Filomat 38:29 (2024), 10391–10403
https://doi.org/10.2298/FIL2429391B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Strongly (p, σ)-absolutely continuous Bloch maps

Amar Bougoutaia[a,*], Amar Belacel[a], Antonio Jiménez-Vargas[b]

[a] *Laboratory of Pure and Applied Mathematics, University Amar Telidji of Laghouat, BP 37G, Ghardaia Road, Laghouat 03000, Algeria*
[b] *Departamento de Matemáticas, Universidad de Almería, Ctra. de Sacramento s/n, 04120 La Cañada de San Urbano, Almería, Spain*

Abstract. Given $1 < p, r < \infty$ and $0 \le \sigma < 1$ such that $1/r + (1-\sigma)/p^* = 1$, we study the Banach normalized Bloch ideal $(\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X), d^{\mathcal{B}}_{p,\sigma})$ formed by all strongly (p, σ)-absolutely continuous Bloch maps from the complex unit open disc $\mathbb{D}$ into a complex Banach space X. Characterizations of such Bloch maps are established in terms of: (i) Pietsch domination, (ii) linearisation on $\mathcal{G}(\mathbb{D})$ (the Bloch-free Banach space over $\mathbb{D}$), (iii) Bloch transposition, and (iv) Pietsch factorization. The invariance of such maps under Möbius transformations of $\mathbb{D}$ and their relation with compact Bloch maps are also addressed. Furthermore, we show that such space can be identified with the dual of the tensor product space $\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}X^*$ equipped with a suitable Bloch reasonable crossnorm $\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}$.

1. Introduction

Given $1 \le p < \infty$ and its conjugate index p^*, the ideal of strongly p-summing operators was introduced by Cohen [6] to analyse the duality properties of the ideal of operators whose adjoints are absolutely p^*-summing. Given $1 \le p < \infty$ and $0 \le \sigma < 1$, Matter [13] introduced the concept of (p, σ)-absolutely continuous operators. This notion serves as a crucial analytical tool for examining properties such as super reflexivity within Banach spaces (see [14]). Its development stems from an interpolation method pioneered by Jarchow and Matter [9]. The class of (p, σ)-absolutely continuous operators can be seen as an intermediate class situated between continuous operators and the well-known class of absolutely p-summing operators, offering a distinctive blend of characteristics from both.

The study of (p, σ)-absolute continuity of maps has been addressed by some authors: for example, by Achour, Rueda and Yahi [2] for Lipschitz maps, by López Molina and Sánchez Pérez [11, 12] and Sánchez Pérez [16] for operators, and, more recently, by the authors of this paper in [4] for Bloch maps. The research on strongly (p, σ)-continuous multilinear operators and strongly (p, σ)-Lipschitz operators has been dealed by Achour, Dahia, Rueda and Sánchez Pérez [1] and by Bougoutaia, Belacel and Macedo [3], respectively.

Let $1 \le p, r < \infty$ and $0 \le \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$. The main objective in this paper is to present and establish the most remarkable properties of a Bloch version of the concept of strongly (p, σ)-continuous linear operator. To be more precise, we introduce (in terms of the concept of a p^*-summing

2020 *Mathematics Subject Classification*. Primary 30H30, 46E15; Secondary 46E40, 47B10, 47B38.
Keywords. Summing operators, vector-valued Bloch maps, Pietsch factorization/domination, compact Bloch maps
Received: 21 March 2024; Revised: 22 May 2024; Accepted: 22 July 2024
Communicated by Ivana Djolović
* Corresponding author: Amar Bougoutaia
Email addresses: amarbou28@gmail.com (Amar Bougoutaia), amarbelacel@yahoo.fr (Amar Belacel), ajimenez@ual.es (Antonio Jiménez-Vargas)

operator) the notion of a strongly (p,σ)-absolutely continuous Bloch map from $\mathbb{D}$ into a complex Banach space X.

The paper is divided into two sections. The first contains some definitions, results and notations used throughout the paper. The second is a complete study on strongly (p,σ)-absolutely continuous Bloch maps from $\mathbb{D}$ into X. Our main result is a characterization of such Bloch maps in terms of a Pietsch domination. Our approach depends essentially on a linearisation process of Bloch maps developed in [10]. Using such a Pietsch domination, we show that strong (p,σ)-absolute continuity of a Bloch map on $\mathbb{D}$ is transferred to its linearisation on $\mathcal{G}(\mathbb{D})$ (the Bloch-free Banach space over $\mathbb{D}$), and vice versa. This linearisation is applied to prove in an easy form that the class of strongly (p,σ)-absolutely continuous Bloch maps, denoted by $[\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}, d^{\mathcal{B}}_{p,\sigma}]$, is an injective Banach normalized Bloch ideal. The invariance of such maps under Möbius group of $\mathbb{D}$ and its inclusion (under a mild condition on X) in the space of compact Bloch maps are also studied. We also show that strong (p,σ)-absolute continuity of a Bloch map from $\mathbb{D}$ to X is inherited by its Bloch transpose from X^* to the normalized Bloch space $\widehat{\mathcal{B}}(\mathbb{D})$, and vice versa. Another characterization of such Bloch maps is established by means of a Pietsch factorization. We conclude the paper introducing a Bloch reasonable crossnorm $\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}$ on the tensor product space $\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}X^*$ and showing that the space $(\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X), d^{\mathcal{B}}_{p,\sigma})$ is isometrically isomorphic to the dual space $(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}} X^*)^*$.

2. Preliminaries

We will recall some concepts and results on the theory of linear operators and holomorphic mappings.

Throughout this paper, X and Y will denote complex Banach spaces and $\mathcal{L}(X,Y)$ will stand for the space of all continuous linear operators of X to Y, under the operator norm. As usual, B_X and X^* will denote the closed unit ball of X and the topological dual of X, respectively. The symbol $\mathcal{P}(B_{X^{**}})$ represents the set of all regular Borel probability measures μ on $B_{X^{**}}$ with the topology w^*.

An operator $T \in \mathcal{L}(X,Y)$ is called *p-summing* with $p \in [1,\infty)$ if there exists a constant $C \geq 0$ such that

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{x^*\in B_{X^*}} \left(\sum_{i=1}^{n} |x^*(x_i)|^p\right)^{\frac{1}{p}}$$

for any $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in X$. The infimum of such constants C is denoted by $\pi_p(T)$, and the Banach space of all p-summing operators of X to Y, under the norm π_p, by $\Pi_p(X,Y)$.

For any $1 < p < \infty$, p^* denotes the Hölder conjugate of p given by $1/p + 1/p^* = 1$. Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ such that $1/r + (1-\sigma)/p^* = 1$. Following [1], an operator $T \in \mathcal{L}(X,Y)$ is called *strongly* (p,σ)*-continuous* if there exist a constant $C > 0$, a Banach space Z, and an operator $S \in \Pi_{p^*}(Y^*,Z)$ such that

$$\left|y^*(T(x))\right| \leq C\,\|x\|\,\left\|y^*\right\|^{\sigma}\,\left\|S(y^*)\right\|^{1-\sigma}$$

for all $x \in X$ and $y^* \in Y^*$. The infimum of all the values $C\pi_{p^*}(S)^{1-\sigma}$ whenever C and S satisfy the inequality above is denoted by $d_{p,\sigma}(T)$ and it defines a complete norm on the linear space $\mathcal{D}_{p,\sigma}(X,Y)$ formed by all strongly (p,σ)-continuous linear operators from X into Y.

