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Power series expansion and decreasing property related to normalized remainders of power series expansion of sine

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Abstract. In the paper, with the aid of a derivative formula for the ratio of two differentiable functions, in view of a monotonicity rule for the ratio of two differentiable functions, in terms of the Hessenberg determinants, the authors present a Maclaurin power series expansion of the logarithm of the normalized remainder of the Maclaurin power series expansion of the sine function, and demonstrate the decreasing property of the ratio of two logarithms of two normalized remainders of the Maclaurin power series expansion of the sine function.

1. Motivations

We first recall some known results, introduce a sequence of functions, and state our main aims.

1.1. *Known results*

In [8, pp. 42 and 55], we looked up the Maclaurin power series expansions

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots, \quad x \in \mathbb{R} \tag{1.1}$$

and

$$\ln \sin x = \ln x - \sum_{k=1}^{\infty} \frac{2^{2k-1}}{(2k)!k} |B_{2k}| x^{2k} = \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots, \quad 0 < x < \pi, \tag{1.2}$$

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where the Bernoulli numbers B_{2k} are generated by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

From the series expansion (1.2), we acquire

$$\ln \frac{\sin x}{x} = -\sum_{k=1}^{\infty} \frac{2^{2k-1}}{k} |B_{2k}| \frac{x^{2k}}{(2k)!} = -\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots, \quad 0 < |x| < \pi. \tag{1.3}$$

In the paper [14], the authors established the following results.

1. The even function

$$Q(x) = \begin{cases} \ln \dfrac{6(x - \sin x)}{x^3}, & 0 < |x| < \infty \\ 0, & x = 0 \end{cases} \tag{1.4}$$

has a Maclaurin power series expansion

$$Q(x) = -\sum_{n=0}^{\infty} \frac{C_{2n}}{(2n)!} x^{2n} = -\frac{1}{20} x^2 - \frac{1}{16800} x^4 + \frac{1}{756000} x^6 + \cdots, \tag{1.5}$$

where the determinant

$$C_{2n} = -\begin{vmatrix} 0 & \binom{0}{0}c_0 & 0 & 0 & \cdots & 0 \\ c_1 & 0 & \binom{1}{1}c_0 & 0 & \cdots & 0 \\ 0 & \binom{2}{0}c_1 & 0 & \binom{2}{2}c_0 & \cdots & 0 \\ c_2 & 0 & \binom{3}{1}c_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \binom{2n-4}{0}c_{n-2} & 0 & \binom{2n-4}{2}c_{n-3} & \cdots & 0 \\ c_{n-1} & 0 & \binom{2n-3}{1}c_{n-2} & 0 & \cdots & 0 \\ 0 & \binom{2n-2}{0}c_{n-1} & 0 & \binom{2n-2}{2}c_{n-1} & \cdots & \binom{2n-2}{2n-2}c_0 \\ c_n & 0 & \binom{2n-1}{1}c_{n-1} & 0 & \cdots & 0 \end{vmatrix}$$
$$= -\left|c_{i,j}\right|_{(2n)\times(2n)},$$
$$c_{i,j} = \begin{cases} \dfrac{1 + (-1)^i}{2} c_{i/2}, & 1 \le i \le 2n, j = 1; \\ \dbinom{i-1}{j-2} \dfrac{1 + (-1)^{i-j+1}}{2} c_{(i-j+1)/2}, & 1 \le i \le 2n, 2 \le j \le 2n, \end{cases}$$

and the scalars

$$c_{m/2} = (-1)^{m/2} \frac{3!m!}{(m+3)!}, \quad m \ge 0.$$

2. The even function

$$R(x) = \begin{cases} \dfrac{\ln \frac{6(x - \sin x)}{x^3}}{\ln \frac{\sin x}{x}}, & |x| \in (0, \pi) \\ \dfrac{3}{10}, & x = 0 \\ 0, & x = \pm\pi \end{cases} \tag{1.6}$$

is decreasing on $[0, \pi]$.

1.2. A sequence of even functions

We now introduce a sequence of even functions, which extend and generalize the functions $Q(x)$ and $R(x)$, as follows.

1.2.1. The first function

The sinc function

$$\operatorname{sinc} x = \begin{cases} \dfrac{\sin x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

arises frequently in signal processing and the theory of Fourier transforms. The Maclaurin power series expansion of the power function $\operatorname{sinc}^r x$ for $r \in \mathbb{R}$, the series expansion (3.5) on Page 10455 below, has been studied in [22] and applied in [6, 7, 10, 14, 21]. On the set

$$S = \bigcup_{k=0}^{\infty} (2k\pi, (2k+1)\pi) \cup (-(2k+1)\pi, -2k\pi), \tag{1.7}$$

the sinc function $\operatorname{sinc} x$ is positive; on the set

$$\bigcup_{k=0}^{\infty} ((2k+1)\pi, (2k+2)\pi) \cup (-(2k+2)\pi, -(2k+1)\pi),$$

the sinc function $\operatorname{sinc} x$ is negative; the points $\pm k\pi$ for $k \ge 1$ are real zeros of $\operatorname{sinc} x$.

The first function we are introducing is

$$Q_0(x) = \begin{cases} \ln \dfrac{\sin x}{x}, & x \in S; \\ 0, & x = 0. \end{cases} \tag{1.8}$$

The function $Q_0(x)$ has the limits

$$\lim_{x \to (\pm 2k\pi)^{\pm}} Q_0(x) = -\infty, \quad k = 1, 2, \dots \quad \text{and} \quad \lim_{x \to (\pm(2k+1)\pi)^{\mp}} Q_0(x) = -\infty, \quad k = 0, 1, 2, \dots .$$

The reciprocal of the function $Q_0(x)$ is defined by

$$\frac{1}{Q_0(x)} = \begin{cases} \dfrac{1}{\ln \frac{\sin x}{x}}, & x \in S; \\ 0, & x = \pm(2k+2)\pi, \pm(2k+1)\pi, k = 0, 1, 2, \dots . \end{cases}$$

The limit $\lim_{x \to 0} \frac{1}{Q_0(x)} = \infty$ is valid.

1.2.2. The second function

We now discuss a generalization of $\operatorname{sinc} x$, which is

$$\operatorname{SinR}_n(x) = \begin{cases} (-1)^n \dfrac{(2n+1)!}{x^{2n+1}} \left[\sin x - \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right], & x \ne 0 \\ 1, & x = 0 \end{cases} \tag{1.9}$$

for $n \ge 0$. It is obvious that $\operatorname{SinR}_0(x) = \operatorname{sinc} x$ for $x \in \mathbb{R}$.

Let

$$\mathrm{SR}_n(x)=\sin x-\sum_{k=0}^{n-1}\frac{(-1)^k}{(2k+1)!}x^{2k+1}=(-1)^nx^{2n+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2n+2k+1)!}x^{2k}$$

for $n\ge1$ and $x\in\mathbb{R}$. It is well known that the quantity $\mathrm{SR}_n(x)$ is called the nth remainder or tail of the series expansion (1.1). It is easy to see that the function

$$\begin{aligned}\mathrm{SinR}_n(x)&=\begin{cases}(-1)^n\dfrac{(2n+1)!}{x^{2n+1}}\,\mathrm{SR}_n(x), & x\ne0\\ 0, & x=0\end{cases}\\&=(2n+1)!\sum_{k=0}^{\infty}\frac{(-1)^k}{(2n+2k+1)!}x^{2k}\end{aligned}\tag{1.10}$$

for $n\ge1$ and $x\in\mathbb{R}$ is even.

