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Invariant nature of Darboux normal curves on a smooth surface under isometry

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Abstract. In this article, we explore the relationships between the Darboux frame and the Frenet frame by investigating the invariant sufficient conditions for the isometric image of a Darboux normal curve when using the Darboux frame instead of the Frenet frame. Additionally, we analyze the deviation in the component of the position vector of a Darboux normal curve on a smooth surface in relation to the provided isometry. The results of this study contribute to the improvement and generalization of certain earlier findings in the literature.

1. Introduction

The notion of the position vector field is a fundamental geometric concept essential for discussing manifold characteristics. It allows us to determine the position of any point on the manifold with a reference point. When investigating curves, the position vector field can be interpreted as illustrating the trajectory of a particle as it moves along the curve. The first and second derivatives of the curve provide insights into the particle's speed and acceleration along the curve, respectively. To study the properties of space curves or curves on surfaces, it is essential to understand their behavior when restricting their position vectors to specific planes associated with the surface. Since each point on a space curve exists in three-dimensional space, such as Euclidean 3 space, a dynamic orthonormal frame known as the Serret-Frenet frame can be established. This frame comprises three vectors, namely, the binormal vector, the principal normal vector, and the unit tangent vector. Each of these vectors corresponds to a specific orthogonal plane—the rectifying plane, the normal plane, and the osculating plane.

In 2003, Chen [1] introduced the concept of a rectifying curve, defining it as a space curve where its position vector consistently resides within its rectifying plane. Chen also established specific defining characteristics for such curves. In 2005, Chen and Dillen [2] explored the relationship between rectifying curves, centrodes, and extremal curves. In [3, 4], the authors examined rectifying curves in three-dimensional Minkowski space, presenting a description similar to Chen's work in 2003. In 2008, Ilarslan and Nesovic

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[5, 6] investigated the characteristics of osculating and rectifying curves in Euclidean spaces. In 2018, Deshmukh et al. [7] characterized rectifying curves by analyzing the centrodes of unit-speed curves in Euclidean space. Shaikh and Ghosh [8, 9, 12, 15] discussed rectifying and osculating curves on smoothly immersed surfaces within three-dimensional Euclidean space by using the Frenet frame and deriving some intriguing results. Shaikh et al. [10, 11, 13, 14] investigated the characterizations of normal, osculating, and rectifying curves under conformal and isometric transformations on smooth surfaces in Euclidean 3-space with the help of the Frenet frame. Lone [16] studied the geometric invariant properties of normal curves under conformal transformation. In [11, 17, 18], authors investigated curves by constraining their position vectors to the osculating, normal, and rectifying planes on a surface, determining how they change under isometry, homothetic, and conformal transformations between surfaces.

Camci et al. [19] investigated specific surface curves by restricting their position vector within three mutually perpendicular planes on the surface, exploring the existence and properties of such curves. For more comprehensive details regarding the characterizations of space curves, one can refer to [20, 21, 25, 26, 28]. In the examination of the geometry of space curves on a continuous surface, researchers frequently encounter the Frenet frame and the Darboux frame, which manifest at non-umbilic points of a surface immersed in Euclidean space. The Darboux frame is named after the French mathematician Jean Gaston Darboux. In [22, 23], the authors delved into the Frenet and Darboux rotation vectors of curves on time-like surfaces, elucidating the Darboux frame of these curves concerning the Lorentzian properties of the surfaces and the curves themselves.

When examining the relationship between two surfaces and their transformations, it becomes apparent that in the case of isometry, both the lengths of curves and the angles between intersecting curves are preserved. In the case of conformal motion, only the angles between curves remain constant, while the distances may undergo changes. In [24], the authors demonstrated that isometric mappings are a subset of conformal mappings where the dilation function equals to one. Recently, several authors [8, 16, 17] have explored osculating curves, rectifying curves, and normal curves on a smooth surface. These studies have established sufficient conditions under which a rectifying curve on such a surface remains unchanged when subjected to isometric transformations and also maintains its conformal invariance when exposed to conformal transformations. These conclusions have been derived using the concept of the Serret-Frenet frame.

