leq \alpha_n\delta\|x_n - p\| + \beta_n\|x_n - p\| + (1-\beta_n)(1 - \frac{\alpha_n}{1-\beta_n}\tau)(1+\theta_n)\|t_n - p\| + \alpha_n\|(f-\mu F)p\| \\ &\leq \alpha_n\delta\|x_n - p\| + \beta_n\|x_n - p\| + (1-\beta_n-\alpha_n\tau)(1 + \frac{\theta_n}{\nu-\theta_n})^2\|z_n - p\| + \alpha_n\|(f-\mu F)p\| \\ &\leq \alpha_n\delta\|x_n - p\| + \beta_n(\|x_n - p\| + \alpha_n M_1) + (1-\beta_n-\alpha_n\tau)(\|x_n - p\| + \alpha_n M_1) + \tilde{\theta}_n\|z_n - p\| + \alpha_n\|(f-\mu F)p\| \\ &\leq [\alpha_n\delta + \beta_n + (1-\beta_n-\alpha_n\tau)]\|x_n - p\| + \alpha_n M_1 + \frac{\alpha_n(\tau-\delta)(\|x_n - p\| + \alpha_n M_1)}{2} + \alpha_n\|(f-\mu F)p\| \\ &\leq [1 - \frac{\alpha_n(\tau-\delta)}{2}]\|x_n - p\| + \alpha_n(2M_1 + \|(f-\mu F)p\|) \\ &\leq \max\{\|x_n - p\|, \frac{2(2M_1 + \|(f-\mu F)p\|)}{\tau-\delta}\}.\end{aligned}$$

By induction, we obtain

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{2(2M_1 + \|(f-\mu F)p\|)}{\tau-\delta}\}, \quad \forall n \geq 1.$$

Therefore, $\{x_n\}$ is bounded, and so are the sequences $\{q_n\}, \{p_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}, \{Ay_n\}, \{W_n x_n\}, \{S^n t_n\}$.

Step 2. We show that

$$\begin{aligned}&(1-\beta_n-\alpha_n\tau)\{(1+\tilde{\theta}_n)^2[(1-\sigma_n)\|p_n - q_n\|^2 + (1-\nu)(\|y_n - z_n\|^2 + \|y_n - q_n\|^2)] \\ &+\nu(1-\nu)\|z_n - S^n t_n\|^2\} \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\alpha_n + \theta_n + \tilde{\theta}_n)M_4\end{aligned}$$

for some $M_4 > 0$. In fact, by inequality (2), Lemma 2.12 and the convexity of the function $\phi(s) = s^2$, $\forall s \in \mathbf{R}$, we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|\alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) + (1-\beta_n)[(I - \frac{\alpha_n}{1-\beta_n}\mu F)S^n t_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)p]\|^2 \\ &\quad + 2\alpha_n\langle (f-\mu F)p, x_{n+1} - p\rangle \\ &\leq [\alpha_n\delta\|x_n - p\| + \beta_n\|x_n - p\| + (1-\beta_n-\alpha_n\tau)(1+\theta_n)\|t_n - p\|]^2 + 2\alpha_n\langle (f-\mu F)p, x_{n+1} - p\rangle \\ &\leq \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)(1+\theta_n)\|t_n - p\|^2 + \alpha_n M_2\end{aligned} \tag{45}$$