If $\mathcal{H}(\mathbb{D},X)$ denotes the space of all holomorphic maps from the complex unit open disc $\mathbb{D}$ into X, a map $f \in \mathcal{H}(\mathbb{D},X)$ is said to be *Bloch* if

$$\rho_{\mathcal{B}}(f) := \sup\left\{(1-|z|^2)\left\|f'(z)\right\| : z \in \mathbb{D}\right\} < \infty.$$

The linear space of all Bloch maps of $\mathbb{D}$ to X, under the Bloch seminorm $\rho_{\mathcal{B}}$, is denoted by $\mathcal{B}(\mathbb{D},X)$, and the normalized Bloch space $\widehat{\mathcal{B}}(\mathbb{D},X)$ is the closed subspace of $\mathcal{B}(\mathbb{D},X)$ formed by all those maps f for which $f(0) = 0$, under the Bloch norm $\rho_{\mathcal{B}}$. For simplicity, we will write $\widehat{\mathcal{B}}(\mathbb{D})$ in place of $\widehat{\mathcal{B}}(\mathbb{D},\mathbb{C})$. Also, $\widehat{\mathcal{B}}(\mathbb{D},\mathbb{D})$ will denote the set of all holomorphic functions h from $\mathbb{D}$ into itself for which $h(0) = 0$. We refer the reader to the book [17] by Zhu for a complete study on Bloch maps.

We may introduce a Bloch version of strongly (p,σ)-continuous linear operators.

Definition 2.1. *Let $1 \leq p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$, and let X be a complex Banach space. A map $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be strongly (p,σ)-absolutely continuous Bloch if there exist a constant $C > 0$, a complex Banach space Y and an operator $S \in \Pi_{p^*}(X^*, Y)$ such that*

$$\left|x^*(f'(z))\right| \leq C\frac{1}{1-|z|^2} \|x^*\|^\sigma \|S(x^*)\|^{1-\sigma}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. The linear space of all strongly (p,σ)–absolutely continuous Bloch maps from $\mathbb{D}$ to X is denoted by $\mathcal{D}^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X)$, and its subspace consisting of all those mappings f so that $f(0) = 0$ by $\widehat{\mathcal{D}^{\mathcal{B}}_{p,\sigma}}(\mathbb{D}, X)$.

We denote by $d^{\mathcal{B}}_{p,\sigma}(f)$ the infimum of all values $C\pi_{p^*}(S)^{1-\sigma}$ whenever C and S vary over all the constants and all p^*-summing linear operators on X^* that fulfill the inequality above.

We also will need some results on the Bloch-free Banach space over $\mathbb{D}$, borrowed from [10].

For each $z \in \mathbb{D}$, a Bloch atom of $\mathbb{D}$ is the function $\gamma_z : \widehat{\mathcal{B}}(\mathbb{D}) \to \mathbb{C}$ defined by $\gamma_z(f) = f'(z)$ for all $f \in \hat{\mathcal{B}}(\mathbb{D})$. Note that $\gamma_z \in \widehat{\mathcal{B}}(\mathbb{D})^*$ with $\|\gamma_z\| = 1/(1-|z|^2)$. The elements of the linear space $\mathrm{lin}(\{\gamma_z : z \in \mathbb{D}\}) \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*$ are referred to as Bloch molecules of $\mathbb{D}$. The Bloch-free Banach space over $\mathbb{D}$ is the Banach space $\mathcal{G}(\mathbb{D}) := \overline{\mathrm{lin}}(\{\gamma_z : z \in \mathbb{D}\}) \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*$.

The following result gathers some needed properties of $\mathcal{G}(\mathbb{D})$.

Theorem 2.2. *[10]*

1. *The map $\Gamma : \mathbb{D} \to \mathcal{G}(\mathbb{D})$, defined by $\Gamma(z) = \gamma_z$ for all $z \in \mathbb{D}$, is holomorphic.*
2. *The space $\widehat{\mathcal{B}}(\mathbb{D})$ is isometrically isomorphic to $\mathcal{G}(\mathbb{D})^*$, via $\Lambda : \widehat{\mathcal{B}}(\mathbb{D}) \to \mathcal{G}(\mathbb{D})^*$ given by*

$$\Lambda(f)(\gamma) = \sum_{k=1}^{n} \lambda_k f'(z_k) \qquad \left(f \in \widehat{\mathcal{B}}(\mathbb{D}),\ \gamma = \sum_{k=1}^{n} \lambda_k \gamma_{z_k} \in \mathrm{lin}(\Gamma(\mathbb{D}))\right).$$

3. *For each function $h \in \widehat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$, there exists a unique operator $\widehat{h} \in \mathcal{L}(\mathcal{G}(\mathbb{D}), \mathcal{G}(\mathbb{D}))$ such that $\widehat{h} \circ \Gamma = h' \cdot (\Gamma \circ h)$. Furthermore, $\|\widehat{h}\| \leq 1$.*
4. *For every complex Banach space X and every map $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$, there exists a unique operator $S_f \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$ such that $S_f \circ \Gamma = f'$. Moreover, $\|S_f\| = \rho_{\mathcal{B}}(f)$.*
5. *The map $f \mapsto S_f$ is an isometric isomorphism from $\widehat{\mathcal{B}}(\mathbb{D}, X)$ onto $\mathcal{L}(\mathcal{G}(\mathbb{D}), X)$.*
6. *For each $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$, the map $f^t : X^* \to \widehat{\mathcal{B}}(\mathbb{D})$, defined by $f^t(x^*) = x^* \circ f$ if $x^* \in X^*$, is in $\mathcal{L}(X^*, \widehat{\mathcal{B}}(\mathbb{D}))$ with $\|f^t\| = \rho_{\mathcal{B}}(f)$ and $f^t = \Lambda^{-1} \circ (S_f)^*$, where $(S_f)^* : X^* \to \mathcal{G}(\mathbb{D})^*$ denotes the adjoint operator of S_f.* □

3. The results

We first present some inclusion relations between $\mathcal{D}^{\mathcal{B}}_{p,\sigma}$-spaces. For two semi-normed spaces (X, ρ_X) and (Y, ρ_Y), the inequality $(X, \rho_X) \leq (Y, \rho_Y)$ will mean that $X \subseteq Y$ and $\rho_Y(x) \leq \rho_X(x)$ for all $x \in X$.

Proposition 3.1. *(Inclusions). Let $1 < p, q, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$ and $1/r + (1-\sigma)/q^* = 1$, and let X be a complex Banach space. If $p < q$, then*

$$(\mathcal{D}^{\mathcal{B}}_{q,\sigma}(\mathbb{D}, X), d^{\mathcal{B}}_{q,\sigma}) \leq (\mathcal{D}^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X), d^{\mathcal{B}}_{p,\sigma}) \leq (\mathcal{B}(\mathbb{D}, X), \rho_{\mathcal{B}}).$$

Proof. If $1 < p < q < \infty$, it is immediate that $q^* < p^*$, and then the relation $(\Pi_{q^*}(X, Y), \pi_{q^*}) \leq (\Pi_{p^*}(X, Y), \pi_{p^*})$, established in [8, Theorem 2.8], yields the first inequality of the statement. For the second, if $f \in \mathcal{D}^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X)$, we can take a constant $C > 0$, a complex Banach space Y and an operator $S \in \Pi_{p^*}(X^*, Y)$ such that

$$\left|x^*(f'(z))\right| \leq C\frac{1}{1-|z|^2} \|x^*\|^\sigma \|S(x^*)\|^{1-\sigma} \leq C\frac{1}{1-|z|^2} \|S\|^{1-\sigma} \|x^*\|$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. Applying the Hahn–Banach Theorem, we deduce that

$$\left\| f'(z) \right\| \leq C \frac{1}{1-|z|^2} \|S\|^{1-\sigma}$$

for all $z \in \mathbb{D}$. Hence $f \in \mathcal{B}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(f) \leq C \|S\|^{1-\sigma}$, and since $\|S\| \leq \pi_{p^*}(S)$ (see [8, p. 31]), taking infimum over all constants C and all operators S satisfying the first inequality above yields that $\rho_{\mathcal{B}}(f) \leq d^{\mathcal{B}}_{p,\sigma}(f)$. □

Next result states a Pietsch domination for strongly (p, σ)-absolutely continuous Bloch maps.