However, when examining a space curve on a flat surface immersed in Euclidean 3-space, we naturally come across to another dynamic orthonormal frame known as the Darboux frame, denoted as $\{T_1, P_1, U_1\}$. In this frame, T_1 represents the unit tangent vector at a specific point on the curve, U_1 signifies the unit normal to the surface, and P_1 is the result of taking the cross product of T_1 and U_1. In [19, 27], authors provided specific descriptions regarding the position vector of a unit-speed curve on a smooth surface immersed in three-dimensional Euclidean space. They established that this position vector consistently lies within three distinct planes defined by $\{T_1, U_1\}$, $\{T_1, P_1\}$, and $\{P_1, U_1\}$, respectively, characterizing their findings by employing the Darboux frame.

The motivation for the study and its findings are quite interesting because the research integrates the characteristics of Darboux normal curves on smoothly immersed surfaces under isometry between the surfaces. The primary objective of this study is to derive sufficient conditions for the isometric image of Darboux normal curves. By employing these conditions, we explore the components of the position vector of Darboux normal curves along any tangent vector, principal normal, and binormal to the surface. Furthermore, we establish the invariant conditions for these components under isometric transformation.

The work presented in this paper is organized into four sections. Sections 1 and 2 provide an introduction and cover the preliminaries necessary for a basic understanding of the topic, establishing relationships between the Frenet frame and Darboux frame in Euclidean space. Section 3 deals with the basics of the Darboux normal curve and obtains the invariant sufficient conditions for Darboux normal curves on smooth surfaces under isometry. Furthermore, we analyze the deviations of the tangential and normal components for the position vector of a Darboux osculating curve in this section. Section 4 concludes the paper and outlines potential future avenues for this study.

2. Preliminaries

In this section, we introduced some fundamental concepts for space curves that are essential for this study. In this paper, we consider $\mathbb{E}^3$ as the 3-dimensional Euclidean space, where $\|.\|$ denotes the Euclidean norm, and $< .,. >$ denotes the Euclidean inner product.

Suppose a unit-speed curve $\delta = \delta(r)$ in $\mathbb{E}^3$, parameterized by the arc length r. Let T_1, N_1, and B_1 represent the vector fields corresponding to the tangent, principal normal, and binormal directions along the curve $\delta(r)$. The Frenet formula can then be expressed as follows

$$\begin{bmatrix} T_1' \\ N_1' \\ B_1' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ B_1 \end{bmatrix}.$$

In this context, we refer to the quantities $\kappa(r)$, greater than zero, and $\tau(r)$ as the curvature and torsion of the curve, respectively. The prime notation indicates differentiation with respect to the arc length parameter r. Let $\{T_1, P_1, U_1\}$ be the Darboux frame of the curve $\delta(r)$, where T_1 is the tangent vector of δ, U_1 is the unit normal to the surface Q, and $P_1 = U_1 \times T_1$. The connection between the Frenet and Darboux frames can be expressed as follows

$$\begin{bmatrix} T_1 \\ P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ B_1 \end{bmatrix}.$$

In this context, there exists a unique angle ϕ such that rotating within the plane defined by N_1 and B_1 results in the pair P_1 and U_1. Consequently, the Darboux formula for $\delta(r)$ can be expressed as follows

$$\begin{bmatrix} T_1' \\ P_1' \\ U_1' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ P_1 \\ U_1 \end{bmatrix},$$

where κ_g, κ_n, and τ_g represent the geodesic curvature, normal curvature, and geodesic torsion, respectively.

Let $\delta : (\mu, \nu) \to Q$ be a unit-speed parametrized curve on the coordinate chart $\sigma : V \to Q$ of the smooth surface Q, where $(\mu, \nu) \subset \mathbb{R}$. This implies that the curve $\delta(r)$ lies within the region covered by the surface patch σ. Thus, we can express it as follows

$$\delta(r) = \sigma(x(r), y(r)). \tag{1}$$

Now, differentiating equation (1) with respect to the parameter r, we obtain

$$T_1(r) = \delta'(r) = \sigma_x x' + \sigma_y y', \tag{2}$$

which again differentiating with respect to the parameter r entails

$$T_1'(r) = \sigma_x x'' + \sigma_y y'' + x'^2 \sigma_{xx} + 2x'y'\sigma_{xy} + y'^2 \sigma_{yy}. \tag{3}$$