(due to $\alpha_n\delta+\beta_n+(1-\beta_n-\alpha_n\tau)(1+\theta_n)\leq 1-\alpha_n(\tau-\delta)+\theta_n\leq 1-\frac{\alpha_n(\tau-\delta)}{2}$), with $\sup_{n\geq 1}2\|(f-\mu F)p\|\|x_n-p\|\leq M_2$ for some $M_2>0$. Using Lemma 3.3, from (45) we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\tau+\theta_n)\|t_n-p\|^2+\alpha_nM_2\\ &= \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\tau)[\nu\|z_n-p\|^2+(1-\nu)\|S^nt_n-p\|^2\\ &\quad -\nu(1-\nu)\|z_n-S^nt_n\|^2]+\theta_n\|t_n-p\|^2+\alpha_nM_2\\ &\leq \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\tau)[\nu\|z_n-p\|^2+(1-\nu)(1+\tilde{\theta}_n)^2\|z_n-p\|^2\\ &\quad -\nu(1-\nu)\|z_n-S^nt_n\|^2]+\theta_n\|t_n-p\|^2+\alpha_nM_2\\ &\leq \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\tau)[(1+\tilde{\theta}_n)^2\|z_n-p\|^2\\ &\quad -\nu(1-\nu)\|z_n-S^nt_n\|^2]+\theta_n\|t_n-p\|^2+\alpha_nM_2\\ &\leq \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\tau)\{(1+\tilde{\theta}_n)^2[\|p_n-p\|^2\\ &\quad -(1-\sigma_n)\|p_n-q_n\|^2-(1-\nu)(\|y_n-z_n\|^2+\|y_n-q_n\|^2)]\\ &\quad -\nu(1-\nu)\|z_n-S^nt_n\|^2\}+\theta_n\|t_n-p\|^2+\alpha_nM_2.\end{aligned} \tag{46}$$

Also, from (21) we have

$$\|p_n-p\|^2\leq(\|x_n-p\|+\alpha_nM_1)^2\leq\|x_n-p\|^2+\alpha_nM_3, \tag{47}$$

where $\sup_{n\geq 1}(2M_1\|x_n-p\|+\alpha_nM_1^2)\leq M_3$ for some $M_3>0$. Combining (46) and (47), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\tau)\{(1+\tilde{\theta}_n)^2[\|x_n-p\|^2\\ &\quad +\alpha_nM_3-(1-\sigma_n)\|p_n-q_n\|^2-(1-\nu)(\|y_n-z_n\|^2+\|y_n-q_n\|^2)]\\ &\quad -\nu(1-\nu)\|z_n-S^nt_n\|^2\}+\theta_n\|t_n-p\|^2+\alpha_nM_2\\ &\leq [1-\alpha_n(\tau-\delta)]\|x_n-p\|^2-(1-\beta_n-\alpha_n\tau)\{(1+\tilde{\theta}_n)^2[(1-\sigma_n)\|p_n-q_n\|^2\\ &\quad +(1-\nu)(\|y_n-z_n\|^2+\|y_n-q_n\|^2)]+\nu(1-\nu)\|z_n-S^nt_n\|^2\}+\theta_n\|t_n-p\|^2\\ &\quad +\alpha_nM_2+\alpha_nM_3+\tilde{\theta}_n(2+\tilde{\theta}_n)(\|x_n-p\|^2+\alpha_nM_3)\\ &\leq \|x_n-p\|^2-(1-\beta_n-\alpha_n\tau)\{(1+\tilde{\theta}_n)^2[(1-\sigma_n)\|p_n-q_n\|^2+(1-\nu)\\ &\quad \times(\|y_n-z_n\|^2+\|y_n-q_n\|^2)]+\nu(1-\nu)\|z_n-S^nt_n\|^2\}+(\alpha_n+\theta_n+\tilde{\theta}_n)M_4,\end{aligned}$$

where $\sup_{n\geq 1}\{\|t_n-p\|^2+M_3+M_2,(2+\tilde{\theta}_n)(\|x_n-p\|^2+\alpha_nM_3)\}\leq M_4$ for some $M_4>0$. This immediately implies that

$$\begin{aligned}&(1-\beta_n-\alpha_n\tau)\{(1+\tilde{\theta}_n)^2[(1-\sigma_n)\|p_n-q_n\|^2+(1-\nu)(\|y_n-z_n\|^2+\|y_n-q_n\|^2)]\\ &\quad +\nu(1-\nu)\|z_n-S^nt_n\|^2\}\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n+\theta_n+\tilde{\theta}_n)M_4.\end{aligned} \tag{48}$$

Step 3. We show that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq [1-\alpha_n(\tau-\delta)]\|x_n-p\|^2+\alpha_n(\tau-\delta)\{\frac{2}{\tau-\delta}\langle(f-\mu F)p,x_{n+1}-p\rangle\\ &\quad +\frac{M}{\tau-\delta}(\frac{\varepsilon_n}{\alpha_n}2\|x_n-x_{n-1}\|+\frac{\theta_n+\tilde{\theta}_nM}{\alpha_n})\}\end{aligned}$$

for some $M>0$. In fact, it is clear that

$$\|p_n-p\|^2\leq\|x_n-p\|^2+\varepsilon_n\|x_n-x_{n-1}\|(2\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|). \tag{49}$$

