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On composition of certain exponential operators

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Abstract. In the present article, we deal with a class of exponential operators, associated with $p(t) = t^r$, $t \in (0, \infty)$, $1 < r < 2$. We consider their composition with the well known Szász-Mirakjan operators and investigate the order of approximation using Taylor's formula. We further discuss the approximation properties of the composition operators by employing the modulus of smoothness and Peetre's K-functional. Moreover, Voronovskaja-type asymptotic theorems are given. Additionally, we provide further compositions and compare their approximation properties graphically.

1. Introduction

In 1978, Ismail and May [13], as well as Volkov [21], individually conducted their studies on the exponential-type operators. Consider an operator of the form

$$(L_\lambda g)(t) = \int_I k_\lambda(t,u) g(u) du,$$

on an infinite interval I. Here, the kernel $k_\lambda(t,u)$ satisfies a homogeneous partial differential equation and the normalization condition, respectively, as follows:

$$\frac{\partial}{\partial t} k_\lambda(t,u) = \frac{\lambda(u-t)}{p(t)} k_\lambda(t,u), \tag{1}$$

and

$$(L_\lambda 1)(t) = \int_I k_\lambda(t,u) du = 1. \tag{2}$$

$p(t)$ is an analytic function with $p(t) \neq 0$ on the interval I. Such operator is called exponential-type operator. May gave some approximation properties of these operators in [16], where $p(t)$ is a function of degree at most 2. Ismail and May, together, captured some existing exponential-type operators including Gauss-Weierstrass, Bernstein, Baskakov and Szász-Mirakjan operators, connected to $p(t) = 1$, $t(1-t)$, $t(1+t)$ and t;

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respectively. These operators preserve constant and linear functions. They have been studied by numerous researchers over last few decades. Abel, in [1], gave complete asymptotic expansion for an exponential-type operator associated with $p(t) = t^3$. Abel et al. studied semi-exponential operators in [2]. Gupta et al. provided convergence estimates of some specific exponential-type operators in [9]. Some exponential operators and their generalizations have been considered in [8, 11, 12, 17]. Recently, Gupta (see [10, Eq.(6)]) introduced a general class of these operators, associated with $p(t) = t^r$, where $1 < r < 2$, and $\lambda \in \mathrm{N}$:

$$(L_\lambda g)(t) = \exp\left(\frac{\lambda}{(r-2)t^{r-2}}\right)\left[g(0) + \sum_{m=1}^{\infty} \frac{\lambda^{\frac{m}{r-1}}}{(2-r)^m m!} \frac{(r-1)^{\frac{(r-2)m}{(r-1)}}}{\Gamma\left(\frac{(2-r)m}{r-1}\right)} \int_0^\infty u^{\frac{(2-r)m}{r-1}-1} \exp\left(\frac{-\lambda u}{(r-1)t^{r-1}}\right) g(u)du\right]. \tag{3}$$

Here, the term $g(0)$ is added, so that the operator in (3) satisfies the normalization condition (2). Another well-known operators of exponential type, associated with $p(t) = t$, given by Szász-Mirakjan [20], are defined as follows:

$$(S_\lambda g)(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} g\left(\frac{k}{\lambda}\right),\ t \in [0, \infty) \subset \mathrm{R} \text{ and } \lambda \in \mathrm{N}. \tag{4}$$

Here, let us introduce a new operator by the composition of the operators defined in (3) and (4), in that order, as follows:

$$(V_\lambda g)(t) := (L_\lambda \circ S_\lambda g)(t).$$

The aim of this paper is to discuss the approximation properties of the operators V_λ. We begin with an explicit formula for the operators and estimate their moments in Sect. 2. In Sect. 3, we provide direct theorems using modulus of smoothness and Peetre's K-functional and also establish Voronovskaja-type theorems. Sect. 4 is devoted to further compositions and their convergence behaviour. Finally, in Sect. 5, we compare the rates of convergence of the operators using graphical examples.

2. Estimation of Moments

In this section, we will calculate the moments and central moments for the operators V_λ, which will aid in discussing their convergence properties. Let us first provide an explicit formula for the operators V_λ.

Theorem 2.1. *For $1 < r < 2$ and $\lambda \in \mathrm{N}$, the new discrete operator V_λ can be represented as follows:*

$$(V_\lambda g)(t) = \sum_{k=0}^{\infty} s_{\lambda,k}(t) g\left(\frac{k}{\lambda}\right),$$

where

$$s_{\lambda,0}(t) = \exp\left(\frac{\lambda}{r-2}\left(t^{2-r} - \left(r - 1 + \frac{1}{t^{r-1}}\right)^{\frac{r-2}{r-1}}\right)\right), \tag{5}$$

and for $k \geq 1$,

$$s_{\lambda,k}(t) = \exp\left(\frac{\lambda}{(r-2)t^{r-2}}\right)\left[\left(\frac{(r-1)t^{r-1}}{1+(r-1)t^{r-1}}\right)^k \sum_{m=1}^{\infty} \binom{\frac{(2-r)m}{(r-1)} + k - 1}{k} \frac{\lambda^m}{(2-r)^m m!} \left(\frac{t^{r-1}}{1+(r-1)t^{r-1}}\right)^{\frac{(2-r)m}{r-1}}\right]. \tag{6}$$