Since $\kappa(r)$ is the curvature of $\delta(r)$ and U_1 represents the normal vector to the surface Q, then the principal normal vector $N_1(r)$ at the point $\delta(r)$ is defined by the following equation

$$N_1(r) = \frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}). \tag{4}$$

The binormal vector $B_1(r)$ can be written as follows

$$B_1(r) = T_1(r) \times N_1(r).$$

On substituting the values of $T_1(r)$ and $N_1(r)$ from equations (2) and (4), we obtain the following

$$\begin{aligned} B_1(r) &= \frac{1}{\kappa(r)}[(\sigma_x x' + \sigma_y y') \times (x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})], \\ &= \frac{1}{\kappa(r)}[(y''x' - y'x'')U_1 + x'^3\sigma_x \times \sigma_{xx} + 2x'^2y'\sigma_x \times \sigma_{xy} + x'y'^2\sigma_x \times \sigma_{yy} \\ &\quad + x'^2y'\sigma_y \times \sigma_{xx} + 2x'y'^2\sigma_y \times \sigma_{xy} + y'^3\sigma_y \times \sigma_{yy}]. \end{aligned} \tag{5}$$

Now, the normal U_1 to the surface Q is defined as follows

$$U_1(r) = \frac{\sigma_x \times \sigma_y}{\|\sigma_x \times \sigma_y\|} = \frac{\sigma_x \times \sigma_y}{\sqrt{EG - F^2}}, \tag{6}$$

where $E = \sigma_x \cdot \sigma_x$, $F = \sigma_x \cdot \sigma_y$, and $G = \sigma_y \cdot \sigma_y$ are the magnitudes of the first fundamental form. As we know that $P_1(r) = U_1(r) \times T_1(r)$, we can employ equations (2) and (6), followed by simplification, to derive

$$P_1(r) = \frac{1}{\sqrt{EG - F^2}}(Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x). \tag{7}$$

3. Darboux normal curve on smooth surface

In this section, we explore Darboux normal curves on a smooth surface and establish a set of conditions that guarantee the invariance of Darboux normal curves under isometries.

Definition 3.1. *Let Q and $\tilde{Q}$ be smooth surfaces in $\mathbb{E}^3$. Then a diffeomorphism J from Q to $\tilde{Q}$ represents an isometry if it preserves the lengths of curves, i.e., mapping curves of the same length from Q to $\tilde{Q}$.*

Definition 3.2. *A curve $\delta(r)$ on a smooth surface Q such that its position vector lies within the $\{U_1, P_1\}$- Darboux normal plane is referred to as a Darboux normal curve.*

Hence, the equation representing a Darboux normal curve is expressed as follows

$$\delta(r) = \lambda_1(r)U_1(r) + \lambda_2(r)P_1(r), \tag{8}$$

for some smooth function $\lambda_1(r)$ and $\lambda_2(r)$.
By virtue of (6) and (7), (8) yields

$$\delta(r) = \lambda_1(r)\frac{(\sigma_x \times \sigma_y)}{\sqrt{EG - F^2}} + \lambda_2(r)\frac{1}{\sqrt{EG - F^2}}\left\{Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\right\}. \tag{9}$$

Next, we examine the representation of the derivative map of $\delta(r)$, denoted as $J_*(\delta(r))$, as the product of a 3×3 matrix J_* and a 3×1 matrix $\delta(r)$.

Theorem 3.3. *Let Q and $\tilde{Q}$ be smooth surfaces and let $J : Q \to \tilde{Q}$ be an isometry. Suppose $\delta(r)$ is a Darboux normal curve on the surface Q. Then $\tilde{\delta}(r)$ is a Darboux normal curve on $\tilde{Q}$ if*

$$\tilde{\delta}(r) = J_*(\delta(r)).$$

Proof. Let J be an isometry between Q and $\tilde{Q}$. Then $J_* : T_pQ \to T_{J(p)}\tilde{Q}$ is such that $J_*\sigma_x = \tilde{\sigma}_x$ and $J_*\sigma_y = \tilde{\sigma}_y$. Since the surfaces Q and $\tilde{Q}$ are isometric, therefore $\tilde{E} = E$, $\tilde{F} = F$ and $\tilde{G} = G$.
Now from equation (9) of Darboux normal curve, we have

