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On the regional controllability and observability for infinite-dimensional conformable systems

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Abstract. This paper focuses on studying regional controllability and observability for fractional linear systems using the conformable derivative of order $0 < \alpha \leq 1$. Utilizing fractional calculus theory and fractional semigroup theory, we provide the necessary and sufficient conditions for exact and approximate regional controllability and observability of the conformable fractional linear system. Finally, we present some examples to illustrate our theoretical results.

1. Introduction

Fractional calculus is a branch of mathematics that aims to extend the concept of classical differentiation to non-integer orders. This field has evolved into one of the most developed areas of mathematical analysis and has proven to be a powerful tool in modeling various phenomena across several fields [11]. Consequently, numerous definitions for fractional derivatives have emerged in the literature to provide more accurate models for real-life phenomena. Some of the most well-known fractional derivatives include Riemann-Liouville, Caputo, Hadamard, Erdèlyi-Kober, Riesz, Grünwald-Letnikov, Marchaud, and others(see [28, 33]).

A recent advancement in fractional calculus, known as the conformable derivative and proposed by Khalil et al. [23], is described by the following formula:

$$\phi^{(\rho)}(s) = \lim_{\mu \to 0} \frac{\phi(s + \mu t^{1-\rho}) - \phi(s)}{\mu}$$

This new fractional derivative has significantly contributed to various disciplines, including engineering, finance, biology, medicine, physics, and applied mathematics [7, 29, 35, 40].

The choice of the conformable derivative in this study is motivated by several key factors. Firstly, the conformable derivative preserves many fundamental properties of classical derivatives, such as the product rule, quotient rule, chain rule, Rolle's theorem, and the mean value theorem. These properties make the conformable derivative particularly attractive for extending classical analysis techniques to the fractional domain. Additionally, the conformable derivative has a straightforward and intuitive definition, which

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simplifies its application across various fields. This simplicity does not compromise its ability to model complex dynamic systems with memory effects and hereditary properties, which are essential in many real-world scenarios. Moreover, the conformable derivative has been shown to be highly effective in control theory. It provides a natural and cohesive framework for addressing issues of controllability, observability, stability, and optimization in both finite and infinite-dimensional systems. This is evident from recent studies that have successfully applied the conformable derivative to various control problems, demonstrating its versatility and robustness. Given these advantages, the conformable derivative is particularly well-suited for our investigation into regional controllability and observability of infinite-dimensional systems. Its properties and ease of use allow us to extend classical control theory results to the fractional context in a coherent manner, thus providing a solid foundation for our theoretical and practical contributions. These properties have been examined by numerous researchers, including those cited in [1, 6, 24, 32], among many others.

In recent years, conformable derivatives have emerged as a competing means of studying control systems. As a result, extensive research has focused on control theory, particularly with respect to the fundamental concepts and classical findings of dynamical systems described by the conformable derivative. This research encompassing topics such as solving the Cauchy problem, applying Gronwall's lemma, studying controllability, observability, stability, stabilization, and solving optimal linear quadratic problems.

The study of conformable fractional dynamical systems in finite dimensions has yielded intriguing results. For example, the solutions of conformable linear and semilinear systems have been treated in several works (see [3, 26]). Several research studies have been conducted to examine the controllability and observability of conformable fractional systems within the framework of finite-dimensional state space. For instance, Xiaowen et al. in [34] have provided sufficient and necessary conditions for the null controllability of conformable linear systems, and in [4], based on the conformable exponential matrix, Al-Zhour discussed the controllability and observability behaviors of a non-homogeneous conformable fractional dynamic system. To explore further, there are additional findings on the controllability and observability of conformable systems (see [8, 36]). We can also cite some works on the stability problems of conformable fractional linear and nonlinear systems [22, 25, 27, 39]. The problem of fractional linear-quadratic optimization for systems governed by conformable fractional derivatives is also a subject of investigation i.e, see [10].

In recent years, researchers have shown a particular interest in conformable fractional dynamic systems in an infinite-dimensional setting. For example, noteworthy works include those of Jaiswal and Bahuguna mentioned in [20], and Rabhi et al. in [31], who provided the solution of the conformable fractional abstract initial value problem using the results discussed by Abdeljawad [1] regarding the theory of conformable fractional groups. Along the same lines, Al Sharif et al. have notably contributed by extending the classical Hille-Yosida theorem to fractional semigroups of operators acting on Hilbert spaces, as exposed in [2]. Das [12] studied the exact controllability of a class of semilinear systems modeled by conformable fractional derivatives of order ($1 < \alpha \leq 2$).