Proof. By definition, we have

$$(V_\lambda g)(t) = \exp\left(\frac{\lambda}{(r-2)t^{r-2}}\right)\left[\sum_{m=1}^{\infty}\frac{\lambda^{\frac{m}{r-1}}}{(2-r)^m m!}\frac{(r-1)^{\frac{(r-2)m}{(r-1)}}}{\Gamma\left(\frac{(2-r)m}{r-1}\right)}\int_0^\infty u^{\frac{(2-r)m}{r-1}-1}\exp\left(\frac{-\lambda u}{(r-1)t^{r-1}}\right)(S_\lambda g)(u)du\right.$$
$$\left.+(S_\lambda g)(0)\right]$$
$$= \exp\left(\frac{\lambda}{(r-2)t^{r-2}}\right)\left[\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}g\left(\frac{k}{\lambda}\right)\sum_{m=1}^{\infty}\frac{\lambda^{\frac{m}{r-1}}}{(2-r)^m m!}\frac{(r-1)^{\frac{(r-2)m}{(r-1)}}}{\Gamma\left(\frac{(2-r)m}{r-1}\right)}\int_0^\infty u^{\frac{(2-r)m}{r-1}+k-1}e^{-\lambda u\left(1+\frac{1}{(r-1)t^{r-1}}\right)}du\right.$$
$$\left.+g(0)\right]$$
$$= \sum_{k=0}^{\infty}s_{\lambda,k}(t)g\left(\frac{k}{\lambda}\right)+\exp\left(\frac{\lambda}{(r-2)t^{r-2}}\right)g(0),$$

where

$$s_{\lambda,0}(t) = \exp\left(\frac{\lambda}{(r-2)t^{r-2}}\right)\sum_{m=1}^{\infty}\frac{\lambda^m}{(2-r)^m m!}\left(\frac{t^{r-1}}{(r-1)t^{r-1}+1}\right)^{\frac{(2-r)m}{r-1}}+\exp\left(\frac{\lambda}{(r-2)t^{r-2}}\right)$$
$$= \exp\left(\frac{\lambda}{r-2}\left(t^{2-r}-\left(r-1+\frac{1}{t^{r-1}}\right)^{\frac{r-2}{r-1}}\right)\right),$$

and for $k \geq 1$,

$$s_{\lambda,k}(t) = \exp\left(\frac{\lambda}{(r-2)t^{r-2}}\right)\left[\frac{1}{k!}\left(\frac{(r-1)t^{r-1}}{1+(r-1)t^{r-1}}\right)^k\sum_{m=1}^{\infty}\frac{\lambda^m}{(2-r)^m m!}\left(\frac{t^{r-1}}{1+(r-1)t^{r-1}}\right)^{\frac{(2-r)m}{r-1}}\frac{\Gamma\left(\frac{(2-r)m}{(r-1)}+k\right)}{\Gamma\left(\frac{(2-r)m}{(r-1)}\right)}\right]$$
$$= \exp\left(\frac{\lambda}{(r-2)t^{r-2}}\right)\left[\left(\frac{(r-1)t^{r-1}}{1+(r-1)t^{r-1}}\right)^k\sum_{m=1}^{\infty}\binom{\frac{(2-r)m}{(r-1)}+k-1}{k}\frac{\lambda^m}{(2-r)^m m!}\left(\frac{t^{r-1}}{1+(r-1)t^{r-1}}\right)^{\frac{(2-r)m}{r-1}}\right].$$

□

Now, we give moments and central moments for the operators V_λ.

Lemma 2.2. *Let us denote by* $\exp_A(u) := e^{Au}$ *and* $e_m(u) := u^m$ $(m = 0,1,2,\ldots)$. *For* $\lambda \in \mathbb{N}$, $t \in [0,\infty)$ *and* $1 < r < 2$, *we have*

$$\left(V_\lambda \exp_A\right)(t) = \exp\left(\frac{\lambda}{r-2}\left(t^{2-r}-\left((r-1)(1-e^{\frac{A}{\lambda}})+t^{1-r}\right)^{\frac{r-2}{r-1}}\right)\right), \tag{7}$$

and

(i) $(V_\lambda e_0)(t) = 1$,
(ii) $(V_\lambda e_1)(t) = t$,
(iii) $(V_\lambda e_2)(t) = t^2+\frac{t^r}{\lambda}+\frac{t}{\lambda}$,
(iv) $(V_\lambda e_3)(t) = t^3+\frac{3t(t+t^r)}{\lambda}+\frac{t+rt^{2r-1}+3t^r}{\lambda^2}$,
(v) $(V_\lambda e_4)(t) = t^4+\frac{6t^2(t+t^r)}{\lambda}+\frac{7t^2+(3+4r)t^{2r}+18t^{1+r}}{\lambda^2}+\frac{t+r(2r-1)t^{3r-2}+7t^r+6rt^{2r-1}}{\lambda^3}$.

Proof. For the operators L_λ, defined in (3), by simple computations we have

$$(L_\lambda \exp_A)(t) = \exp\left(\frac{\lambda}{r-2}\left(\frac{1}{t^{r-2}}-\left(\frac{(r-1)}{\lambda}\left(-A+\frac{\lambda}{(r-1)t^{r-1}}\right)\right)^{\frac{r-2}{r-1}}\right)\right),\ 1 < r < 2.$$

Also, for the Szász-Mirakjan operators, defined in (4), we have

$$\left(S_\lambda \exp_A\right)(t) = \exp\left(\lambda t(e^{\frac{A}{\lambda}} - 1)\right).$$

Now, the MGF (moment-generating function) for the operators V_λ is obtained as

$$\begin{aligned}\left(V_\lambda \exp_A\right)(t) &= (L_\lambda \circ S_\lambda \exp_A)(t)\\ &= \exp\left(\frac{\lambda}{r-2}\left(t^{2-r} - \left((r-1)(1-e^{\frac{A}{\lambda}}) + t^{1-r}\right)^{\frac{r-2}{r-1}}\right)\right),\ 1 < r < 2.\end{aligned}$$