Combining (21), (45) and (49), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\tau)(1+\theta_n)\|t_n-p\|^2+2\alpha_n\langle(f-\mu F)p,x_{n+1}-p\rangle\\
&\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\tau)\|t_n-p\|^2+\theta_n\|t_n-p\|^2\\
&\quad+2\alpha_n\langle(f-\mu F)p,x_{n+1}-p\rangle\\
&\le \alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\tau)(1+\tilde\theta_n)\|z_n-p\|^2+\theta_n\|t_n-p\|^2\\
&\quad+2\alpha_n\langle(f-\mu F)p,x_{n+1}-p\rangle\\
&\le \alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)(1+\tilde\theta_n)[\|x_n-p\|^2+\varepsilon_n\|x_n-x_{n-1}\|(2\|x_n-p\|\\
&\quad+\varepsilon_n\|x_n-x_{n-1}\|)]+\theta_n\|t_n-p\|^2+2\alpha_n\langle(f-\mu F)p,x_{n+1}-p\rangle\\
&\le \alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)[\|x_n-p\|^2+\varepsilon_n\|x_n-x_{n-1}\|(2\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|)]\\
&\quad+\tilde\theta_n(\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|)^2+\theta_n\|t_n-p\|^2+2\alpha_n\langle(f-\mu F)p,x_{n+1}-p\rangle\\
&\le [1-\alpha_n(\tau-\delta)]\|x_n-p\|^2+\varepsilon_n\|x_n-x_{n-1}\|(2\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|)\\
&\quad+\theta_n\|t_n-p\|^2+\tilde\theta_n(\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|)^2+2\alpha_n\langle(f-\mu F)p,x_{n+1}-p\rangle\\
&\le [1-\alpha_n(\tau-\delta)]\|x_n-p\|^2+(\varepsilon_n\|x_n-x_{n-1}\|2+\theta_n+\tilde\theta_nM)M+2\alpha_n\langle(f-\mu F)p,x_{n+1}-p\rangle\\
&= [1-\alpha_n(\tau-\delta)]\|x_n-p\|^2+\alpha_n(\tau-\delta)[\frac{2\langle(f-\mu F)p,x_{n+1}-p\rangle}{\tau-\delta}+\frac{M}{\tau-\delta}(\frac{\varepsilon_n}{\alpha_n}2\|x_n-x_{n-1}\|\\
&\quad+\frac{\theta_n+\tilde\theta_nM}{\alpha_n})],\end{aligned}\tag{50}$$

where $\sup_{n\ge1}\{\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|,\|t_n-p\|^2\}\le M$ for some $M>0$.

Step 4. We show that $\{x_n\}$ converges strongly to the unique solution $x^*\in\Omega$ of the HVI (15). In fact, putting $p=x^*$, we deduce from (50) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le [1-\alpha_n(\tau-\delta)]\|x_n-x^*\|^2+\alpha_n(\tau-\delta)[\frac{2\langle(f-\mu F)x^*,x_{n+1}-x^*\rangle}{\tau-\delta}\\
&\quad+\frac{M}{\tau-\delta}(\frac{\varepsilon_n}{\alpha_n}2\|x_n-x_{n-1}\|+\frac{\theta_n+\tilde\theta_nM}{\alpha_n})].\end{aligned}\tag{51}$$

Putting $\mathbf{\Gamma}_n=\|x_n-x^*\|^2$, we show the convergence of $\{\mathbf{\Gamma}_n\}$ to zero by the following two cases.