Recently, Ennouari et al. [16, 17] utilized conformable calculus, specifically the notion of fractional semigroups, to establish exact and approximate controllability and observability conditions for conformable linear systems in an infinite-dimensional setting, represented by the following equations:

$$\begin{cases} \omega^{(\alpha)}(t) = \Lambda\omega(t) + \Gamma u(t) \\ z(t) = C\omega(t) \end{cases} \tag{1}$$

These conditions encompass both necessary and sufficient criteria. Their work also presented examples where controllability and observability were not attained in both the exact and approximate senses. Furthermore, recent research by Jneid et al. [21] delved into the partial approximate controllability of fractional control systems in Hilbert spaces utilizing conformable derivatives. Such findings are not uncommon, as in classical cases, the studied concepts are not always realized. Hence, exploring alternative methods to guide the state of systems towards desired states and to reconstruct the state of systems based on observed states becomes a natural course of action.

In this paper, we delve into the regional observability and controllability of conformable fractional dynamical linear systems within an infinite-dimensional context. Regional controllability entails the capability to guide a dynamical system from an initial state to a desired final state within a specified time frame using admissible controls limited to a subregion of its evolution domain. Initially addressed by El Jai et al. [14], this notion holds paramount importance in practical applications. For instance, consider the regional controllability scenario in an industrial furnace, where control is tasked with maintaining temperature within a prescribed subregion of the furnace.

Conversely, regional observability pertains to estimating and reconstructing the initial state of a particular system over a designated subregion ω within the larger system domain Ω. Introduced by El Jai et al. in their seminal works [5, 13], this concept emerged to address practical challenges encountered in real-world applications. One such challenge is exemplified by the energy exchange problem, which involves determining energy transfer within a plasma striking a flat target oriented perpendicular to the flow direction, as detailed in [37].

As evident from the cited literature [16, 17], it is increasingly evident that many conformable systems cannot be fully observed and controlled throughout their evolution. This observation motivates a deeper exploration of regional controllability and observability for fractional systems utilizing the conformable derivative. Several researchers have investigated these notions using various fractional derivatives, including the Riemann-Liouville, Caputo, and others [9, 15, 18, 19, 38].

The structure of our paper is as follows: Section 2 provides a review of basic definitions and preliminary results on conformable derivatives. In Section 3, we present necessary and sufficient conditions to establish the regional controllability of conformable linear systems. Section 4 outlines the necessary and sufficient conditions for regional observability. To illustrate the effectiveness of our theoretical approach, several examples are included in Section 5. Finally, Section 6 summarizes the main conclusions of our research.

2. Preliminary concepts

The purpose of this section is to give a brief overview of some concepts and results of the conformable fractional derivative theory. We begin by defining the conformable fractional derivative

Definition 2.1. *Let X be a Banach space. The conformable fractional derivative of a X valued function $\phi : [0, +\infty[\longrightarrow X$ of order $\alpha \in]0, 1]$, at $t > 0$ is defined as follows*

$$\phi^{(\alpha)}(t) = \lim_{\epsilon \to 0} \frac{\phi(t + \epsilon t^{1-\alpha}) - \phi(t)}{\epsilon} \tag{2}$$

Furthermore, we say ϕ is $\alpha-$differentiable at t when the limit exists.

If ϕ is $\alpha-$differentiable in some interval $]0, a]$, $a > 0$ and $\lim_{t \to 0^+} \phi^{(\alpha)}(t)$ exists in X, we define $\phi^{(\alpha)}(0)$ by $\phi^{(\alpha)}(0) = \lim_{t \to 0^+} \phi^{(\alpha)}(t)$.
The following result is established in [31].

Theorem 2.2. *If a function $\phi : [0, +\infty[\longrightarrow X$ is $\alpha-$differentiable at $t_0 > 0$, $\alpha \in]0, 1]$, then ϕ is continuous at t_0. In addition, if ϕ is differentiable, then $\phi^{(\alpha)}(t) = t^{1-\alpha} \dfrac{d\phi}{dt}(t)$.*

The $\alpha-$fractional integral of a function ϕ is given by

$$I^a_\alpha(\phi)(t) = \int_a^t \phi(s) d_\alpha s = \int_a^t \frac{\phi(s)}{s^{1-\alpha}} ds \text{ where } a \geq 0 \tag{3}$$

In the upcoming discussion, we expound upon the concept of the C_0-α-semigroup, an idea originally introduced by Abdeljawad et al. in their work [1].

