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Geometric properties of a manifold associated with a generalized quarter-symmetric non-metric connection

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Abstract. The present paper investigates the fundamental physical invariants under some transform groups, and confirms some geometric characteristics of a manifold associated with some connections. This article firstly introduces a generalized quarter-symmetric non-metric connection family and studies systematically its geometrical properties. The present paper also arrives at the interesting projective invariant of the generalized quarter-symmetric non-metric connection family.

1. Introduction

The concept of the semi-symmetric connection was introduced by Fridman and Schouten in [5] for the first time. H. A. Hayden in [11] introduced the metric connection with torsion, and K. Yano in [18] introduced a semi-symmetric metric connection and studied its geometrical properties of a manifold with this class of connections. De, Han and Zhao in [2] investigated recently and confirmed some interesting geometries of a manifold with the semi-symmetric non-metric connection. A quarter-symmetric connection was defined and studied by S. Golab in [7]. Afterwards, several types of a quarter-symmetric metric connection were studied (one can see [4, 8, 16, 17, 19, 22] for details).

On the one hand, the Schur's theorem of a semi-symmetric non-metric connection is well known ([9, 12, 14, 15]) based only on the second Bianchi identity. Is is well know that a semi-symmetric non-metric connection that is just as a geometric model for scalar-tensor theories of gravitation [3, 13]. And a projective, conformal and projective conformal semi-symmetric connection were also studied ([15, 21, 23, 24]). And the physical characteristics of a manifold associated with a non-metric connection were studied ([1, 6, 10, 23]).

Recently the Ricci quarter-symmetric connection family and the projective Ricci quarter-symmetric connection family and the Schur's theorem of these connection families were studied ([13, 17, 20]). And Tang, Ho, Fu and Zhao in [16] defined a generalized quarter-symmetric metric recurrent connection and investigated its properties of a manifold, and obtained the Schur's theorem of a manifold associated with these connections. In fact, there were few results about quarter-symmetric non-metric connections because of its formal complexity and computational difficulty.

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Motivated by the forgoing researches we define newly in the present paper a generalized quarter-symmetric non-metric connection family and study its properties and its mutual connection family. And we find a fundamental projective invariant for the generalized quarter-symmetric non-metric connection family and study the generalized projective conformal quarter-symmetric non-metric connection family.

The organizations of this paper are as follows. Section 1 considered the previous study results. Section 2 newly defined a generalized quarter-symmetric non-metric connection family and studied its geometrical properties. Section 3 studied the projective invariant for the generalized quarter-symmetric non-metric connection family. Section 4 considered the generalized projective conformal quarter-symmetric non-metric connection family.

2. A generalized quarter-symmetric non-metric connection family

Let (M, g) be a Riemannian manifold ($dimM \geq 3$), g be the Riemannian metric on M and $\overset{\circ}{\nabla}$ be the Levi-Civita connection with respect to g. Let TM denote the collection of all vector fields on M.

Definition 2.1. *A connection familay $\overset{t}{\nabla}$ is called a generalized quarter-symmetric non-metric connection family if it satisfies the relation*

$$\begin{aligned}(\overset{t}{\nabla}_Z g)(X,Y) &= -2(t-1)2\omega(Z)U(X,Y) - \omega(X)U(Y,Z) - t\omega(Y)U(X,Z)\\ T(X,Y) &= \pi(Y)\varphi X - \pi(X)\varphi Y\end{aligned} \tag{2.1}$$

where φ is a (1,1)-type tensor field and ω, π are 1-form respectively and $U(X,Y) = \frac{1}{2}[\varphi(X,Y) + \varphi(Y,X)]$, $V(X,Y) = \frac{1}{2}[\varphi(X,Y) - \varphi(Y,X)]$ and $t \in R$ is a parameter of the connection family.

Let (x^i) be the local coordinate, then $g, \overset{\circ}{\nabla}, \nabla, \omega, \pi, \varphi, U, V, T$ have the local expressions g_{ij}, $\{^k_{ij}\}$, Γ^k_{ij}, $\omega_i, \pi_i, \varphi^k_i$ U^k_i, V^k_i, T^k_{ij} respectively. At the same time the expression (2.1) can be rewritten as

$$\overset{t}{\nabla}_k g_{ji} = -2(t-1)\omega_k U_{ij} - t\omega_i U_{jk} - t\omega_j U_{ik}, \qquad T^k_{ji} = \pi_j \varphi^k_i - \pi_i \varphi^k_j \tag{2.2}$$

The coefficient of $\overset{t}{\nabla}$ is given as

$$\overset{t}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + (t-1)\omega_i U^k_j + [(t-1)\omega_j + \pi_j]U^k_i + U_{ij}(\omega^k - \pi^k) - \pi_i V^k_j \tag{2.3}$$

where $U_{ij} = \frac{1}{2}(\varphi_{ij} + \varphi_{ji})$, $V_{ij} = \frac{1}{2}(\varphi_{ij} - \varphi_{ji})$.

Remark 2.1. *When $\omega = \pi$ and $\varphi = U$, then this connection family was studied in [20]. When $U_{ij} = g_{ij}$, then this connection family was studied in [16]. When U_{ij} is Ricci tensor, then this connection family was studied in [22].*

From the expression (2.3), the curvature tensor of $\overset{t}{\nabla}$, by a direct computation, is

$$\begin{aligned}\overset{t}{R}{}^l_{ijk} &= K^l_{ijk} + U^l_j a_{ik} - U^l_i a_{jk} + U_{jk} b^l_i - U_{ik} b^l_j + (U^l_{ij} - U^l_{ji})[(t-1)\omega_k + \pi_k]\\ &+ (U_{ijk} - U_{jik})(\omega^l - \pi^l) + (t-1)(U^l_{ik}\omega_j - U^l_{jk}\omega_i) + (t-1)U^l_k \omega_{ij} + \pi_i V^l_{jk} - \pi_j V^l_{ik} - V^l_k \pi_{ij}\end{aligned} \tag{2.4}$$

where K^l_{ijk} is the curvature tensor of the Levi-Civita connection $\overset{\circ}{\nabla}$ and the other notations are given as follows

$$a_{ik} = \overset{\circ}{\nabla}_i[(t-1)\omega_k + \pi_k] - (t-1)\omega_i U^p_k[(t-1)\omega_p + \pi_p] - U^p_i[(t-1)\omega_p + \pi_p][(t-1)\omega_k + \pi_k]$$

$$b_{ik} = \overset{\circ}{\nabla}_i(\omega_k - \pi_k) + (t-1)U_{kp}(\omega_p - \pi_p) + U_{ip}(\omega^p - \pi^p)(\omega_k - \pi_k) \\ +U_{ik}[(t-1)\omega_p + \pi_p](\omega^p - \pi^p)$$

$$U^l_{ij} = \overset{\circ}{\nabla}_i U^l_j$$

$$U^l_{ik} = \overset{\circ}{\nabla}_i U^l_k + U_{ip}U^p_k(\omega^l - \pi^l) - U^p_i U^l_p[(t-1)\omega_k + \pi_k] \tag{2.5}$$

$$\begin{aligned} V^l_{ik} &= \overset{\circ}{\nabla}_i V^l_k + (t-1)\omega_i V^p_k U^l_p + U^l_i V^p_k[(t-1)\omega_p + \pi_p] + U_{ip}V^p_k(\omega^l - \pi^l) \\ &- (t-1)\omega_i U^p_k V^l_p - U^p_i V^l_p[(t-1)\omega_k + \pi_k] - U_{ik}V^l_p(\omega^p - \pi^p) \end{aligned}$$

$$\omega_{ij} = \overset{\circ}{\nabla}_i\omega_j - \overset{\circ}{\nabla}_j\omega_i$$

$$\pi_{ij} = \overset{\circ}{\nabla}_i\pi_j - \overset{\circ}{\nabla}_j\pi_i$$

Suppose

$$\alpha_i = U^k_i\omega_k, \quad \overline{\alpha}_i = U^k_k\omega_i, \quad \beta_i = U^k_i\pi_k, \quad \overline{\beta}_i = U^k_k\pi_i, \quad \gamma_i = V^k_i\pi_k \tag{2.6}$$

where α_i, $\overline{\alpha}_i$, β_i, $\overline{\beta}_i$ and γ_i are a component of 1-form α, $\overline{\alpha}$, β, $\overline{\beta}$ and γ respectively.