The moments, now, can be obtained by using the following relation between the MGF and moments of the operator V_λ:

$$(V_\lambda e_m)(t) = \left[\frac{\partial^m}{\partial A^m}\left\{\exp\left(\frac{\lambda}{r-2}\left(t^{2-r} - \left((r-1)(1-e^{\frac{A}{\lambda}}) + t^{1-r}\right)^{\frac{r-2}{r-1}}\right)\right)\right\}\right]_{A=0},\ m = 0,1,2,\ldots. \tag{8}$$

□

Remark 2.3. *For $\lambda \in \mathbb{N}$ and $1 < r < 2$, we have $\lim_{\lambda\to\infty}(V_\lambda e_k)(t) = t^k$ for $k = 0,1,2,\ldots$.*

Lemma 2.4. *Let $\mu_{\lambda,m}(t) := (V_\lambda(e_1 - te_0)^m)(t), m \in \mathbb{N}\cup\{0\}$ denote the m-th order central moment for the operator V_λ, then for $\lambda \in \mathbb{N}$ and $1 < r < 2$, we have*

$$\mu_{\lambda,m}(t) = \left[\frac{\partial^m}{\partial A^m}\left\{\exp\left(\frac{\lambda}{r-2}\left(t^{2-r} - \left((r-1)(1-e^{\frac{A}{\lambda}}) + t^{1-r}\right)^{\frac{r-2}{r-1}}\right) - At\right)\right\}\right]_{A=0},\ m = 0,1,2,\ldots. \tag{9}$$

In particular, the first few central moments are

(i) $\mu_{\lambda,0}(t) = 1$,
(ii) $\mu_{\lambda,1}(t) = 0$,
(iii) $\mu_{\lambda,2}(t) = \dfrac{t + t^r}{\lambda}$,
(iv) $\mu_{\lambda,3}(t) = \dfrac{t^2 + rt^{2r} + 3t^{1+r}}{t\lambda^2}$,
(v) $\mu_{\lambda,4}(t) = \dfrac{r(-1+2r)t^{3r} + 3t^{1+2r}(2r + t\lambda) + t^3(1 + 3t\lambda) + t^{2+r}(7 + 6t\lambda)}{t^2\lambda^3}$.

In general, we notice that that for $s \in \mathbb{N}\cup\{0\}$, as $\lambda \to \infty$

$$\mu_{\lambda,s}(t) = O\left(\lambda^{-\lfloor\frac{s+1}{2}\rfloor}\right), \tag{10}$$

where $\lfloor a\rfloor$ denotes the integral part of a.

Remark 2.5. *Presupposing that the operators V_λ are of exponential-type, associated with $p(t) = t + t^r$, where $1 < r < 2$. Then their moments must satisfy the following recurrence relation*

$$(V_\lambda e_{k+1})(t) = \frac{(t + t^r)}{\lambda}(V_\lambda e_k)'(t) + t(V_\lambda e_k)(t),\ k = 0,1,2,3,\ldots,$$

which can be obtained by using the differential equation $\frac{\partial}{\partial t}s_{\lambda,k}(t) = \left(\frac{k-\lambda t}{t+t^r}\right)s_{\lambda,k}(t)$. Here, we observe that the moments obtained in Lemma 2.2, do not satisfy this recurrence relation for 3rd and higher order. Therefore, we conclude that the operators V_λ are not of exponential-type.

3. Estimation of Convergence

In this section, we will give direct theorems using modulus of continuity and K-functional. Let $C_B[0,\infty)$ denote the class of continuous functions, which are bounded on the positive real axis. Also, let $C_B^s[0,\infty)$ denote the class of functions that are in $C_B[0,\infty)$, and whose first s derivatives exist and are bounded, continuous on $[0,\infty)$ for $s = 0,1,2,\dots$.

Theorem 3.1. *If $g \in C_B[0,\infty)$ and $m \geq 1$, then the operator V_λ satisfies the following property with the operator S_λ, defined in* (4)*:*

$$\lim_{\lambda\to\infty} \left(V_{m\lambda} g\left(\lambda u\right)\right)\left(\frac{t}{\lambda}\right) = \left(S_m g(u)\right)(t).$$

Moreover,

$$\lim_{\lambda\to\infty} \left(V_\lambda g\right)(t) = g(t).$$

Proof. Let $\lambda \in \mathrm{N}$ and $1 < r < 2$.

(i) By simple calculations, we have

$$\begin{aligned}\lim_{\lambda\to\infty} \left(V_{m\lambda} \exp_{is\lambda}\right)\left(\frac{t}{\lambda}\right) &= \lim_{\lambda\to\infty} \exp\left(\frac{m\lambda}{r-2}\left(\left(\frac{t}{\lambda}\right)^{2-r} - \left((r-1)\left(1-e^{is/m}\right) + \left(\frac{t}{\lambda}\right)^{1-r}\right)^{\frac{r-2}{r-1}}\right)\right)\\ &= \exp\left(mt(e^{is/m}-1)\right) = \left(S_m \exp_{is}\right)(t),\end{aligned}$$

where $\exp_{is\lambda}(u) = \cos(s\lambda u) + i\sin(s\lambda u)$, $s \in \mathbb{R}$ and $i = \sqrt{-1}$.