Theorem 3.2. *(Pietsch domination). Let $1 \leq p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$. Given a complex Banach space X and $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$, the following are equivalent:*

1. $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$.
2. *There exist a constant $C > 0$ and a measure $\mu \in \mathcal{P}(B_{X^{**}})$ such that*

$$\left| x^*(f'(z)) \right| \leq C \frac{1}{1-|z|^2} \left(\int_{B_{X^{**}}} \left(\left| \varphi(x^*) \right|^{1-\sigma} \|x^*\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi) \right)^{\frac{1-\sigma}{p^*}}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$.

3. *There exists a constant $C > 0$ such that*

$$\sum_{i=1}^{n} |\lambda_i| \left| x_i^*(f'(z_i)) \right| \leq C \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2} \right)^{r} \right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(\left| \varphi(x_i^*) \right|^{1-\sigma} \left\| x_i^* \right\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}}$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $z_1, \ldots, z_n \in \mathbb{D}$ and $x_1^*, \ldots, x_n^* \in X^*$.

In such a case,

$$d^{\mathcal{B}}_{p,\sigma}(f) = \inf \{ C > 0 \textit{ satisfying (ii)} \} = \inf \{ C > 0 \textit{ satisfying (iii)} \}.$$

Proof. $(i) \Rightarrow (ii)$: If $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$, then there exist a constant $C' > 0$, a complex Banach space Y and an operator $S \in \Pi_{p^*}(X^*, Y)$ such that

$$\left| x^*(f'(z)) \right| \leq C' \frac{1}{1-|z|^2} \|x^*\|^{\sigma} \|S(x^*)\|^{1-\sigma}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. Applying [8, Theorem 2.12] to S, we have a measure $\mu \in \mathcal{P}(B_{X^{**}})$ so that

$$\|S(x^*)\| \leq \pi_{p^*}(S) \left(\int_{B_{X^{**}}} \left| \varphi(x^*) \right|^{p^*} d\mu(\varphi) \right)^{\frac{1}{p^*}}$$

for all $x^* \in X^*$, and taking $C = C' \pi_{p^*}(S)^{1-\sigma}$, we obtain

$$\left| x^*(f'(z)) \right| \leq C \frac{1}{1-|z|^2} \left(\int_{B_{X^{**}}} \left(\left| \varphi(x^*) \right|^{1-\sigma} \|x^*\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi) \right)^{\frac{1-\sigma}{p^*}}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. Moreover, $d^{\mathcal{B}}_{p,\sigma}(f) = C$, and so $\inf\{C > 0 \text{ satisfying } (ii)\} \leq d^{\mathcal{B}}_{p,\sigma}(f)$.

$(ii) \Rightarrow (iii)$: If (ii) holds, given $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $z_1, \ldots, z_n \in \mathbb{D}$ and $x_1^*, \ldots, x_n^* \in X^*$, Hölder's Inequality gives

$$\begin{aligned}
\sum_{i=1}^{n} |\lambda_i| \left| x_i^*(f'(z_i)) \right| &\leq C \sum_{i=1}^{n} |\lambda_i| \frac{1}{1-|z_i|^2} \left(\int_{B_{X^{**}}} \left(\left| \varphi(x_i^*) \right|^{1-\sigma} \left\| x_i^* \right\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi) \right)^{\frac{1-\sigma}{p^*}} \\
&\leq C \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2} \right)^{r} \right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} \int_{B_{X^{**}}} \left(\left| \varphi(x_i^*) \right|^{1-\sigma} \left\| x_i^* \right\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi) \right)^{\frac{1-\sigma}{p^*}} \\
&\leq C \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2} \right)^{r} \right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} \left(\left\| x_i^* \right\|^{1-\sigma} \left\| x_i^* \right\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}} \\
&= C \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2} \right)^{r} \right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} \left(\left| \varphi_i(x_i^*) \right|^{1-\sigma} \left\| x_i^* \right\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}} \\
&\leq C \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2} \right)^{r} \right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(\left| \varphi(x_i^*) \right|^{1-\sigma} \left\| x_i^* \right\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}},
\end{aligned}$$

where we have taken $\varphi_i \in B_{X^{**}}$ with $\varphi_i(x_i^*) = \left\| x_i^* \right\|$ for each $i = 1, \ldots, n$ by the Hahn–Banach Theorem. Moreover, note that C (that was a constant satisfying the inequality in (ii)) now verifies the inequality in (iii), and thus $\inf\{C > 0 \text{ satisfying } (iii)\} \leq \inf\{C > 0 \text{ satisfying } (ii)\}$.

$(iii) \Rightarrow (i)$: We will apply a general Pietsch domination theorem stated in [15, Theorem 4.6]. Define the functions

$$\begin{aligned}
&R_1 \colon B_{X^{**}} \times \mathbb{D} \times \mathbb{R} \to [0, \infty[, \qquad R_1(\varphi, z, \lambda) = \frac{|\lambda|}{1-|z|^2}, \\
&R_2 \colon B_{X^{**}} \times \mathbb{D} \times X^* \to [0, \infty[, \qquad R_2(\varphi, z, x^*) = \left| \varphi(x^*) \right|^{1-\sigma} \|x^*\|^{\sigma}, \\
&S \colon \widehat{\mathcal{B}}(\mathbb{D}, X) \times \mathbb{D} \times \mathbb{R} \times X^* \to [0, \infty[, \qquad S(f, z, \lambda, x^*) = |\lambda| \left| x^*(f'(z)) \right|.
\end{aligned}$$

Notice that R_1, R_2 and S satisfy the conditions (1) and (2) preceding to Definition 4.4 in [15]:

(1) For each $z \in \mathbb{D}$, $\lambda \in \mathbb{R}$ and $x^* \in X^*$, the maps

$$\begin{aligned}
&(R_1)_{z,\lambda} \colon B_{X^{**}} \to [0, \infty[\qquad (R_1)_{z,\lambda}(\varphi) = R_1(\varphi, z, \lambda), \\
&(R_2)_{z,x^*} \colon B_{X^{**}} \to [0, \infty[\qquad (R_2)_{z,x^*}(\varphi) = R_2(\varphi, z, x^*),
\end{aligned}$$

are continuous.

(2) The equalities

$$\begin{aligned}
R_1(\varphi, z, \beta_1 \lambda) &= \beta_1 R_1(\varphi, z, \lambda), \\
R_2(\varphi, z, \beta_2 x^*) &= \beta_2 R_2(\varphi, z, x^*), \\
S(f, z, \beta_1 \lambda, \beta_2 x^*) &= \beta_1 \beta_2 S(f, z, \lambda, x^*),
\end{aligned}$$

hold for all $\varphi \in B_{X^{**}}$, $z \in \mathbb{D}$, $\lambda \in \mathbb{R}$, $x^* \in X^*$ and $\beta_1, \beta_2 \in [0, 1]$.

We now prove that the map f is R_1, R_2-S-abstract $(r, p^*/(1-\sigma))$-summing. Indeed, let $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $z_1, \ldots, z_n \in \mathbb{D}$ and $x_1^*, \ldots, x_n^* \in X^*$. By (iii), we have a constant $C > 0$ so that

$$\sum_{i=1}^{n} |\lambda_i| \left| x_i^*(f'(z_i)) \right| \leq C \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2} \right)^{r} \right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(\left| \varphi(x_i^*) \right|^{1-\sigma} \left\| x_i^* \right\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}},$$

and so we get

$$\begin{aligned}\sum_{i=1}^{n} S(f,z_i,\lambda_i,x_i^*) &= \sum_{i=1}^{n} |\lambda_i| \left|x_i^*(f'(z_i))\right| \\ &\leq C\left(\sum_{i=1}^{n}\left(\frac{|\lambda_i|}{1-|z_i|^2}\right)^r\right)^{\frac{1}{r}} \sup_{\varphi\in B_{X^{**}}}\left(\sum_{i=1}^{n}\left(\left|\varphi(x_i^*)\right|^{1-\sigma}\left\|x_i^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}} \\ &= C\left(\sum_{i=1}^{n} R_1(\varphi_i,z_i,\lambda_i)^r\right)^{\frac{1}{r}} \sup_{\varphi\in B_{X^{**}}}\left(\sum_{i=1}^{n} R_2(\varphi,z_i,x_i^*)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}} \\ &\leq C \sup_{\varphi\in B_{X^{**}}}\left(\sum_{i=1}^{n} R_1(\varphi,z_i,\lambda_i)^r\right)^{\frac{1}{r}} \sup_{\varphi\in B_{X^{**}}}\left(\sum_{i=1}^{n} R_2(\varphi,z_i,x_i^*)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}.\end{aligned}$$