Theorem 2.1. *For a Riemannian manifold (M, g), if 1-form α and $\overline{\alpha}$ are closed forms then the volume curvature tensor of $\overset{t}{\nabla}$ is zero, namely*

$$\overset{t}{P}_{ij} = 0 \tag{2.7}$$

where $\overset{t}{P}_{ij} = \overset{t}{R}{}^{\,k}_{ijk}$ is a volume curvature tensor of $\overset{t}{\nabla}$.

Proof. Contracting the indices k and l of the expression (2.4), then we have

$$\begin{aligned} \overset{t}{P}_{ij} &= K_{ij} + U^k_j a_{ik} - U^k_i a_{jk} + U_{kj}b^k_i - U_{ki}b^k_j + (U^k_{ij} - U^k_{ji})[(t-1)\omega_k + \pi_k] + (U_{ijk} - U_{jik})(\omega^k - \pi^k) \\ &+ (t-1)(U^k_{ik}\omega_j - U^k_{jk}\omega_i) + (t-1)U^k_k\omega_{ij} + \pi_i V^k_{jk} - \pi_j V^k_{ik} - V^k_k\pi_{ij} \end{aligned}$$

where $K_{ij} = K^k_{ijk}$ is a volume curvature tensor of $\overset{\circ}{\nabla}$ of g_{ij}.

On the other hand

$$U^k_j a_{ik} - U^k_i a_{jk} = U^k_j\overset{\circ}{\nabla}_i[(t-1)\omega_k + \pi_k] - U^k_i\overset{\circ}{\nabla}_j[(t-1)\omega_k + \pi_k] - (t-1)(\omega_i U^k_j - \omega_j U^k_i)U^p_k[(t-1)\omega_p + \pi_p]$$

$$U_{jk}b^k_i - U_{ik}b^k_j = U_{jk}\overset{\circ}{\nabla}_i[\omega^k + \pi^k] - U_{jk}\overset{\circ}{\nabla}_i(\omega^k + \pi^k) + (t-1)(\omega_i U_{jk} - \omega_j U_{ik})U^k_p(\omega^p - \pi^p)$$

$$(U^k_{ij} - U^k_{ji})[(t-1)\omega_k + \pi_k] = (\overset{\circ}{\nabla}_i U^k_j - \overset{\circ}{\nabla}_j U^k_i)[(t-1)\omega_k + \pi_k]$$

$$(U_{ijk} - U_{jik})(\omega^k - \pi^k) = (\overset{\circ}{\nabla}_i U_{jk} - \overset{\circ}{\nabla}_j U_{ik})(\omega^k - \pi^k)$$

$$\begin{aligned} (t-1)(U^k_{ik}\omega_j - U^k_{jk}\omega_i) &= (t-1)[\overset{\circ}{\nabla}_i(U^k_k\omega_j - \overset{\circ}{\nabla}_j(U^k_k\omega_i)] - (t-1)(\omega_i U_{jp} - \omega_j U_{ip})U^p_k(\omega^k - \pi^k) \\ &+ (t-1)(\omega_i U^p_j - \omega_j U^p_i)U^k_p[(t-1)\omega_k + \pi_k] \end{aligned}$$

$$(t-1)U^k_k\omega_{ij} = (t-1)(U^k_k\overset{\circ}{\nabla}_i\omega_j - U^k_k\overset{\circ}{\nabla}_j\omega_i)$$

$$\begin{aligned} V^k_{ik} &= \overset{\circ}{\nabla}_i V^k_k + (t-1)\omega_i V^p_k U^k_p + U^k_i V^p_k[(t-1)\omega_p + \pi_p]U_{ip}V^p_k(\omega^k - \pi^k) \\ &- (t-1)\omega_i U^p_k V^k_p - U^p_i V^k_p[(t-1)\omega_k + \pi_k] - U^p_{ik}V^k_p(\omega^p - \pi^p) \\ &= 0 \end{aligned}$$

$$V^k_k = 0, \ \ K_{ij} = 0$$

Hence using these expressions and the expression (2.6), we obtain

$$\overset{t}{P}_{ij} = t(\overset{\circ}{\nabla}_i\alpha_j - \overset{\circ}{\nabla}_j\alpha_i) + (t-1)(\overset{\circ}{\nabla}_i\overline{\alpha}_j - \overset{\circ}{\nabla}_j\overline{\alpha}_i) \tag{2.8}$$

If a 1-form α and $\overline{\alpha}$ are closed, then $\overset{\circ}{\nabla}_i\alpha_j - \overset{\circ}{\nabla}_j\alpha_i = 0$ and $\overset{\circ}{\nabla}_i\overline{\alpha}_j - \overset{\circ}{\nabla}_j\overline{\alpha}_i = 0$. Hence from expression (2.8), we obtain the expression (2.7). □

Remark 2.2. *Theorem 2.1 shows that the volume flat condition of the generalized quarter-symmetric non-metric connection family* $\overset{t}{\nabla}$ *is independent of 1-form* α *and that it is dependent only on 1-form* ω.

By (2.2) and (2.3), it is obvious that there holds the following.

When $t = 0$ we denote $\overset{t}{\nabla}$ as $\overset{\circ}{D}$, then the connection $\overset{\circ}{D}$ satisfies

$$\overset{\circ}{D}_k g_{ij} = 2\omega_k U_{ij}, \quad T^k_{ij} = \pi_j\varphi^k_i - \pi_i\varphi^k_j$$

and its coefficient is

$$\overset{t}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - \omega_i U^k_j - (\omega_j - \pi_j)U^k_i + U_{ij}(\omega^k - \pi^k) - \pi_i V^k_j$$

When $t = 1$ we denote $\overset{t}{\nabla}$ as $\overset{1}{D}$, then the connection $\overset{1}{D}$ satisfies

$$\overset{1}{D}_k g_{ij} = -\omega_i U_{jk} - \omega_j U_{ik}, \quad T^k_{ij} = \pi_j\varphi^k_i - \pi_i\varphi^k_j$$

and its coefficient is

$$\overset{1}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \pi_j U^k_i + U_{ij}(\omega^k - \pi^k) - \pi_i V^k_j$$

When $t = 2$ we denote $\overset{t}{\nabla}$ as D, Then the connection D satisfies

$$D_k g_{ij} = -2\omega_k U_{ij} - 2\omega_i U_{jk} - 2\omega_j U_{ik}, \quad T^k_{ij} = \pi_j\varphi^k_i - \pi_i\varphi^k_j \tag{2.9}$$

and its coefficient is

$$\Gamma^k_{ij} = \{^k_{ij}\} + \omega_i U^k_j + (\omega_j + \pi_j)U^k_i + U_{ij}(\omega^k - \pi^k) - \pi_i V^k_j$$

From the expression (2.8), it is easy to see that there holds the following corollary for the generalized quarter-symmetric non-metric connection.