$$\begin{aligned}
\tilde{\delta}(r) &= J_*(\delta(r)) \\
&= \lambda_1(r)\frac{(J_*\sigma_x \times J_*\sigma_y)}{||J_*\sigma_x \times J_*\sigma_y||} + \frac{\lambda_2(r)}{||J_*\sigma_x \times J_*\sigma_y||}\left\{Ex'J_*\sigma_y + F(y'J_*\sigma_y - x'J_*\sigma_x) - Gy'J_*\sigma_x\right\}, \\
&= \lambda_1(r)\frac{(\tilde{\sigma}_x \times \tilde{\sigma}_y)}{||\tilde{\sigma}_x \times \tilde{\sigma}_y||} + \frac{\lambda_2(r)}{||\tilde{\sigma}_x \times \tilde{\sigma}_y||}\left\{Ex'\tilde{\sigma}_y + F(y'\tilde{\sigma}_y - x'\tilde{\sigma}_x) - Gy'\tilde{\sigma}_x\right\}, \\
&= \lambda_1(r)\frac{(\tilde{\sigma}_x \times \tilde{\sigma}_y)}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}} + \frac{\lambda_2(r)}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}\left\{Ex'\tilde{\sigma}_y + F(y'\tilde{\sigma}_y - x'\tilde{\sigma}_x) - Gy'\tilde{\sigma}_x\right\}, \\
&= \tilde{\lambda}_1(r)\tilde{U}_1(r) + \tilde{\lambda}_2(r)\tilde{P}_1(r),
\end{aligned}$$

where $\tilde{\lambda}_1(r) = \lambda_1(r)$ and $\tilde{\lambda}_2(r) = \lambda_2(r)$ are smooth functions. Thus $\tilde{\delta}(r)$ is the linear combination of $\tilde{U}_1(r)$ and $\tilde{P}_1(r)$. This proves that $\tilde{\delta}(r)$ is a Darboux normal curve on the surface $\tilde{Q}$. □

Theorem 3.4. *Let Q and $\tilde{Q}$ be smooth surfaces and let $J : Q \to \tilde{Q}$ be an isometry. Suppose $\delta(r)$ and $\tilde{\delta}(r)$ are Darboux normal curves on surfaces Q and $\tilde{Q}$ respectively. Then under the isometry, the component of the position vector of the curve $\delta(r)$ in the direction of any tangent vector $T(r) = a\sigma_x + b\sigma_y$, where a and b are real numbers, to the surface Q is invariant, i.e.,*

$$\tilde{\delta}(r) \cdot (\tilde{T}(r) = (a\tilde{\sigma}_x + b\tilde{\sigma}_y)) = \delta(r) \cdot (T(r) = (a\sigma_x + b\sigma_y)).$$

Proof. Let $T(r) = a\sigma_x + b\sigma_y$ be an arbitrary tangent vector to the surface Q at the point $\delta(r)$, where a and b are real numbers. Then we can write

$$\begin{aligned}
\delta(r) \cdot T(r) &= (\lambda_1(r)\frac{(\sigma_x \times \sigma_y)}{\sqrt{EG - F^2}} + \frac{\lambda_2(r)}{\sqrt{EG - F^2}}\left\{Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\right\}) \cdot (a\sigma_x + b\sigma_y), \\
\Rightarrow \delta(r) \cdot T(r) &= \frac{\lambda_2(r)}{\sqrt{EG - F^2}}\left\{Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\right\} \cdot (a\sigma_x + b\sigma_y), \\
\Rightarrow \delta(r) \cdot T(r) &= \frac{\lambda_2(r)}{\sqrt{EG - F^2}}\left\{bx'EG + ay'F^2 - bx'F^2 - ay'EG\right\}, \\
\Rightarrow \delta(r) \cdot T(r) &= \frac{\lambda_2(r)}{\sqrt{EG - F^2}}(EG - F^2)(x'b - ay'), \\
\Rightarrow \delta(r) \cdot T(r) &= \lambda_2(r)(\sqrt{EG - F^2})(x'b - ay'). \qquad (10)
\end{aligned}$$

