Filomat 38:29 (2024), 10373–10389
https://doi.org/10.2298/FIL2429373C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Complete moment convergence of CWOD sequences and moving average processes based on CWOD in Hilbert spaces

Mengmeng Chang[a,b,*], Yu Miao[b,c]

[a]*College of Mathematics and Information Science, Anyang Institute of Technology, Henan Province, 455000, China*
[b]*College of Mathematics and Information Science, Henan Normal University, Henan Province, 453007, China*
[c]*Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, Henan Normal University, Henan Province, 453007, China*

Abstract. In this paper, the concept of coordinatewise widely orthant dependent (CWOD) random vectors is introduced. Some results on complete moment convergence of CWOD sequence and the convergence properties of moving average processes based on CWOD are established in real separable Hilbert spaces. In particular, the complete integral convergence and the complete convergence of moving average processes based on CWOD are derived in Theorems 2.7 and 2.8, both of which guarantee the corresponding complete moment convergence result (see Theorem 2.10). Theses conclusions are generalizations of [6], [15], [16], [22] and [26].

1. Introduction

In many stochastic models, the assumption that random variables are independent is not reasonable. One of the important dependent structure is widely orthant dependent (WOD) random variables which was introduced by Wang et al. [27] as follows.

Definition 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, if there exists a sequence of finite real numbers $\{g_U(n), n \geq 1\}$ such that for all $x_i \in \mathbb{R}$, $1 \leq i \leq n$,*

$$\mathbb{P}(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leq g_U(n) \prod_{i=1}^{n} \mathbb{P}(X_i > x_i),$$

then we say that random variables $\{X_n, n \geq 1\}$ are widely upper orthant dependent (WUOD). If there exits a sequence of finite real numbers $\{g_L(n), n \geq 1\}$ such that for all $x_i \in \mathbb{R}$, $1 \leq i \leq n$,

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) \leq g_L(n) \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i),$$

2020 *Mathematics Subject Classification*. Primary 60F15.
Keywords. Complete moment convergence; complete integral convergence; complete convergence; moving average process; coordinatewise widely orthant dependent random vectors; Hilbert space.
Received: 02 April 2024; Accepted: 09 June 2024
Communicated by Biljana Popović
Research supported by National Natural Science Foundation of China (NSFC-11971154).
* Corresponding author: Mengmeng Chang
Email addresses: aygxycmm@163.com (Mengmeng Chang), yumiao728@gmail.com (Yu Miao)

then we say that random variables $\{X_n, n \geq 1\}$ *are widely lower orthant dependent (WLOD). If they are both WUOD and WLOD, then we say that the random variables* $\{X_n, n \geq 1\}$ *are widely orthant dependent (WOD), and* $g_U(n), g_L(n), n \geq 1$ *are called dominating coefficients.*

From the definition of WOD random variables, many other dependent structure can be induced. When $g_U(n) = g_L(n) = M$ for some positive constant M, the random variables $\{X_n, n \geq 1\}$ are called extended negatively dependent (END). The concept of extended negative dependence was introduced by Liu [20]. In the case $M = 1$ the notion of END random variables reduced to the so called negative orthant dependence (NOD) or negative dependence (ND) random variables which was introduced by Lehmann [18]. Meanwhile, Lehmann also introduced that random variables $\{X_n, n \geq 1\}$ are said to be pairwise negatively dependent (PND) if for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$\mathbb{P}(X_i \leq x, X_j \leq y) \leq \mathbb{P}(X_i \leq x)\mathbb{P}(X_j \leq y),$$

which is equivalent to for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$\mathbb{P}(X_i > x, X_j > y) \leq \mathbb{P}(X_i > x)\mathbb{P}(X_j > y).$$

Joag-Dev and Proschan [10] pointed out that negatively associated (NA) random variables must be NOD and NOD is not necessarily NA. Negatively superadditive dependence (NSD) is a notion proposed by Hu [8], he emphasized that NSD implies NOD. Christofides and Vaggelatou [2] indicated that NA implies NSD. From the above statements, we can conclude that independent random variables, NA random variables, NSD random variables, NOD random variables and END random variables all belong to WOD random variables. Therefore it is of great value to study the convergence behaviours of the WOD group.

As usual, let H denote a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle\cdot,\cdot\rangle$ and let $\{e_j, j \in B\}$ (B is a set of positive integers) be an orthonormal basis in H. For any random vector X in H, define $X^{(j)} = \langle X, e_j\rangle$ to denote the jth component of X, then $X = \sum_{j\in B} X^{(j)} e_j$.

In [6], Hien et al. introduced the notions of coordinatewise negative dependence (CND) and pairwise and coordinatewise negative dependence (PCND) for random vectors in Hilbert spaces, which are extensions of the concepts of ND and PND to real separable Hilbert spaces. Since WOD structure is a natural extension of ND and PND, now we extend the concept of WOD to real separable Hilbert spaces and give the definition of coordinatewise widely orthant dependence (CWOD).

Definition 1.2. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random vectors in H. If for all* $j \in B$*, the sequences of random variables* $\{\langle X_n, e_j\rangle, n \geq 1\}$ *are WOD with the common dominating coefficients* $g_U(n)$ *and* $g_L(n)$*, then random vectors* $\{X_n, n \geq 1\}$ *are said to be CWOD,* $g_U(n)$ *and* $g_L(n)$ *are called the dominating coefficients of* $\{X_n, n \geq 1\}$.

In the definition of CWOD, between two different coordinates, there are no WOD requirements and repetitions are even rational. The following examples can help us better understand this concept. Consider the real separable Hilbert space ℓ_2 of all square summable real sequences with inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i \text{ for } x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots).$$

Let $\{e_n, n \geq 1\}$ denote the orthonormal basis of ℓ_2 with 1 in its nth position and 0 elsewhere. Let $\{X_i, i \geq 1\}$ be a sequence of real-valued WOD random variables and let $p = (p_1, p_2, \cdots, p_n, \cdots) \in \ell_2$, then $\{Y_n = (p_1 X_n, p_2 X_n, \cdots, p_n X_n, \cdots), n \geq 1\}$ is a sequence of CWOD random vectors in ℓ_2. More generally, if for any $j \geq 1$, $\{X_{nj}, n \geq 1\}$ is a sequence of real-valued WOD random variables and for each $n \geq 1$, $\sum_{j=1}^{\infty}\left(X_{nj}\right)^2 < \infty$ a.s., then $\{X_n = \sum_{j\in B} X_{nj} e_j, n \geq 1\}$ is a sequence of CWOD random vectors in ℓ_2.

It is well known that complete convergence and complete moment convergence play important roles in the probability limit theory and mathematical statistics. A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to random variable X if for any $\varepsilon > 0$,

$$\sum_{n\geq 1} \mathbb{P}(|X_n - X| > \varepsilon) < \infty.$$

This concept was introduced by Hsu and Robbins [7]. Moreover, they proved that the sequence of arithmetic means of independent identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. The Borel-Cantelli lemma ensures that complete convergence implies almost sure convergence. The concept of complete moment convergence can be found in Chow [1] as follows.

Definition 1.3. *Let $\{X_n, n \geq 1\}$ be sequence of random variables and $a_n > 0, b_n > 0, q > 0$. If for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} a_n \mathbb{E}\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty,$$

then $\{X_n, n \geq 1\}$ is said to be complete moment convergence.

It can be easily proved that complete moment convergence implies complete convergence. Recently, some researchers investigated complete moment convergence in Hilbert spaces. For example, Ko [11, 14] established the complete moment convergence of CNA random vectors, Wu et al. [30] and Ko [15] established the complete moment convergence of the weighted sums of ρ^*−mixing random vectors, Ko [13] derived the complete moment convergence for CAANA random vectors, Ko [17] investigated the complete moment convergence for weighted sums of CANA random vectors.

Definition 1.4. *Suppose that $\{Y_i, -\infty < i < \infty\}$ is a doubly infinite sequence of random variables and $\{a_i, -\infty < i < \infty\}$ is a sequence of absolutely summable real numbers. A moving average process generated by $\{Y_i, -\infty < i < \infty\}$ is defined as*

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}. \tag{1}$$

Moving average processes is a concept with extensive applications in electronics, financial mathematics, time series and so on. So it is of great significance to investigate its limiting properties. On the complete moment convergence part, Qu [22] investigated the complete moment convergence of moving average processes under END random variables, Li and Zhang [19] showed the complete moment convergence of moving average processes under NA random variables, Wang and Shen [26] and Guan et al. [4] established the complete moment convergence of moving average processes generated by m-WOD random variables, Zhou [31] investigated the complete moment convergence of moving average processes based on φ−mixing random variables, Zhou and Lin [32] established the complete moment convergence of moving average processes based on ρ−mixing random variables and so on.