Corollary 2.2. *For the Riemannian manifold* (M, g)*, if 1-form* $\overline{\alpha}$ *is closed form, then volume curvature tensor of* $\overset{\circ}{D}$ *is zero, namely*

$$\overset{\circ}{P}_{ij} = 0$$

where $\overset{\circ}{P}_{ij} = \overset{\circ}{R}{}^k_{ijk}$ *is a volume curvature tensor of* $\overset{\circ}{D}$.

Corollary 2.3. *For the Riemannian manifold (M, g), if 1-form α is closed form, then volume curvature tensor of $\overset{1}{D}$ is zero, namely*

$$\overset{1}{P}_{ij} = 0$$

where $\overset{1}{P}_{ij} = \overset{1}{R}{}_{ijk}^{\ k}$ is a volume curvature tensor of $\overset{1}{D}$.

Let be

$$\begin{aligned} A_{ijk}^{l} &= U_j^l a_{ik} + U_{jk} b_i^l + U_{ij}^l[(t-1)\omega_k + \pi_k] + U_{ijk}(\omega^l - \pi^l) + (t-1)U_{ik}^l \omega_j \\ &+ \pi_i V_{jk}^l + (t-1)U_k^l \overset{\circ}{\nabla}_i \omega_j - V_k^l \overset{\circ}{\nabla}_i \pi_j \end{aligned}$$

Then, from (2.4) we get

$$\overset{t}{R}{}_{ijk}^{\ l} = K_{ijk}^{\ l} + A_{ijk}^{\ l} - A_{jik}^{\ l}$$

So there exists the following.

Theorem 2.4. *When $A_{ijk}^{\ l} = A_{jik}^{\ l}$, then the curvature tensor will keep unchanged under the connection transformation $\overset{\circ}{\nabla} \to \overset{t}{\nabla}$.*

It is well known that if a sectional curvature at a point p in a Riemannian manifold is independent of E (a 2-dimensional subspace of T_pM), the curvature tensor is

$$R_{ijk}^{\ l} = k(p)(\delta_i^l g_{jk} - \delta_j^l g_{ik}) \tag{2.10}$$

In this case, if $k(p) = const$, then the Riemannian manifold is a constant curvature manifold.

Theorem 2.5. *(The Schur's theorem of the generalized quarter-symmetric non-metric connection D) Suppose that (M, g) $(dimM \geq 3)$ is a connected Riemannian manifold associated with an isotropic generalized quarter-symmetric non-metric connection D. If*

$$s_h = 0 \tag{2.11}$$

then the Riemannian manifold (M, g, D) is a constant curvature manifold, where $s_h = T_{hp}^p$.

Proof. Substituting the expression (2.10) into the second Bianchi identity of the curvature tensor of the generalized quarter-symmetric non-metric connection D, we get

$$D_h R_{ijk}^{\ l} + D_i R_{jhk}^{\ l} + D_j R_{hik}^{\ l} = T_{hi}^p R_{jpk}^{\ l} + T_{ij}^p R_{hpk}^{\ l} + T_{jh}^p R_{ipk}^{\ l}$$

□

and using the expression (2.9), then we have

$$\begin{aligned} & D_h k(\delta_i^l g_{jk} - \delta_j^l g_{ik}) + D_i k(\delta_j^l g_{hk} - \delta_h^l g_{jk}) + D_j k(\delta_h^l g_{ik} - \delta_i^l g_{hk}) \\ & = k[(\pi_i \varphi_h^p - \pi_h \varphi_i^p)(\delta_j^l g_{pk} - \delta_p^l g_{jk}) + (\pi_j \varphi_i^p - \pi_i \varphi_j^p)(\delta_h^l g_{pk} - \delta_p^l g_{hk})] \\ & + (\pi_h \varphi_j^p - \pi_j \varphi_h^p)(\delta_i^l g_{pk} - \delta_p^l g_{ik})] \end{aligned}$$

Contracting the indices i, l of this expression, then we obtain

$$(n-2)(D_h k g_{jk} - D_j k g_{hk})$$

$$= k[(n-3)(\pi_h \varphi_{jk} - \pi_j \varphi_{hk}) + \varphi_i^i(\pi_h g_{jk} - \pi_j g_{hk}) + g_{hk}\pi_i\varphi_j^i - g_{jk}\pi_i\varphi_h^i]$$

Multiplying both sides of this expression again by g^{jk}, then we have

$$(n-1)(n-2)D_h k = 2(n-2)k(\pi_h\varphi_i^i - \pi_i\varphi_h^i)$$

From this equation above we obtain

$$D_h k + \frac{2k}{n-1}s_h = 0$$

where $s_h = T_{hi}^i = \pi_i\varphi_h^i - \pi_h\varphi_i^i$. According to $dimM \geq 3$ if $s_h = 0$, then $k = const$. The connected condition implies theorem2.3 is tenable.

From the expression (2.2) and (2.3), the coefficient of the mutual connection family $\overset{tm}{\nabla}$ of the generalized quarter-symmetric non-metric connection family $\overset{t}{\nabla}$ is

$$\overset{tm}{\Gamma}_{ij} = \{^k_{ij}\} + [(t-1)\omega_i + \pi_i]U_j^k + (t-1)\omega_j U_i^k + U_{ij}(\omega^k - \pi^k) - \pi_j V_i^k \tag{2.12}$$

And from this expression the curvature tensor of $\overset{tm}{\nabla}$, by a direct computation, is

$$\begin{aligned}\overset{tm}{R}{}^{\,l}_{ijk} &= K^{\,l}_{ijk} + U_j^l\overset{m}{a}_{ik} - U_i^l\overset{m}{a}_{jk} + U_{jk}\overset{m}{b}{}^{\,l}_{i} - U_{ik}\overset{m}{b}{}^{\,l}_{j} + (t-1)(U_{ij}^l - U_{ji}^l)\omega_k \\ &+(U_{ijk} - U_{jik})(\omega^l - \pi^l) + \overset{m}{U}{}^{\,l}_{ik}[(t-1)\omega_j + \pi_j] - \overset{m}{U}{}^{\,l}_{jk}[(t-1)\omega_i + \pi_i] \\ &+(t-1)U_k^l\omega_{ij} + U_k^l\pi_{ij} + V_i^l f_{jk} - V_j^l f_{ik} - (\overset{m}{V}{}^{\,l}_{ij} - \overset{m}{V}{}^{\,l}_{ij})\pi_k\end{aligned} \tag{2.13}$$

where the other notations are given as

$$\begin{aligned}\overset{m}{a}_{ik} &= (t-1)\{\overset{\circ}{\nabla}_i\omega_k - [(t-1)\omega_i + \pi_i]U_k^p\omega_k - (t-1)U_i^p\omega_p\omega_k - U_{ik}\omega_p(\omega^p - \pi^p)\} \\ \overset{m}{b}_{ik} &= \overset{\circ}{\nabla}_i(\omega_k - \pi_k) + [(t-1)\omega_i + \pi_i]U_{kp}(\omega^p - \pi^p) + U_{ip}(\omega^p - \pi^p)(\omega_k - \pi_k) \\ \overset{m}{U}{}^{\,l}_{ik} &= \overset{\circ}{\nabla}_i U_k^l + U_{ip}U_k^p(\omega^l - \pi^l) - (t-1)U_i^p U_p^l\omega_k \\ f_{ik} &= \overset{\circ}{\nabla}_i\pi_k - [(t-1)\omega_i + \pi_i]U_k^p\pi_p - (t-1)U_i^p\pi_p\omega_k - U_{ik}\pi_p(\omega^p - \pi^p) + V_i^p\omega_p\pi_k \\ \overset{m}{V}{}^{\,l}_{ij} &= \overset{\circ}{\nabla}_i V_j^l + [(t-1)\omega_i + \pi_i]V_j^p U_p^l + U_{ip}V_j^p(\omega^l - \pi^l) + (t-1)U_i^l V_j^p\omega_p\end{aligned} \tag{2.14}$$