By [15, Theorem 4.6], we have measures $\mu_1,\mu_2\in\mathcal{P}(B_{X^{**}})$ such that

$$S(f,z,\lambda,x^*) \leq C\left(\int_{B_{X^{**}}} R_1(\varphi,z,\lambda)^r d\mu_1(\varphi)\right)^{\frac{1}{r}}\left(\int_{B_{X^{**}}} R_2(\varphi,z,x^*)^{\frac{p^*}{1-\sigma}} d\mu_2(\varphi)\right)^{\frac{1-\sigma}{p^*}}$$

for all $(z,\lambda,x^*)\in\mathbb{D}\times\mathbb{R}\times X^*$. It follows that

$$\left|x^*(f'(z))\right| \leq C\frac{1}{1-|z|^2}\,\|x^*\|^{\sigma}\left(\int_{B_{X^{**}}}\left|\varphi(x^*)\right|^{p^*} d\mu_2(\varphi)\right)^{\frac{1-\sigma}{p^*}}$$

for all $(z,x^*)\in\mathbb{D}\times X^*$. Finally, take the Banach space $Y=L_{p^*}(\mu_2)$ and the operator $S=I_{\infty,p^*}\circ j_\infty\circ\iota_{X^*}\colon X^*\to Y$, where $I_{\infty,p^*}\colon L_\infty(\mu_2)\to L_{p^*}(\mu_2)$ and $j_\infty\colon C(B_{X^{**}})\to L_\infty(\mu_2)$ are the formal inclusion operators and $\iota_{X^*}\colon X^*\to C(B_{X^{**}})$ is the isometric linear embedding given by

$$\iota_{X^*}(x^*)(\varphi)=\varphi(x^*)\quad(\varphi\in B_{X^{**}},\ x^*\in X^*).$$

Then we can write

$$\begin{aligned}\left|x^*(f'(z))\right| &\leq C\frac{1}{1-|z|^2}\,\|x^*\|^{\sigma}\left(\int_{B_{X^{**}}}\left|S(x^*)(\varphi)\right|^{p^*} d\mu_2(\varphi)\right)^{\frac{1-\sigma}{p^*}} \\ &= C\frac{1}{1-|z|^2}\,\|x^*\|^{\sigma}\,\|S(x^*)\|^{1-\sigma}\end{aligned}$$

for all $(z,x^*)\in\mathbb{D}\times X^*$, and since $S\in\Pi_{p^*}(X^*,Y)$ with $\pi_{p^*}(S)\leq 1$ by [8, 2.4 and 2.9], we conclude that $f\in\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X)$ with $d^{\mathcal{B}}_{p,\sigma}(f)\leq C$, and thus $d^{\mathcal{B}}_{p,\sigma}(f)\leq\inf\{C>0$ satisfying $(iii)\}$. □

We now show that strong (p,σ)-absolute continuity of a Bloch map on $\mathbb{D}$ is transferred to its linearisation on $\mathcal{G}(\mathbb{D})$, and vice versa.

Theorem 3.3. *(Linearisation). Let $1<p,r<\infty$ and $0\leq\sigma<1$ be such that $1/r+(1-\sigma)/p^*=1$, let X be a complex Banach space and let $f\in\widehat{\mathcal{B}}(\mathbb{D},X)$. Then $f\in\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X)$ if and only if $S_f\in\mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}),X)$. In this case, $d^{\mathcal{B}}_{p,\sigma}(f)=d_{p,\sigma}(S_f)$.*

Proof. Suppose that $f\in\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X)$. Let $\gamma\in\operatorname{lin}(\Gamma(\mathbb{D}))$ and $x^*\in X^*$. If $\sum_{i=1}^{n}\lambda_i\gamma_{z_i}$ is a representation of γ, Theorem 3.2 provides a measure $\mu\in\mathcal{P}(B_{X^{**}})$ such that

$$\begin{aligned}\left|x^*(S_f(\gamma))\right| &\leq \sum_{i=1}^{n}|\lambda_i|\left|x^*(f'(z_i))\right| \\ &\leq d^{\mathcal{B}}_{p,\sigma}(f)\sum_{i=1}^{n}|\lambda_i|\,\frac{1}{1-|z_i|^2}\left(\int_{B_{X^{**}}}\left(\left|\varphi(x^*)\right|^{1-\sigma}\|x^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi)\right)^{\frac{1-\sigma}{p^*}}.\end{aligned}$$

Taking the infimum over all such representations of γ and using [10, Lemma 3.1], we obtain

$$\left|x^*(S_f(\gamma))\right| \leq d^{\mathcal{B}}_{p,\sigma}(f)\left\|\gamma\right\|\left(\int_{B_{X^{**}}}\left(\left|\varphi(x^*)\right|^{1-\sigma}\|x^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi)\right)^{\frac{1-\sigma}{p^*}}.$$

Since $\mathrm{lin}(\Gamma(\mathbb{D}))$ is norm-dense in $\mathcal{G}(\mathbb{D})$, we deduce

$$\left|x^*(S_f(\gamma))\right| \leq d^{\mathcal{B}}_{p,\sigma}(f)\left\|\gamma\right\|\left(\int_{B_{X^{**}}}\left(\left|\varphi(x^*)\right|^{1-\sigma}\|x^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi)\right)^{\frac{1-\sigma}{p^*}}$$

whenever $\gamma \in \mathcal{G}(\mathbb{D})$. Now, Pietsch's domination for operators in $\mathcal{D}_{p,\sigma}$ (see [1, Theorem 3.2]) shows that $S_f \in \mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}), X)$ and $d_{p,\sigma}(S_f) \leq d^{\mathcal{B}}_{p,\sigma}(f)$.

Conversely, suppose that $S_f \in \mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}), X)$. By [1, Theorem 3.2], there exists a measure $\mu \in \mathcal{P}(B_{X^{**}})$ such that

$$\begin{aligned}\left|x^*(f'(z))\right| &= \left|x^*(S_f(\gamma_z))\right| \\ &\leq d_{p,\sigma}(S_f)\left\|\gamma_z\right\|\left(\int_{B_{X^{**}}}\left(\left|\varphi(x^*)\right|^{1-\sigma}\|x^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi)\right)^{\frac{1-\sigma}{p^*}} \\ &= d_{p,\sigma}(S_f)\frac{1}{1-|z|^2}\left(\int_{B_{X^{**}}}\left(\left|\varphi(x^*)\right|^{1-\sigma}\|x^*\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}} d\mu(\varphi)\right)^{\frac{1-\sigma}{p^*}}\end{aligned}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. Hence $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with $d^{\mathcal{B}}_{p,\sigma}(f) \leq d_{p,\sigma}(S_f)$ by Theorem 3.2. □

We now present new examples of Banach normalized Bloch ideal (see [10, Definition 5.11]).

Proposition 3.4. *(Banach Bloch ideal property). Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$. Then $[\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}, d^{\mathcal{B}}_{p,\sigma}]$ is an injective Banach normalized Bloch ideal.*

Proof. Let X be a complex Banach space.

(N1): $(\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X), d^{\mathcal{B}}_{p,\sigma})$ is a Banach space and $\rho_{\mathcal{B}}(f) \leq d^{\mathcal{B}}_{p,\sigma}(f)$ for all $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$.