We point out the results above were established in classical probability space. To our best knowledge, no one considered the complete moment convergence of moving average processes in Hilbert spaces. In this paper, we will obtain some convergence conclusions about CWOD random vectors and the moving average processes generated by CWOD random vectors in real separable Hilbert spaces. In particular, the complete integral convergence and the complete convergence of moving average processes based on CWOD are derived in Theorems 2.7 and 2.8, both of which guarantee the complete moment convergence of moving average processes based on CWOD (see Theorem 2.10). Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each of its appearance.

2. Main results

Before giving the main results, we need to recall two domination conditions in Hilbert spaces. Let $\{X_n, n \geq 1\}$ be a sequence of H-valued random vectors, Wu et al. [30] introduced that sequence $\{X_n, n \geq 1\}$ is coordinatewise stochastically upper dominated by a random vector X, if there exists a positive constant C such that the following inequality holds for all $j \in B$, $n \geq 1$ and $t \geq 0$,

$$\mathbb{P}\left(\left|X_n^{(j)}\right| > t\right) \leq C\mathbb{P}\left(\left|X^{(j)}\right| > t\right). \tag{2}$$

Rosalsky and Thanh [23] proved that (2) is equivalent to

$$\mathbb{P}\left(\left|X_n^{(j)}\right| > t\right) \leq \mathbb{P}\left(\left|X^{(j)}\right| > t\right). \tag{3}$$

Another domination condition was introduced by Huan et al. [9]. Let $\{X_n, n \geq 1\}$ be a sequence of H-valued random vectors, we say that sequence $\{X_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector X, if there exists a positive constant C such that the following inequality holds for all $j \in B$, $n \geq 1$ and $t \geq 0$,

$$\frac{1}{n}\sum_{k=1}^{n} \mathbb{P}\left(\left|X_k^{(j)}\right| > t\right) \leq C\mathbb{P}\left(\left|X^{(j)}\right| > t\right). \tag{4}$$

Thanh [25] proved that (4) is equivalent to

$$\frac{1}{n}\sum_{k=1}^{n} \mathbb{P}\left(\left|X_k^{(j)}\right| > t\right) \leq \mathbb{P}\left(\left|X^{(j)}\right| > t\right). \tag{5}$$

From Example 2.1 showed in Gut [5], it is obvious that (4) is weaker than (2), and both of them are weaker than identical distribution.

In addition, we still need to recall the concept of slowly varying function, which was introduced by Seneta [24]. A real-valued function $h(\cdot)$ is said to be regularly varying with index of regular variation $\rho \in \mathbb{R}$ if it is a positive and measurable function on $[A, \infty)$ for some $A > 0$, and for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \lambda^{\rho}.$$

A regularly varying function with the index of regular variation $\rho = 0$ is called to be slowly varying. A useful conclusion related to slowly varying function is that

$$\begin{cases} \sum_{n=1}^{k} n^{\alpha} h(n) = O\left(k^{\alpha+1}h(k)\right), & \text{if } \alpha > -1, \\ \sum_{n=k}^{\infty} n^{\alpha} h(n) = O\left(k^{\alpha+1}h(k)\right), & \text{if } \alpha < -1. \end{cases} \tag{6}$$

2.1. Complete moment convergence of CWOD random vectors

Theorem 2.1. *Let p be a positive real number such that $1 \leq p < 2$. Let $\{Y_i, i \geq 1\}$ be a sequence of CWOD random vectors with dominating coefficient $g(n) = O(n^{\delta})$ for some $0 \leq \delta < \frac{2-r}{p}$, where $1 < r < 2$ and $h(\cdot)$ be a slowly varying function. Assume that $\{Y_i, i \geq 1\}$ is coordinatewise weakly upper bounded by a random vector Y. If*

$$\sum_{j\in B} \mathbb{E}\left|Y^{(j)}\right|^{r+\delta p} h\left(\left|Y^{(j)}\right|^{p}\right) < \infty, \tag{7}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \mathbb{E}\left(\left\|\sum_{k=1}^{n} Y_k\right\| - \varepsilon n^{\frac{1}{p}}\right)_+ < \infty. \tag{8}$$

Corollary 2.2. *Under the conditions of Theorem 2.1, if $r = p$, i.e. $0 \leq \delta < \frac{2}{p}-1$, then $\sum_{j\in B} \mathbb{E}\left|Y^{(j)}\right|^{(1+\delta)p} h\left(\left|Y^{(j)}\right|^{p}\right) < \infty$ implies*

$$\sum_{n=1}^{\infty} n^{-1-\frac{1}{p}} h(n) \mathbb{E}\left(\left\|\sum_{k=1}^{n} Y_k\right\| - \varepsilon n^{\frac{1}{p}}\right)_+ < \infty. \tag{9}$$

Remark 2.3. *Qu [22] considered complete moment convergence of moving average processes based on END assumption as*

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n)\mathbb{E}\left(\left|\sum_{k=1}^{n} X_k\right| - \varepsilon n^{\frac{1}{p}}\right)_+ < \infty, \tag{10}$$

where $\{X_k, k \geq 1\}$ is a moving average process defined as $X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}$, $\{a_i, -\infty < i < \infty\}$ is a sequence of absolutely summable real numbers and $\{Y_i, -\infty < i < \infty\}$ is a sequence of identically distributed END random variables with $\mathbb{E}Y_1 = 0$. We extend (10) to CWOD random vectors in Hilbert spaces, but here we do not consider the moving average processes based on CWOD or using the method provided in [22], the primary cause is that according to Qu's method,

$$\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty}\sum_{k=1}^{n} a_{i+k} Y_i = \sum_{i=-\infty}^{\infty} a_{ni} Y_i,$$

where $a_{ni} = \sum_{k=1}^{n} a_{i+k}$, but inequality (23) does not apply to $\{Y_i, -\infty < i < \infty\}$, $\{Y_{ni}, -\infty < i < \infty\}$ or $\{Z_{ni}, -\infty < i < \infty\}$ (refer to the proof of Theorem 2.1 for the definitions of Y_{ni} and Z_{ni}).

Theorem 2.4. *Let p be a positive real number such that $1 < p < 2$. Let $\{Y_i, i \geq 1\}$ be a sequence of mean zero CWOD random vectors with dominating coefficient $g(n) = O\left(n^{\delta} \log^{-2} n\right)$ for some $0 \leq \delta < \frac{2-r}{p}$, where $1 < r < 2$ and $h(\cdot)$ be a slowly varying function. Assume that $\{Y_i, i \geq 1\}$ is coordinatewise weakly upper bounded by a random vector Y. If (7) holds, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n)\mathbb{E}\left(\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s} Y_k\right\| - \varepsilon n^{\frac{1}{p}}\right)_+ < \infty. \tag{11}$$

Corollary 2.5. *Under the conditions of Theorem 2.4, if $r = p$, i.e. $0 \leq \delta < \frac{2}{p}-1$, then $\sum_{j\in B} \mathbb{E}\left|Y^{(j)}\right|^{(1+\delta)p} h\left(\left|Y^{(j)}\right|^{p}\right) < \infty$ implies*

$$\sum_{n=1}^{\infty} n^{-1-\frac{1}{p}} h(n)\mathbb{E}\left(\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s} Y_k\right\| - \varepsilon n^{\frac{1}{p}}\right)_+ < \infty. \tag{12}$$

Remark 2.6. *When $\sum_{j\in B} \mathbb{E}|Y^{(j)}|^p < \infty$, Ko [15] established the complete moment convergence of H-valued ρ^*-mixing random vectors*

$$\sum_{n=1}^{\infty} n^{-1-\frac{1}{p}} \mathbb{E}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} a_{ni} Y_i\right\| - \varepsilon n^{\frac{1}{p}}\right)_+ < \infty, \tag{13}$$

where $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is a set of real numbers such that $\sup_{n\geq 1} n^{-1}\sum_{i=1}^{n} |a_{ni}|^{\alpha} < \infty$. Obviously, this inequality is satisfied for $a_{ni} \equiv 1$. Lemma 2.2 in [15] showed that ρ^*-mixing random vectors $\{Y_i, i \geq 1\}$ satisfy the maximal inequality

$$\mathbb{E}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} Y_i\right\|^2\right) \leq C\sum_{i=1}^{n} \mathbb{E}\|Y_i\|^2,$$

which means $g(n) = 1$ compared with (23). But Ko did not consider the slowly varying function, so Theorem 2.4 extends Ko's result from ρ^*-mixing random vectors to CWOD random vectors in a certain sense.