(ii) Similarly,

$$\begin{aligned}\lim_{\lambda\to\infty} \left(V_\lambda \exp_{is}\right)(t) &= \exp\left(\frac{\lambda}{r-2}\left(t^{2-r} - \left((r-1)(1-e^{\frac{is}{\lambda}}) + t^{1-r}\right)^{\frac{r-2}{r-1}}\right)\right)\\ &= e^{ist} = \left(Id \exp_{is}\right)(t), \text{ where } Id \text{ is the identity operator.}\end{aligned}$$

Now, the conclusion follows from [3, Theorem 1.1] and [4, Theorem 2.1]. □

Theorem 3.2. *For any function $g \in C_B^2[0,\infty)$ and $1 < r < 2$, we have*

$$\lim_{\lambda\to\infty} \left(V_\lambda g\right)(t) = g(t). \tag{11}$$

Proof. For any function $g \in C_B^2[0,\infty)$, the Taylor's expansion at any point $t \in [0,\infty)$ is given by

$$g\left(\frac{k}{\lambda}\right) = g(t) + \left(\frac{k}{\lambda}-t\right)g'(t) + \frac{1}{2!}\left(\frac{k}{\lambda}-t\right)^2 g''(t) + \left(\frac{k}{\lambda}-t\right)^2 \xi\left(\frac{k}{\lambda};t\right), \tag{12}$$

where ξ is a bounded function and $\xi\left(\frac{k}{\lambda};t\right) = \xi\left(\frac{k}{\lambda}-t\right)$ with $\lim_{u\to 0}\xi(u) = 0$.
Operating V_λ on $g(t)$ and using Lemma 2.4, we get

$$\left(V_\lambda g\right)(t) = g(t) + \frac{(t+t^r)}{2\lambda}g''(t) + \sum_{k=0}^{\infty} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^2 \xi\left(\frac{k}{\lambda};t\right),$$

where $s_{\lambda,k}(t)$ denotes the kernel of the operator V_λ, defined by (5) and (6).
Set the remainder term as

$$R_\lambda^2(t) := \sum_{k=0}^{\infty} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^2 \xi\left(\frac{k}{\lambda};t\right),$$

or

$$R_\lambda^2(t) = \sum_{|\frac{k}{\lambda}-t|\geq\delta} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^2 \xi\left(\frac{k}{\lambda};t\right) + \sum_{|\frac{k}{\lambda}-t|<\delta} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^2 \xi\left(\frac{k}{\lambda};t\right)$$
$$:= A_1 + A_2.$$

ξ, being bounded and $\lim_{u\to 0}\xi(u) = 0$, we have that for a pre-assigned $\epsilon > 0$, there must be a $\delta > 0$ such that $|\xi(u)| \leq \epsilon$ for all u such that $|u| \leq \delta$. For the first term from above, choosing a positive constant M such that $\left|\xi\left(\frac{k}{\lambda};t\right)\right| \leq M$, we get

$$|A_1| \leq M \sum_{|\frac{k}{\lambda}-t|\geq\delta} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^2 = o(\lambda^{-1}),$$

as $\lambda \to \infty$. Moreover, for the second term, by applying Cauchy–Schwarz inequality and using Lemma 2.4,

$$\begin{aligned}|A_2| &\leq \epsilon \sum_{|\frac{k}{\lambda}-t|<\delta} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^2 \\ &\leq \sqrt{\epsilon^2 \sum_{|\frac{k}{\lambda}-t|<\delta} s_{\lambda,k}(t)} \sqrt{\sum_{|\frac{k}{\lambda}-t|<\delta} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^4} \\ &\leq \epsilon\sqrt{\mu_{\lambda,4}(t)} = o(\lambda^{-1}),\end{aligned}$$

as $\lambda \to \infty$.
Thus,

$$\lim_{\lambda\to\infty} R_\lambda^2(t) = 0,$$

and hence, (11) follows. □

Theorem 3.3. *Let $g \in C_B^s[0,\infty)$, then for $s \in \mathbb{N}\cup\{0\}$ and $1 < r < 2$, we have*

$$\lim_{\lambda\to\infty}\left[(V_\lambda g)(t) - \sum_{j=0}^{s} \frac{g^{(j)}(t)}{j!}\mu_{\lambda,j}(t)\right] = 0. \tag{13}$$

Proof. Using Taylor's formula, we have the following expansion for $g(t)$:

$$g\left(\frac{k}{\lambda}\right) = \sum_{j=0}^{s}\left(\frac{k}{\lambda}-t\right)^j \frac{g^{(j)}(t)}{j!} + \left(\frac{k}{\lambda}-t\right)^s \xi\left(\frac{k}{\lambda};t\right), \tag{14}$$

where ξ is, as defined in the previous Theorem 3.2.
Multiplying by $s_{\lambda,k}(t)$ and taking summation over $k \in \mathbb{N}\cup\{0\}$, we get

$$\begin{aligned}(V_\lambda g)(t) &= \sum_{j=0}^{s}\frac{g^{(j)}(t)}{j!}\sum_{k=0}^{\infty} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^j + \sum_{k=0}^{\infty} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^s \xi\left(\frac{k}{\lambda};t\right) \\ &= \sum_{j=0}^{s}\frac{g^{(j)}(t)}{j!}\mu_{\lambda,j}(t) + \sum_{k=0}^{\infty} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^s \xi\left(\frac{k}{\lambda};t\right).\end{aligned} \tag{15}$$

We set the remainder term as

$$\begin{aligned} R_\lambda^s(t) &:= \sum_{k=0}^{\infty} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^s \xi\left(\frac{k}{\lambda};t\right) \\ &= \sum_{|\frac{k}{\lambda}-t|\geq\delta} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^s \xi\left(\frac{k}{\lambda};t\right) + \sum_{|\frac{k}{\lambda}-t|<\delta} s_{\lambda,k}(t)\left(\frac{k}{\lambda}-t\right)^s \xi\left(\frac{k}{\lambda};t\right) \\ &:= B_1 + B_2. \end{aligned}$$

Utilizing the fact that ξ is bounded and $\lim_{u\to 0}\xi(u) = 0$; for a given $\epsilon > 0$, there must be a $\delta > 0$ such that $|\xi(u)| \leq \epsilon$ for all u satisfying $|u| \leq \delta$. We can choose a positive constant D such that $\left|\xi\left(\frac{k}{\lambda};t\right)\right| \leq D$, so that

$$|B_1| \leq D|\mu_{\lambda,s}(t)| = o\left(\lambda^{-\lfloor\frac{s+1}{2}\rfloor}\right),$$

as $\lambda \to \infty$.