Let $\lambda \in \mathbb{C}$ and $f, g \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$. We have:

$$\begin{aligned}&d^{\mathcal{B}}_{p,\sigma}(\lambda f) = d_{p,\sigma}(S_{\lambda f}) = d_{p,\sigma}(\lambda S_f) = |\lambda|\, d_{p,\sigma}(S_f) = |\lambda|\, d^{\mathcal{B}}_{p,\sigma}(f), \\ &d^{\mathcal{B}}_{p,\sigma}(f+g) = d_{p,\sigma}(S_{f+g}) = d_{p,\sigma}(S_f + S_g) \leq d_{p,\sigma}(S_f) + d_{p,\sigma}(S_g) = d^{\mathcal{B}}_{p,\sigma}(f) + d^{\mathcal{B}}_{p,\sigma}(g), \\ &d^{\mathcal{B}}_{p,\sigma}(f) = 0 \Rightarrow d_{p,\sigma}(S_f) = 0 \Rightarrow S_f = 0 \Rightarrow f' = S_f \circ \Gamma = 0 \Rightarrow f = 0,\end{aligned}$$

by using Theorem 2.2 and 3.3. Applying also both theorems, it is immediate that $f \mapsto S_f$ is an isometric isomorphism of $(\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X), d^{\mathcal{B}}_{p,\sigma})$ onto $(\mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}), X), d_{p,\sigma})$, and

$$\rho_{\mathcal{B}}(f) = \left\|S_f\right\| \leq d_{p,\sigma}(S_f) = d^{\mathcal{B}}_{p,\sigma}(f)$$

by using also that $[\mathcal{D}_{p,\sigma}, d_{p,\sigma}]$ is a Banach operator ideal.

(N2): Let $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x \in X$. Then $g \cdot x \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with $d^{\mathcal{B}}_{p,\sigma}(g \cdot x) = \rho_{\mathcal{B}}(g)\,\|x\|$.

Since $\Lambda(g) \cdot x \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$ and

$$(g \cdot x)'(z) = g'(z)x = \Lambda(g)(\gamma_z)x = (\Lambda(g) \cdot x)(\gamma_z) = (\Lambda(g) \cdot x \circ \Gamma)(z)$$

for all $z \in \mathbb{D}$, Theorem 2.2 gives $S_{g\cdot x} = \Lambda(g) \cdot x$. By the operator ideal property of $[\mathcal{D}_{p,\sigma}, d_{p,\sigma}]$ (see [1, Corollary 4.6]), it follows that $S_{g\cdot x} \in \mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}), X)$ with $d_{p,\sigma}(S_{g\cdot x}) = \left\|\Lambda(g)\right\|\|x\| = \rho_{\mathcal{B}}(g)\,\|x\|$. Hence $g \cdot x \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with $d^{\mathcal{B}}_{p,\sigma}(g \cdot x) = \rho_{\mathcal{B}}(g)\,\|x\|$ by Theorem 3.3.

(N3): Let $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$, $T \in \mathcal{L}(X, Y)$ and $h \in \widehat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$. Then $T \circ f \circ h \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, Y)$ with $d^{\mathcal{B}}_{p,\sigma}(T \circ f \circ h) \leq \|T\| \, d^{\mathcal{B}}_{p,\sigma}(f)$.

Since $T \circ S_f \circ \widehat{h} \in \mathcal{L}(\mathcal{G}(\mathbb{D}), Y)$ and

$$\begin{aligned}(T \circ f \circ h)' &= T \circ [h' \cdot (f' \circ h)] = T \circ [h' \cdot (S_f \circ \Gamma \circ h)] \\ &= T \circ [S_f(h' \cdot (\Gamma \circ h))] = T \circ [S_f \circ (\widehat{h} \circ \Gamma)] \\ &= (T \circ S_f \circ \widehat{h}) \circ \Gamma\end{aligned}$$

we deduce that $S_{T\circ f\circ h} = T \circ S_f \circ \widehat{h}$ by Theorem 2.2. Since $S_f \in \mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}), X)$ by Theorem 3.3, we get that $S_{T\circ f\circ h} \in \mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}), Y)$ with $d_{p,\sigma}(S_{T\circ f\circ h}) \leq \|T\| \, d_{p,\sigma}(S_f)\|\widehat{h}\|$ by the operator ideal property of $[\mathcal{D}_{p,\sigma}, d_{p,\sigma}]$, and thus $T \circ f \circ h \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, Y)$ with $d^{\mathcal{B}}_{p,\sigma}(T \circ f \circ h) \leq \|T\| \, d^{\mathcal{B}}_{p,\sigma}(f)$ by Theorem 3.3.

(I): Let $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ and let $\iota \colon X \to Y$ be a linear isometry so that $\iota \circ f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, Y)$. Then $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with $d^{\mathcal{B}}_{p,\sigma}(f) = d^{\mathcal{B}}_{p,\sigma}(\iota \circ f)$.

Note that $\iota \circ S_f = S_{\iota\circ f} \in \mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}), Y)$. Since the operator ideal $[\mathcal{D}_{p,\sigma}, d_{p,\sigma}]$ is injective, it follows that $S_f \in \mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}), X)$ with $\|S_f\| = \|\iota \circ S_f\|$ or, equivalently, $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with $d^{\mathcal{B}}_{p,\sigma}(f) \leq d^{\mathcal{B}}_{p,\sigma}(\iota \circ f)$. The reverse inequality follows from (N3). □

The Möbius group of $\mathbb{D}$, denoted $\mathrm{Aut}(\mathbb{D})$, consists of all biholomorphic bijections from $\mathbb{D}$ onto itself. Let us recall that a linear space $\mathcal{A}(\mathbb{D}, X) \subseteq \mathcal{B}(\mathbb{D}, X)$, under a seminorm $\rho_{\mathcal{A}}$, is Möbius-invariant if: (i) there is $C > 0$ such that $\rho_{\mathcal{B}}(f) \leq C\rho_{\mathcal{A}}(f)$ for all $f \in \mathcal{A}(\mathbb{D}, X)$; and (ii) $f \circ \phi \in \mathcal{A}(\mathbb{D}, X)$ with $\rho_{\mathcal{A}}(f \circ \phi) = \rho_{\mathcal{A}}(f)$ for all $\phi \in \mathrm{Aut}(\mathbb{D})$ and $f \in \mathcal{A}(\mathbb{D}, X)$.

Invariance of strongly (p, σ)-absolutely continuous Bloch maps by Möbius transformations over $\mathbb{D}$ can be now derived.

Proposition 3.5. *(Möbius invariance). Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1 - \sigma)/p^* = 1$, and let X be a complex Banach space. Then $(\mathcal{D}^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X), d^{\mathcal{B}}_{p,\sigma})$ is Möbius-invariant.*

Proof. (i) Proposition 3.1 yields $(\mathcal{D}^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X), d^{\mathcal{B}}_{p,\sigma}) \leq (\mathcal{B}(\mathbb{D}, X), \rho_{\mathcal{B}})$.

(ii) A reading of the proof of (N3) above shows that $f \circ \phi \in \mathcal{D}^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X)$ with $d^{\mathcal{B}}_{p,\sigma}(f \circ \phi) \leq d^{\mathcal{B}}_{p,\sigma}(f)$ if $f \in \mathcal{D}^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X)$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, and from this we also deduce that $d^{\mathcal{B}}_{p,\sigma}(f) = d^{\mathcal{B}}_{p,\sigma}((f\circ\phi)\circ\phi^{-1}) \leq d^{\mathcal{B}}_{p,\sigma}(f\circ\phi)$. □

In clear parallelism with Theorem 3.3, strong (p, σ)-absolute continuity of a Bloch map from $\mathbb{D}$ to X is inherited by its Bloch transpose from X^* to $\widehat{\mathcal{B}}(\mathbb{D})$, and vice versa.