2.2. Complete moment convergence of moving average processes based on CWOD

Firstly, we will give a complete integral convergence result for a moving average process generated by CWOD random vectors.

Theorem 2.7. *Let p be a positive real number such that $1 < p < 2$. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ be a moving average process generated by a sequence of mean zero CWOD random vectors $\{Y_i, -\infty < i < \infty\}$ with dominating coefficient $g(n) = O(n^\delta)$ for some $\delta \geq 0$. Assume that $\{Y_i, -\infty < i < \infty\}$ is coordinatewise stochastically upper dominated by a random vector Y. If*

$$\sum_{j\in B} \mathbb{E}\left|Y^{(j)}\right|^p < \infty, \tag{14}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}} \int_{(n^{1+\delta}\log^2 n)^{1/p}}^{\infty} \mathbb{P}\left(\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s} X_k\right\| > t\right) < \infty. \tag{15}$$

Theorem 2.8 is a complete convergence result for a moving average process generated by CWOD random vectors.

Theorem 2.8. *Let p be a positive real number such that $1 \leq p < 2$. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ be a moving average process generated by a sequence of mean zero CWOD random vectors $\{Y_i, -\infty < i < \infty\}$ with dominating coefficient $g(n) = O(n^\delta)$ for some $\delta \geq 0$. Assume that $\{Y_i, -\infty < i < \infty\}$ is coordinatewise stochastically upper dominated by a random vector Y. If (14) holds, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{n}\mathbb{P}\left\{\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s} X_k\right\| > \varepsilon(n^{1+\delta}\log^2 n)^{1/p}\right\} < \infty. \tag{16}$$

Remark 2.9. *Theorem 3.3 in Hien et al. [6] considered (16) under the case that $\{X_n, n \geq 1\}$ is a sequence of identically distributed mean zero PCND random vectors and $\delta = 0$, so Theorem 2.8 is a generalization of the work of Hien et al. from PCND random vectors to moving average processes under CWOD random vectors.*

Combing Theorems 2.7 and 2.8, we finally achieve the following complete moment convergence conclusion.

Theorem 2.10. *Under the conditions of Theorem 2.7 (or Theorem 2.8), we can also get*

$$\sum_{n=1}^{\infty} \frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}} \mathbb{E}\left(\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s} X_k\right\| - \varepsilon\left(n^{1+\delta}\log^2 n\right)^{1/p}\right)_+ < \infty. \tag{17}$$

Remark 2.11. *Ko [16] considered (15) and (17) under the condition that $\{X_n, n \geq 1\}$ is a sequence of coordinatewise PNQD random vectors and $\delta = 0$. Therefore, Theorems 2.7 and 2.10 extend Ko's work from coordinatewise PNQD random vectors to moving average processes generated by CWOD random vectors.*

3. Lemmas

In this section, we present some lemmas which will be used to prove the main results of the paper. Firstly, we give three lemmas about WOD random variables, which can be found in Ding et al. [3] and Wang et al. [28].

Lemma 3.1. ([28]) *Let $\{X_n, n \geq 1\}$ be WLOD (WUOD) with dominating coefficients $g_L(n), n \geq 1$ ($g_U(n), n \geq 1$). If functions $\{f_n(\cdot), n \geq 1\}$ are nondecreasing, then $\{f_n(X_n), n \geq 1\}$ are still WLOD (WUOD) with dominating coefficients $g_L(n), n \geq 1$ ($g_U(n), n \geq 1$); if functions $\{f_n(\cdot), n \geq 1\}$ are nonincreasing, then $\{f_n(X_n), n \geq 1\}$ are still WLOD (WUOD) with dominating coefficients $g_L(n), n \geq 1$ ($g_U(n), n \geq 1$).*

Lemma 3.2. *([28]) Let $p \geq 1$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $\mathbb{E}|X_n|^p < \infty$ for each $n \geq 1$. Define $g(n) = \max\{g_U(n), g_L(n)\}$. Assume that $\mathbb{E}X_n = 0$ for each $n \geq 1$ when $p \geq 2$. Then there exist positive constants $c_1(p)$ and $c_2(p)$ depending only on p such that*

$$\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^p \leq [c_1(p) + c_2(p)g(n)] \sum_{i=1}^{n} \mathbb{E}|X_i|^p, \ \textit{for } 1 \leq p \leq 2, \tag{18}$$

and

$$\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^p \leq c_1(p) \sum_{i=1}^{n} \mathbb{E}|X_i|^p + c_2(p)g(n) \left(\sum_{i=1}^{n} \mathbb{E}|X_i|^2\right)^{p/2}, \ \textit{for } p \geq 2. \tag{19}$$

Lemma 3.3. *([3]) Let $p \geq 1$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $\mathbb{E}X_n = 0$ and $\mathbb{E}|X_n|^p < \infty$ for each $n \geq 1$. Define $g(n) = \max\{g_U(n), g_L(n)\}$. Then there exist positive constants $c_1(p)$ and $c_2(p)$ depending only on p such that*

$$\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|\right)^p \leq [c_1(p) + c_2(p)g(n)] \log^p n \sum_{i=1}^{n} \mathbb{E}|X_i|^p, \ \textit{for } 1 \leq p \leq 2, \tag{20}$$

and

$$\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|\right)^p \leq c_1(p) \log^p n \sum_{i=1}^{n} \mathbb{E}|X_i|^p + c_2(p)g(n) \log^p n \left(\sum_{i=1}^{n} \mathbb{E}|X_i|^2\right)^{p/2}, \ \textit{for } p \geq 2. \tag{21}$$

We extend Lemmas 3.2 and 3.3 to CWOD random vectors in Hilbert spaces under the condition $1 \leq p \leq 2$.

Lemma 3.4. *Let $1 \leq p \leq 2$ and $\{X_n, n \geq 1\}$ be a sequence of CWOD random vectors in H with $\mathbb{E}\left|X_n^{(j)}\right|^p < \infty$ for each $n \geq 1$ and each $j \in B$. Define $g(n) = \max\{g_U(n), g_L(n)\}$. Then there exist positive constants $c_1(p)$ and $c_2(p)$ depending only on p such that*

$$\mathbb{E}\left\|\sum_{i=1}^{n} X_i\right\|^p \leq [c_1(p) + c_2(p)g(n)] \sum_{j\in B} \sum_{i=1}^{n} \mathbb{E}|X_i^{(j)}|^p, \ \textit{for } 1 \leq p \leq 2. \tag{22}$$

In particular,

$$\mathbb{E}\left\|\sum_{i=1}^{n} X_i\right\|^2 \leq [c_1 + c_2 g(n)] \sum_{i=1}^{n} \mathbb{E}\|X_i\|^2. \tag{23}$$

Proof. For $1 \leq p \leq 2$, from C_r inequality and Lemma 3.2, we have

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=1}^{n} X_i\right\|^p &= \mathbb{E}\left(\sum_{j\in B}\left|\sum_{i=1}^{n} X_i^{(j)}\right|^2\right)^{p/2} \\
&\leq \sum_{j\in B} \mathbb{E}\left(\left|\sum_{i=1}^{n} X_i^{(j)}\right|^2\right)^{p/2} \\
&= \sum_{j\in B} \mathbb{E}\left|\sum_{i=1}^{n} X_i^{(j)}\right|^p \\
&\leq [c_1(p) + c_2(p)g(n)] \sum_{j\in B} \sum_{i=1}^{n} \mathbb{E}\left|X_i^{(j)}\right|^p.
\end{aligned}$$

The proof is completed. □

Lemma 3.5. *Let $1 \le p \le 2$ and $\{X_n, n \ge 1\}$ be a sequence of CWOD random vectors in H with $\mathbb{E}X_n = 0$ and $\mathbb{E}\left|X_n^{(j)}\right|^p < \infty$ for each $n \ge 1$ and each $j \in B$. Define $g(n) = \max\{g_U(n), g_L(n)\}$. Then there exist positive constants $c_1(p)$ and $c_2(p)$ depending only on p such that*

$$\mathbb{E}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} X_i\right\|\right)^p \le [c_1(p) + c_2(p)g(n)]\log^p n \sum_{j\in B}\sum_{i=1}^{n}\mathbb{E}|X_i^{(j)}|^p,\ for\ 1 \le p \le 2. \tag{24}$$

In particular,

$$\mathbb{E}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{n} X_i\right\|\right)^2 \le [c_1 + c_2 g(n)]\log^2 n \sum_{i=1}^{n}\mathbb{E}\|X_i\|^2. \tag{25}$$

Proof. The proof of Lemma 3.5 is similar to that of Lemma 3.4, so we omit it. □

According to the properties of stochastic domination shown in Wu [29] and using the same proof procedure as Lemma 3.1 in [21], we have the following lemma.