We call the quantity $\mathrm{SinR}_n(x)$ defined for $n\ge1$ by (1.9) the normalized remainder of the Maclaurin power series expansion of the sine function $\sin x$. In [29], the normalized remainder $\mathrm{SinR}_n(x)$ for $n\ge1$ was proved to be positive and decreasing in $x\in(0,\infty)$ and to be concave in $x\in(0,\pi)$; see also [11, Remark 7].

Considering the positivity of $\mathrm{SinR}_n(x)$ for $n\ge1$ on $(0,\infty)$, we introduce the second even function

$$Q_n(x)=\begin{cases}\ln\mathrm{SinR}_n(x), & 0<|x|<\infty\\ 0, & x=0\end{cases}\tag{1.11}$$

for $n\ge1$. It is obvious that $Q_1(x)=Q(x)$ defined by (1.4) on $(-\infty,\infty)$.

1.2.3. The third function

Basing on the domains of $Q_0(x)$ and $Q_n(x)$ for $n\ge1$, we now introduce the third even function

$$R_{m,n}(x)=\begin{cases}\dfrac{Q_n(x)}{Q_0(x)}, & x\in\overline{S}\setminus\{0\},\quad m=0,\quad n\ge1;\\[2mm] \dfrac{Q_n(x)}{Q_m(x)}, & 0<|x|<\infty,\quad n>m\ge1;\\[2mm] \dfrac{(m+1)(2m+3)}{(n+1)(2n+3)}, & x=0,\quad n>m\ge0,\end{cases}\tag{1.12}$$

where $\overline{S}$ denotes the closure of the set S defined by (1.7).

It is obvious that $R_{0,1}(x)=R(x)$ on $[-\pi,\pi]$, which is defined by (1.6).

1.3. Aims of this paper

In this paper, we will consider the following problems:

1. What are the monotonicity and concavity of the functions $Q_0(x)$ on $(0,\pi)$ and $Q_n(x)$ on $(0,\infty)$ for $n\ge1$?
2. Expand the functions $Q_n(x)$ for $n\ge0$ into a Maclaurin power series at the origin $x=0$.
3. What is the monotonicity of the functions $R_{m,n}(x)$ for $n>m\ge0$?

The motivations and reasons why we consider these three problems are purely due to our hobby and interest in mathematics, rather than due to their history, backgrounds, and applicability in mathematical sciences.

By virtue of properties of the normalized remainder $\mathrm{SinR}_n(x)$ for $n\ge1$ in [29], we see easily that the function $Q_n(x)$ for $n\ge1$ is decreasing and concave in $x\in(0,\infty)$, because a concave function must be a logarithmically concave function but the converse is not true. It is straightforward that

$$Q_0'(x)=\frac{1}{\tan x}-\frac{1}{x}<0\quad\text{and}\quad Q_0''(x)=\frac{1}{x^2}-\frac{1}{\sin^2x}<0\tag{1.13}$$

on $(0,\pi)$. Hence, the function $Q_0(x)$ is decreasing and concave on $(0,\pi)$. On the other hand, the series expansion (1.3) shows us that the function $Q_0(x)$ is decreasing and concave on $(0,\pi)$. In a word, the first problem has been completely solved.

The Maclaurin power series expansions of the functions $Q_0(x)$ and $Q_1(x)$ around $x=0$ are just the series expansions (1.3) and (1.5).

The decreasing property of $R_{0,1}(x)$ on $(0,\pi)$ has been proved in [14, Theorem 2].

In this paper, we will completely solve the second problem by expanding $Q_n(x)$ for $n\ge 0$ into a Maclaurin power series at the origin $x=0$, and partially solve the third problem by showing the decreasing property of $R_{0,2}(x)$ on $(0,\pi)$.

2. Lemmas

For smoothly proceeding, we need the following lemmas.

Lemma 1. *Let $u(x)$ and $v(x)\ne 0$ be two n-time differentiable functions on an interval I for a given integer $n\ge 0$. Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ is*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{u(x)}{v(x)}\right]=(-1)^n\frac{\left|W_{(n+1)\times(n+1)}(x)\right|}{v^{n+1}(x)},\quad n\ge 0, \tag{2.1}$$

where the matrix

$$W_{(n+1)\times(n+1)}(x)=\begin{pmatrix}U_{(n+1)\times1}(x) & V_{(n+1)\times n}(x)\end{pmatrix}_{(n+1)\times(n+1)},$$

the matrix $U_{(n+1)\times1}(x)$ is an $(n+1)\times 1$ matrix whose elements satisfy $u_{k,1}(x)=u^{(k-1)}(x)$ for $1\le k\le n+1$, the matrix $V_{(n+1)\times n}(x)$ is an $(n+1)\times n$ matrix whose elements are

$$v_{\ell,j}(x)=\begin{cases}\dbinom{\ell-1}{j-1}v^{(\ell-j)}(x), & \ell-j\ge 0\\ 0, & \ell-j<0\end{cases}$$

for $1\le \ell\le n+1$ and $1\le j\le n$, and the notation $|W_{(n+1)\times(n+1)}(x)|$ denotes the determinant of the $(n+1)\times(n+1)$ matrix $W_{(n+1)\times(n+1)}(x)$.

The formula (2.1) is a reformulation of [4, p. 40, Exercise 5)]. See also the papers [19, 20] and those papers collected at https://qifeng618.wordpress.com/2020/03/22/some-papers-authored-by-dr-prof-feng-qi-and-utilizing-a-general-derivative-formula-for-the-ratio-of-two-differentiable-functions.

Lemma 2 (Monotonicity rule for the ratio of two functions [2, Theorem 1.25])**.** *For $a,b\in\mathbb{R}$ with $a<b$, let $\lambda(x)$ and $\mu(x)$ be continuous on $[a,b]$, differentiable on (a,b), and $\mu'(x)\ne 0$ on (a,b). If the ratio $\frac{\lambda'(x)}{\mu'(x)}$ is increasing on (a,b), then both $\frac{\lambda(x)-\lambda(a)}{\mu(x)-\mu(a)}$ and $\frac{\lambda(x)-\lambda(b)}{\mu(x)-\mu(b)}$ are increasing in $x\in(a,b)$.*

3. Maclaurin power series expansion

In this section, with the aid of the derivative formula (2.1), we expand the functions $Q_n(x)$ defined by (1.8) and (1.11) for $n\ge 0$ into a Maclaurin power series around the origin $x=0$.