Moreover, in view of (10), we have

$$|B_2| \leq \epsilon|\mu_{\lambda,s}(t)| = o\left(\lambda^{-\lfloor\frac{s+1}{2}\rfloor}\right),$$

as $\lambda \to \infty$.

Thus,

$$\lim_{\lambda\to\infty} R_\lambda^s(t) = 0,$$

and hence, (13) follows. □

Corollary 3.4. *Let $g \in C_B^2[0,\infty)$, then for $1 < r < 2$, we have*

$$\lim_{\lambda\to\infty} \lambda[(V_\lambda g)(t) - g(t)] = \frac{(t+t^r)}{2} g''(t). \tag{16}$$

Proof. Substituting $s = 2$ in (15), we have

$$(V_\lambda g)(t) - g(t) = \frac{g''(t)}{2}\mu_{\lambda,2}(t) + o\left(\lambda^{-\lfloor\frac{2+1}{2}\rfloor}\right),$$

as $\lambda \to \infty$. Then, in view of Lemma 2.4, we immediately get (16). □

Let $f \in C_B[0,\infty)$ and $\delta > 0$. Consider Peetre's K functional defined as (see [18])

$$K_m(f;\delta) = \inf_{p\in C_B^m[0,\infty)} \{\|f-p\|_\infty + \delta\|p^{(m)}\|_\infty\},\ m = 1,2,\ldots .$$

Also, we have the following relations (see [5, 14]):

$$C_1\omega_1(f;\sqrt{\delta}) \leq K_1(f;\delta) \leq C_2\omega_1(f;\sqrt{\delta}),$$

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}),$$

where

$$\omega_1(f;\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{t\geq 0} |f(t+h) - f(t)| \tag{17}$$

and

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{t\geq 0} |f(t+2h) - 2f(t+h) + f(t)|$$

are the first- and second-order moduli of continuity [6], respectively and C, C_1 and C_2 are positive constants. For $s \geq 1$, let $f \in C^s(J)$, then the Taylor's series expansion for $t_1, t_2 \in J \subseteq R$ is given by

$$f(t_1) = \sum_{j=0}^{s} \frac{f^{(j)}(t_2)}{j!}(t_1 - t_2)^j + R_s(f; t_1, t_2),$$

and the remainder term $R_s(f; t_1, t_2) := (t_1 - t_2)^s \xi(t_1; t_2)$ satisfies (see [7]):

$$|R_s(f; t_1, t_2)| \leq 2\frac{|t_1 - t_2|^s}{s!} K_1\left(f^{(s)}; \frac{|t_1 - t_2|}{2(s+1)}\right). \tag{18}$$

Now, we present a Voronovskaja-type asymptotic formula with the help of K-functional.

Theorem 3.5. *For $g \in C_B^2[0, \infty)$ and $1 < r < 2$, we have*

$$\left|\lambda\left[(V_\lambda g)(t) - g(t)\right] - \frac{(t + t^r)}{2} g''(t)\right| \leq (t + t^r) K_1\left(g''; \frac{\left|t + rt^{2r-1} + 3t^r\right|}{6\,|t + t^r|\,\lambda}\right). \tag{19}$$

Proof. For $g \in C_B^2[0, \infty)$, we have

$$(V_\lambda g)(t) = g(t) + \frac{g''(t)}{2}\mu_{\lambda,2}(t) + \sum_{k=0}^{\infty} s_{\lambda,k}(t)\left(\frac{k}{\lambda} - t\right)^2 \xi\left(\frac{k}{\lambda}; t\right),$$

where ξ is, as defined in the Theorem 3.2. Let $R_2(g; t_1, t_2) := (t_1 - t_2)^2\, \xi(t_1; t_2)$, then

$$\left|\lambda\left\{(V_\lambda g)(t) - g(t) - \mu_{\lambda,2}(t)\frac{g''(t)}{2}\right\}\right| \leq \lambda \sum_{k=0}^{\infty} s_{\lambda,k}(t)\left|R_2\left(g; t, \frac{k}{\lambda}\right)\right|.$$

In view of (18), we have

$$\left|\lambda\left\{(V_\lambda g)(t) - g(t) - \mu_{\lambda,2}(t)\frac{g''(t)}{2}\right\}\right| \leq \lambda \sum_{k=0}^{\infty} s_{\lambda,k}(t)\left(\frac{k}{\lambda} - t\right)^2 K_1\left(g''; \frac{\left|t - \frac{k}{\lambda}\right|}{6}\right).$$

Let $h \in C_B^3[0, \infty)$ be fixed. Then

$$\begin{aligned}\left|\lambda\left\{(V_\lambda g)(t) - g(t) - \mu_{\lambda,2}(t)\frac{g''(t)}{2}\right\}\right| &\leq \lambda \sum_{k=0}^{\infty} s_{\lambda,k}(t)\left(\frac{k}{\lambda} - t\right)^2 \left\{\|(g - h)''\|_\infty + \frac{\left|t - \frac{k}{\lambda}\right|}{6}\|h'''\|_\infty\right\} \\ &= \lambda\|(g - h)''\|_\infty \left|\mu_{\lambda,2}(t)\right| + \frac{\lambda\|h'''\|_\infty}{6}\left|\mu_{\lambda,3}(t)\right|.\end{aligned}$$