Lemma 3.6. *If H-valued sequence $\{X_n, n \ge 1\}$ is coordinatewise stochastically upper dominated by a random vector X, then for all $j \in B$, $p > 0$, $M > 0$ and for any $k \ge 1$, we have*

(i) $\mathbb{E}\left|X_k^{(j)}\right|^p I\left\{\left|X_k^{(j)}\right| \le M\right\} \le \mathbb{E}\left|X^{(j)}\right|^p I\left\{\left|X^{(j)}\right| \le M\right\} + M^p\mathbb{P}\left(\left|X^{(j)}\right| > M\right)$,

(ii) $\mathbb{E}\left|X_k^{(j)}\right|^p I\left\{\left|X_k^{(j)}\right| > M\right\} \le \mathbb{E}\left|X^{(j)}\right|^p I\left\{\left|X^{(j)}\right| > M\right\}$.

Lemma 3.7. *If H-valued sequence $\{X_n, n \ge 1\}$ is coordinatewise weakly upper bounded by a random vector X in H, then for all $j \in B$, $p > 0$, we have*

(i) if $\mathbb{E}\left|X^{(j)}\right|^p < \infty$, *then* $n^{-1}\sum_{k=1}^{n}\mathbb{E}\left|X_k^{(j)}\right|^p \le \mathbb{E}\left|X^{(j)}\right|^p$,

(ii) for any $M > 0$,

$$n^{-1}\sum_{k=1}^{n}\mathbb{E}\left(\left|X_k^{(j)}\right|^p I\left\{\left|X_k^{(j)}\right| \le M\right\}\right) \le \mathbb{E}\left(\left|X^{(j)}\right|^p I\left\{\left|X^{(j)}\right| \le M\right\}\right) + M^p\mathbb{P}\left(\left|X^{(j)}\right| > M\right),$$

(iii) $n^{-1}\sum_{k=1}^{n}\mathbb{E}\left(\left|X_k^{(j)}\right|^p I\left\{\left|X_k^{(j)}\right| > M\right\}\right) \le \mathbb{E}\left(\left|X^{(j)}\right|^p I\left\{\left|X^{(j)}\right| > M\right\}\right)$.

Lemma 3.8. *([30]) If $\{Y_i, 1 \le i \le n\}$ and $\{Z_i, 1 \le i \le n\}$ are sequences of random vectors, then for any $q > r > 0$, $\varepsilon > 0$ and $a > 0$, the following inequalities hold:*

$$\mathbb{E}\left(\left\|\sum_{i=1}^{n}(Y_i + Z_i)\right\| - \varepsilon a\right)_+^r \le C_r\left(\varepsilon^{-q} + \frac{r}{q-r}\right)a^{r-q}\mathbb{E}\left\|\sum_{i=1}^{n} Y_i\right\|^q + C_r\mathbb{E}\left\|\sum_{i=1}^{n} Z_i\right\|^r,$$

$$\mathbb{E}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}(Y_i + Z_i)\right\| - \varepsilon a\right)_+^r \le C_r\left(\varepsilon^{-q} + \frac{r}{q-r}\right)a^{r-q}\mathbb{E}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{n} Y_i\right\|^q\right) + C_r\mathbb{E}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{n} Z_i\right\|^r\right),$$

where

$$Cr = \begin{cases} 1, & if\ 0 < r \le 1, \\ 2^{r-1}, & if\ r > 1. \end{cases} \tag{26}$$

4. Proof of main results

Proof. **[Proof of Theorem 2.1]** For any $n > 0$ and $j \in B$, set

$$\begin{cases} Y_{ni}^{(j)} = -n^{1/p} I\left\{Y_i^{(j)} < -n^{1/p}\right\} + Y_i^{(j)} I\left\{\left|Y_i^{(j)}\right| \le n^{1/p}\right\} + n^{1/p} I\left\{Y_i^{(j)} > n^{1/p}\right\}, \\ Y_{ni} = \sum_{j\in B} Y_{ni}^{(j)} e_j; \end{cases}$$

and

$$\begin{cases} Z_{ni}^{(j)} = Y_i^{(j)} - Y_{ni}^{(j)} = \left(Y_i^{(j)} + n^{1/p}\right) I\left\{Y_i^{(j)} < -n^{1/p}\right\} + \left(Y_i^{(j)} - n^{1/p}\right) I\left\{Y_i^{(j)} > n^{1/p}\right\}, \\ Z_{ni} = \sum_{j\in B} Z_{ni}^{(j)} e_j. \end{cases}$$

By the definition of CWOD, for any $j \in B$, sequence $\left\{Y_i^{(j)}, 1 \le i \le n\right\}$ is WOD, then sequence $\left\{Y_{ni}^{(j)}, n \ge 1, 1 \le i \le n\right\}$ is WOD from Lemma 3.1, so sequence $\{Y_{ni}, n \ge 1, 1 \le i \le n\}$ is CWOD. Applying Lemmas 3.8 and 3.4 to (8) gives

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \mathbb{E}\left(\left\|\sum_{i=1}^{n} Y_i\right\| - \varepsilon n^{\frac{1}{p}}\right)_+ \\
=&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \mathbb{E}\left(\left\|\sum_{i=1}^{n} (Y_{ni} + Z_{ni})\right\| - \varepsilon n^{\frac{1}{p}}\right)_+ \\
\le& C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) n^{-\frac{1}{p}} \mathbb{E}\left\|\sum_{i=1}^{n} Y_{ni}\right\|^2 + C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \mathbb{E}\left\|\sum_{i=1}^{n} Z_{ni}\right\| \\
\le& C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{2}{p}} h(n) g(n) \sum_{i=1}^{n} \mathbb{E}\|Y_{ni}\|^2 + C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \sum_{i=1}^{n} \mathbb{E}\,\|Z_{ni}\| \\
\le& C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{2}{p}} h(n) g(n) \sum_{i=1}^{n} \sum_{j\in B} \mathbb{E}\left|Y_i^{(j)}\right|^2 I\left\{\left|Y_i^{(j)}\right| \le n^{1/p}\right\} \\
&+ C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2} h(n) g(n) \sum_{i=1}^{n} \sum_{j\in B} \mathbb{P}\left\{\left|Y_i^{(j)}\right| > n^{1/p}\right\} \\
&+ C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \sum_{i=1}^{n} \sum_{j\in B} \mathbb{E}\left|Y_i^{(j)}\right| I\left\{\left|Y_i^{(j)}\right| > n^{1/p}\right\} \\
=& I_1 + I_2 + I_3.
\end{aligned}$$

For I_1 , I_2 and I_3, Lemma 3.7 and standard calculation will show that

$$\begin{aligned}
I_1 =& C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{2}{p}} h(n) g(n) \sum_{i=1}^{n} \sum_{j\in B} \mathbb{E}\left|Y_i^{(j)}\right|^2 I\left\{\left|Y_i^{(j)}\right| \le n^{1/p}\right\} \\
\le& C\sum_{n=1}^{\infty} n^{\frac{r}{p}-1-\frac{2}{p}+\delta} h(n) \sum_{j\in B} \mathbb{E}\left|Y^{(j)}\right|^2 I\left\{\left|Y^{(j)}\right| \le n^{1/p}\right\} \\
&+ C\sum_{n=1}^{\infty} n^{\frac{r}{p}-1+\delta} h(n) \sum_{j\in B} \mathbb{P}\left\{\left|Y^{(j)}\right| > n^{1/p}\right\} \\
=& C\sum_{n=1}^{\infty} n^{\frac{r}{p}-1-\frac{2}{p}+\delta} h(n) \sum_{j\in B} \mathbb{E}\left|Y^{(j)}\right|^2 \sum_{k=1}^{n} I\left\{(k-1)^{1/p} < \left|Y^{(j)}\right| \le k^{1/p}\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{n=1}^{\infty} n^{\frac{r}{p}-1+\delta}h(n)\sum_{j\in B}\sum_{k=n}^{\infty}\mathbb{P}\left\{k^{1/p}<\left|Y^{(j)}\right|\le (k+1)^{1/p}\right\}\\
=&C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|^2\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} n^{\frac{r}{p}-1-\frac{2}{p}+\delta}h(n)\right)I\left\{(k-1)^{1/p}<\left|Y^{(j)}\right|\le k^{1/p}\right\}\\
&+ C\sum_{j\in B}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k} n^{\frac{r}{p}-1+\delta}h(n)\right)\mathbb{P}\left\{k^{1/p}<\left|Y^{(j)}\right|\le (k+1)^{1/p}\right\}\\
\le&C\sum_{j\in B}\sum_{k=1}^{\infty} k^{\frac{r-2}{p}+\delta}h(k)\mathbb{E}\left|Y^{(j)}\right|^2 I\left\{(k-1)^{1/p}<\left|Y^{(j)}\right|\le k^{1/p}\right\}\\
&+ C\sum_{j\in B}\sum_{k=1}^{\infty} k^{\frac{r}{p}+\delta}h(k)\mathbb{P}\left\{k^{1/p}<\left|Y^{(j)}\right|\le (k+1)^{1/p}\right\}\\
\le&C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|^{r+\delta p}h\left(\left|Y^{(j)}\right|^p\right)<\infty,
\end{aligned}
$$