Theorem 2.6. *For a Riemannian manifold (M, g), if 1-form α, $\overline{\alpha}$, β, $\overline{\beta}$, are closed forms, then the volume curvature tensor $\overset{tm}{P}_{ij}$ of the mutual connection family $\overset{m}{\nabla}$ is zero, namely*

$$\overset{tm}{P}_{ij} = 0 \tag{2.15}$$

where $\overset{tm}{P}_{ij}$ is said to be the volume curvature tensor of $\overset{tm}{\nabla}$ defined by $\overset{tm}{P}_{ij} = \overset{tm}{P}{}^{\,k}_{ijk}$.

Proof. Contracting the indices k and l of the expression (2.13), then we have

$$\begin{aligned}\overset{tm}{P}_{ij} &= K_{ij} + U_j^k \overset{m}{a}_{ik} - U_i^k \overset{m}{a}_{jk} + U_{jk} \overset{m}{b}{}_i^k - U_{ik} \overset{m}{b}{}_j^k + (t-1)(U_{ij}^k - U_{ji}^k)\omega_k + (U_{ijk} - U_{jik})(\omega^k - \pi^k) \\ &+ \overset{m}{U}{}_{ik}^k[(t-1)\omega_j + \pi_j] - \overset{m}{U}{}_{jk}^k[(t-1)\omega_i + \pi_i] + (t-1)U_k^k \omega_{ij} + U_k^k \pi_{ij} \\ &- V_i^k f_{jk} - V_j^k f_{ik} - (\overset{m}{V}{}_{ij}^k - \overset{m}{V}{}_{ji}^k)\pi_k\end{aligned} \tag{2.16}$$

Using the expression (2.14), we obtain

$$\begin{aligned}U_j^k \overset{m}{a}_{ik} - U_i^k \overset{m}{a}_{jk} &= (t-1)(U_j^k \overset{\circ}{\nabla}_i \omega_k - U_i^k \overset{\circ}{\nabla}_j \omega_k) - (t-1)^2(\omega_i U_j^k - \omega_j U_i^k)U_k^p \omega_p \\ &- (t-1)(\pi_i U_j^k - \pi_j U_i^k)U_k^p \omega_p\end{aligned}$$

$$\begin{aligned}U_{jk} \overset{m}{b}{}_i^k - U_{ik} \overset{m}{b}{}_j^k &= U_{jk} \overset{\circ}{\nabla}_i(\omega^k - \pi^k) - U_{ik} \overset{\circ}{\nabla}_j(\omega^k - \pi^k) + (t-1)(\omega_i U_{jk} - \omega_j U_{ik})U_k^p(\omega^p - \pi^p) \\ &+ (\pi_i U_{jk} - \pi_j U_{ik})U_p^k(\omega^p - \pi^p)\end{aligned}$$

$$(t-1)(U_{ij}^k - U_{ji}^k)\omega_k + (U_{ijk} - U_{jik})(\omega^k - \pi^k) = t(\overset{\circ}{\nabla}_i U_j^k - \overset{\circ}{\nabla}_j U_i^k)\omega_k - (\overset{\circ}{\nabla}_i U_{jk} - \overset{\circ}{\nabla}_j U_{ik})\pi^k$$

$$\overset{m}{U}{}_{ik}^k[(t-1)\omega_j - \pi_j] - \overset{m}{U}{}_{jk}^k[(t-1)\omega_i - \pi_i]$$

$$= -(t-1)(\omega_i \overset{\circ}{\nabla}_j U_k^k - \omega_j \overset{\circ}{\nabla}_i U_k^k) - (t-1)(\omega_i U_{jp} - \omega_j U_{ip})U_k^p(\omega^k - \pi^k) + (t-1)^2(\omega_i U_j^p - \omega_j U_i^p)U_p^k \omega_k$$

$$-(\pi_i \overset{\circ}{\nabla}_j U_k^k - \pi_j \overset{\circ}{\nabla}_i U_k^k) - (\pi_i U_{jp} - \pi_j U_{ip})U_k^p(\omega^k - \pi^k)$$

$$+(t-1)(\pi_i U_j^p - \pi_j U_i^p)U_p^k \omega_k$$

$$V_i^k f_{jk} - V_j^k f_{ik} - (\overset{m}{V}{}_{ij}^k - \overset{m}{V}{}_{ji}^k)\pi_k = -[\overset{\circ}{\nabla}_i(V_j^k \pi_k) - \overset{\circ}{\nabla}_j(V_i^k \pi_k)]$$

$$(t-1)U_k^k \omega_{ij} = (t-1)(U_k^k \overset{\circ}{\nabla}_i \omega_j - U_k^k \overset{\circ}{\nabla}_j \omega_i)$$

$$U_k^k \pi_{ij} = U_k^k \overset{\circ}{\nabla}_i \pi_j - U_k^k \overset{\circ}{\nabla}_j \pi_i$$

$$K_{ij} = 0$$

Substituting these expressions above into (2.16) and using (2.6), we have

$$\begin{aligned}\overset{m}{P}_{ij} &= t(\overset{\circ}{\nabla}_i \alpha_j - \overset{\circ}{\nabla}_j \alpha_i) + (t-1)(\overset{\circ}{\nabla}_i \overline{\alpha}_j - \overset{\circ}{\nabla}_j \overline{\alpha}_i) - (\overset{\circ}{\nabla}_i \beta_j - \overset{\circ}{\nabla}_j \beta_i) \\ &+ (\overset{\circ}{\nabla}_i \overline{\beta}_j - \overset{\circ}{\nabla}_j \overline{\beta}_i) - (\overset{\circ}{\nabla}_i \gamma_j - \overset{\circ}{\nabla}_j \gamma_i)\end{aligned} \tag{2.17}$$

If a 1-form α, $\overline{\alpha}$, β, $\overline{\beta}$, γ are closed, then there holds $\overset{\circ}{\nabla}_i \alpha_j - \overset{\circ}{\nabla}_j \alpha_i = 0$, $\overset{\circ}{\nabla}_i \overline{\alpha}_j - \overset{\circ}{\nabla}_j \overline{\alpha}_i = 0$, $\overset{\circ}{\nabla}_i \beta_j - \overset{\circ}{\nabla}_j \beta_i = 0$, $\overset{\circ}{\nabla}_i \overline{\beta}_j - \overset{\circ}{\nabla}_j \overline{\beta}_i = 0$, $\overset{\circ}{\nabla}_i \gamma_j - \overset{\circ}{\nabla}_j \gamma_i = 0$. From expression (2.17), it is not hard to see that (2.15) is tenable. □