Definition 2.3. *Let X be a Banach space and $\alpha \in]0, 1]$. A family of bounded linear operators $(S(t))_{t \geq 0}$ on X, is called a fractional C_0-α-semigroup of operators if*

1. $S(0) = I$,

2. $S(t+s)^{\frac{1}{\alpha}} = S(t^{\frac{1}{\alpha}})S(s^{\frac{1}{\alpha}}),\ t,s \geq 0$
3. $\lim_{t\to 0^+} S(t)x = x$, *pour tout* $x \in X$.

If $\alpha = 1$, *then the* $1-$*semigroup is just the usual* C_0-*semigroup.*

Definition 2.4. *Let* $(S(t))_{t\geq 0}$ *be a* C_0-α-*semigroup on a Banach space* X. *We call the* α-*infinitesimal generator of* $(S(t))_{t\geq 0}$ *the operator* $(A, D(A))$ *given by*

$$\begin{aligned} D(A) &= \{x \in X \lim_{t\to 0^+} S^{(\alpha)}(t)x \quad \textit{exists}\} \\ Ax &= \lim_{t\to 0^+} S^{(\alpha)}(t)x,\ x \in D(A) \end{aligned}$$

The following theorems are shown in [31]

Theorem 2.5. *Let* $(S(t))_{t\geq 0}$ *be a* C_0-α-*semigroup on the Banach space* X *and let* A *be its* α-*infinitesimal generator. Then*

1. *For* $x \in X$
$$\lim_{\epsilon\to 0}\int_t^{t+\epsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} S(s)xds = S(t)x \quad \textit{for every } t > 0.$$
2. *For* $x \in X$, $\int_0^t \frac{1}{s^{1-\alpha}} S(s)xds \in D(A)$ *and*
$$A\Big(\int_0^t \frac{1}{s^{1-\alpha}} S(s)xds\Big) = S(t)x - x$$
3. *For* $x \in D(A)$, $S(t)x \in D(A)$ *and*
$$S^{(\alpha)}(t)x = AS(t)x = S(t)Ax.$$
4. *For* $x \in D(A)$
$$\begin{aligned} S(t)x - S(s)x &= \int_s^t \frac{1}{u^{1-\alpha}} S(u)Axdu \\ &= \int_s^t \frac{1}{u^{1-\alpha}} AS(u)xdu \end{aligned}$$

Theorem 2.6. *Let* $((S(t))_{t\geq 0}$ *be a* C_0-α-*semigroup on the Banach space* X. *There exist constants* $M \geq 1$ *and* $\omega \geq 0$ *such that*
$$\|S(t)\| \leq Me^{\omega t^{\alpha}}.$$

Corollary 2.7. *If* $(S(t))_{t\geq 0}$ *be a* $C_0 - \alpha-$*semigroup, then for every* $x \in X$, $t \to S(t)x$ *is a continuous function from* $[0, +\infty[$ *into* X.

Next, We will recall the following space:

$$L^p_\alpha([0,a];X) := \{\phi : [0,a] \to X$$
$$\text{is measurable function such that } \int_0^a \|\phi(s)\|^p d_\alpha s < \infty\}$$

Under the norm,

$$\|\phi\|_\alpha = \Big(\int_0^a \|\phi(s)\|^p d_\alpha s\Big)^{\frac{1}{p}} \tag{4}$$

$L^p_\alpha([0,a];X)$ is a Banach space.
The space $L^2_\alpha([0,a];X)$ is a Hilbert space with the inner product

$$< f, g >:= \int_0^a f(s)g(s)d_\alpha s,\ f, g \in L^2_\alpha([0,a];X)$$

3. Regional controllability

In this section, we study the regional controllability of conformable linear systems. We start by describing the conformable linear system and recalling its mild solution. We consider the following abstract conformable fractional linear system:

$$\begin{cases} x^{(\alpha)}(t) = Ax(t) + Bu(t) \quad t \in I = [0, t_f] \\ x(0) = x_0 \in D(A) \end{cases} \tag{5}$$