$$
\begin{aligned}
I_2 =&C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2}h(n)g(n)\sum_{i=1}^{n}\sum_{j\in B}\mathbb{P}\left\{\left|Y_i^{(j)}\right|>n^{1/p}\right\}\\
\le&C\sum_{n=1}^{\infty} n^{\frac{r}{p}-1+\delta}h(n)\sum_{j\in B}\mathbb{P}\left\{\left|Y^{(j)}\right|>n^{1/p}\right\}\\
<&\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 =&C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\sum_{i=1}^{n}\sum_{j\in B}\mathbb{E}\left|Y_i^{(j)}\right|I\left\{\left|Y_i^{(j)}\right|>n^{1/p}\right\}\\
\le&C\sum_{j\in B}\sum_{n=1}^{\infty} n^{\frac{r}{p}-1-\frac{1}{p}}h(n)\sum_{k=n}^{\infty}\mathbb{E}\left|Y^{(j)}\right|I\left\{k^{1/p}<\left|Y^{(j)}\right|\le (k+1)^{1/p}\right\}\\
=&C\sum_{j\in B}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k} n^{\frac{r}{p}-1-\frac{1}{p}}h(n)\right)\mathbb{E}\left|Y^{(j)}\right|I\left\{k^{1/p}<\left|Y^{(j)}\right|\le (k+1)^{1/p}\right\}\\
\le&C\sum_{j\in B}\sum_{k=1}^{\infty} k^{\frac{r-1}{p}}h(k)\mathbb{E}\left|Y^{(j)}\right|I\left\{k^{1/p}<\left|Y^{(j)}\right|\le (k+1)^{1/p}\right\}\\
\le&C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|^{r}h\left(\left|Y^{(j)}\right|^p\right)<\infty.
\end{aligned}
$$

Thus, (8) holds and the proof of Theorem 2.1 is ended. □

Proof. **[Proof of Theorem 2.4]** We only show the difference between the proof of Theorems 2.1 and 2.4. Note that $\mathbb{E}Y_i = 0$, applying Lemmas 3.8 and 3.5 to (11) gives

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\mathbb{E}\left(\max_{1\le s\le n}\left\|\sum_{i=1}^{s}Y_i\right\|-\varepsilon n^{\frac{1}{p}}\right)_+\\
=&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\mathbb{E}\left(\max_{1\le s\le n}\left\|\sum_{i=1}^{s}(Y_i-\mathbb{E}Y_i)\right\|-\varepsilon n^{\frac{1}{p}}\right)_+
\end{aligned}
$$

$$
\begin{aligned}
\leq & C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) n^{-\frac{1}{p}} \mathbb{E}\left(\max_{1\leq s\leq n}\left\|\sum_{i=1}^{s}(Y_{ni}-\mathbb{E}Y_{ni})\right\|^2\right) \\
& + C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \mathbb{E}\left(\max_{1\leq s\leq n}\left\|\sum_{i=1}^{s}(Z_{ni}-\mathbb{E}Z_{ni})\right\|\right) \\
\leq & C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{2}{p}} h(n) g(n) \log^2 n \sum_{i=1}^{n} \mathbb{E}\|Y_{ni}\|^2 + C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \sum_{i=1}^{n} \mathbb{E}\|Z_{ni}\| \\
\leq & C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{2}{p}} h(n) g(n) \log^2 n \sum_{i=1}^{n}\sum_{j\in B} \mathbb{E}\left|Y_i^{(j)}\right|^2 I\left\{\left|Y_i^{(j)}\right| \leq n^{1/p}\right\} \\
& + C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2} h(n) g(n) \log^2 n \sum_{i=1}^{n}\sum_{j\in B} \mathbb{P}\left\{\left|Y_i^{(j)}\right| > n^{1/p}\right\} \\
& + C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \sum_{i=1}^{n}\sum_{j\in B} \mathbb{E}\left|Y_i^{(j)}\right| I\left\{\left|Y_i^{(j)}\right| > n^{1/p}\right\} \\
\leq & C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{2}{p}+\delta} h(n) \sum_{i=1}^{n}\sum_{j\in B} \mathbb{E}\left|Y_i^{(j)}\right|^2 I\left\{\left|Y_i^{(j)}\right| \leq n^{1/p}\right\} \\
& + C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2+\delta} h(n) \sum_{i=1}^{n}\sum_{j\in B} \mathbb{P}\left\{\left|Y_i^{(j)}\right| > n^{1/p}\right\} \\
& + C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \sum_{i=1}^{n}\sum_{j\in B} \mathbb{E}\left|Y_i^{(j)}\right| I\left\{\left|Y_i^{(j)}\right| > n^{1/p}\right\} \\
< & \infty. \text{ (by the proof of Theorem 2.1)}
\end{aligned}
$$

□

Proof. **[Proof of Theorem 2.7]** It is not hard to observe that

$$\sum_{k=1}^{s} X_k = \sum_{i=-\infty}^{\infty}\sum_{k=1}^{s} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{k=i+1}^{i+s} Y_k. \tag{27}$$

Considering that $\{a_i, -\infty < i < \infty\}$ is absolutely summable, we assume that $\sum_{-\infty<i<\infty}|a_i| \leq 1$. For any $t > 0$ and $j \in B$, set

$$\begin{cases} Y_{kt}^{(j)} = -tI\left\{Y_k^{(j)} < -t\right\} + Y_k^{(j)} I\left\{\left|Y_k^{(j)}\right| \leq t\right\} + tI\left\{Y_k^{(j)} > t\right\}, \\ Y_{kt} = \sum_{j\in B} Y_{kt}^{(j)} e_j; \end{cases}$$

and

$$\begin{cases} Z_{kt}^{(j)} = Y_k^{(j)} - Y_{kt}^{(j)} = \left(Y_k^{(j)} - t\right) I\left\{Y_k^{(j)} > t\right\} + \left(Y_k^{(j)} + t\right) I\left\{Y_k^{(j)} < -t\right\}, \\ Z_{kt} = \sum_{j\in B} Z_{kt}^{(j)} e_j. \end{cases}$$

Since

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\le s\le n}\left\|\sum_{k=1}^{s}X_k\right\|>t\right)\\
=&\mathbb{P}\left(\max_{1\le s\le n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}Y_k\right\|>t\right)\\
=&\mathbb{P}\left(\max_{1\le s\le n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Y_k-\mathbb{E}Y_k)\right\|>t\right)\\
\le&\mathbb{P}\left(\max_{1\le s\le n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Y_{kt}-\mathbb{E}Y_{kt})\right\|>t/2\right)+\mathbb{P}\left(\max_{1\le s\le n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Z_{kt}-\mathbb{E}Z_{kt})\right\|>t/2\right),
\end{aligned} \tag{28}$$

we need to prove

$$I_1=\sum_{n=1}^{\infty}\frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}}\int_{(n^{1+\delta}\log^2 n)^{1/p}}^{\infty}\mathbb{P}\left(\max_{1\le s\le n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Y_{kt}-\mathbb{E}Y_{kt})\right\|>t/2\right)dt<\infty, \tag{29}$$

and

$$I_2=\sum_{n=1}^{\infty}\frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}}\int_{(n^{1+\delta}\log^2 n)^{1/p}}^{\infty}\mathbb{P}\left(\max_{1\le s\le n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Z_{kt}-\mathbb{E}Z_{kt})\right\|>t/2\right)dt<\infty. \tag{30}$$