Case 1. Suppose that there exists an integer $n_0\ge1$ such that $\{\mathbf{\Gamma}_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty}\mathbf{\Gamma}_n=d<+\infty$ and $\lim_{n\to\infty}(\mathbf{\Gamma}_n-\mathbf{\Gamma}_{n+1})=0$. Putting $p=x^*$, from (48) and $\{\beta_n\}\subset[a,b]\subset(0,1)$ we obtain

$$\begin{aligned}&(1-b-\alpha_n\tau)\{(1+\tilde\theta_n)^2[(1-\sigma_n)\|p_n-q_n\|^2+(1-\nu)(\|y_n-z_n\|^2+\|y_n-q_n\|^2)]\\
&\quad+\nu(1-\nu)\|z_n-S^nt_n\|^2\}\le(1-\beta_n-\alpha_n\tau)\{(1+\tilde\theta_n)^2[(1-\sigma_n)\|p_n-q_n\|^2\\
&\quad+(1-\nu)(\|y_n-z_n\|^2+\|y_n-q_n\|^2)]+\nu(1-\nu)\|z_n-S^nt_n\|^2\}\\
&\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+(\alpha_n+\theta_n+\tilde\theta_n)M_4=\mathbf{\Gamma}_n-\mathbf{\Gamma}_{n+1}+(\alpha_n+\theta_n+\tilde\theta_n)M_4.\end{aligned}$$

Noticing $0<\nu<1,\ 0<\liminf_{n\to\infty}(1-\sigma_n)$, $\alpha_n\to0,\ \theta_n\to0,\ \tilde\theta_n\to0$ and $\mathbf{\Gamma}_n-\mathbf{\Gamma}_{n+1}\to0$, one has

$$\lim_{n\to\infty}\|p_n-q_n\|=\lim_{n\to\infty}\|y_n-z_n\|=\lim_{n\to\infty}\|y_n-q_n\|=\lim_{n\to\infty}\|z_n-S^nt_n\|=0.\tag{52}$$

Thus, we get

$$\|p_n-z_n\|\le\|p_n-q_n\|+\|q_n-y_n\|+\|y_n-z_n\|\to0\quad(n\to\infty).\tag{53}$$

Also, from (2) and (21), we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_n(x_n-x^*)+(1-\beta_n)(S^nt_n-x^*)+\alpha_n(f(x_n)-\mu FS^nt_n)\|^2\\ &\le \|\beta_n(x_n-x^*)+(1-\beta_n)(S^nt_n-x^*)\|^2+2\langle\alpha_n(f(x_n)-\mu FS^nt_n),x_{n+1}-x^*\rangle\\ &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|S^nt_n-x^*\|^2-\beta_n(1-\beta_n)\|x_n-S^nt_n\|^2\\ &\quad+2\|\alpha_n(f(x_n)-\mu FS^nt_n)\|\|x_{n+1}-x^*\|\\ &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)(1+\tilde{\theta}_n)^2\|z_n-x^*\|^2-\beta_n(1-\beta_n)\|x_n-S^nt_n\|^2\\ &\quad+2\alpha_n(\|f(x_n)\|+\|\mu FS^nt_n\|)\|x_{n+1}-x^*\|\\ &\le \beta_n(1+\tilde{\theta}_n)^2(\|x_n-x^*\|+\alpha_nM_1)^2+(1-\beta_n)(1+\tilde{\theta}_n)^2(\|x_n-x^*\|+\alpha_nM_1)^2\\ &\quad-\beta_n(1-\beta_n)\|x_n-S^nt_n\|^2+2\alpha_n(\|f(x_n)\|+\|\mu FS^nt_n\|)\|x_{n+1}-x^*\|\\ &= (1+\tilde{\theta}_n)^2(\|x_n-x^*\|+\alpha_nM_1)^2-\beta_n(1-\beta_n)\|x_n-S^nt_n\|^2\\ &\quad+2\alpha_n(\|f(x_n)\|+\|\mu FS^nt_n\|)\|x_{n+1}-x^*\|\\ &= (1+\tilde{\theta}_n)^2\|x_n-x^*\|^2+(1+\tilde{\theta}_n)^2\alpha_nM_1[2\|x_n-x^*\|+\alpha_nM_1]\\ &\quad-\beta_n(1-\beta_n)\|x_n-S^nt_n\|^2+2\alpha_n(\|f(x_n)\|+\|\mu FS^nt_n\|)\|x_{n+1}-x^*\|,\end{aligned}$$