Where, $x^{(\alpha)}$ is the conformable derivative of x of order α, A is the α−infinitesimal generator of a C_0-α-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space $X = L^2_\alpha(\Omega)$ (where Ω be an open bounded set of $\mathbb{R}^n$)
The operator B is defined as a bounded linear operator from the control space U to X, i.e., $B \in \mathcal{L}(U, X)$, where U is a Hilbert space representing the control input space.
Under these conditions, the conformable linear system (5) admits a mild solution which is written as

$$x(t) = T(t)x_0 + \int_0^t T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} Bu(s) d_\alpha s \tag{6}$$

(see [31])
In this section, our aim is to identify conditions that ensure regional controllability of the system described by equation (5).

To begin, let's establish the definition of regional controllability. For this purpose, let ω be a subregion of Ω, and we'll define the restriction operator in ω as follows

$$\begin{aligned} \chi_\omega : L^2_\alpha(\Omega) &\longrightarrow L^2_\alpha(\omega) \\ x &\longrightarrow x_{|_\omega}. \end{aligned} \tag{7}$$

The adjoint operator of χ_ω can be expressed as

$$(\chi^*_\omega x)(z) := \begin{cases} x(z), & z \in \omega; \\ 0, & z \in \Omega \setminus \omega. \end{cases}$$

Definition 3.1. 1. *The system (5) is said to be exactly regionally controllable to $x_d \in L^2_\alpha(\omega)$ if there exists a control $u \in U$ such that*

$$\chi_\omega x(t_f) = x_d \tag{8}$$

2. *The system (5) is said to be approximately regionally controllable to $x_d \in L^2_\alpha(\omega)$ if, for all $\varepsilon > 0$, there exists a control $u \in U$ such that*

$$\|\chi_\omega x(t_f) - x_d\|_{L^2_\alpha(\omega)} \leq \varepsilon \tag{9}$$

3. *The system (5) is said to be exactly (approximately) regionally controllable if it is regionally controllable to all (a dense set of) $L^2_\alpha(\omega)$.*

Let

$$H : L^2_\alpha([0, t_f], \mathbb{R}^n) \longrightarrow X \tag{10}$$

to be defined as

$$Hu = \int_0^t T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} Bu(s) d_\alpha s \qquad for\ t > 0. \tag{11}$$

The following result is crucial in the following paragraphs; it is provided in [16]

Lemma 3.2. *The operator H satisfies the following properties:*

1. $H \in \mathcal{L}(L^2_\alpha([0,t_f];V),X)$
2. $(H^*x)(s) = B^*T((t_f^\alpha - s^\alpha)^{\frac{1}{\alpha}})x$ *for* $s \in [0,t_f]$
3. $(HH^*x)(t) = \int_0^{t_f} T((t_f^\alpha - s^\alpha)^{\frac{1}{\alpha}})BB^*T^*((t_f^\alpha - s^\alpha)^{\frac{1}{\alpha}})x d_\alpha s$

By using the properties of the operator H given in Lemma 3.2, we obtain the following result, which provides a detailed characterization of regional controllability (exact and approximate) for the conformable system described by equation (5).

Theorem 3.3. *The following properties are equivalent:*

1. *The system (5) is exactly regionally controllable on ω at time t_f.*
2. *$Im\, \chi_\omega H = L^2_\alpha(\omega)$.*
3. *$ker\, \chi_\omega + Im\, H = X$.*
4. *For every $x \in L^2_\alpha(\omega)$, there exists a $\delta > 0$ such that*

$$\|x\|_{L^2_\alpha(\omega)} \leq \delta \|H^*\chi_\omega^* x\|_{L^2_\alpha([0,t_f];\mathbb{R}^n)}. \tag{12}$$

Proof. It is easy to see that (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (3): For any $x \in L^2_\alpha(\omega)$, let $\tilde{x}$ be its extension to $L^2_\alpha(\Omega)$. Since $\mathrm{Im}\, \chi_\omega H = L^2_\alpha(\omega)$, there exist $u \in L^2_\alpha([0,t_f];\mathbb{R}^n)$ and $x_1 \in \ker \chi_\omega$ such that $\tilde{x} = x_1 + Hu$.