Remark 2.3. *Theorem 2.4 shows that the volume flat condition of the quarter-symmetric family $\overset{tm}{\nabla}$ is different from the volume flat condition of the generalized quarter-symmetric non-metric connection family $\overset{t}{\nabla}$.*

3. A generalized projective quarter-symmetric non-metric connection family

Definition 3.1. *A connection family $\overset{P}{\nabla}$ is called a generalized projective quarter-symmetric non-metric connection family, if $\overset{P}{\nabla}$ is a projective equivalent to the generalized quarter-symmetric non-metric connection family $\overset{t}{\nabla}$.*

In a Riemannian manifold (M, g), a generalized projective quarter-symmetric non-metric connection family $\overset{t}{\nabla}$ satisfies the relation

$$\overset{P}{\nabla}_k g_{ij} = -2\psi_k g_{ij} - \psi_i g_{jk} - \psi_j g_{ik} - 2(t-1)\omega_k U_{ij} - t\omega_i U_{jk} - t\omega_j U_{ik} \tag{3.1}$$

$$T^k_{ij} = \pi_j \varphi^k_i - \pi_i \varphi^k_j$$

and the coefficient of this connection family is

$$\overset{P}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \psi_i \delta^k_j + \psi_j \delta^k_i + (t-1)\omega_i U^k_j + [(t-1)\omega_j + \pi_j]U^k_i + U_{ij}(\omega^k - \pi^k) - \pi_i V^k_j \tag{3.2}$$

where ψ_i is a projective component of $\overset{P}{\nabla}$.

From the expression (3.2), we find that the curvature tensor of $\overset{P}{\nabla}$ is

$$\begin{aligned} \overset{P}{R}{}^l_{ijk} &= K^l_{ijk} + \delta^l_j c_{ik} - \delta^l_i c_{jk} + U^l_j a_{ik} - U^l_i a_{jk} + U_{jk} b^l_i - U_{ik} b^l_j + (U^l_{ij} - U^l_{ji})[(t-1)\omega_k + \pi_k] \\ &+ (U_{ijk} - U_{jik})(\omega^l - \pi^l) + (t-1)(U^l_{ik}\omega_j - U^l_{jk}\omega_i) + (t-1)U^l_k \omega_{ij} + \pi_i V^l_{jk} - \pi_j V^l_{ik} \\ &+ \delta^l_k \psi_{ij} + T^l_{ij}\psi_k \end{aligned} \tag{3.3}$$

where

$$c_{ik} = \overset{\circ}{\nabla}_i \psi_k - \psi_i \psi_k - (t-1)\omega_i U^p_k \psi_p - U^p_i \psi_p[(t-1)\omega_k + \pi_k] - U_{ik}\psi_p(\omega^p - \pi^p) + \pi_i V^p_k \psi_p$$

$$\psi_{ij} == \overset{\circ}{\nabla}_i \psi_j - \overset{\circ}{\nabla}_i \psi_i \tag{3.4}$$

Using the expression (2.4), the expression (3.3) becomes

$$\overset{P}{R}{}^l_{ijk} = \overset{t}{R}{}^l_{ijk} + \delta^l_j c_{ik} - \delta^l_i c_{jk} + \delta^l_k \psi_{ij} + T^l_{ij}\psi_k \tag{3.5}$$

Theorem 3.1. *In a Riemannian manifold (M, g), if 1-form ψ, α and $\overline{\alpha}$ are closed forms, then the volume curvature tensor of $\overset{P}{\nabla}$ is zero, namely*

$$\overset{P}{P}_{ij} = 0 \tag{3.6}$$

where $\overset{P}{P}_{ij} = \overset{P}{R}{}^k_{ijk}$ is a volume curvature tensor of $\overset{P}{\nabla}$.

Proof. Contracting the indices k and l of the expression (3.5), then we obtain

$$\overset{P}{P}_{ij} = \overset{t}{P}_{ij} + c_{ij} - c_{ji} + T^k_{ij}\psi_k + n\psi_{ij}$$

On the other hand, from the expression (3.4), we have

$$c_{ij} - c_{ji} = \psi_{ij} - T^p_{ij}\psi_p$$

Substituting this expression into the above expression, we obtain

$$\overset{p}{P}_{ij} = \overset{t}{P}_{ij} + (n+1)\psi_{ij} \tag{3.7}$$

If a 1-form ψ, α and $\overline{\alpha}$ are closed, then $\overset{t}{P}_{ij} = 0$(theorem 2.1) and $\psi_{ij} = 0$. Hence from the expression (3.7), we obtain the expression (3.6). □

From the expressions (3.1) and (3.2), the coefficient of the mutual connection family $\overset{pm}{\nabla}$ of the generalized projective quarter-symmetric non-metric connection family $\overset{p}{\nabla}$ is

$$\overset{pm}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + \psi_i\delta^k_j + \psi_j\delta^k_i + [(t-1)\omega_i + \pi_i]U^k_j + (t-1)\omega_j U^k_i + U_{ij}(\omega^k - \pi^k) - \pi_j V^k_i \tag{3.8}$$

and from this expression the curvature tensor of $\overset{pm}{\nabla}$, by a direct computation is

$$\begin{aligned} \overset{pm}{R}{}^{l}_{ijk} &= K^{l}_{ijk} + \delta^l_j \overset{m}{c}_{ik} - \delta^l_i \overset{m}{c}_{jk} + U^l_j \overset{m}{a}_{ik} - U^l_i \overset{m}{a}_{jk} + U_{jk}\overset{m}{b}{}^{l}_{i} - U_{ik}\overset{m}{b}{}^{l}_{j} + (t-1)(U^l_{ij} - U^l_{ji})\omega_k \\ &+ (U_{ijk} - U_{jik})(\omega^l - \pi^l) + \overset{m}{U}{}^{l}_{ik}[(t-1)\omega_j + \pi_j] - \overset{m}{U}{}^{l}_{jk}[(t-1)\omega_i + \pi_i] \\ &+ (t-1)U^l_k\omega_{ij} + U^l_k\pi_{ij} + V^l_i f_{jk} - V^l_j f_{ik} - (\overset{m}{V}{}^{l}_{ij} - \overset{m}{V}{}^{l}_{ji})\pi_k + \delta^l_k\psi_{ij} - T^l_{ij}\psi_k \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \overset{m}{c}_{ik} &= \overset{\circ}{\nabla}_i\psi_j - \psi_i\psi_j - [(t-1)\omega_i + \pi_i]U^p_k\psi_p + (t-1)U^p_i\pi_p\omega_k \\ &- U_{ik}\pi_p(\omega^p - \pi^p) + V^p_i\pi_p\pi_k \end{aligned} \tag{3.10}$$

Using the expression (2.13), the expression (3.9) becomes

$$\overset{pm}{R}{}^{l}_{ijk} = \overset{tm}{R}{}^{l}_{ijk} + \delta^l_j\overset{m}{c}_{ik} - \delta^l_i\overset{m}{c}_{jk} - T^l_{ij}\psi_k + \delta^l_k\psi_{ij} - T^l_{ij}\psi_k \tag{3.11}$$

Theorem 3.2. *In a Riemannian manifold (M, g), if 1-form ψ, α, $\overline{\alpha}$, β, $\overline{\beta}$ and γ are closed, then a volume curvature tensor of $\overset{pm}{\nabla}$ is zero, namely*

$$\overset{pm}{P}_{ij} = 0 \tag{3.12}$$

where $\overset{pm}{P}_{ij} = \overset{pm}{R}{}^{k}_{ijk}$ is the volume curvature tensor of $\overset{pm}{\nabla}$.