Taking the infimum over $h'' \in C_B^1[0, \infty)$,

$$\left|\lambda\left\{(V_\lambda g)(t) - g(t) - \mu_{\lambda,2}(t)\frac{g''(t)}{2}\right\}\right| \leq \lambda\left|\mu_{\lambda,2}(t)\right| K_1\left(g''; \frac{\left|\mu_{\lambda,3}(t)\right|}{6\left|\mu_{\lambda,2}(t)\right|}\right),$$

and so the result (19) follows. □

Theorem 3.6. *For $g \in C_B^s[0, \infty)$ and $1 < r < 2$, we have*

$$\left|(V_\lambda g)(t) - g(t) - \sum_{j=1}^{s} \frac{g^{(j)}(t)}{j!}\mu_{\lambda,j}(t)\right| \leq \frac{2\left|\mu_{\lambda,s}(t)\right|}{s!} K_1\left(g^{(s)}; \frac{\left|\mu_{\lambda,s+1}(t)\right|}{2(s+1)\left|\mu_{\lambda,s}(t)\right|}\right). \tag{20}$$

Proof. By using the Taylor's formula upto s-th order terms,

$$\left|\left(V_\lambda g\right)(t)-g(t)-\sum_{j=1}^{s}\frac{g^{(j)}(t)}{j!}\mu_{\lambda,j}(t)\right|\leq\sum_{k=0}^{\infty}s_{\lambda,k}(t)\left|R_s\left(g;t,\frac{k}{\lambda}\right)\right|,$$

where $R_s(g;t_1,t_2):=(t_1-t_2)^s\xi(t_1;t_2)$. In accordance with (18), we have

$$\left|\left(V_\lambda g\right)(t)-g(t)-\sum_{j=1}^{s}\frac{g^{(j)}(t)}{j!}\mu_{\lambda,j}(t)\right|\leq\sum_{k=0}^{\infty}s_{\lambda,k}(t)2\frac{\left|t-\frac{k}{\lambda}\right|^s}{s!}K_1\left(g^{(s)};\frac{\left|t-\frac{k}{\lambda}\right|}{2(s+1)}\right).$$

Let $h\in C_B^{s+1}[0,\infty)$ be fixed. Then

$$\begin{aligned}\left|\left(V_\lambda g\right)(t)-g(t)-\sum_{j=1}^{s}\frac{g^{(j)}(t)}{j!}\mu_{\lambda,j}(t)\right|&\leq2\sum_{k=0}^{\infty}s_{\lambda,k}(t)\frac{\left|t-\frac{k}{\lambda}\right|^s}{s!}\left\{\left\|(g-h)^{(s)}\right\|_\infty+\frac{\left|t-\frac{k}{\lambda}\right|}{2(s+1)}\left\|h^{(s+1)}\right\|_\infty\right\}\\&=2\frac{\|(g-h)^{(s)}\|_\infty}{s!}\left|\mu_{\lambda,s}(t)\right|+\frac{\|h^{(s+1)}\|_\infty}{(s+1)!}\left|\mu_{\lambda,s+1}(t)\right|.\end{aligned}$$

Taking the infimum over $h^{(s)}\in C_B^1[0,\infty)$ gives the desired result (20). □

Theorem 3.7. *Let $g\in C^2[0,\infty)$ be defined on a compact interval $[0,b]\subset[0,\infty)$ with $b>0$, then for $1<r<2$,*

$$\left|\left(V_\lambda g\right)(t)-g(t)\right|\leq(1+b+b^r)\,\omega_1\left(g;\frac{1}{\sqrt{\lambda}}\right),$$

and

$$\begin{aligned}&\lambda\left|(V_\lambda g)(t)-\sum_{j=0}^{2}\frac{g^{(j)}(t)}{j!}\mu_{\lambda,j}(t)\right|\\&\leq\frac{1}{2}\left(b+b^r+r(2r-1)b^{3r-2}+6rb^{2r-1}+3b^{2r}+b+3b^2+7b^r+6b^{1+r}\right)\omega_1\left(g'';\frac{1}{\sqrt{\lambda}}\right),\end{aligned}$$

where $C^2[0,\infty)$ is the space of functions f such that f,f' and f'' are continuous on $[0,\infty)$. Also, ω_1 is the first order modulus of continuity, defined in (17).

Proof. Let $T_{\lambda,p}(t):=\lambda^p\mu_{\lambda,p}(t),p=0,1,2,\dots$, then we have $\lim_{\lambda\to\infty}\frac{T_{\lambda,2}(t)}{\lambda}=t+t^r$ and $\lim_{\lambda\to\infty}\frac{T_{\lambda,4}(t)}{\lambda^2}=3t^{2r}+3t^2+6t^{1+r}$. Also, for $t\in[0,b]$, we have the bounds $\frac{T_{\lambda,2}(t)}{\lambda}\leq b+b^r$ and $\frac{T_{\lambda,4}(t)}{\lambda^2}\leq r(2r-1)b^{3r-2}+6rb^{2r-1}+3b^{2r}+b+3b^2+7b^r+6b^{1+r}$. Utilizing the above terms and following Pop [19, Theorem 3.1], we get the desired result. □

4. Further composition

The composition of the operators V_λ, defined in (3) and Szász-Mirakjan operators, defined in (4) yields a new operator, which we denote by $\widetilde{V}_\lambda$, so that

$$\left(\widetilde{V}_\lambda g\right)(t):=\left(V_\lambda\circ S_\lambda g\right)(t).$$

Lemma 4.1. *The MGF of the operators $\widetilde{V}_\lambda$ is given as*

$$\left(\widetilde{V}_\lambda \exp_A\right)(t) = \exp\left(\frac{\lambda}{r-2}\left(t^{2-r} - \left((r-1)\left(1-e^{e^{A/\lambda}-1}\right) + t^{1-r}\right)^{\frac{r-2}{r-1}}\right)\right).$$