(3) $\Rightarrow$ (2): For any $\tilde{x} \in X$, decompose $\tilde{x} = x_1 + x_2$ where $x_1 \in \ker \chi_\omega$ and $x_2 \in \mathrm{Im}\, H$. Then there exists $u \in L^2_\alpha([0,t_f];\mathbb{R}^n)$ such that $Hu = x_2$. Hence, by definition of χ_ω, $\mathrm{Im}\, \chi_\omega H = L^2_\alpha(\omega)$.

The following general result [30] can be used to deduce the equivalence between (1) and (4):

Let E, F, and G be reflexive Hilbert spaces, and $f \in \mathcal{L}(E,G)$, $g \in \mathcal{L}(E,G)$. The following properties are equivalent:

1. $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$,
2. There exists $\delta > 0$ such that $\|f^*x^*\|_{E^*} \leq \delta \|g^*x^*\|_{F^*}$ for all $x^* \in G$.

By choosing $E = G = L^2_\alpha(\omega)$, $F = L^2_\alpha([0,t_f];\mathbb{R}^n)$, $f = \mathrm{Id}_{L^2_\alpha(\omega)}$, and $g = \chi_\omega H$, we conclude the proof. □

Theorem 3.4. *The following properties are equivalent:*

1. *The system (5) is approximately regionally controllable on ω at time t_f.*
2. *$\overline{Im\chi_\omega H} = L^2_\alpha(\omega)$.*
3. *$ker\chi_\omega + \overline{ImH} = X$*
4. *The operator $\chi_\omega HH^*\chi_\omega^*$ is positive definite.*

Proof. According to Theorem 3.3, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Finally, we demonstrate that (2) $\Leftrightarrow$ (4). Indeed, it is widely acknowledged that

$$\overline{Im\chi_\omega H} = L^2_\alpha(\omega) \Leftrightarrow < \chi_\omega Hu, x >= 0,\ \forall u \in L^2_\alpha([0,t_f],\mathbb{R}^n) \text{ implies } x = 0.$$

Let $u = H^*\chi_\omega^* x$. After that, we notice that

$$\overline{Im\chi_\omega H} = L^2_\alpha(\omega) \Leftrightarrow < \chi_\omega HH^*\chi_\omega^* x, x >= 0, \text{ implies } x = 0,\ \forall x \in L^2_\alpha(\omega)$$

In other words, the operator $\chi_\omega HH^*\chi_\omega^*$ is positive definite, and the proof is complete. □

Remark 3.5. *the results presented above demonstrate that regional controllability can be ensured under the specified conditions, thus offering a new method to target specific subsets of the state space in fractional dynamic systems.*

In the following section, we will delve into the specific conditions that are both necessary and sufficient for achieving regional observability. This exploration will emphasize how crucial and practical this concept is in gaining a deep understanding of intricate systems and effectively steering them.

4. Regional observability

In this section, we will focus on the regional reconstruction of the initial state of linear systems given in the following form

$$\begin{cases} x^{(\alpha)}(t) = Ax(t) \quad t \in I = [0, t_f];\ x(0) = x_0 \in D(A) \\ y(t) = Cx(t) \end{cases} \tag{13}$$

where X and Y are Banach spaces, and $C : X \to Y$ is a bounded linear operator. In the following, we assume that x_0 is unknown on ω, and therefore the goal of regional observability is to reconstruct x_0 in ω from the output equation. Now, let

$$K : L^2_\alpha(\Omega) \to L^2_\alpha(I; Y)$$

defined by $K(t) = CT(t)$ and we have;

$$K^* y = \int_0^{t_f} T^*(s)C^*(s)y(s)d_\alpha s$$

Definition 4.1. 1. *The system* (13) *is said to be exactly regionally observable if*

$$\operatorname{Im} \chi_\omega K^* = L^2_\alpha(\omega) \tag{14}$$

2. *The system* (13) *is said to be approximately regionally observable if*

$$\overline{\operatorname{Im} \chi_\omega K^*} = L^2_\alpha(\omega) \tag{15}$$

Since A^*, the adjoint operator of A, generates the C_0-α-semigroup $T^*(t)$ (where $T^*(t)$ is simply the adjoint of $T(t)$ for all $t \geq 0$) on the Hilbert space X (see [2]). Ennouari et al. demonstrated in [17] that observability of the systems (13) is equivalent to controllability of the following system:

$$\begin{cases} \tilde{x}^{(\alpha)}(t) = A^*\tilde{x}(t) + C^*v(t),\ 0 \leq t \leq t_f \\ \tilde{x}(0) = x_0 = 0, \end{cases} \tag{16}$$