Proof. Contracting the indices and of the expression (3.11), then we have

$$\overset{pm}{P}_{ij} = \overset{tm}{P}_{ij} + \overset{m}{c}_{ij} - \overset{m}{c}_{ji} + T^k_{ij}\psi_k + n\psi_{ij}$$

On the other hand from the expression (3.4), we have

$$\overset{m}{c}_{ij} - \overset{m}{c}_{ji} = \psi_{ij} - T^p_{ij}\psi_p$$

Substituting this expression into the above expression, we obtain

$$\overset{pm}{P}_{ij} = \overset{tm}{P}_{ij} + (n+1)\psi_{ij} \tag{3.13}$$

If a 1-form ψ, α, $\overline{\alpha}$, β, $\overline{\beta}$ and γ are closed form, then $\overset{tm}{P}_{ij} = 0$(theorem2.4) and and $\psi_{ij} = 0$. Hence from the expression (3.13), we obtain the expression (3.12). □

Theorem 3.3. *In a Riemannian manifold (M, g), if a 1-form ψ is closed, the tensor below*

$$\overset{t}{W}{}^{l}_{ijk} + \overset{tm}{W}{}^{l}_{ijk} \tag{3.14}$$

is an invariant under the projective connection transformation $\overset{t}{\nabla} \to \overset{P}{\nabla}$, $\overset{tm}{\nabla} \to \overset{pm}{\nabla}$ where

$$\begin{aligned} \overset{t}{W}{}^{l}_{ijk} &= \overset{t}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik}) \\ \overset{tm}{W}{}^{l}_{ijk} &= \overset{tm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{tm}{R}_{jk} - \delta^l_j \overset{tm}{R}_{ik}) \end{aligned} \tag{3.15}$$

where $\overset{t}{W}{}^{l}_{ijk}$ and $\overset{tm}{W}{}^{l}_{ijk}$ are the Weyl projective curvature tensor of $\overset{t}{\nabla}$ and $\overset{tm}{\nabla}$ respectively.

Proof. Adding the expressions (3.5) and (3.11), we obtain

$$\overset{p}{R}{}^{l}_{ijk} + \overset{pm}{R}{}^{l}_{ijk} = \overset{t}{R}{}^{l}_{ijk} + \overset{tm}{R}{}^{l}_{ijk} + \delta^l_j \alpha_{ik} - \delta^l_i \alpha_{jk} + 2\delta^l_k \psi_{ij} \tag{3.16}$$

where $a_{ik} = c_{ik} + \overset{m}{c}_{ik}$, If 1-form ψ is closed, then $\psi_{ij} = \overset{\circ}{\nabla}_i \psi_j - \overset{\circ}{\nabla}_j \psi_i = 0$. From this fact, the expression (3.16) is

$$\overset{p}{R}{}^{l}_{ijk} + \overset{pm}{R}{}^{l}_{ijk} = \overset{t}{R}{}^{l}_{ijk} + \overset{tm}{R}{}^{l}_{ijk} + \delta^l_j \alpha_{ik} - \delta^l_i \alpha_{jk} \tag{3.17}$$

Contracting the indices i, l of (3.17), we get

$$\overset{p}{R}_{jk} + \overset{pm}{R}_{jk} = \overset{t}{R}_{jk} + \overset{tm}{R}_{jk} - (n-1)\alpha_{jk}$$

From this expression above we obtain

$$\alpha_{jk} = \frac{1}{n-1}(\overset{t}{R}_{jk} + \overset{tm}{R}_{jk} - \overset{p}{R}_{jk} - \overset{pm}{R}_{jk})$$

Substituting this expression into (3.17) and putting

$$\begin{aligned} \overset{p}{W}{}^{l}_{ijk} &= \overset{p}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{p}{R}_{jk} - \delta^l_j \overset{p}{R}_{ik}) \\ \overset{pm}{W}{}^{l}_{ijk} &= \overset{pm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{pm}{R}_{jk} - \delta^l_j \overset{pm}{R}_{ik}) \end{aligned} \tag{3.18}$$

then by a direct computation, we obtain

$$\overset{p}{W}{}^{l}_{ijk} + \overset{pm}{W}{}^{l}_{ijk} = \overset{t}{W}{}^{l}_{ijk} + \overset{tm}{W}{}^{l}_{ijk} \tag{3.19}$$

where $\overset{p}{W}{}^{l}_{ijk}$ and $\overset{pm}{W}{}^{l}_{ijk}$ are the Weyl projective curvature tensor of $\overset{p}{\nabla}$ and $\overset{pm}{\nabla}$. □

Theorem 3.4. *In a Riemannian manifold (M, g), the tensor below*

$$\overline{\overset{t}{W}}{}^{l}_{ijk} + \overline{\overset{tm}{W}}{}^{l}_{ijk} \tag{3.20}$$

is an invariant under the projective connection transformation $\overset{t}{\nabla} \to \overset{P}{\nabla}, \overset{tm}{\nabla} \to \overset{pm}{\nabla}$, *where*

$$\begin{aligned} \overset{t}{\overline{W}}{}^{l}_{ijk} &= \overset{t}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik}) \\ &\quad + \frac{1}{n^2-1}[\delta^l_i(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) - \delta^l_j(\overset{t}{R}_{ik} - \overset{t}{R}_{ki}) + (n-1)\delta^l_k(\overset{t}{R}_{ij} - \overset{t}{R}_{ji})] \\ \overset{tm}{\overline{W}}{}^{l}_{ijk} &= \overset{tm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{tm}{R}_{jk} - \delta^l_j \overset{tm}{R}_{ik}) \\ &\quad + \frac{1}{n^2-1}[\delta^l_i(\overset{tm}{R}_{jk} - \overset{tm}{R}_{kj}) - \delta^l_j(\overset{tm}{R}_{ik} - \overset{tm}{R}_{ki}) + (n-1)\delta^l_k(\overset{tm}{R}_{ij} - \overset{tm}{R}_{ji})] \end{aligned} \tag{3.21}$$

(tensors $\overset{t}{\overline{W}}{}^{l}_{ijk}$ *and* $\overset{tm}{\overline{W}}{}^{l}_{ijk}$ *are the generalized Weyl projective curvature tensors of* $\overset{t}{\nabla}$ *and* $\overset{tm}{\nabla}$*, respectively).*

Proof. Contracting the indices i, l of expression (3.16) and using $\psi_{ij} = -\psi_{ji}$, we get

$$\overset{p}{R}_{jk} + \overset{pm}{R}_{jk} = \overset{t}{R}_{jk} + \overset{tm}{R}_{jk} - (n-1)\alpha_{jk} - 2\psi_{jk} \tag{3.22}$$

Alternating the indices j and k of this expression and using $\alpha_{jk} - \alpha_{kj} = 2\psi_{jk}$, we obtain

$$\overset{p}{R}_{jk} - \overset{p}{R}_{kj} + \overset{pm}{R}_{jk} - \overset{pm}{R}_{kj} = \overset{t}{R}_{jk} - \overset{t}{R}_{kj} + \overset{tm}{R}_{jk} - \overset{tm}{R}_{kj} - 2(n+1)\psi_{jk}$$