For I_1, by Markov's inequality, Cauchy's inequality, Lemmas 3.5 and 3.6, we have

$$\begin{aligned}
I_1\le&C\sum_{n=1}^{\infty}\frac{1}{n\left(n^{1+\delta}\log^2 n\right)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\mathbb{E}\left(\max_{1\le s\le n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Y_{kt}-\mathbb{E}Y_{kt})\right\|\right)^2\frac{dt}{t^2}\\
\le&C\sum_{n=1}^{\infty}\frac{1}{n\left(n^{1+\delta}\log^2 n\right)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\mathbb{E}\left(\sum_{i=-\infty}^{\infty}|a_i|^{1/2}\cdot|a_i|^{1/2}\max_{1\le s\le n}\left\|\sum_{k=i+1}^{i+s}(Y_{kt}-\mathbb{E}Y_{kt})\right\|\right)^2\frac{dt}{t^2}\\
\le&C\sum_{n=1}^{\infty}\frac{1}{n\left(n^{1+\delta}\log^2 n\right)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)\cdot\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}\left(\max_{1\le s\le n}\left\|\sum_{k=i+1}^{i+s}(Y_{kt}-\mathbb{E}Y_{kt})\right\|\right)^2\frac{dt}{t^2}\\
\le&C\sum_{n=1}^{\infty}\frac{g(n)\log^2 n}{n(n^{1+\delta}\log^2 n)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\sum_{i=-\infty}^{\infty}|a_i|\sum_{k=i+1}^{i+n}\mathbb{E}\left\|Y_{kt}\right\|^2\frac{dt}{t^2}\\
\le&C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{n\left(n^{1+\delta}\log^2 n\right)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\sum_{i=-\infty}^{\infty}|a_i|\sum_{k=i+1}^{i+n}\mathbb{P}\left(\left|Y_k^{(j)}\right|>t\right)dt\\
&+C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{n\left(n^{1+\delta}\log^2 n\right)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\sum_{i=-\infty}^{\infty}|a_i|\sum_{k=i+1}^{i+n}\mathbb{E}\left|Y_k^{(j)}\right|^2 I\left\{\left|Y_k^{(j)}\right|\le t\right\}\frac{dt}{t^2}\\
\le&C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\mathbb{P}\left(\left|Y^{(j)}\right|>t\right)dt\\
&+C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\mathbb{E}\left|Y^{(j)}\right|^2 I\left\{\left|Y^{(j)}\right|\le t\right\}\frac{dt}{t^2}
\end{aligned}$$

$$=I_{11}+I_{12}.$$

The standard computation will show that

$$\begin{aligned}
I_{11} =&C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\mathbb{P}\left(\left|Y^{(j)}\right|>t\right)dt\\
\le&C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\mathbb{E}\left|Y^{(j)}\right|I\left\{\left|Y^{(j)}\right|>\left(n^{1+\delta}\log^2 n\right)^{1/p}\right\}\\
=&C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\sum_{i=n}^{\infty}\mathbb{E}\left|Y^{(j)}\right|I\left\{\left(i^{1+\delta}\log^2 i\right)^{1/p}<\left|Y^{(j)}\right|\le\left((i+1)^{1+\delta}\log^2(i+1)\right)^{1/p}\right\}\\
=&C\sum_{j\in B}\sum_{i=1}^{\infty}\left(\sum_{n=1}^{i}\frac{n^{\delta}\log^2 n}{(n^{1+\delta}\log^2 n)^{1/p}}\right)\mathbb{E}\left|Y^{(j)}\right|I\left\{\left(i^{1+\delta}\log^2 i\right)^{1/p}<\left|Y^{(j)}\right|\le\left((i+1)^{1+\delta}\log^2(i+1)\right)^{1/p}\right\}\\
\le&C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|\sum_{i=1}^{\infty}\left(i^{1+\delta}\log^2 i\right)^{1-1/p}I\left\{\left(i^{1+\delta}\log^2 i\right)^{1/p}<\left|Y^{(j)}\right|\le\left((i+1)^{1+\delta}\log^2(i+1)\right)^{1/p}\right\}\\
\le&C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|^p<\infty,
\end{aligned}$$

and

$$\begin{aligned}
I_{12} =&C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\mathbb{E}\left|Y^{(j)}\right|^2 I\left\{\left|Y^{(j)}\right|\le t\right\}\frac{dt}{t^2}\\
\le&C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\sum_{i=n}^{\infty}\int_{\left(i^{1+\delta}\log^2 i\right)^{1/p}}^{\left((i+1)^{1+\delta}\log^2(i+1)\right)^{1/p}}\mathbb{E}\left|Y^{(j)}\right|^2 I\left\{\left|Y^{(j)}\right|\le t\right\}\frac{dt}{t^2}\\
\le&C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\sum_{i=n}^{\infty}\frac{1}{i\left(i^{1+\delta}\log^2 i\right)^{1/p}}\mathbb{E}\left|Y^{(j)}\right|^2 I\left\{\left|Y^{(j)}\right|^p\le(i+1)^{1+\delta}\log^2(i+1)\right\}\\
\le&C\sum_{j\in B}\sum_{i=1}^{\infty}\left(\sum_{n=1}^{i}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\right)\frac{1}{i\left(i^{1+\delta}\log^2 i\right)^{1/p}}\mathbb{E}\left|Y^{(j)}\right|^2 I\left\{\left|Y^{(j)}\right|^p\le(i+1)^{1+\delta}\log^2(i+1)\right\}\\
\le&C\sum_{j\in B}\sum_{i=1}^{\infty}\frac{i^{\delta}\log^2 i}{(i^{1+\delta}\log^2 i)^{2/p}}\mathbb{E}\left|Y^{(j)}\right|^2 I\left\{\left|Y^{(j)}\right|^p\le(i+1)^{1+\delta}\log^2(i+1)\right\}\\
=&C\sum_{j\in B}\sum_{i=1}^{\infty}\frac{i^{\delta}\log^2 i}{(i^{1+\delta}\log^2 i)^{2/p}}\mathbb{E}\left|Y^{(j)}\right|^2\sum_{m=1}^{i}I\left\{m^{1+\delta}\log^2 m<\left|Y^{(j)}\right|^p\le(m+1)^{1+\delta}\log^2(m+1)\right\}\\
=&C\sum_{j\in B}\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty}\frac{i^{\delta}\log^2 i}{(i^{1+\delta}\log^2 i)^{2/p}}\right)\mathbb{E}\left|Y^{(j)}\right|^2 I\left\{m^{1+\delta}\log^2 m<\left|Y^{(j)}\right|^p\le(m+1)^{1+\delta}\log^2(m+1)\right\}\\
\le&C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|^2\sum_{i=1}^{\infty}\left(i^{1+\delta}\log^2 i\right)^{1-2/p}I\left\{i^{1+\delta}\log^2 i<\left|Y^{(j)}\right|^p\le(i+1)^{1+\delta}\log^2(i+1)\right\}\\
\le&C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|^p<\infty.
\end{aligned}$$

Hence, the proof of (29) is completed.

For I_2, by Markov's inequality and the proof process of I_{11}, we can derive

$$\begin{aligned}
I_2 \leq & C\sum_{n=1}^{\infty}\frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\mathbb{E}\left(\max_{1\leq s\leq n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Z_{kt}-\mathbb{E}Z_{kt})\right\|\right)\frac{dt}{t}\\
\leq & C\sum_{n=1}^{\infty}\frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\sum_{i=-\infty}^{\infty}|a_i|\sum_{k=i+1}^{i+n}\mathbb{E}\left\|Z_{kt}\right\|\frac{dt}{t}\\
\leq & C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{1}{(n^{1+\delta}\log^2 n)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\mathbb{E}\left|Y^{(j)}\right|I\left\{\left|Y^{(j)}\right|>t\right\}\frac{dt}{t}\\
\leq & C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{1}{(n^{1+\delta}\log^2 n)^{1/p}}\sum_{i=n}^{\infty}\int_{\left(i^{1+\delta}\log^2 i\right)^{1/p}}^{\left((i+1)^{1+\delta}\log^2(i+1)\right)^{1/p}}\mathbb{E}\left|Y^{(j)}\right|I\left\{\left|Y^{(j)}\right|>\left(i^{1+\delta}\log^2 i\right)^{1/p}\right\}\frac{dt}{t}\\
\leq & C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{1}{(n^{1+\delta}\log^2 n)^{1/p}}\sum_{i=n}^{\infty}\frac{1}{i}\mathbb{E}\left|Y^{(j)}\right|I\left\{\left|Y^{(j)}\right|>\left(i^{1+\delta}\log^2 i\right)^{1/p}\right\}\\
\leq & C\sum_{j\in B}\sum_{i=1}^{\infty}\left(\sum_{n=1}^{i}\frac{n^{\delta}\log^2 n}{(n^{1+\delta}\log^2 n)^{1/p}}\right)\frac{1}{i}\mathbb{E}\left|Y^{(j)}\right|I\left\{\left|Y^{(j)}\right|>\left(i^{1+\delta}\log^2 i\right)^{1/p}\right\}\\
\leq & C\sum_{j\in B}\sum_{i=1}^{\infty}\frac{i^{\delta}\log^2 i}{(i^{1+\delta}\log^2 i)^{1/p}}\mathbb{E}\left|Y^{(j)}\right|I\left\{\left|Y^{(j)}\right|>\left(i^{1+\delta}\log^2 i\right)^{1/p}\right\}\\
\leq & C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|^p<\infty.
\end{aligned}$$