Theorem 1. *For $n\ge 0$, the function $Q_n(x)$ can be expanded into the Maclaurin power series*

$$Q_n(x)=-\sum_{m=1}^{\infty}\left|W_{(2m)\times(2m)}(0;n)\right|\frac{x^{2m}}{(2m)!},\quad x\in\begin{cases}(-\pi,\pi), & n=0;\\ (-\infty,\infty), & n\ge 1,\end{cases} \tag{3.1}$$

where the determinant $\left|W_{(2m)\times(2m)}(0;n)\right|$ *is given by*

$$\left|W_{(2m)\times(2m)}(0;n)\right| = \left|U_{(2m)\times 1}(0;n) \quad V_{(2m)\times(2m-1)}(0;n)\right|_{(2m)\times(2m)}$$

with the $(2m)\times 1$ *matrix*

$$U_{(2m)\times 1}(0;n) = \left(\frac{1+(-1)^k}{2}\frac{(-1)^{k/2}}{\binom{2n+k+1}{k}}\right)_{1\le k\le 2m}$$

and the $(2m)\times(2m-1)$ *matrix*

$$V_{(2m)\times(2m-1)}(0;n) = \left(\frac{1+(-1)^{q-j}}{2}(-1)^{(q-j)/2}\frac{\binom{q-1}{j-1}}{\binom{2n+q-j+1}{q-j}}\right)_{\substack{1\le q\le 2m\\ 1\le j\le 2m-1}}.$$

Proof. It is immediate that

$$Q_n'(x) = \frac{\mathrm{SinR}_n'(x)}{\mathrm{SinR}_n(x)} \to 0, \quad x\to 0.$$

Utilizing the derivative formula (2.1) and considering the even property of $Q_n(x)$ for $n \ge 1$, we acquire $Q_n^{(2m+1)}(0) = 0$ and

$$\lim_{x\to 0} Q_n^{(2m)}(x) = \lim_{x\to 0}\left[\frac{\mathrm{SinR}_n'(x)}{\mathrm{SinR}_n(x)}\right]^{(2m-1)} = -\frac{\left|W_{(2m)\times(2m)}(0;n)\right|}{\mathrm{SinR}_n^{2m}(0)} = -\left|W_{(2m)\times(2m)}(0;n)\right|, \quad m\ge 1,$$

where

$$W_{(2m)\times(2m)}(0;n) = \left(U_{(2m)\times 1}(0;n) \quad V_{(2m)\times(2m-1)}(0;n)\right)_{(2m)\times(2m)}, \quad U_{(2m)\times 1}(0;n) = \left(u_{k,1}(0;n)\right)_{(2m)\times 1},$$
$$u_{k,1}(0;n) = \mathrm{SinR}_n^{(k)}(0), \quad 1\le k\le 2m,$$
$$V_{(2m)\times(2m-1)}(0;n) = \left(v_{q,j}(0;n)\right)_{(2m)\times(2m-1)},$$
$$v_{q,j}(0;n) = \binom{q-1}{j-1}\mathrm{SinR}_n^{(q-j)}(0), \quad 1\le q\le 2m, \quad 1\le j\le 2m-1.$$

Employing the series expression (1.10), we derive $\mathrm{SinR}_n^{(2\ell+1)}(0) = 0$ for $\ell \ge 0$ and

$$\mathrm{SinR}_n^{(2\ell)}(0) = \frac{(-1)^\ell}{\binom{2n+2\ell+1}{2\ell}}, \quad \ell\ge 0.$$

Hence, it follows that

$$\begin{aligned} u_{k,1}(0;n) &= \begin{cases} 0, & 1\le k = 2\ell-1\le 2m-1\\ \mathrm{SinR}_n^{(2\ell)}(0), & 1\le k=2\ell\le 2m \end{cases}\\ &= \begin{cases} 0, & 1\le k = 2\ell-1\le 2m-1\\ \dfrac{(-1)^\ell}{\binom{2n+2\ell+1}{2\ell}}, & 1\le k=2\ell\le 2m \end{cases}\\ &= \begin{cases} 0, & 1\le k = 2\ell-1\le 2m-1\\ \dfrac{(-1)^{k/2}}{\binom{2n+k+1}{k}}, & 1\le k=2\ell\le 2m \end{cases} \end{aligned}$$

$$= \frac{1+(-1)^k}{2} \frac{(-1)^{k/2}}{\binom{2n+k+1}{k}}$$

for $1 \le k \le 2m$ and

$$\begin{aligned} v_{q,j}(0;n) &= \begin{cases} 0, & q-j<0 \\ 0, & 1 \le q-j = 2\ell-1 \le 2m-1 \\ \binom{q-1}{j-1} \operatorname{SinR}_n^{(q-j)}(0), & 0 \le q-j = 2\ell-2 \le 2m-1 \end{cases} \\ &= \begin{cases} 0, & q-j<0 \\ 0, & 1 \le q-j = 2\ell-1 \le 2m-1 \\ \binom{q-1}{j-1} \operatorname{SinR}_n^{(2\ell-2)}(0), & 0 \le q-j = 2\ell-2 \le 2m-1 \end{cases} \\ &= \begin{cases} 0, & q-j<0 \\ 0, & 1 \le q-j = 2\ell-1 \le 2m-1 \\ (-1)^{(q-j)/2} \frac{\binom{q-1}{j-1}}{\binom{2n+q-j+1}{q-j}}, & 0 \le q-j = 2\ell-2 \le 2m-1 \end{cases} \\ &= \frac{1+(-1)^{q-j}}{2} (-1)^{(q-j)/2} \frac{\binom{q-1}{j-1}}{\binom{2n+q-j+1}{q-j}} \end{aligned}$$

for $1 \le q \le 2m$ and $1 \le j \le 2m-1$.

In conclusion, we obtain

$$Q_n(x) = \sum_{m=0}^{\infty} Q_n^{(m)} \frac{x^m}{m!} = \sum_{m=1}^{\infty} Q_n^{(2m)} \frac{x^{2m}}{(2m)!} = -\sum_{m=1}^{\infty} \left| W_{(2m)\times(2m)}(0;n) \right| \frac{x^{2m}}{(2m)!}$$

for $n \ge 0$ and $x \in (-\pi, \pi)$ or $x \in (-\infty, \infty)$. The proof of Theorem 1 is complete. □

Remark 1. The determinant $\left| W_{(2m)\times(2m)}(0;n) \right|$ can be alternatively formulated as

$$\begin{vmatrix} 0 & \frac{\binom{0}{0}}{\binom{2n+1}{0}} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{\binom{2n+3}{2}} & 0 & \frac{\binom{1}{1}}{\binom{2n+1}{0}} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-\binom{2}{0}}{\binom{2n+3}{2}} & 0 & \frac{\binom{2}{2}}{\binom{2n+1}{0}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\binom{2n+5}{4}} & 0 & \frac{-\binom{3}{1}}{\binom{2n+3}{2}} & 0 & \frac{\binom{3}{3}}{\binom{2n+1}{0}} & 0 & \cdots & 0 \\ 0 & \frac{\binom{4}{0}}{\binom{2n+5}{4}} & 0 & \frac{-\binom{4}{2}}{\binom{2n+3}{2}} & 0 & \frac{\binom{4}{4}}{\binom{2n+1}{0}} & \cdots & 0 \\ \frac{-1}{\binom{2n+7}{6}} & 0 & \frac{\binom{5}{1}}{\binom{2n+5}{4}} & 0 & \frac{-\binom{5}{3}}{\binom{2n+3}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{(-1)^{m-1}\binom{2m-2}{0}}{\binom{2n+2m-1}{2m-2}} & 0 & \frac{(-1)^{m-2}\binom{2m-2}{2}}{\binom{2n+2m-3}{2m-4}} & 0 & \frac{(-1)^{m-3}\binom{2m-2}{4}}{\binom{2n+2m-5}{2m-6}} & \cdots & \frac{\binom{2m-2}{2m-2}}{\binom{2n+1}{0}} \\ \frac{(-1)^m}{\binom{2n+2m+1}{2m}} & 0 & \frac{(-1)^{m-1}\binom{2m-1}{1}}{\binom{2n+2m-1}{2m-2}} & 0 & \frac{(-1)^{m-2}\binom{2m-1}{3}}{\binom{2n+2m-3}{2m-4}} & 0 & \cdots & 0 \end{vmatrix}.$$