Building upon this equivalence and the results obtained in the previous section, we derive the following characterizations for the two notions of regional observability defined earlier

Corollary 4.2. *The following properties are equivalent:*

1. *The system* (13) *is said to be exactly regionally observable on ω at time t_f.*
2. $\operatorname{Im} \chi_\omega K^* = L^2_\alpha(\omega)$.
3. $\ker \chi_\omega + \operatorname{Im} K^* = Y$.
4. *For $x \in L^2_\alpha(\omega)$, if there exists a positive constant δ_ω such that*

$$\|x\|_{L^2_\alpha(\omega)} \leq \delta_\omega \|K\chi^*_\omega x\|_{L^2_\alpha(I;Y)} \tag{17}$$

Corollary 4.3. *The following properties are equivalent:*

1. *The system* (13) *is said to be approximately regionally observable on ω at time t_f.*
2. $\overline{\operatorname{Im} \chi_\omega K^*} = L^2_\alpha(\omega)$.
3. $\ker \chi_\omega + \overline{\operatorname{Im} K^*} = Y$.
4. *The operator $\chi_\omega K^* K \chi^*_\omega$ is positive definite.*

The conditions established here allow for efficient verification of regional observability, which is crucial for the design of more precise monitoring and control systems.

Remark 4.4. *The concepts of regional controllability and regional observability in conformable systems are highly practical for analyzing real systems. This practicality is evident for the following reasons:*

- *The definitions 3.1-4.1 are general and can be applied even when $\omega = \Omega$.*
- *There exist systems that are not controllable (observable) across the entire domain Ω, but they exhibit controllability (observability) within a subregion ω of Ω. This is justified by the following examples, where we demonstrated that the system is not approximately controllable (respectively, approximately observable) over the entire domain. However, we have identified a sub-region where the system is controllable (respectively observable).*

5. Examples

Example 5.1. *Let us consider the following one dimension fractional order sub-diffusion system with $Bu = \mathbb{1}_{[a,b]}u$, $0 \le a \le b \le 1$ for $0 < \alpha \le 1$:*

$$\begin{aligned}\frac{\partial^\alpha z}{\partial t^\alpha}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t) + \mathbb{1}_{[a,b]}u(x,t), \quad z(x,0) = 0,\\ z(0,t) &= z(1,t) = 0, \quad t \in]0, t_f[\end{aligned}$$

This system can be formulated in $Z = L^2_\alpha([0,1], \mathbb{R})$ as follows:

$$\frac{d^\alpha z}{dt^\alpha} = Az(t) + Bu(t), \quad z(0) = z_0,$$

where the operator A is defined by, $Az = z''$ with domain

$$D(A) = \{z \in Z \mid z, \frac{\partial z}{\partial x} \quad \text{are} \quad \text{absolutely} \quad \text{continuous}, \quad \frac{\partial^2 z}{\partial x^2}, \in Z\}$$

It is given in [14] that A is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\ge 0}$ which is defined on Z as,

$$S(t)z = \sum_{n\ge 1} 2e^{-n^2\pi^2 t} < z, \sin(n\pi.) >_{L^2_\alpha([0,1])} \sin(n\pi z)$$

As a consequence A is the $\alpha-$infinitesimal generator of a $\alpha-$semigroup given by

$$T(t) = S(\frac{t^\alpha}{\alpha})$$

The operator H defined in (11) is such that

$$(H^* z)(t) = B^* T^* (t_f^\alpha - t^\alpha)^{\frac{1}{\alpha}} z = \sum_{n\ge 1} 2e^{-n^2\pi^2(t_f^\alpha - t^\alpha)} < z, \sin(n\pi.) >_{L^2_\alpha([0,1])} \int_a^b \sin(n\pi y)dy$$

So, for all a, b such that $b - a \in \mathbb{Q}$, the considered system is not approximately controllable (because $KerH^* \neq \{0\}$ see [16]).

In the following we are looking for a sub-region $[x_1, x_2] \subset]0,1[$, for a convenient x_1 and x_2 so that the system is approximately controllable in this subregion.