From this expression above we have

$$\psi_{jk} = \frac{1}{2(n+1)}[(\overset{t}{R}_{jk} - \overset{t}{R}_{kj} + \overset{tm}{R}_{jk} - \overset{tm}{R}_{kj}) - (\overset{p}{R}_{jk} - \overset{p}{R}_{kj} + \overset{pm}{R}_{jk} - \overset{pm}{R}_{kj})]$$

Using this expression from the expression (3.22), we find

$$\begin{aligned} \alpha_{jk} &= \frac{1}{n-1}\{(\overset{t}{R}_{jk} + \overset{tm}{R}_{jk}) - (\overset{p}{R}_{jk} + \overset{pm}{R}_{jk}) \\ &\quad - \frac{1}{n+1}[(\overset{t}{R}_{jk} - \overset{t}{R}_{kj} + \overset{tm}{R}_{jk} - \overset{tm}{R}_{kj}) - (\overset{p}{R}_{jk} - \overset{p}{R}_{kj} + \overset{pm}{R}_{jk} - \overset{pm}{R}_{kj})]\} \end{aligned}$$

Substituting the above two expressions into (3.16) and putting

$$\begin{aligned} \overset{p}{\overline{W}}{}^{l}_{ijk} &= \overset{p}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{p}{R}_{jk} - \delta^l_j \overset{p}{R}_{ik}) \\ &\quad + \frac{1}{n^2-1}[\delta^l_i(\overset{p}{R}_{jk} - \overset{p}{R}_{kj}) - \delta^l_j(\overset{p}{R}_{ik} - \overset{p}{R}_{ki}) + (n-1)\delta^l_k(\overset{p}{R}_{ij} - \overset{p}{R}_{ji})] \\ \overset{pm}{\overline{W}}{}^{l}_{ijk} &= \overset{pm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{pm}{R}_{jk} - \delta^l_j \overset{pm}{R}_{ik}) \\ &\quad + \frac{1}{n^2-1}[\delta^l_i(\overset{pm}{R}_{jk} - \overset{pm}{R}_{kj}) - \delta^l_j(\overset{pm}{R}_{ik} - \overset{pm}{R}_{ki}) + (n-1)\delta^l_k(\overset{pm}{R}_{ij} - \overset{pm}{R}_{ji})] \end{aligned} \tag{3.23}$$

then by a direct computation, we obtain

$$\overset{p}{\overline{W}}{}^{l}_{ijk} + \overset{pm}{\overline{W}}{}^{l}_{ijk} = \overset{t}{\overline{W}}{}^{l}_{ijk} + \overset{tm}{\overline{W}}{}^{l}_{ijk} \tag{3.24}$$

where $\overset{p}{\overline{W}}{}^{l}_{ijk}$ and $\overset{pm}{\overline{W}}{}^{l}_{ijk}$ are the generalized Weyl projective curvature tensors of $\overset{p}{\nabla}$ and $\overset{pm}{\nabla}$, respectively. □

Remark 3.1. *Theorem 3.3 and theorem 3.4 show that the proposed projective invariant is independent of parameter t.*

4. A generalized projective conformal quarter-symmetric non-metric connection family

Definition 4.1. *A connection family ∇ is called a generalized projective conformal quarter-symmetric non-metric connection family, if ∇ is conformal equivalent to the generalized projective conformal quarter-symmetric non-metric connection family $\overset{p}{\nabla}$.*

In the Riemannian manifold (M, g), a generalized projective conformal quarter-symmetric non-metric connection family ∇ satisfies the relation

$$\begin{aligned} \nabla_k g_{ij} &= -2(\psi_k + \sigma_k)g_{ij} - \psi_i g_{jk} - 2(t-1)\omega_k U_{ij} - t\omega_i U_{jk} - t\omega_j U_{ik} \\ T^k_{ij} &= \psi_j \varphi^k_i - \psi_i \varphi^k_j \end{aligned} \tag{4.1}$$

and the coefficient of this connection family is

$$\begin{aligned} \Gamma^k_{ij} = &\{^k_{ij}\} + (\psi_i + \sigma_i)\delta^k_j + (\psi_j + \sigma_j)\delta^k_i - g_{ij}\sigma^k + (t-1)\omega_i U^k_j \\ &+ [(t-1)\omega_j + \pi_j]U^k_i + U_{ij}(\omega^k - \pi^k) - \pi_i V^k_j \end{aligned} \tag{4.2}$$

From the expression (4.2), the curvature tensor of ∇, by a direct computation, is

$$\begin{aligned} R^l_{ijk} = &K^{\ l}_{ijk} + \delta^l_j d_{ik} - \delta^l_i d_{jk} + g_{ik}e^l_j - g_{jk}e^l_i + U^l_j a_{ik} - U^l_i a_{jk} + U_{jk}b^l_i - U_{ik}b^l_j \\ &+ (U^l_{ij} - U^l_{ji})[(t-1)\omega_k + \pi_k] + (U_{ijk} - U_{jik})(\omega^l - \pi^l) + (t-1)(U^l_{ik}\omega_j - U^l_{jk}\omega_i) \\ &+ \delta^l_k \psi_{ij} + (t-1)U^l_k \omega_{ij} - V^l_k \pi_{ij} + \pi_i V^l_{jk} - \pi_j V^l_{ik} - (t-2)(\omega_j U_{ik} - \omega_i U_{jk})\sigma^l \\ &+ T^l_{ij}(\psi_k + \sigma_k) - T_{ijk}\sigma^l \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} d_{ik} = &\overset{\circ}{\nabla}_i(\psi_k + \sigma_k) - (\psi_i + \sigma_i)(\psi_k + \sigma_k) - g_{ik}(\psi_p + \sigma_p)\sigma^p - (t-1)\omega_i U^p_k(\psi_p + \sigma_p) \\ &- U^p_i(\psi_p + \sigma_p)[(t-1)\omega_k + \pi_k] - U_{ik}(\psi_p + \sigma_p)(\omega^p - \pi^p) + \pi_i V^p_k(\psi_p + \sigma_p) \\ e_{ik} = &\overset{\circ}{\nabla}_i\sigma_k - \sigma_i\sigma_k + (t-1)\omega_i U_{kp}\sigma^p + U_{ik}[(t-1)\omega_p + \pi_p]\sigma^p + U_{ip}\sigma^p(\omega_k - \pi_k) + \pi_i V_{kp}\sigma^p \end{aligned} \tag{4.4}$$

Using the expression (2.4), the expression (4.3) becomes

$$\begin{aligned} R^l_{ijk} = &\overset{t}{R}{}^{\,l}_{ijk}\delta^l_j d_{ik} - \delta^l_i d_{jk} + g_{ik}e^l_j - g_{jk}e^l_i + \delta^l_k\psi_{ij} \\ &- (t-2)(\omega_j U_{ik} - \omega_i U_{jk})\sigma^l + T^l_{ij}(\psi_k + \sigma_k) - T_{ijk}\sigma^l \end{aligned} \tag{4.5}$$

Theorem 4.1. *In a Riemannian manifold (M, g), if 1-form ψ, α and $\overline{\alpha}$ are closed forms, then the volume curvature tensor of ∇ is zero, namely*

$$P_{ij} = 0 \tag{4.6}$$

where $P_{ij} = R^k_{ijk}$ is a volume curvature tensor of ∇.