Consequently, (30) is proved and the proof of Theorem 2.7 is complete. □

Proof. **[Proof of Theorem 2.8]** We also assume that $\sum_{-\infty<i<\infty}|a_i|\leq 1$. For any $t>0$ and $j\in B$, set

$$\begin{cases}Y_{in}^{(j)}=-n^{1+\delta}\log^2 nI\left\{Y_i^{(j)}<-n^{1+\delta}\log^2 n\right\}+Y_i^{(j)}I\left\{\left|Y_i^{(j)}\right|\leq n^{1+\delta}\log^2 n\right\}\\ \qquad +n^{1+\delta}\log^2 nI\left\{Y_i^{(j)}>n^{1+\delta}\log^2 n\right\},\\ Y_{in}=\sum_{j\in B}Y_{in}^{(j)}e_j;\end{cases}$$

and

$$\begin{cases}Z_{in}^{(j)}=Y_i^{(j)}-Y_{in}^{(j)}=\left(Y_i^{(j)}+n^{1+\delta}\log^2 n\right)I\left\{Y_i^{(j)}<-n^{1+\delta}\log^2 n\right\}\\ \qquad +\left(Y_i^{(j)}-n^{1+\delta}\log^2 n\right)I\left\{Y_i^{(j)}>n^{1+\delta}\log^2 n\right\},\\ Z_{in}=\sum_{j\in B}Z_{in}^{(j)}e_j.\end{cases}$$

Similar to (28), (16) can be estimated as

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}\left\{\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s}X_k\right\|>\varepsilon\left(n^{1+\delta\log^2 n}\right)^{1/p}\right\}\\
\leq&\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}\left\{\max_{1\leq s\leq n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Y_{kn}-\mathbb{E}Y_{kn})\right\|>\varepsilon/2\left(n^{1+\delta}\log^2 n\right)^{1/p}\right\}\\
&+\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}\left\{\max_{1\leq s\leq n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Z_{kn}-\mathbb{E}Z_{kn})\right\|>\varepsilon/2\left(n^{1+\delta}\log^2 n\right)^{1/p}\right\}\\
=&I_1+I_2
\end{aligned}\tag{31}$$

Markov's inequality, Cauchy's inequality, Lemma 3.5-3.6 and the proof method shown in Theorem 2.7 will guarantee that

$$\begin{aligned}
I_1 \leq & C\sum_{n=1}^{\infty}\frac{1}{n\left(n^{1+\delta}\log^2 n\right)^{2/p}}\mathbb{E}\left(\max_{1\leq s\leq n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Y_{kn}-\mathbb{E}Y_{kn})\right\|\right)^2 \\
\leq & C\sum_{n=1}^{\infty}\frac{1}{n\left(n^{1+\delta}\log^2 n\right)^{2/p}}\mathbb{E}\left(\sum_{i=-\infty}^{\infty}|a_i|^{1/2}\cdot|a_i|^{1/2}\max_{1\leq s\leq n}\left\|\sum_{k=i+1}^{i+s}(Y_{kn}-\mathbb{E}Y_{kn})\right\|\right)^2 \\
\leq & C\sum_{n=1}^{\infty}\frac{g(n)\log^2 n}{n\left(n^{1+\delta}\log^2 n\right)^{2/p}}\sum_{i=-\infty}^{\infty}|a_i|\sum_{k=i+1}^{i+n}\mathbb{E}\left\|Y_{kn}\right\|^2 \\
\leq & C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{2/p}}\mathbb{E}\left|Y^{(j)}\right|^2 I\left\{\left|Y^{(j)}\right|\leq\left(n^{1+\delta}\log^2 n\right)^{1/p}\right\} \\
& + C\sum_{j\in B}\sum_{n=1}^{\infty}n^{\delta}\log^2 n\mathbb{P}\left\{|Y^{(j)}|>\left(n^{1+\delta}\log^2 n\right)^{1/p}\right\} \\
\leq & C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|^p<\infty,
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
I_2 \leq & C\sum_{n=1}^{\infty}\frac{1}{n\left(n^{1+\delta}\log^2 n\right)^{1/p}}\mathbb{E}\left(\max_{1\leq s\leq n}\left\|\sum_{i=-\infty}^{\infty}a_i\sum_{k=i+1}^{i+s}(Z_{kn}-\mathbb{E}Z_{kn})\right\|\right) \\
\leq & C\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{n\left(n^{1+\delta}\log^2 n\right)^{1/p}}\sum_{i=-\infty}^{\infty}|a_i|\sum_{k=i+1}^{i+n}\mathbb{E}\left\|Z_{kn}\right\| \\
\leq & C\sum_{j\in B}\sum_{n=1}^{\infty}\frac{n^{\delta}\log^2 n}{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\mathbb{E}\left|Y^{(j)}\right| I\left\{\left|Y^{(j)}\right|>\left(n^{1+\delta}\log^2 n\right)^{1/p}\right\} \\
\leq & C\sum_{j\in B}\mathbb{E}\left|Y^{(j)}\right|^p<\infty.
\end{aligned} \tag{33}$$

The proof of Theorem 2.8 is completed. □

Proof. **[Proof of Theorem 2.10]** By Theorems 2.7 and 2.8,

$$\begin{aligned}
& \sum_{n=1}^{\infty}\frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}}\mathbb{E}\left(\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s}X_k\right\|-\varepsilon\left(n^{1+\delta}\log^2 n\right)^{1/p}\right)_+ \\
= & \sum_{n=1}^{\infty}\frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}}\int_0^{\infty}\mathbb{P}\left(\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s}X_k\right\|>\varepsilon\left(n^{1+\delta}\log^2 n\right)^{1/p}+t\right)dt \\
= & \sum_{n=1}^{\infty}\frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}}\int_0^{\left(n^{1+\delta}\log^2 n\right)^{1/p}}\mathbb{P}\left(\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s}X_k\right\|>\varepsilon\left(n^{1+\delta}\log^2 n\right)^{1/p}+t\right)dt \\
& + \sum_{n=1}^{\infty}\frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}}\int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty}\mathbb{P}\left(\max_{1\leq s\leq n}\left\|\sum_{k=1}^{s}X_k\right\|>\varepsilon\left(n^{1+\delta}\log^2 n\right)^{1/p}+t\right)dt
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\left(\max_{1\leq s\leq n} \left\| \sum_{k=1}^{s} X_k \right\| > \varepsilon \left(n^{1+\delta} \log^2 n \right)^{1/p} \right)$$
$$+ \sum_{n=1}^{\infty} \frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}} \int_{\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty} \mathbb{P}\left(\max_{1\leq s\leq n} \left\| \sum_{k=1}^{s} X_k \right\| > t \right) dt$$
$$<\infty.$$

□

5. Examples

In this section, we present an example to illustrate the sharpness of Theorem 2.10. This example is inspired by Example 3.4 of Ko [16], it will show that the converse of Theorem 2.10 is not true.