Moreover, let us denote the m-th order moment as $\left(\widetilde{V}_\lambda e_m\right)(t)$, where $e_m(u) := u^m,\ m = 0,1,2,\ldots$, then the moments are obtained by applying the formula (8) for $\widetilde{V}_\lambda$, as in Lemma 2.2. First few moments are as follows:

(i) $\left(\widetilde{V}_\lambda e_0\right)(t) = 1,$

(ii) $\left(\widetilde{V}_\lambda e_1\right)(t) = t,$

(iii) $\left(\widetilde{V}_\lambda e_2\right)(t) = t^2 + \frac{2t}{\lambda} + \frac{t^r}{\lambda},$

(iv) $\left(\widetilde{V}_\lambda e_3\right)(t) = t^3 + \frac{5t + 6t^r + rt^{2r-1}}{\lambda^2} + \frac{6t^2 + 3t^{1+r}}{\lambda}.$

Lemma 4.2. *Let us denote the central moment of m-th order for the operators $\widetilde{V}_\lambda$ by $\tilde{\mu}_{\lambda,m}(t) := \left(\widetilde{V}_\lambda (e_1 - te_0)^m\right)(t), m \in \mathbb{N} \cup \{0\}$. These are obtained by applying the formula (9) for $\widetilde{V}_\lambda$. First few central moments are*

(i) $\tilde{\mu}_{\lambda,0}(t) = 1,$

(ii) $\tilde{\mu}_{\lambda,1}(t) = 0,$

(iii) $\tilde{\mu}_{\lambda,2}(t) = \frac{2t + t^r}{\lambda},$

(iv) $\tilde{\mu}_{\lambda,3}(t) = \frac{5t + rt^{2r-1} + 6t^r}{\lambda^2}.$

The corresponding approximation results for the operators $\widetilde{V}_\lambda$ are outlined below, with formulations similar to the Theorem 3.2 and Theorem 3.5 for the operators V_λ, respectively:

Theorem 4.3. *For any function $g \in C_B^2[0,\infty)$ and $1 < r < 2$, we have*

$$\lim_{\lambda\to\infty} \left(\widetilde{V}_\lambda g\right)(t) = g(t). \tag{21}$$

Theorem 4.4. *For $g \in C_B^2[0,\infty)$ and $1 < r < 2$, we have*

$$\left|\lambda\left[\left(\widetilde{V}_\lambda g\right)(t) - g(t)\right] - \frac{(2t+t^r)}{2} g''(t)\right| \leq (2t+t^r) K_1\left(g''; \frac{\left|5t + rt^{2r-1} + 6t^r\right|}{6\left|2t + t^r\right|\lambda}\right). \tag{22}$$

The composition of the operators $\widetilde{V}_\lambda$ and Szász-Mirakjan operators yields a new operator, which we denote by $\widetilde{\widetilde{V}}_\lambda$, so that

$$\left(\widetilde{\widetilde{V}}_\lambda g\right)(t) := (\widetilde{V}_\lambda \circ S_\lambda g)(t).$$

Lemma 4.5. *The MGF of the operators $\widetilde{\widetilde{V}}_\lambda$ is given as*

$$\left(\widetilde{\widetilde{V}}_\lambda \exp_A\right)(t) = \exp\left(\frac{\lambda}{r-2}\left(t^{2-r} - \left((r-1)\left(1-e^{e^{e^{A/\lambda}-1}-1}\right) + t^{1-r}\right)^{\frac{r-2}{r-1}}\right)\right).$$

Also, let the m-th order moment be denoted by $\left(\widetilde{\widetilde{V}}_\lambda e_m\right)(t)$, where $e_m(u) := u^m,\ m = 0,1,2,\ldots$. First few moments are as follows:

(i) $\left(\widetilde{\widetilde{V}}_\lambda e_0\right)(t) = 1,$

(ii) $\left(\widetilde{\widetilde{V}}_\lambda e_1\right)(t) = t,$

(iii) $\left(\widetilde{\widetilde{V}}_\lambda e_2\right)(t) = t^2 + \frac{3t}{\lambda} + \frac{t^r}{\lambda},$

(iv) $\left(\widetilde{\widetilde{V}}_\lambda e_3\right)(t) = t^3 + \frac{12t + 9t^r + rt^{2r-1}}{\lambda^2} + \frac{9t^2 + 3t^{1+r}}{\lambda}.$

Lemma 4.6. *Let the central moment of m-th order for the operators $\widetilde{\widetilde{V}}_\lambda$ be denoted by $\tilde{\tilde{\mu}}_{\lambda,m}(t) := \left(\widetilde{\widetilde{V}}_\lambda (e_1 - te_0)^m\right)(t), m \in \mathbb{N} \cup \{0\}$. First few central moments are*

(i) $\tilde{\tilde{\mu}}_{\lambda,0}(t) = 1,$

(ii) $\tilde{\tilde{\mu}}_{\lambda,1}(t) = 0,$

(iii) $\tilde{\tilde{\mu}}_{\lambda,2}(t) = \frac{3t + t^r}{\lambda},$

(iv) $\tilde{\tilde{\mu}}_{\lambda,3}(t) = \frac{9t^r + 12t + rt^{2r-1}}{\lambda^2}.$

Theorem 4.7. *For any function $g \in C_B^2[0,\infty)$ and $1 < r < 2$, we have*

$$\lim_{\lambda\to\infty} \left(\widetilde{\widetilde{V}}_\lambda g\right)(t) = g(t). \tag{23}$$