Indeed. Let $I := \{i / i(b-a) = 2n,\ n \in \mathbb{N}\} \neq \emptyset$ (since $b - a \in \mathbb{Q}$), and note that $\varphi_i(x) = \sqrt{2}\sin(i\pi x)$ and $\lambda_n = -n^2\pi^2$. Suppose that $(\varphi_i)_{i\in I}$ are the eigenfunctions in $KerH^*$, so

$$KerH^* = span\{(\varphi_i)_{i\in I}\}$$

Now, let the subregion $[x_1, x_2]$ *such that* $x_2 = x_1 + b - a$. *We have for all* $i \in I$

$$\int_{x_1}^{x_2} \varphi_i^2(x)dx = x_2 - x_1$$

and for $i \neq j$, $i, j \in I$

$$\int_{x_1}^{x_2} \varphi_i(x)\varphi_j dx = 0$$

- *Let* $i_0 \in I$, *so* $\varphi_{i_0} \in \ker H^*$, *and therefore* $\varphi_{i_0} \in L^2_\alpha(0,1)$ *is not approximately controllable.*
- *Let us show that* $\varphi_{i_0} \in L^2_\alpha(\omega)$ *is regionally approximately controllable for a subregion* $\omega = [x_1, x_2]$. *That is* $\chi_\omega \varphi_{i_0} \notin \ker H^* \chi_\omega^*$
Indeed.

$$\begin{array}{rcl} H^*\chi_\omega^*(\chi_\omega\varphi_{i_0}) & = & \displaystyle\sum_{n=1}^{\infty} e^{-n^2\pi^2(t_f^\alpha - t^\alpha)} < \varphi_{i_0}, \varphi_n >_{L^2_\alpha(\omega)} \int_a^b \varphi_n(y)dy \\ & = & \displaystyle\sum_{n \notin I} e^{-n^2\pi^2(t_f^\alpha - t^\alpha)} < \varphi_{i_0}, \varphi_n >_{L^2_\alpha(\omega)} \int_a^b \varphi_n(y)dy \\ & \neq & 0 \end{array}$$

Hence, φ_{i_0} *is regionally approximately controllable on* $[x_1, x_2]$.

This example illustrates the practical application of the theoretical conditions for regional controllability, showing how they can be implemented in real-world scenarios.

Example 5.2. *Consider the following system for* $0 < \alpha \leq 1$*:*

$$\begin{array}{rcl} \dfrac{\partial^\alpha z}{\partial t^\alpha}(x,t) & = & \dfrac{\partial^2 z}{\partial x^2}(x,t),]0,1[\times]0,t_f[, \\ z(0,t) & = & z(1,t) = 0, \quad t \in]0,t_f[\\ z(x,0) & = & z_0(x) \text{ unknown in } \Omega =]0,1[\end{array}$$

Augmented by the output equation

$$y(t) = z(L,t), \quad L \in \Omega$$

and we have

$$y(t) = \sum_{n \geq 1} e^{\frac{\lambda_n t^\alpha}{\alpha}} < y_0, \varphi_n >_{L^2_\alpha(\Omega)} \varphi_n(L) = K(t)y_0.$$

where λ_n *and* φ_n *are given in the previous example.*

If $L \in Q$, *we have*

$$KerK(t) \neq \{0\}$$

So, this system is not approximately observable in Ω *(see [17]).*
In the following, we are looking for a subregion $\omega \subset \Omega$ *such that the studied systems is regionally observable in this subregion. Indeed, by following the same calculus in the previous example and if we take*

$$I = \{n \mid nL = 2k, \ k \in \mathbb{N}^*\}$$

we have for all $n_0 \in I$, φ_{n_0} *is not observable in* Ω *but is observable on the subregion* $\omega = [\frac{1}{4}, \frac{3}{4}]$.

6. Conclusion

In this study, we delved into the regional controllability and observability of fractional linear systems using the conformable derivative of order $0 < \alpha < 1$. By employing fractional calculus theory and semigroup theory, we established the necessary and sufficient conditions for exact and approximate regional controllability and observability of the conformal fractional linear system. Our analysis provided a solid conceptual framework to assess the feasibility and effectiveness of control and observation in these specific systems. The regional observability and controllability of conformable infinite-dimensional semilinear systems is under investigation.

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