Proposition 3.6. *(Bloch transposition). Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1 - \sigma)/p^* = 1$, let X be a complex Banach space and let $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$. Then $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ if and only if $f^t \in \Pi_{p^*,\sigma}(X^*, \widehat{\mathcal{B}}(\mathbb{D}))$. In this case, $d^{\mathcal{B}}_{p,\sigma}(f) = \pi_{p^*,\sigma}(f^t)$.*

Proof. Applying Theorem 3.3, [1, Remark 3.3] and [8, Theorem 2.4], we have

$$\begin{aligned}f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X) &\Leftrightarrow S_f \in \mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}), X) \\ &\Leftrightarrow (S_f)^* \in \Pi_{p^*,\sigma}(X^*, \mathcal{G}(\mathbb{D})^*) \\ &\Leftrightarrow f^t = \Lambda^{-1} \circ (S_f)^* \in \Pi_{p^*,\sigma}(X^*, \widehat{\mathcal{B}}(\mathbb{D})),\end{aligned}$$

with

$$d^{\mathcal{B}}_{p,\sigma}(f) = d_{p,\sigma}(S_f) = \pi_{p^*,\sigma}((S_f)^*) = \pi_{p^*,\sigma}(f^t).$$

□

We now relate strong (p,σ)-absolute continuity and compactness of Bloch maps. Following [10, Definition 5.1], a map $f \in \mathcal{H}(\mathbb{D},X)$ is called *compact Bloch* if its Bloch range

$$\operatorname{rang}_{\mathcal{B}}(f) := \left\{(1-|z|^2)f'(z) \colon z \in \mathbb{D}\right\}$$

is a relatively compact subset of X.

Proposition 3.7. *(Bloch compactness). Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$ and let X be a reflexive complex Banach space. Every strongly (p,σ)-absolutely continuous Bloch map $f \colon \mathbb{D} \to X$ is compact Bloch.*

Proof. Let $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X)$. Then $f^t \in \Pi_{p^*,\sigma}(X^*, \widehat{\mathcal{B}}(\mathbb{D}))$ by Proposition 3.6. Hence f^t is a compact linear operator by [7, Corollary 5.2] and, equivalently, f is compact Bloch by [10, Theorem 5.19]. □

Our next goal is to get a result on Pietsch factorization for strongly (p,σ)-absolutely continuous Bloch maps. Its proof is based on some results of [7, Section 3.2] which we recall next.

Given a complex Banach space X, let $\iota_X \colon X \to C(B_{X^*})$ be the isometric linear embedding defined by

$$\iota_X(x)(\varphi) = \varphi(x) \quad (\varphi \in B_{X^*},\ x \in X).$$

Given $\mu \in \mathcal{P}(B_{X^*})$, define the seminorm

$$\left\|f\right\|_{p,\sigma} = \inf\left\{\sum_{k=1}^{n} \left\|f_k\right\|^{\sigma}_{\iota_X(X)} \left(\int_{B_{X^*}} \left|f_k(\varphi)\right|^p \, d\mu(\varphi)\right)^{\frac{1-\sigma}{p}}\right\} \quad (f \in \iota_X(X)),$$

being the infimum taken over all decompositions of f as $f = \sum_{k=1}^n f_k$ in $\iota_X(X)$. Let $L_{p,\sigma}(\mu)$ be the completion of the quotient normed space $\iota_X(B_X)/\left\|\cdot\right\|^{-1}_{p,\sigma}(\{0\})$ with the quotient norm $\|\cdot\|_{p,\sigma}$, let $J_{p,\sigma} \colon \iota_X(X) \to L_{p,\sigma}(\mu)$ be the canonical projection, and let $\widetilde{J}_{p,\sigma}$ denote the operator $J_{p,\sigma}$ considered from $C(B_{X^*})$ into $L_{p,\sigma}(\mu)$.

Theorem 3.8. *(Pietsch factorization). Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$, let X be a complex Banach space and let $f \in \widehat{\mathcal{B}}(\mathbb{D},X)$. Then $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X)$ if and only if there exist a measure $\mu \in \mathcal{P}(B_{\mathcal{G}(\mathbb{D})^*})$, a map $g \in \widehat{\mathcal{B}}(\mathbb{D}, L_{p,\sigma}(\mu))$ and an operator $T \in \mathcal{L}(L_{p,\sigma}(\mu),X)$ such that $f' = T \circ g'$.*

Furthermore, $d^{\mathcal{B}}_{p,\sigma}(f) = \inf\{\|T\|\, \rho_{\mathcal{B}}(g)\}$, the infimum being taken over all such factorizations of f' as above, and this infimum is attained.

Proof. Assume that $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X)$. Then $S_f \in \mathcal{D}_{p,\sigma}(\mathcal{G}(\mathbb{D}),X)$ with $d_{p,\sigma}(S_f) = d^{\mathcal{B}}_{p,\sigma}(f)$ by Theorem 3.3. By a version of the Pietsch factorization theorem for (p,σ)-absolutely continuous linear operators [7, Theorem 3.5], there exist a measure $\mu \in \mathcal{P}(B_{\mathcal{G}(\mathbb{D})^*})$, an operator $\widetilde{J}_{p,\sigma} \in \mathcal{D}_{p,\sigma}(C(B_{\mathcal{G}(\mathbb{D})}), L_{p,\sigma}(\mu))$ with $\pi_{p,\sigma}(\widetilde{J}_{p,\sigma}) \leq 1$ (see Lemma 3.4 and the comment which follows in [7]) and an operator $T \in \mathcal{L}(L_{p,\sigma}(\mu),X)$ with $\|T\| \leq d_{p,\sigma}(S_f)$ such that $S_f = T \circ \widetilde{J}_{p,\sigma} \circ \iota_{\mathcal{G}(\mathbb{D})}$. Although in [7, Theorem 3.5] the factorization is given through a subspace $X_{p,\sigma}$ of $L_{p,\sigma}(\mu)$, a quick look to the proof shows that $X_{p,\sigma} = L_{p,\sigma}(\mu)$ (see comment in [1, p. 14]). By [5, Lemma 1.5], we can find a map $g \in \widehat{\mathcal{B}}(\mathbb{D}, L_{p,\sigma}(\mu))$ with $\rho_{\mathcal{B}}(g) = 1$ such that $g' = \widetilde{J}_{p,\sigma} \circ \iota_{\mathcal{G}(\mathbb{D})} \circ \Gamma$. Hence $f' = S_f \circ \Gamma = T \circ g'$ with $\|T\|\, \rho_{\mathcal{B}}(g) \leq d^{\mathcal{B}}_{p,\sigma}(f)$.

Conversely, assume that there are a measure $\mu \in \mathcal{P}(B_{\mathcal{G}(\mathbb{D})^*})$, a map $g \in \widehat{\mathcal{B}}(\mathbb{D}, L_{p,\sigma}(\mu))$ and an operator $T \in \mathcal{L}(L_{p,\sigma}(\mu),X)$ such that $f' = T \circ g'$. We can assume $g \neq 0$. For any $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $z_1, \ldots, z_n \in \mathbb{D}$

and $x_1^*, \dots, x_n^* \in X^*$, Hölder's Inequality yields

$$\begin{aligned}
\sum_{i=1}^n |\lambda_i| \left|x_i^*(f'(z_i))\right| &= \sum_{i=1}^n |\lambda_i| \left|x_i^*(T(g'(z_i))\right| \\
&\le \|T\| \sum_{i=1}^n |\lambda_i| \left\|x_i^*\right\| \left\|g'(z_i)\right\| \\
&\le \|T\| \rho_{\mathcal{B}}(g) \left(\sum_{i=1}^n \left(\frac{|\lambda_i|}{1-|z_i|^2}\right)^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^n \left(\left\|x_i^*\right\|^{1-\sigma} \left\|x_i^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}} \\
&= \|T\| \rho_{\mathcal{B}}(g) \left(\sum_{i=1}^n \left(\frac{|\lambda_i|}{1-|z_i|^2}\right)^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^n \left(\left|\varphi_i(x_i^*)\right|^{1-\sigma} \left\|x_i^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}} \\
&\le \|T\| \rho_{\mathcal{B}}(g) \left(\sum_{i=1}^n \left(\frac{|\lambda_i|}{1-|z_i|^2}\right)^r\right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^n \left(\left|\varphi(x_i^*)\right|^{1-\sigma} \left\|x_i^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}},
\end{aligned}$$

by taking $\varphi_i \in B_{X^{**}}$ with $\varphi_i(x_i^*) = \left\|x_i^*\right\|$ for each $i = 1, \dots, n$ by the Hahn–Banach Theorem. Hence $f \in \mathcal{D}_{p,\sigma}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ with $d_{p,\sigma}^{\mathcal{B}}(f) \le \|T\| \rho_{\mathcal{B}}(g)$ by Theorem 3.2. Taking the infimum over all such factorizations of f', we deduce that $d_{p,\sigma}^{\mathcal{B}}(f) \le \inf\{\|T\| \rho_{\mathcal{B}}(g)\}$. □