which immediately arrives at

$$\begin{aligned}\beta_n(1-\beta_n)\|x_n-S^nt_n\|^2 &\le (1+\tilde{\theta}_n)^2\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad+(1+\tilde{\theta}_n)^2\alpha_nM_1[2\|x_n-x^*\|+\alpha_nM_1]+2\alpha_n(\|f(x_n)\|+\|\mu FS^nt_n\|)\|x_{n+1}-x^*\|\\ &\le (1+\tilde{\theta}_n)^2\mathbf{\Gamma}_n-\mathbf{\Gamma}_{n+1}+(1+\tilde{\theta}_n)^2\alpha_nM_1[2\mathbf{\Gamma}_n^{\frac{1}{2}}+\alpha_nM_1]+2\alpha_n(\|f(x_n)\|+\|\mu FS^nt_n\|)\mathbf{\Gamma}_{n+1}^{\frac{1}{2}}.\end{aligned}$$

Since $\{\beta_n\}\subset[a,b]\subset(0,1)$, $\tilde{\theta}_n\to0$, $\alpha_n\to0$, $\mathbf{\Gamma}_n-\mathbf{\Gamma}_{n+1}\to0$ and $\lim_{n\to\infty}\mathbf{\Gamma}_n=d<+\infty$, from the boundedness of $\{f(x_n)\},\{S^nt_n\}$, we infer that

$$\lim_{n\to\infty}\|x_n-S^nt_n\|=0,$$

which together with (52), leads to

$$\|x_n-y_n\|\le\|x_n-S^nt_n\|+\|S^nt_n-z_n\|+\|z_n-y_n\|\to0\quad(n\to\infty). \tag{54}$$

In addition, it follows from (43) that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_nf(x_n)+(1-\beta_n)(S^nt_n-x_n)-\alpha_n\mu FS^nt_n\|\\ &\le \|S^nt_n-x_n\|+\alpha_n(\|f(x_n)\|+\|\mu FS^nt_n\|)\to0\quad(n\to\infty).\end{aligned} \tag{55}$$

Using the similar arguments to those of (37), we obtain

$$\limsup_{n\to\infty}\langle(f-\mu F)x^*,x_{n+1}-x^*\rangle\le0.$$

Note that $\{\alpha_n(\tau-\delta)\}\subset[0,1]$, $\sum_{n=1}^{\infty}\alpha_n(\tau-\delta)=\infty$, and

$$\limsup_{n\to\infty}\Big[\frac{2\langle(f-\mu F)x^*,x_{n+1}-x^*\rangle}{\tau-\delta}+\frac{M}{\tau-\delta}\Big(\frac{\varepsilon_n}{\alpha_n}2\|x_n-x_{n-1}\|+\frac{\theta_n+\tilde{\theta}_nM}{\alpha_n}\Big)\Big]\le0.$$

Consequently, applying Lemma 2.9 to (51), one has $\lim_{n\to\infty}\|x_n-x^*\|^2=0$.

Case 2. Suppose that $\exists\{\mathbf{\Gamma}_{n_k}\}\subset\{\mathbf{\Gamma}_n\}$ s.t. $\mathbf{\Gamma}_{n_k}<\mathbf{\Gamma}_{n_k+1}$, $\forall k\in\mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\psi:\mathcal{N}\to\mathcal{N}$ by

$$\psi(n):=\max\{k\le n:\mathbf{\Gamma}_k<\mathbf{\Gamma}_{k+1}\}.$$

By Lemma 2.11, we get

$$\mathbf{\Gamma}_{\psi(n)}\le\mathbf{\Gamma}_{\psi(n)+1}\quad\text{and}\quad\mathbf{\Gamma}_n\le\mathbf{\Gamma}_{\psi(n)+1}.$$

In the remainder of the proof, using the same arguments as in Case 2 of Step 4 in the proof of Theorem 3.5, we obtain the desired assertion. □

Acknowledgments

This research was supported by the 2020 Shanghai Leading Talents Program of the Shanghai Municipal Human Resources and Social Security Bureau (20LJ2006100), the Innovation Program of Shanghai Municipal Education Commission (15ZZ068) and the Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).

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