Proof. Contracting the indices k and l of the expression (4.4), then we have

$$P_{ij} = \overset{t}{P}_{ij} + d_{ij} - d_{ji} + e_{ij} - e_{ji} + n\psi_{ij} - (t-2)(\omega_j U_{ik} - \omega_i U_{jk})\sigma^k + T^k_{ij}(\psi_k + \sigma_k) - T_{ijk}\sigma^k$$

On the other hand, from the expression (4.4), we have

$$d_{ij} - d_{ji} = \psi_{ij} - T^p_{ij}(\psi_p + \sigma_p)$$

$$e_{ij} - e_{ji} = (t-2)(\omega_j U_{ip} - \omega_i U_{jp})\sigma^p + T_{ijp}\sigma^p$$

Substituting these expressions into the above expression, we obtain

$$\overset{p}{P}_{ij} = \overset{t}{P}_{ij} + (n+1)\psi_{ij} \tag{4.7}$$

If a 1-form ψ, α and $\overline{\alpha}$ are closed, then $\overset{p}{P}_{ij} = 0$(theorem 2.1) and $\psi_{ij} = 0$. Hence from the expression (4.7), we obtain the expression (4.6). □

Remark 4.1. *The expression (4.7) is not different from the expression (3.7). This shows that volume flat condition of* $\overset{p}{\nabla}$ *and* ∇ *is equal.*

From the expressions (4.1) and (4.2), the coefficient of the mutual connection family $\overset{m}{\nabla}$ of the generalized projective conformal quarter-symmetric non-metric connection family ∇ is

$$\begin{aligned}\overset{m}{\Gamma}{}^k_{ij} &= \{^k_{ij}\} + (\psi_i + \sigma_i)\delta^k_j + (\psi_j + \sigma_j)\delta^k_i - g_{ij}\sigma^k + [(t-1)\omega_i + \pi_i]U^k_j \\ &\quad + (t-1)\omega_j U^k_i + U_{ij}(\omega^k - \pi^k) - \pi_j V^k_i\end{aligned} \tag{4.8}$$

And from this expression the curvature tensor of $\overset{m}{\nabla}$, by a direct computation is

$$\begin{aligned}\overset{m}{R}{}^{\,l}_{ijk} &= K^{\,l}_{ijk} + \delta^l_j \overset{m}{d}_{ik} - \delta^l_i \overset{m}{d}_{jk} + g_{ik}\overset{m}{e}{}^{\,l}_{j} - g_{jk}\overset{m}{e}{}^{\,l}_{i} + U^l_j \overset{m}{a}_{ik} - U^l_i \overset{m}{a}_{jk} + U_{jk}\overset{m}{b}{}^{\,l}_{i} - U_{ik}\overset{m}{b}{}^{\,l}_{j} \\ &\quad + (t-1)(U^l_{ij} - U^l_{ji})\omega_k + (U_{ijk} - U_{jik})(\omega^l - \pi^l) + \overset{m}{U}{}^{\,l}_{ik}[(t-1)\omega_j + \pi_j] \\ &\quad - \overset{m}{U}{}^{\,l}_{jk}[(t-1)\omega_i + \pi_i] + \delta^l_k \psi_{ij} + (t-1)U^l_k \omega_{ij} + U^l_k \pi_{ij} + V^l_i f_{jk} - V^l_j f_{ik} + (\overset{m}{V}{}^{\,l}_{ij} - \overset{m}{V}{}^{\,l}_{ji})\pi_k \\ &\quad - (t-2)(\omega_j U_{ik} - \omega_i U_{jk})\sigma^l - 2(\pi_j U_{ik} - \pi_i U_{jk})\sigma^l - T^l_{ij}(\psi_k + \sigma_k) - 2V_{ij}\pi_k \sigma^l\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}\overset{m}{d}_{ik} &= \overset{\circ}{\nabla}_i(\psi_k + \sigma_k) - (\psi_i + \sigma_i)(\psi_k + \sigma_k) + g_{ik}(\psi_p + \sigma_p)\sigma^p - [(t-1)\omega_i + \pi_k]U^p_k(\psi_p + \sigma_p) \\ &\quad - (t-1)U^p_i(\psi_p + \sigma_p)\omega_k - U_{ik}(\psi_p + \sigma_p)(\omega^p - \pi^p) + V^p_i(\psi_p + \sigma_p) \\ \overset{m}{e}_{ik} &= \overset{\circ}{\nabla}_i \sigma_k - \sigma_i \sigma_k + [(t-1)\omega_i + \pi_i]U_{kp}\sigma^p + (t-1)U_{ik}\omega_p \sigma^p + U_{ip}\sigma^p(\omega_k - \pi_k) + V_{ik}\pi_p \sigma^p\end{aligned} \tag{4.10}$$

Using the expression (2.13), the expression (4.9) becomes

$$\begin{aligned}\overset{pm}{R}{}^{\,l}_{ijk} &= \overset{pm}{R}{}^{\,l}_{ijk} + \delta^l_j \overset{m}{d}_{ik} - \delta^l_i \overset{m}{d}_{jk} + g_{ik}\overset{m}{e}{}^{\,l}_{j} - g_{jk}\overset{m}{e}{}^{\,l}_{i} + \delta^l_k \psi_{ij} - (t-2)(\omega_j U_{ik} - \omega_i U_{jk})\sigma^l \\ &\quad - 2(\pi_j U_{ik} - \pi_i U_{jk})\sigma^l - T^l_{ij}(\psi_k + \sigma_k) - 2V_{ij}\pi_k \sigma^l\end{aligned} \tag{4.11}$$

Theorem 4.2. *In a Riemannian manifold (M, g), if 1-form ψ, α, $\overline{\alpha}$, β, $\overline{\beta}$ and γ are closed, then a volume curvature tensor of $\overset{m}{\nabla}$ is zero, namely*

$$\overset{m}{P}_{ij} = 0 \tag{4.12}$$

where $\overset{m}{P}_{ij} = \overset{m}{R}{}^{\,k}_{ijk}$ is the volume curvature tensor of $\overset{m}{\nabla}$.

Proof. Contracting the indices k and l of the expression (4.11), then we have

$$\overset{pm}{P}_{ij} = \overset{tm}{P}_{ij} + \overset{m}{d}_{ij} - \overset{m}{d}_{ji} + \overset{m}{e}_{ji} - \overset{m}{e}_{ij} + n\psi_{ij} - (t-2)(\omega_j U_{ik} - \omega_i U_{jk})\sigma^k - 2(\pi_j U_{ik} - \pi_i U_{jk})\sigma^k - 2V_{ij}\pi_k\sigma^k - T^k_{ij}(\psi_k + \sigma_k)$$

On the other hand from the expression (4.4), we have

$$\overset{m}{d}_{ij} - \overset{m}{d}_{ji} = \psi_{ij} + T^p_{ij}\psi_p$$

$$\overset{m}{e}_{ji} - \overset{m}{e}_{ij} = (t-2)(\omega_j U_{ip} - \omega_i U_{jp})\sigma^p + 2(\pi_j U_{ip} - \pi_i U_{jp})\sigma^p + 2V_{ij}\pi_p\sigma^p$$

Substituting this expression into the above expression, we obtain

$$\overset{m}{P}_{ij} = \overset{tm}{P}_{ij} + (n+1)\psi_{ij} \tag{4.13}$$

If a 1-form ψ, α, $\overline{\alpha}$, β, $\overline{\beta}$ and γ are closed form, then $\overset{tm}{P}_{ij} = 0$(theorem 2.4) and $\psi_{ij} = 0$. Hence from the expression (4.13), we obtain the expression (4.12). □

Ackonowedement

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