In particular, when $m = 3, 2, 1$, we obtain

$$
\begin{aligned}
\left|W_{6\times 6}(0;n)\right| &= \begin{vmatrix}
0 & \frac{1}{\binom{2n+1}{0}} & 0 & 0 & 0 & 0 \\
\frac{-1}{\binom{2n+3}{2}} & 0 & \frac{\binom{1}{1}}{\binom{2n+1}{0}} & 0 & 0 & 0 \\
0 & \frac{-1}{\binom{2n+3}{2}} & 0 & \frac{\binom{2}{2}}{\binom{2n+1}{0}} & 0 & 0 \\
\frac{1}{\binom{2n+5}{4}} & 0 & \frac{-\binom{3}{1}}{\binom{2n+3}{2}} & 0 & \frac{\binom{3}{3}}{\binom{2n+1}{0}} & 0 \\
0 & \frac{1}{\binom{2n+5}{4}} & 0 & \frac{-\binom{4}{2}}{\binom{2n+3}{2}} & 0 & \frac{\binom{4}{4}}{\binom{2n+1}{0}} \\
\frac{-1}{\binom{2n+7}{6}} & 0 & \frac{\binom{5}{1}}{\binom{2n+5}{4}} & 0 & \frac{-\binom{5}{3}}{\binom{2n+3}{2}} & 0
\end{vmatrix} \\
&= \frac{\left(2n^2+5n+3\right)^3\left(2n^2+9n+10\right) - 30\left(4n^2+6n-1\right)\binom{2n+7}{6}}{(n+1)^3(n+2)(2n+3)^3(2n+5)\binom{2n+7}{6}}, \\
\left|W_{4\times 4}(0;n)\right| &= \begin{vmatrix}
0 & \frac{1}{\binom{2n+1}{0}} & 0 & 0 \\
\frac{-1}{\binom{2n+3}{2}} & 0 & \frac{\binom{1}{1}}{\binom{2n+1}{0}} & 0 \\
0 & \frac{-1}{\binom{2n+3}{2}} & 0 & \frac{\binom{2}{2}}{\binom{2n+1}{0}} \\
\frac{1}{\binom{2n+5}{4}} & 0 & \frac{-\binom{3}{1}}{\binom{2n+3}{2}} & 0
\end{vmatrix} \\
&= -\frac{3\left(2n^2+n-4\right)}{(n+1)^2(n+2)(2n+3)^2(2n+5)},
\end{aligned}
$$

and

$$
\left|W_{2\times 2}(0;n)\right| = \begin{vmatrix}
0 & \frac{1}{\binom{2n+1}{0}} \\
\frac{-1}{\binom{2n+3}{2}} & 0
\end{vmatrix} = \frac{1}{2n^2+5n+3}.
$$

Remark 2. Comparing the series expansion (1.3) with the series expansion (3.1) for $n = 0$ yields a determinantal representation

$$
|B_{2k}| = \frac{k}{2^{2k-1}}\left|W_{(2k)\times(2k)}(0;0)\right|, \quad k \ge 1 \tag{3.2}
$$

for the Bernoulli numbers B_{2k}, that is,

$$
|B_{2k}| = \frac{k}{2^{2k-1}}\begin{vmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{-1}{3} & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & \frac{-1}{3} & 0 & 1 & 0 & 0 & \cdots & 0 \\
\frac{1}{5} & 0 & -1 & 0 & 1 & 0 & \cdots & 0 \\
0 & \frac{1}{5} & 0 & -2 & 0 & 1 & \cdots & 0 \\
\frac{-1}{7} & 0 & 1 & 0 & \frac{-10}{3} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \frac{(-1)^{k-1}}{2k-1} & 0 & \frac{(-1)^{k-2}\binom{2k-2}{2}}{\binom{0+2k-3}{2k-4}} & 0 & \frac{(-1)^{k-3}\binom{2k-2}{4}}{\binom{0+2k-5}{2k-6}} & \cdots & 1 \\
\frac{(-1)^k}{2k+1} & 0 & (-1)^{k-1} & 0 & \frac{(-1)^{k-2}\binom{2k-1}{3}}{\binom{0+2k-3}{2k-4}} & 0 & \cdots & 0
\end{vmatrix}
$$

for $k \ge 1$. In particular, we obtain

$$
|B_2| = \frac{\left|W_{2\times 2}(0;0)\right|}{2} = \frac{1}{6}, \quad |B_4| = \frac{\left|W_{4\times 4}(0;0)\right|}{4} = \frac{1}{30}, \quad |B_6| = \frac{3}{2^5}\left|W_{6\times 6}(0;0)\right| = \frac{1}{42}.
$$

Remark 3. Letting $n = 0$ in (3.1) and differentiating give

$$Q_0'(x) = -\sum_{m=1}^{\infty} \left|W_{(2m)\times(2m)}(0;0)\right| \frac{x^{2m-1}}{(2m-1)!}, \quad x \in (-\pi, \pi)$$

and

$$Q_0''(x) = -\sum_{m=0}^{\infty} \left|W_{(2m+2)\times(2m+2)}(0;0)\right| \frac{x^{2m}}{(2m)!}, \quad x \in (-\pi, \pi).$$

Comparing these two with the quantities in (1.13) results in

$$\frac{x}{\tan x} = 1 - \sum_{m=1}^{\infty} \frac{\left|W_{(2m)\times(2m)}(0;0)\right|}{(2m-1)!} x^{2m}, \quad x \in (-\pi, \pi) \tag{3.3}$$

and

$$\left(\frac{x}{\sin x}\right)^2 = 1 + \sum_{m=0}^{\infty} \frac{\left|W_{(2m+2)\times(2m+2)}(0;0)\right|}{(2m)!} x^{2m+2}, \quad x \in (-\pi, \pi). \tag{3.4}$$

In [1, p. 75, Entry 4.3.70], it was listed that

$$\cot z = \frac{1}{z} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2^{2k}}{(2k)!} B_{2k} z^{2k-1}, \quad |z| < \pi.$$

Comparing this series expansion with (3.3) recovers the determinantal expression (3.2) for the Bernoulli numbers B_{2k}.

Remark 4. The central factorial numbers of the second kind $T(n,k)$ for $n \ge k \ge 0$ can be generated [5, 15] by

$$\frac{1}{k!}\left(2\sinh\frac{x}{2}\right)^k = \sum_{n=k}^{\infty} T(n,k)\frac{x^n}{n!}.$$

In [5, Proposition 2.4, (xii)] and [23, Chapter 6, Eq. (26)], it was established that $T(0,0) = 1$ and

$$T(n,k) = \frac{1}{k!}\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell}\left(\frac{k}{2} - \ell\right)^n$$

for $n \ge k \ge 0$ but $(n,k) \ne (0,0)$.