We now introduce a Bloch reasonable crossnorm $\varrho_{p,\sigma}^{\widehat{\mathcal{B}}}$ on $\mathcal{G}(\mathbb{D}) \widehat{\otimes} X^*$ (the completion of the tensor product space $\mathcal{G}(\mathbb{D}) \otimes X^*$) whose dual represents the space $(\mathcal{D}_{p,\sigma}^{\widehat{\mathcal{B}}}(\mathbb{D}, X), d_{p,\sigma}^{\mathcal{B}})$.

Towards this end, consider the space

$$\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^* := \mathrm{lin}\left(\{\gamma_z \otimes x^* : z \in \mathbb{D},\ x^* \in X^*\}\right) \subseteq \widehat{\mathcal{B}}(\mathbb{D}, X)^*,$$

where $\gamma_z \otimes x^* : \widehat{\mathcal{B}}(\mathbb{D}, X) \to \mathbb{C}$ is the functional given by

$$(\gamma_z \otimes x^*)(f) = x^*(f'(z)) \qquad (f \in \widehat{\mathcal{B}}(\mathbb{D}, X)).$$

Each element $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^*$ is of the form $\gamma = \sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i^*$ for some $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $z_i \in \mathbb{D}$ and $x_i^* \in X^*$ for $i = 1, \dots, n$, and its action comes given as

$$\gamma(f) = \sum_{i=1}^n \lambda_i x_i^*(f'(z_i)) \qquad (f \in \widehat{\mathcal{B}}(\mathbb{D}, X)).$$

Definition 3.9. *Let $1 < p, r < \infty$ and $0 \le \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$, and let X be a complex Banach space. For each $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^*$, we set*

$$\varrho_{p,\sigma}^{\widehat{\mathcal{B}}}(\gamma) = \inf\left\{\left(\sum_{i=1}^n \left(\frac{|\lambda_i|}{1-|z_i|^2}\right)^r\right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^n \left(\left|\varphi(x_i^*)\right|^{1-\sigma} \left\|x_i^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right\},$$

where the infimum is taken over all representations of γ as $\gamma = \sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i^*$.

According to [5, Definition 2.5], a norm α on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ is a *Bloch reasonable crossnorm* if it holds: (i) $\alpha(\gamma_z \otimes x) \le \left\|\gamma_z\right\| \|x\|$ for all $z \in \mathbb{D}$ and $x \in X$; and (ii) Given $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$, the linear functional $g \otimes x^* : \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X \to \mathbb{C}$ given by $(g \otimes x^*)(\gamma_z \otimes x) = g'(x)x^*(x)$ is bounded on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$ with $\left\|g \otimes x^*\right\| \le \rho_{\mathcal{B}}(g) \|x^*\|$.

Proposition 3.10. *Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$, and let X be a complex Banach space. Then $\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}$ is a Bloch reasonable crossnorm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^*$.*

Proof. Using a standard reasoning (see, for example, the proof of [4, Theorem 6.2]), it can be shown that $\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}$ is a norm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^*$, but to be safe, we check that $\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}$ is a Bloch reasonable crossnorm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^*$:

(i) Given $z \in \mathbb{D}$ and $x^* \in X^*$, we have

$$\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_z \otimes x^*) \leq \frac{1}{1-|z|^2} \sup_{\varphi \in B_{X^{**}}} \left(\left(\left|\varphi(x^*)\right|^{1-\sigma} \|x^*\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}} = \frac{\|x^*\|}{1-|z|^2} = \left\|\gamma_z\right\| \|x^*\|.$$

(ii) For any $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^{**} \in X^{**}$, an application of Hahn–Banach Theorem and Hölder's Inequality yield

$$\begin{aligned}
\left|(g \otimes x^{**})(\gamma)\right| &= \left| \sum_{i=1}^{n} \lambda_i (g \otimes x^{**})(\gamma_{z_i} \otimes x_i^*) \right| = \left| \sum_{i=1}^{n} \lambda_i g'(z_i) x^{**}(x_i^*) \right| \\
&\leq \sum_{i=1}^{n} |\lambda_i| \left|g'(z_i)\right| \left|x^{**}(x_i^*)\right| \leq \rho_{\mathcal{B}}(g) \|x^{**}\| \sum_{i=1}^{n} \frac{|\lambda_i|}{1-|z_i|^2} \left\|x_i^*\right\| \\
&= \rho_{\mathcal{B}}(g) \|x^{**}\| \sum_{i=1}^{n} \frac{|\lambda_i|}{1-|z_i|^2} \left|\varphi_i(x_i^*)\right| = \rho_{\mathcal{B}}(g) \|x^{**}\| \sum_{i=1}^{n} \frac{|\lambda_i|}{1-|z_i|^2} \left|\varphi_i(x_i^*)\right|^{1-\sigma} \left\|x_i^*\right\|^{\sigma} \\
&\leq \rho_{\mathcal{B}}(g) \|x^{**}\| \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2} \right)^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} \left(\left|\varphi_i(x_i^*)\right|^{1-\sigma} \left\|x_i^*\right\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}} \\
&\leq \rho_{\mathcal{B}}(g) \|x^{**}\| \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2} \right)^r \right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(\left|\varphi(x_i^*)\right|^{1-\sigma} \left\|x_i^*\right\|^{\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}},
\end{aligned}$$

where, for each $i = 1, \ldots, n$, we have taken a functional $\varphi_i \in B_{X^{**}}$ such that $\left|\varphi_i(x_i^*)\right| = \left\|x_i^*\right\|$. Passing to the infimum over all the representations of γ, we obtain

$$\left|(g \otimes x^{**})(\gamma)\right| \leq \rho_{\mathcal{B}}(g) \|x^{**}\| \varrho^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma).$$

Hence $g \otimes x^{**} \in (\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_{\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}} X^*)^*$ and $\left\|g \otimes x^{**}\right\| \leq \rho_{\mathcal{B}}(g) \|x^{**}\|$. □

We are now ready to study the duality of the space of strongly (p, σ)-absolutely continuous Bloch maps from $\mathbb{D}$ into a complex Banach space X.

Theorem 3.11. *(Duality). Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$ be such that $1/r + (1-\sigma)/p^* = 1$, and let X be a complex Banach space. Then the space $\left(\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X), d^{\mathcal{B}}_{p,\sigma}\right)$ is isometrically isomorphic to $\left(\mathrm{lin}(\Gamma(\mathbb{D})) \widehat{\otimes}_{\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}} X^*\right)^*$.*

Proof. It is easy to see that the map $\Lambda\colon \left(\mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X), d^{\mathcal{B}}_{p,\sigma}\right) \to \left(\mathrm{lin}(\Gamma(\mathbb{D})) \widehat{\otimes}_{\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}} X^*\right)^*$, defined by

$$\Lambda(f)(\gamma_z \otimes x^*) = x^*(f'(z)) \qquad \left(f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X),\ z \in \mathbb{D},\ x^* \in X^*\right),$$

is linear and injective. Fix $f \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$. For $\gamma = \sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i^* \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^*$, an application of Theorem 3.2 gives

$$\begin{aligned}\left|\Lambda(f)(\gamma)\right| &\leq \sum_{i=1}^{n} |\lambda_i| \left|x_i^*(f'(z_i))\right| \\ &\leq d^{\mathcal{B}}_{p,\sigma}(f) \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2}\right)^r\right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(\left|\varphi(x_i^*)\right|^{1-\sigma} \left\|x_i^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}.\end{aligned}$$

Taking the infimun over all the representation of γ, we get $\left|\Lambda(f)(\gamma)\right| \leq d^{\mathcal{B}}_{p,\sigma}(f) \varrho^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma)$, and therefore $\left\|\Lambda(f)\right\| \leq d^{\mathcal{B}}_{p,\sigma}(f)$.