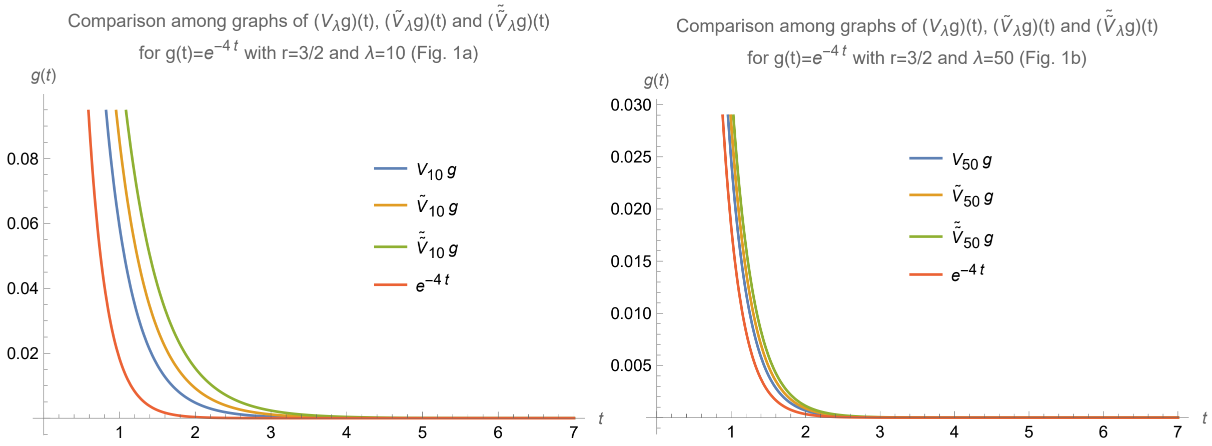
Theorem 4.8. *For $g \in C_B^2[0,\infty)$ and $1 < r < 2$, we have*

$$\left|\lambda\left[\left(\widetilde{\widetilde{V}}_\lambda g\right)(t) - g(t)\right] - \frac{(3t + t^r)}{2} g''(t)\right| \leq (3t + t^r) K_1\left(g''; \frac{|9t^r + 12t + rt^{2r-1}|}{6|3t + t^r|\lambda}\right). \tag{24}$$

Remark 4.9. *Upon comparison of the approximation properties of the operators $V_\lambda, \widetilde{V}_\lambda$ and $\widetilde{\widetilde{V}}_\lambda$, it is evident from* (19), (22) *and* (24) *that higher order compositions produce less precise approximations.*

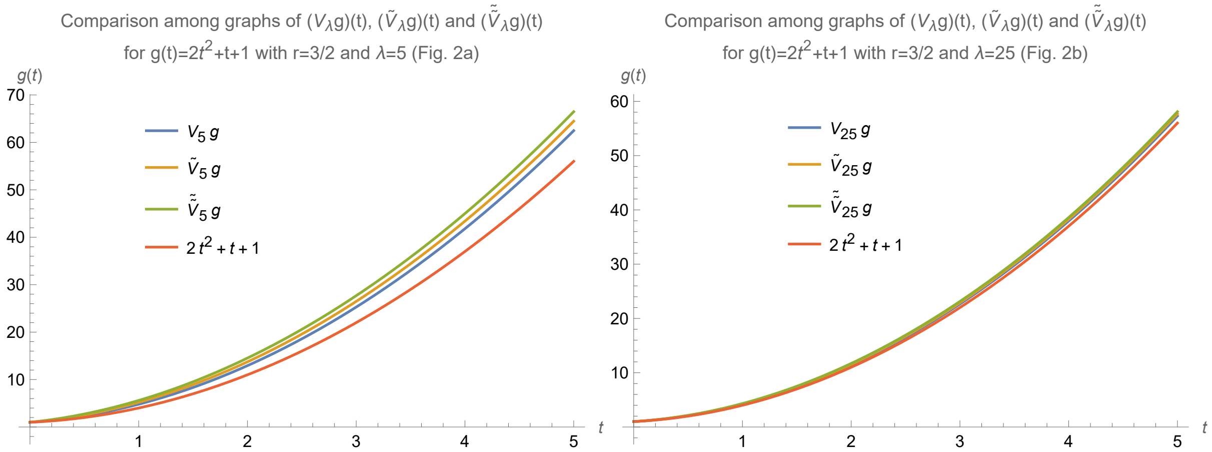
5. Graphical comparison among V_λ, $\widetilde{V}_\lambda$ and $\widetilde{\widetilde{V}}_\lambda$

Here, we provide a comparison among the approximations of functions through the operators $V_\lambda, \widetilde{V}_\lambda$ and $\widetilde{\widetilde{V}}_\lambda$, with the help of following graphs.
In Figure 1, the graphs (Fig. 1a, Fig. 1b) present the comparison among the approximations of $g(t) = e^{-4t}$ by these operators. Likewise, in Figure 2, the graphs (Fig. 2a, Fig. 2b) present the comparison among the approximations of $g(t) = 2t^2 + t + 1$. We observe that V_λ provides better approximation compared to $\widetilde{V}_\lambda$ and $\widetilde{\widetilde{V}}_\lambda$ for both functions. Moreover, as λ increases, the approximations become more precise.



(a) 1a (b) 1b

Figure 1: Comparison among the graphs of V_λ, $\widetilde{V}_\lambda$ and $\widetilde{\widetilde{V}}_\lambda$ for $g(t) = e^{-4t}$



(a) 2a (b) 2b

Figure 2: Comparison among the graphs of V_λ, $\widetilde{V}_\lambda$ and $\widetilde{\widetilde{V}}_\lambda$ for $g(t) = 2t^2 + t + 1$

6. Conclusion

In this paper, we introduced composition operator of certain class of exponential operators with Szász-Mirakjan operators. We estimated their moments and gave the approximation properties of the new operators by means of the modulus of continuity and Peetre's K- functional. We also pointed out that the new operator is not of exponential type. Also, the Voronovskaja-type results were discussed. Finally, we discussed further compositions and compared the approximation properties of these operators with the help of graphs. We observed that higher order compositions increase the approximation error.

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