In [22, Theorem 4.1], it was established that,

1. when $r < 0$ is a real number, the series expansion

$$\left(\frac{\sin x}{x}\right)^r = 1 + \sum_{m=1}^{\infty} (-1)^m \left[\sum_{k=1}^{2m} \frac{(-r)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2m+j, j)}{\binom{2m+j}{j}}\right] \frac{(2x)^{2m}}{(2m)!} \tag{3.5}$$

is convergent in $x \in (-\pi, \pi)$;

2. when $r \ge 0$, the series expansion (3.5) is convergent in $x \in (-\infty, \infty)$;

where the rising factorial $(r)_k$ is defined by

$$(r)_k = \prod_{\ell=0}^{k-1} (r+\ell) = \begin{cases} r(r+1)\cdots(r+k-1), & k \ge 1; \\ 1, & k = 0. \end{cases}$$

Letting $r=-2$ in (3.5) gives

$$\left(\frac{x}{\sin x}\right)^2 = 1+\sum_{m=1}^{\infty}(-1)^m\left[\sum_{k=1}^{2m}(k+1)\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2m+j,j)}{\binom{2m+j}{j}}\right]\frac{(2x)^{2m}}{(2m)!},\quad x\in(-\pi,\pi). \tag{3.6}$$

Comparing (3.4) with (3.6) and equating lead to

$$\left|W_{(2m)\times(2m)}(0;0)\right| = \frac{(-1)^m}{(2m)(2m-1)}\sum_{k=1}^{2m}(k+1)\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2m+j,j)}{\binom{2m+j}{j}},\quad m\ge 1.$$

Further making use of the relation (3.2) arrives at

$$|B_{2m}| = \frac{(-1)^m}{2^{2m}(2m-1)}\sum_{k=1}^{2m}(k+1)\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2m+j,j)}{\binom{2m+j}{j}},\quad m\ge 1.$$

This formula is similar to, but different from, the identity

$$B_{2m} = \frac{2^{2m-1}}{2^{2m-1}-1}\sum_{k=1}^{2m}\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\frac{T(2m+j,j)}{\binom{2m+j}{j}},\quad m\ge 1$$

in [7, Theorem 4.1], which was applied in the proof of Corollary 1 in [9].

Remark 5. Theorem 1 in this paper is a generalization of [14, Theorem 1].

4. Decreasing property

In this section, by virtue of the monotonicity rule recited in Lemma 2, we prove that the function $R_{0,2}(x)$ is decreasing on $(0,\pi)$.

Theorem 2. *The function $R_{0,2}(x)$ is decreasing on $(0,\pi)$.*

Proof. From the definition in (1.12), it follows that

$$R_{0,2}(x) = \frac{\ln\left[\frac{5!}{x^5}\left(\sin x - x + \frac{x^3}{3!}\right)\right]}{\ln\frac{\sin x}{x}},\quad 0<|x|<\pi.$$

It is straightforward that

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\ln\left[\frac{5!}{x^5}\left(\sin x - x + \frac{x^3}{3!}\right)\right] = \frac{6x\cos x - 30\sin x - 2x^3 + 24x}{6x\sin x + x^4 - 6x^2}.$$

Making use of the first derivative in (1.13) yields

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\,x}\ln\left[\frac{5!}{x^5}\left(\sin x - x + \frac{x^3}{3!}\right)\right]}{\frac{\mathrm{d}}{\mathrm{d}\,x}\ln\frac{\sin x}{x}} = \frac{2(x^3-12x+15\sin x-3x\cos x)\sin x}{(x^3-6x+6\sin x)(\sin x - x\cos x)}$$

and

$$\left[\frac{(x^3-12x+15\sin x-3x\cos x)\sin x}{(x^3-6x+6\sin x)(\sin x - x\cos x)}\right]' = \frac{-T(x)}{4(x^3-6x+6\sin x)^2(\sin x - x\cos x)^2},$$

where

$$\begin{aligned}
T(x) &= 4x^7 - 72x^5 + 264x^3 + 144x - 9x^5\cos x - 9x^3\cos x + 198x\cos x + 24x^3\cos(2x) - 144x\cos(2x)\\
&\quad - 3x^5\cos(3x) + 81x^3\cos(3x) - 198x\cos(3x) + 93x^4\sin x - 549x^2\sin x - 54\sin x - 2x^6\sin(2x)\\
&\quad + 60x^4\sin(2x) - 144x^2\sin(2x) - 72\sin(2x) + 21x^4\sin(3x) - 81x^2\sin(3x) + 18\sin(3x) + 36\sin(4x)\\
&= \frac{1}{54}\sum_{k=8}^{\infty}(-1)^k S(k)\frac{x^{2k+1}}{(2k+1)!}\\
&= \frac{1}{54}\sum_{k=4}^{\infty}\left[(4k+2)(4k+3)\frac{S(2k)}{S(2k+1)} - x^2\right]\frac{S(2k+1)}{(4k+3)!}x^{4k+1}
\end{aligned}$$

and

$$\begin{aligned}
S(k) &= 243\times 2^{4k+5} - 8\times 3^{2k}\Big(8k^5 - 104k^4 + 580k^3 - 703k^2 + 2163k + 972\Big)\\
&\quad + 27\times 2^{2k}\Big(8k^6 - 36k^5 + 290k^4 - 351k^3 + 503k^2 - 198k - 576\Big)\\
&\quad - 648\Big(24k^5 - 184k^4 + 148k^3 - 137k^2 - 163k - 12\Big)\\
&= 7776\times 3^{2k}\left[\left(\frac{4}{3}\right)^{2k} - \frac{8k^5 - 104k^4 + 580k^3 - 703k^2 + 2163k + 972}{972}\right]\\
&\quad + 648\Big(8k^6 - 36k^5 + 290k^4 - 351k^3 + 503k^2 - 198k - 576\Big)\\
&\quad \times\left(\frac{2^{2k}}{24} - \frac{24k^5 - 184k^4 + 148k^3 - 137k^2 - 163k - 12}{8k^6 - 36k^5 + 290k^4 - 351k^3 + 503k^2 - 198k - 576}\right).
\end{aligned}$$

It is easy to verify that

$$\begin{aligned}
&8k^6 - 36k^5 + 290k^4 - 351k^3 + 503k^2 - 198k - 576\\
&= 8(k-8)^6 + 348(k-8)^5 + 6530(k-8)^4 + 67809(k-8)^3 + 410639(k-8)^2 + 1369962(k-8) + 1955664\\
&> 0, \quad k \ge 8.
\end{aligned}$$

By induction, we can arrive at

$$\left(\frac{4}{3}\right)^{2k} - \frac{8k^5 - 104k^4 + 580k^3 - 703k^2 + 2163k + 972}{972} > 0, \quad k \ge 10$$

and

$$\frac{2^{2k}}{24} - \frac{24k^5 - 184k^4 + 148k^3 - 137k^2 - 163k - 12}{8k^6 - 36k^5 + 290k^4 - 351k^3 + 503k^2 - 198k - 576} > 0, \quad k \ge 8.$$

Moreover, numerical computation gives

$$S(8) = 215348170752 \quad \text{and} \quad S(9) = 14719784361984.$$

Consequently, the sequence $S(k)$ is positive for $k \ge 8$.