In order to establish the reverse inequality and the surjectivity of Λ, let $\phi \in \left(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}} X^*\right)^*$. Define $F_\phi \colon \mathbb{D} \to X$ by

$$x^*(F_\phi(z)) = \phi(\gamma_z \otimes x^*) \qquad (z \in \mathbb{D},\ x^* \in X^*).$$

A look at the proof of [5, Proposition 2.4] shows that $F_\phi \in \mathcal{H}(\mathbb{D}, X)$ and $F_\phi = f'_\phi$ for a convenient map $f_\phi \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(f_\phi) \leq \left\|\phi\right\|$.

To prove that $f_\phi \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$, let $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$ and $z_i \in \mathbb{D}$ for $i = 1, \ldots, n$. For each $i \in \{1, \ldots, n\}$, we can take a functional $x_i^* \in X^*$ with $\left\|x_i^*\right\| = 1$ so that $|x_i^*(f'_\phi(z_i))| = \|f'_\phi(z_i)\|$. Obviously, the function $T \colon \mathbb{C}^n \to \mathbb{C}$ defined by

$$T(t_1, \ldots, t_n) = \sum_{i=1}^{n} t_i \lambda_i \left\|f'_\phi(z_i)\right\|, \qquad (t_1, \ldots, t_n) \in \mathbb{C}^n,$$

is in $(\mathbb{C}^n, \|\cdot\|_\infty)^*$ and $\|T\| = \sum_{i=1}^{n} |\lambda_i| \|f'_\phi(z_i)\|$. For any $(t_1, \ldots, t_n) \in \mathbb{C}^n$ with $\|(t_1, \ldots, t_n)\|_\infty \leq 1$, we get

$$\begin{aligned}|T(t_1, \ldots, t_n)| &= \left|\phi\left(\sum_{i=1}^{n} t_i \lambda_i \gamma_{z_i} \otimes x_i^*\right)\right| \leq \left\|\phi\right\| \varrho^{\widehat{\mathcal{B}}}_{p,\sigma}\left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes t_i x_i^*\right) \\ &\leq \left\|\phi\right\| \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2}\right)^r\right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(\left|\varphi(x_i^*)\right|^{1-\sigma} \left\|t_i x_i^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}} \\ &\leq \left\|\phi\right\| \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2}\right)^r\right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(\left|\varphi(x_i^*)\right|^{1-\sigma} \left\|x_i^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}},\end{aligned}$$

and therefore

$$\sum_{i=1}^{n} |\lambda_i| \left|x_i^*(f'_\phi(z_i))\right| \leq \left\|\phi\right\| \left(\sum_{i=1}^{n} \left(\frac{|\lambda_i|}{1-|z_i|^2}\right)^r\right)^{\frac{1}{r}} \sup_{\varphi \in B_{X^{**}}} \left(\sum_{i=1}^{n} \left(\left|\varphi(x_i^*)\right|^{1-\sigma} \left\|x_i^*\right\|^{\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}.$$

Hence Theorem 3.2 assures that $f_\phi \in \mathcal{D}^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ and $d^{\mathcal{B}}_{p,\sigma}(f_\phi) \leq \left\|\phi\right\|$.

Now, for any $\gamma = \sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i^* \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X^*$, we have

$$\Lambda(f_\phi)(\gamma) = \sum_{i=1}^{n} \lambda_i x_i^*(f'_\phi(z_i)) = \sum_{i=1}^{n} \lambda_i \phi(\gamma_{z_i} \otimes x_i^*) = \phi\left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i^*\right) = \phi(\gamma),$$

and so $\Lambda(f_\phi) = \phi$ on $\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{\varrho^{\widehat{\mathcal{B}}}_{p,\sigma}} X^*$. Hence $d^{\mathcal{B}}_{p,\sigma}(f_\phi) \leq \left\|\Lambda(f_\phi)\right\|$ and the proof is complete. □

Acknowledgment

The authors extend their sincere thanks to the reviewers for their insightful comments and constructive feedback, which significantly improved the quality of this paper.

Furthermore, the third author acknowledges the partial support from Ministerio de Ciencia e Innovación grant PID2021-122126NB-C31 funded by MICIU/AEI/10.13039/501100011033 and ERDF/EU, as well as support from the Junta de Andalucía grant FQM194.

References

[1] D. Achour, E. Dahia, P. Rueda and E.A. Sánchez Pérez, Factorization of strongly (p,σ)-continuous multilinear operators, Linear Multilinear Algebra **62** (2014), no. 12, 1649–1670.
[2] D. Achour, P. Rueda and R. Yahi, (p,σ)-Absolutely Lipschitz operators, Ann. Funct. Anal. **8** (2017), no. 1, 38–50.
[3] A. Bougoutaia, A. Belacel and R. Macedo, Strongly (p,σ)-Lipschitz operators, Advances in Operator Theory, (2023) 8:20.
[4] A. Bougoutaia, A. Belacel, O. Djeribia and A. Jiménez-Vargas, (p,σ)-Absolute continuity of Bloch maps, Banach J. Math. Anal. (2024) 18:29.
[5] M.G. Cabrera-Padilla, A. Jiménez-Vargas and D. Ruiz-Casternado, p-Summing Bloch mappings on the complex unit disc, Banach J. Math. Anal. (2024) 18:9.
[6] J. S. Cohen, Absolutely p-summing, p-nuclear operators and their conjugates, Math. Ann. **201** (1973), 177–200.
[7] E. Dahia, D. Achour and E.A. Sánchez Pérez, Absolutely continuous multilinear operators, J. Math. Anal. Appl. **397** (2013), no. 1, 205–224.
[8] J. Diestel, H. Jarchow and A. Tonge, Absolutely Summing Operators, Cambridge University Press, Cambridge (1995).
[9] H. Jarchow and U. Matter, Interpolative constructions for operator ideals. Note di Matematica, **VIII**, no. 1, 45–56 (1988).
[10] A. Jiménez-Vargas and D. Ruiz-Casternado, Compact Bloch mappings on the complex unit disc, http://arxiv.org/abs/2308.02461.
[11] J.A. López Molina and E.A. Sánchez Pérez, Ideales de operadores absolutamente continuos, Rev. R. Acad. Cienc. Exactas Fís. Nat. Madrid. **87** (1993), no. 2-3, 349–378.
[12] J.A. López Molina and E.A. Sánchez Pérez, On operator ideals related to (p,σ)-absolutely continuous operators, Studia Math. **138** (2000), no. 1, 25–40
[13] U. Matter, Absolute continuous operators and super-reflexivity, Math. Nachr. **130** (1987), 193–216.
[14] U. Matter, Factoring trough interpolation spaces and super-reflexive Banach spaces, Rev. Roumaine Math. Pures Appl. **34** (1989), 147–156.
[15] D. Pellegrino, J. Santos and J. Seoane-Sepúlveda, Some techniques on nonlinear analysis and applications, Adv. Math. **229** (2012), 1235–1265.
[16] E.A. Sánchez Pérez, On the structure of tensor norms related to (p,σ)-absolutely continuous operators, Collect. Math. **47** (1996), no. 1, 35–46.
[17] K. Zhu, Operator theory in function spaces, 2nd ed., Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, Providence, RI, 2007.