Example 5.1. *Consider the real separable Hilbert space ℓ_2 of all square summable real sequences with inner product*

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i \text{ for } x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots).$$

Let $\{Y, Y_n, n \geq 1\}$ be a sequence of i.i.d ℓ_2-valued CWOD random vectors with

$$\mathbb{P}\left(Y^{(j)} = -j^{-1/p}\right) = \mathbb{P}\left(Y^{(j)} = j^{-1/p}\right) = \frac{1}{2} \text{ for all } j \in B.$$

And $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ is a moving average process generated by $\{Y, Y_n, n \geq 1\}$ with $\sum_{i=-\infty}^{\infty} |a_i| = 1$. Then (17) holds, but (14) fails.
In fact, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}} \mathbb{E}\left(\max_{1\leq s\leq n} \left\| \sum_{k=1}^{s} X_k \right\| - \varepsilon \left(n^{1+\delta} \log^2 n \right)^{1/p} \right)_+$$
$$= \sum_{n=1}^{\infty} \frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}} \int_0^{\infty} \mathbb{P}\left(\max_{1\leq s\leq n} \left\| \sum_{k=1}^{s} X_k \right\| \geq \varepsilon \left(n^{1+\delta} \log^2 n \right)^{1/p} + u \right) du$$
$$= \sum_{n=1}^{\infty} \frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}} \int_{\varepsilon\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty} \mathbb{P}\left(\max_{1\leq s\leq n} \left\| \sum_{k=1}^{s} X_k \right\| > t \right) dt$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{n(n^{1+\delta}\log^2 n)^{1/p}} \int_{\varepsilon\left(n^{1+\delta}\log^2 n\right)^{1/p}}^{\infty} \frac{dt}{t^2} \mathbb{E}\left(\max_{1\leq s\leq n} \left\| \sum_{k=1}^{s} X_k \right\| \right)^2$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{n(n^{1+\delta}\log^2 n)^{2/p}} \mathbb{E}\left(\max_{1\leq s\leq n} \left\| \sum_{i=-\infty}^{\infty} a_i \sum_{k=i+1}^{i+s} Y_k \right\| \right)^2 \text{ (by the proof of } I_{12} \text{ in Theorem 2.7)}$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{n(n^{1+\delta}\log^2 n)^{2/p}} \sum_{i=-\infty}^{\infty} |a_i| \sum_{k=i+1}^{i+n} \mathbb{E}\,\|Y_k\|^2 \text{ (by the proof of } I_1 \text{ in Theorem 2.7)}$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{(n^{1+\delta}\log^2 n)^{2/p}} \mathbb{E}\|Y\|^2$$
$$\leq C \sum_{n=1}^{\infty} \frac{1}{(n^{1+\delta}\log^2 n)^{2/p}} \sum_{j\in B} \mathbb{E}\left|Y^{(j)}\right|^2$$

$$\leq C \sum_{n=1}^{\infty} \frac{1}{n^{2/p} \log^{4/p} n} \left(\sum_{j \in B} \frac{1}{j^{2/p}} \right) < \infty,$$

but

$$\sum_{j \in B} \mathbb{E} \left| Y^{(j)} \right|^p = \sum_{j \in B} \frac{1}{j} = \infty.$$

References

[1] Y. S. Chow, On the rates of moment convegence of sample sums and etremes, *Bulletin of the institute of Mathematics Academia Sinica*, **16** (1988), 177–201.
[2] T. C. Christofides, and E. Vaggelatou, *A connection between supermodular ordering and positive/negative association*, Journal of Multivariate Analysis. **88** (2004), 138–151.
[3] Y. Ding, Y. Wu, S. L. Ma, X. R. Tao, X. J. Wang, *Complete convergence and complete moment convergence for widely orthant-dependent random variables*, Communications in Statistics-Theory and Methods, **46** (2017), 8278–8294.
[4] L. H. Guan, Y. S. Xiao, Y. N. Zhao, *Complete moving average processes for m-WOD sequence*, J. Inequal. Appl. (2021), doi:10.1186/s13660-021-02546-6.
[5] A. Gut, *Complete convergence for arrays*, Period. Math. Hung. **25** (1992), 51–75.
[6] N. T. T. Hien, L. V. Thanh, V. T. H. Van, *On the negative dependnece in Hilbert spaces with applications*, Appl. Math. **64** (2019), 45–59.
[7] P. L. Hsu, H. Robbins, *Complete convergence and the law of large numbers*, Proc. Natl. Acad. Sci. U.S.A. **33** (1947), 25–31.
[8] T. Z. Hu, *Negatively superadditive dependence of random variables with applications*, Chinese J. Appl. Probab. Statist. **16** (2000), 133–144.
[9] N. V. Huan, N. V. Quang, N. T. Thuan, *Baum-Katz type theorems for coordinatewise negatively associated random vectors in Hilbert spaces*, Acta Math. Hung. **144** (2014), 132–149.
[10] K. Joag-Dev, F. Proschan, *Negative association of random variables with applications*, Ann. Statist. **11** (1983), 286–295.
[11] M. H. Ko, *The complete moment convergence for CNA random vectors in Hilbert spaces*, J. Inequal. Appl. (2017), doi:s13660-017-1566-x.
[12] M. H. Ko, *Complete convergence for coordinatewise asymptotically negatively associated random vectors in Hilbert spaces*, Comm. Statist. Theory Methods. **47** (2018), 671–680.
[13] M. H. Ko, *On complete moment convergence for CAANA random vectors in Hilbert spaces*, Statist. Probab. Lett. **138** (2018), 104–110.
[14] M. H. Ko, *A note on complete moment convergence for coordinatewise negatively associated random vectors in Hilbert spaces*, Comm. Statist. Theory Methods (2019), doi: 10.1080/03610926.2019.1565833.
[15] M. H. Ko, *Complete and complete moment convergence of the wighted sums of ρ^*-mxing random vectors in Hilbert spaces*, East Asian J. Appl. Math. **12** (2021), 617–627.
[16] M. H. Ko, *The complete moment convergence for coordinatewise pairwise negatively quadrant dependent random vectors in Hilbert space*, Comm. Statist. Theory Methods (2022). doi: 10.1080/03610926.2022.2064500.
[17] M. H. Ko, *The complete moment convergence for weighted sums of coordinatewise ANA random vectors in Hilbert space*, Comm. Statist. Theory Methods (2022). doi: 10.1080/03610926.2022.2026395.
[18] E. Lehmann, *Some concepts of dependence*, Ann. Statist. **37** (1966), 1137–1153.
[19] Y. X. Li, L. X. Zhang, *Complete moment convergence of moving average processes under dependence assumptions*, Statist. Probab. Lett. **70** (2004), 191–197.
[20] L. Liu, *Precise large deviations for dependent random variables with heavy tails*, Statist. Probab. Lett. **79** (2009), 1290–1298.
[21] Y. Miao, G. Y. Yang, G. Stoica, *On the rate of convergence in the strong law of large numbers for martingales*, Stochastics. **87** (2015), 185–198.
[22] X. M. Qu, *Complete moment convergence of moving average processes under END assumptions*, Comm. Statist. Theory Methods (2020). doi:10.1080/03610926.2020.1767138.
[23] A. Rosalsky , L. V. Thanh. *A note on the stochastic domination condition and uniform integrability with applications to the strong law of large numbers*, Statist. Probab. Lett. **178** (2021), Paper No. 109181, 1–10.
[24] E. Seneta, *Regularly varying functions*. Lecture Notes in Mathematics, (Vol. 508), Springer-Verlag, Berlin-New York, 1976.
[25] L. V. Thanh, *On a new concept of stochastic domination and the laws of large numbers*, TEST (2022). doi: 10.1007/s11749-022-00827-w.
[26] R. Wang, A. T. Shen, *Complete moment convergence for moving average processes based on m-WOD random variables*, Comm. Statist. Theory Methods (2022). doi: 10.1080/03610926.2022.2037649.
[27] K. Y. Wang, Y. B. Wang, Q. W. Gao, *Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate*, Methodol. Comput. Appl. Probab. **15** (2013), 109–124.
[28] X. J. Wang, C. Xu, T. C. Hu, A. Volodin, S. H. Hu, *On complete convergence for widely orthant-dependent random variables and its applications in nonparametric regression models*, TEST. **23** (2014), 607–629.
[29] Q. Y. Wu, *Probability limit theory for mixing sequences*, Science Press of China, Beijing, 2006.
[30] Y. Wu, F. Zhang, X. Wang, *Convergence properties for weighted sums of weakly dependent random vectors in Hilbert spaces*, Stochatics (2020). doi:10.1080/17442508.2019.1652607.
[31] X. C. Zhou, *Complete moment convergence of moving average process under $\varphi-$mixing assumption*, Statist. Probab. Lett. **80** (2010), 285–292.
[32] X. C. Zhou, J. G Lin, Complete moment convergence of moving average processes under $\rho-$mixing assumption, Math. Slovoca. **61** (2011), 979–992.