We claim that the sequence

$$(4k+2)(4k+3)\frac{S(2k)}{S(2k+1)}, \quad k \ge 4 \tag{4.1}$$

is increasing, that is,

$$(4k+6)(4k+7)\frac{S(2k+2)}{S(2k+3)} > (4k+2)(4k+3)\frac{S(2k)}{S(2k+1)}, \quad k \ge 4.$$

This inequality can be reformulated as

$$(2k+3)(4k+7)S(2k+1)S(2k+2) > (2k+1)(4k+3)S(2k)S(2k+3) \tag{4.2}$$

for $k \ge 4$, that is, the sequence

$$\begin{aligned}
&6^{4k}\Big[5877462609408(k-4)^9 + 39472831259904(k-4)^8 + 196342134897472(k-4)^7 \\
&+730097994024712(k-4)^6 + 2014535981948876(k-4)^5 + 4007358824520542(k-4)^4 \\
&+5376959126388333(k-4)^3 + 4133484135723492(k-4)^2 + 868746111722577(k-4) \\
&-707080673557332\Big] + 27\times 2^{8k+3}\Big(524288k^{13} + 3735552k^{12} + 18546688k^{11} + 65630208k^{10} \\
&+174176256k^9 + 388783872k^8 + 1039877632k^7 + 89567376k^6 + 117719408k^5 + 988449720k^4 \\
&+556139052k^3 + 170461233k^2 + 110225934k + 30779595\Big) + 81\times 2^{4k}\Big[4110967986688(k-4)^9 \\
&+30920619750144(k-4)^8 + 170234785243200(k-4)^7 + 691384393694904(k-4)^6 \\
&+2057371425719188(k-4)^5 + 4383401348467698(k-4)^4 + 6380703192978083(k-4)^3 \\
&+5789501374332684(k-4)^2 + 2661504373407231(k-4) + 286553966419476\Big] \\
&+1944\Big[1133133824(k-4)^9 + 13971296256(k-4)^8 + 113369713152(k-4)^7 \\
&+634689396096(k-4)^6 + 2497060922976(k-4)^5 + 6891012178308(k-4)^4 \\
&+13049925954580(k-4)^3 + 16138319947155(k-4)^2 + 11749819839327(k-4) \\
&+3846043491534\Big] + 2^{8k+3}3^{4k+2}\Big[1254400(k-14)^7 + 102037760(k-14)^6 \\
&+3460657408(k-14)^5 + 62705578352(k-14)^4 + 641678111260(k-14)^3 \\
&+3538255310240(k-14)^2 + 8452271478129(k-14) + 1859190048117\Big] \\
&+2187\times 2^{12k+17}(8k+9)\left[2^{4k} - \frac{\begin{pmatrix} 92160k^8 - 303104k^7 + 311808k^6 - 933440k^5 \\ -2725992k^4 - 2651276k^3 - 2019162k^2 - 854289k - 66879 \end{pmatrix}}{18432(8k+9)}\right] \\
&+16\times 3^{4k+1}\begin{bmatrix} 131072(k-4)^{11} + 5357568(k-4)^{10} + 99696640(k-4)^9 + 1118859264(k-4)^8 \\ +8449651200(k-4)^7 + 45287008128(k-4)^6 + 176490543328(k-4)^5 \\ +501911255076(k-4)^4 + 1024728330068(k-4)^3 \\ +1440024128235(k-4)^2 + 1270102753719(k-4) + 544466994750 \end{bmatrix} \\
&\times\left[\frac{9\times 3^{4k}}{2} - \frac{\begin{pmatrix} 62914560(k-4)^{12} + 2902589440(k-4)^{11} + 61055680512(k-4)^{10} \\ +774376570880(k-4)^9 + 6595463086080(k-4)^8 + 39726293521920(k-4)^7 \\ +173317513116288(k-4)^6 + 550395775653920(k-4)^5 \\ +1255946637183108(k-4)^4 + 1987551374800804(k-4)^3 \\ +2028126908873955(k-4)^2 + 1144415164087359(k-4) + 236105909426862 \end{pmatrix}}{\begin{pmatrix} 131072(k-4)^{11} + 5357568(k-4)^{10} + 99696640(k-4)^9 + 1118859264(k-4)^8 \\ +8449651200(k-4)^7 + 45287008128(k-4)^6 + 176490543328(k-4)^5 \\ +501911255076(k-4)^4 + 1024728330068(k-4)^3 \\ +1440024128235(k-4)^2 + 1270102753719(k-4) + 544466994750 \end{pmatrix}}\right]
\end{aligned}$$

is positive for $k \ge 4$. By induction, we prove that the sequences

$$2^{4k} - \frac{\begin{pmatrix} 92160k^8 - 303104k^7 + 311808k^6 - 933440k^5 - 2725992k^4 \\ -2651276k^3 - 2019162k^2 - 854289k - 66879 \end{pmatrix}}{18432(8k+9)}$$

and

$$\frac{9\times 3^{4k}}{2}-\frac{\begin{pmatrix}62914560(k-4)^{12}+2902589440(k-4)^{11}+61055680512(k-4)^{10}+774376570880(k-4)^{9}\\+6595463086080(k-4)^{8}+39726293521920(k-4)^{7}+173317513116288(k-4)^{6}\\+550395775653920(k-4)^{5}+1255946637183108(k-4)^{4}+1987551374800804(k-4)^{3}\\+2028126908873955(k-4)^{2}+1144415164087359(k-4)+236105909426862\end{pmatrix}}{\begin{pmatrix}131072(k-4)^{11}+5357568(k-4)^{10}+99696640(k-4)^{9}+1118859264(k-4)^{8}\\+8449651200(k-4)^{7}+45287008128(k-4)^{6}+176490543328(k-4)^{5}\\+501911255076(k-4)^{4}+1024728330068(k-4)^{3}+1440024128235(k-4)^{2}\\+1270102753719(k-4)+544466994750\end{pmatrix}}$$

are positive for $k \ge 4$, so the inequality (4.2) is valid for $k \ge 14$. Furthermore, numerical computation shows that the inequality (4.2) is valid for $4 \le k \le 14$. Therefore, the inequality (4.2) is valid for $k \ge 4$. Consequently, the sequence (4.1) is increasing in $k \ge 4$, with

$$\lim_{k\to 4}\left[(4k+2)(4k+3)\frac{S(2k)}{S(2k+1)}\right]=\frac{2937}{587}<\pi^2 \quad\text{and}\quad \lim_{k\to 4}\left[(4k+2)(4k+3)\frac{S(2k)}{S(2k+1)}\right]=\frac{3697387}{225268}>\pi^2.$$

The first three terms of the Maclaurin power series expansion for $54T(x)$ is

$$\sum_{k=4}^{6}\left[(4k+2)(4k+3)\frac{S(2k)}{S(2k+1)}-x^2\right]\frac{S(2k+1)}{(4k+3)!}x^{4k+1}=\frac{x^{17}P\left(x^2\right)}{2439237690489600000},$$

where

$$\begin{aligned}P\left(x^2\right)&=-2462313709x^{10}+74884614633x^{8}-1728814760640x^{6}\\&\quad+28375522583760x^{4}-295162342675200x^{2}+1476817377235200.\end{aligned}$$

It is straightforward that

$$\begin{aligned}P(u)&=-2462313709u^{5}+74884614633u^{4}-1728814760640u^{3}\\&\quad+28375522583760u^{2}-295162342675200u+1476817377235200,\\P'(u)&=-12311568545u^{4}+299538458532u^{3}-5186444281920u^{2}\\&\quad+56751045167520u-295162342675200,\\P''(u)&=-4\left(12311568545u^{3}-224653843899u^{2}+2593222140960u-14187761291880\right).\end{aligned}$$

The polynomial $P''(u)$ of degree 3 has a unique real zero

$$\begin{aligned}u_0&=\frac{3\sqrt[3]{19\times\begin{pmatrix}584360753784815184016105681949\\+10\sqrt{82635903278860936800448804452642049133164965437040238389 10}\end{pmatrix}}}{12311568545}\\&\quad-\frac{883246527172043212 23\times 19^{2/3}}{12311568545\sqrt[3]{\begin{matrix}584360753784815184016105681949\\+10\sqrt{8263590327886093680044880445264204913316496543704023838910}\end{matrix}}}\\&\quad+\frac{74884614633}{12311568545}\\&=9.0456\ldots\end{aligned}$$

which is the unique maximum point of $P'(u)$ and

$$P'(u_0)=\frac{4617}{1866122555439895734412857478625}$$

$$
\begin{aligned}
&\times\left[\frac{\sqrt[3]{19}\begin{pmatrix}19807343890326281222822705724805786470462800629872464942480 3\\ +116872150756963036803221136389 80\\ \times\sqrt{826359032788609368004488044526420491331649654370402383891 0}\end{pmatrix}}{\begin{pmatrix}584360753784815184016105681949\\ +10\sqrt{8263590327886093680044880445264204913316496543704023838910}\end{pmatrix}^{2/3}}\right.\\
&\quad-\frac{\begin{bmatrix}29441550905734773741\times 19^{2/3}\\ \times\begin{pmatrix}18507915044804815482372031463008988473679273150395294172068 03\\ +23374430151392607360644227277960\\ \times\sqrt{8263590327886093680044880445264204913316496543704023838910}\end{pmatrix}\end{bmatrix}}{\begin{pmatrix}584360753784815184016105681949\\ +10\sqrt{8263590327886093680044880445264204913316496543704023838910}\end{pmatrix}^{4/3}}\\
&\quad\left.-39643993535095422440165533699011283683 83\right]\\
&=-10^{13}\times 6.691\cdots.
\end{aligned}
$$

Accordingly, the polynomial $P'(u)$ of degree 4 is negative, and then the polynomial $P(u)$ of degree 5 is decreasing. Numerical computation gives

$$
\begin{aligned}
P\left(\pi^2\right)&=1476817377235200-295162342675200\pi^2+28375522583760\pi^4\\
&\quad-1728814760640\pi^6+74884614633\pi^8-2462313709\pi^{10}\\
&=10^{14}\times 1.456\cdots.
\end{aligned}
$$

As a result, the polynomial $P\left(x^2\right)$ of degree 10 is positive in $x\in[0,\pi]$. Consequently, the first three terms of the Maclaurin power series expansion for $54T(x)$ is positive in $x\in(0,\pi]$.

In a word, the function $T(x)$ is positive on $(0,\pi]$.

By virtue of the monotonicity rule in Lemma 2, we obtain that the function $R_{0,2}(x)$ is decreasing on $(0,\pi)$. The required proof is complete. □

Remark 6. Can one find a new and more effective method to prove the decreasing property of the function $R_{n,m}(x)$ for all $n>m\ge 0$?

5. Connections with generalized hypergeometric functions

For $\alpha_1\in\mathbb{C}$ and $\beta_1,\beta_2\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, the generalized hypergeometric function ${}_1F_2(\alpha_1;\beta_1,\beta_2;z)$ is defined [8, p. 1020] by

$$
{}_1F_2(\alpha_1;\beta_1,\beta_2;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n}{(\beta_1)_n(\beta_2)_n}\frac{z^n}{n!},\quad z\in\mathbb{C}.
$$

In [29, p. 16], Qi and his coauthors derived the relation

$$
\operatorname{SinR}_n(x)={}_1F_2\left(1;n+1,n+\frac{3}{2};-\frac{x^2}{4}\right),\quad n\in\mathbb{N}. \tag{5.1}
$$

As done in [11, Remark 7] and [16, Section 5], by the relation (5.1), we can restate main results of this paper in terms of the generalized hypergeometric function ${}_1F_2$ as follows.

1. The logarithmic function

$$
\ln\operatorname{SinR}_n(x)=\ln\left[{}_1F_2\left(1;n+1,n+\frac{3}{2};-\frac{x^2}{4}\right)\right],\quad n\in\mathbb{N}
$$

can be expanded into

$$\ln\left[{}_1F_2\left(1; n+1, n+\frac{3}{2}; -\frac{x^2}{4}\right)\right] = -\sum_{m=1}^{\infty} \left|W_{(2m)\times(2m)}(0;n)\right| \frac{x^{2m}}{(2m)!}, \quad x \in (-\infty, \infty).$$

2. The ratio

$$\frac{\ln \operatorname{SinR}_2(x)}{\ln \frac{\sin x}{x}} = \frac{\ln\left[{}_1F_2\left(1; 3, \frac{7}{2}; -\frac{x^2}{4}\right)\right]}{\ln \frac{\sin x}{x}}$$

is decreasing on $(0, \pi)$.

6. Conclusions

In this paper, the logarithm of the normalized remainder $\operatorname{SinR}_n(x)$ for $n \ge 0$, that is, the function $Q_n(x)$ for $n \ge 0$, was expanded into a Maclaurin power series (3.1) and the function $R_{0,2}(x)$ was proved to be decreasing on $(0, \pi)$.

The concept and idea of normalized remainders, also known as normalized tails, of the Maclaurin power series expansions of analytic functions originated from Qi and his joint paper [12]. Since then, Qi and his coauthors set up and investigated normalized remainders of the Maclaurin power series expansions of several elementary functions. We now systematically sum up as follows.

1. The normalized remainders of the Maclaurin power series expansion of the tangent function $\tan x$ were introduced and studied in the paper [12] and another two manuscripts by Qi and his coauthors.
2. The normalized remainders of the Maclaurin power series expansion of the square $\tan^2 x$ of the tangent function $\tan x$ were defined and investigated in [27].
3. The normalized remainders of the Maclaurin power series expansion of the function $\frac{x}{e^x - 1}$, the generating function of the Bernoulli numbers B_n for $n \ge 0$, were defined and considered in the paper [28]. Due to the study in [28], we discovered in [25, 28] some properties of the ratios of two Bernoulli polynomials $B_n(t)$ on $(0, \frac{1}{2})$ for $n \ge 1$.
4. The normalized remainders of the Maclaurin power series expansions of the sine function $\sin x$ were researched in this paper, in the articles [14, 16, 29], and in [11, Remark 7].
5. The normalized remainders of the Maclaurin power series expansions of the cosine function $\cos x$ were discussed in this paper, in the articles [13, 16, 17, 24, 26, 29], and in [11, Remark 7].
6. The normalized remainders of the Maclaurin power series expansions of the inverse sine, cosine, and tangent functions $\arcsin x$, $\arccos x$, and $\arctan x$ were looked into in a forthcoming manuscript by Qi and his coauthors.
7. The normalized remainders of the Maclaurin power series expansions of the exponential function e^x were researched in [3, 18].
8. The normalized remainders of the Maclaurin power series expansions of the power function $\left(\frac{\arcsin z}{z}\right)^q$ for $q \in \mathbb{N}$ were investigated in a forthcoming manuscript by Qi and his coauthors.

We believe that normalized remainders of the Maclaurin power series expansions of analytic functions, initially and originally invented by Qi, will be interesting objects attracting mathematicians in the world to further deeply and intrinsically investigate.

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.

[2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[3] Z.-H. Bao, R. P. Agarwal, F. Qi, and W.-S. Du, *Some properties on normalized tails of Maclaurin power series expansion of exponential function*, Symmetry **16** (2024), no. 8, Art. 989, 15 pages; available online at `https://doi.org/10.3390/sym16080989`.

[4] N. Bourbaki, *Elements of Mathematics: Functions of a Real Variable: Elementary Theory*, Translated from the 1976 French original by Philip Spain. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004; available online at `https://doi.org/10.1007/978-3-642-59315-4`.

[5] P. L. Butzer, M. Schmidt, E. L. Stark, and L. Vogt, *Central factorial numbers; their main properties and some applications*, Numer. Funct. Anal. Optim. **10** (1989), no. 5-6, 419–488, DOI: `https://doi.org/10.1080/01630568908816313`.

[6] J. Cao, J. L. López-Bonilla, and F. Qi, *Three identities and a determinantal formula for differences between Bernoulli polynomials and numbers*, Electron. Res. Arch. **32** (2024), no. 1, 224–240; available online at `https://doi.org/10.3934/era.2024011`.

[7] X.-Y. Chen, L. Wu, D. Lim, and F. Qi, *Two identities and closed-form formulas for the Bernoulli numbers in terms of central factorial numbers of the second kind*, Demonstr. Math. **55** (2022), no. 1, 822–830; available online at `https://doi.org/10.1515/dema-2022-0166`.

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Eighth edition, Revised from the seventh edition, Elsevier/Academic Press, Amsterdam, 2015; available online at `https://doi.org/10.1016/B978-0-12-384933-5.00013-8`.

[9] C.-Y. He and F. Qi, *Reformulations and generalizations of Hoffman's and Genčev's combinatorial identities*, Results Math. **79** (2024), no. 4, Paper No. 131, 17 pages; available online at `https://doi.org/10.1007/s00025-024-02160-0`.

[10] W.-H. Li, Q.-X. Shen, and B.-N. Guo, *Several double inequalities for integer powers of the sinc and sinhc functions with applications to the Neuman–Sándor mean and the first Seiffert mean*, Axioms **11** (2022), no. 7, Art. 304, 12 pages; available online at `https://doi.org/10.3390/axioms11070304`.

[11] Y.-W. Li and F. Qi, *A new closed-form formula of the Gauss hypergeometric function at specific arguments*, Axioms **13** (2024), no. 5, Art. 317, 24 pages; available online at `https://doi.org/10.3390/axioms13050317`.

[12] Y.-W. Li, F. Qi, and W.-S. Du, *Two forms for Maclaurin power series expansion of logarithmic expression involving tangent function*, Symmetry **15** (2023), no. 9, Art. 1686, 18 pages; available online at `https://doi.org/10.3390/sym15091686`.

[13] Y.-F. Li and F. Qi, *A series expansion of a logarithmic expression and a decreasing property of the ratio of two logarithmic expressions containing cosine*, Open Math. **21** (2023), no. 1, Paper No. 20230159, 12 pages; available online at `https://doi.org/10.1515/math-2023-0159`.

[14] X.-L. Liu, H.-X. Long, and F. Qi, *A series expansion of a logarithmic expression and a decreasing property of the ratio of two logarithmic expressions containing sine*, Mathematics **11** (2023), no. 14, Art. 3107, 12 pages; available online at `https://doi.org/10.3390/math11143107`.

[15] M. Merca, *Connections between central factorial numbers and Bernoulli polynomials*, Period. Math. Hungar. **73** (2016), no. 2, 259–264; available online at `https://doi.org/10.1007/s10998-016-0140-5`.

[16] D.-W. Niu and F. Qi, *Monotonicity results of ratios between normalized tails of Maclaurin power series expansions of sine and cosine*, Mathematics **12** (2024), no. 12, Art. 1781, 20 pages; available online at `https://doi.org/10.3390/math12121781`.

[17] W.-J. Pei and B.-N. Guo, *Monotonicity, convexity, and Maclaurin series expansion of Qi's normalized remainder of Maclaurin series expansion with relation to cosine*, Open Math. **22** (2024), no. 1, Paper No. 20240095, 11 pages; available online at `https://doi.org/10.1515/math-2024-0095`.

[18] F. Qi, *Absolute monotonicity of normalized tail of power series expansion of exponential function*, Mathematics **12** (2024), no. 18, Art. 2859, 11 pages; available online at `https://doi.org/10.3390/math12182859`.

[19] F. Qi, *Determinantal expressions and recursive relations of Delannoy polynomials and generalized Fibonacci polynomials*, J. Nonlinear Convex Anal. **22** (2021), no. 7, 1225–1239.

[20] F. Qi, *On signs of certain Toeplitz–Hessenberg determinants whose elements involve Bernoulli numbers*, Contrib. Discrete Math. **18** (2023), no. 2, 48–59; available online at `https://doi.org/10.55016/ojs/cdm.v18i2.73022`.

[21] F. Qi, G. V. Milovanović, and D. Lim, *Specific values of partial Bell polynomials and series expansions for real powers of functions and for composite functions*, Filomat **37** (2023), no. 28, 9469–9485; available online at `https://doi.org/10.2298/FIL2328469Q`.

[22] F. Qi and P. Taylor, *Series expansions for powers of sinc function and closed-form expressions for specific partial Bell polynomials*, Appl. Anal. Discrete Math. **18** (2024), no. 1, 92–115; available online at `https://doi.org/10.2298/AADM230902020Q`.

[23] J. Riordan, *Combinatorial Identities*, Reprint of the 1968 original, Robert E. Krieger Publishing Co., Huntington, N.Y., 1979.

[24] A. Wan and F. Qi, *Power series expansion, decreasing property, and concavity related to logarithm of normalized tail of power series expansion of cosine*, Electron. Res. Arch. **32** (2024), no. 5, 3130–3144; available online at `https://doi.org/10.3934/era.2024143`.

[25] Z.-H. Yang and F. Qi, *Monotonicity and inequalities for the ratios of two Bernoulli polynomials*, arXiv preprint (2024), available online at `https://arxiv.org/abs/2405.05280` or `https://doi.org/10.48550/arxiv.2405.05280`.

[26] H.-C. Zhang, B.-N. Guo, and W.-S. Du, *On Qi's normalized remainder of Maclaurin power series expansion of logarithm of secant function*, Axioms **13** (2024), no. 12, Article 860, 11 pages; available online at `https://doi.org/10.3390/axioms13120860`.

[27] G.-Z. Zhang and F. Qi, *On convexity and power series expansion for logarithm of normalized tail of power series expansion for square of tangent*, J. Math. Inequal. **18** (2024), no. 3, 937–952; available online at `https://doi.org/10.7153/jmi-2024-18-51`.

[28] G.-Z. Zhang, Z.-H. Yang, and F. Qi, *On normalized tails of series expansion of generating function of Bernoulli numbers*, Proc. Amer. Math. Soc. **153** (2025), no. 1, 131–141; available online at `https://doi.org/10.1090/proc/16877`.

[29] T. Zhang, Z.-H. Yang, F. Qi, and W.-S. Du, *Some properties of normalized tails of Maclaurin power series expansions of sine and cosine*, Fractal Fract. **8** (2024), no. 5, Art. 257, 17 pages. `https://doi.org/10.3390/fractalfract8050257`.