Similarly, we get

$$\tilde{\delta}(r) \cdot \tilde{T}(r) = \tilde{\lambda}_2(r)(\sqrt{EG - F^2})(x'b - ay'). \qquad (11)$$

In view of (10) and (11), we get

$$\tilde{\delta}(r) \cdot \tilde{T}(r) - \delta(r) \cdot T(r) = (\tilde{\lambda}_2(r) - \lambda_2(r))(\sqrt{EG - F^2})(x'b - ay').$$

Since $\delta(r)$ and $\tilde{\delta}(r)$ are respectively Darboux normal curves on the surface Q and $\tilde{Q}$, we can write $\tilde{\lambda}_2(r) = \lambda_2(r)$. Therefore, $\tilde{\delta}(r) \cdot \tilde{T}(r) = \delta(r) \cdot T(r)$. □

Theorem 3.5. *Let $J : Q \to \tilde{Q}$ be an isometry between smooth surfaces Q and $\tilde{Q}$. If $\delta(r)$ and $\tilde{\delta}(r)$ are Darboux normal curves on the surfaces Q and $\tilde{Q}$ respectively, then under the isometry, the component of the position vector of the curves $\delta(r)$ and $\tilde{\delta}(r)$ along the principal normals $N_1(r)$ and $\tilde{N}_1(r)$ to the curves, the following relation holds:*

$$\tilde{\delta}(r) \cdot \tilde{N}_1(r) - \delta(r) \cdot N_1(r) = \lambda_1(r)\left\{\frac{\tilde{\kappa}_n}{\tilde{\kappa}(r)} - \frac{\kappa_n}{\kappa(r)}\right\} + \lambda_2(r)\zeta(E, F, G, E_x, E_y, F_x, F_y, G_x, G_y)\left\{\frac{1}{\tilde{\kappa}(r)} - \frac{1}{\kappa(r)}\right\},$$

where $\zeta(E, F, G, E_x, E_y, F_x, F_y, G_x, G_y)$ *are defined by equation* (18).

Proof. Let J be an isometric transformation between the surfaces Q and $\tilde{Q}$, and also let δ be a Darboux normal curve on the surface Q. Then, under the isometry between Q and $\tilde{Q}$, we have

$$\tilde{E} = E, \ \tilde{F} = F, \ and \ \tilde{G} = G. \tag{12}$$

Since $E = (\sigma_x \cdot \sigma_x)$, $F = (\sigma_x \cdot \sigma_y)$ and $G = (\sigma_y \cdot \sigma_y)$, upon differentiation with respect to x and y, we obtain

$$\begin{aligned} E_x = (\sigma_x \cdot \sigma_x)_x &= 2\sigma_{xx} \cdot \sigma_x \\ \Rightarrow \quad \sigma_{xx} \cdot \sigma_x &= \frac{E_x}{2}. \end{aligned} \tag{13}$$

Likewise, we can determine

$$\left.\begin{aligned} \sigma_{xx} \cdot \sigma_y = F_x - \frac{E_y}{2}, \ \sigma_{xy} \cdot \sigma_x = \frac{E_y}{2}, \ \sigma_{xy} \cdot \sigma_y = \frac{G_x}{2}, \\ \sigma_{yy} \cdot \sigma_x = F_y - \frac{G_x}{2}, \ \sigma_{yy} \cdot \sigma_y = \frac{G_y}{2}. \end{aligned}\right\} \tag{14}$$

Now, from equation (12), we can write

$$\tilde{E}_x = E_x, \ \tilde{E}_y = E_y, \ \tilde{F}_x = F_x, \ \tilde{F}_y = F_y, \ \tilde{G}_x = G_x, \ \tilde{G}_y = G_y. \tag{15}$$

The component of $\delta(r)$ in the direction normal to the curve yields

$$\begin{aligned} \delta(r) \cdot N_1(r) &= \{\lambda_1(r)\frac{(\sigma_x \times \sigma_y)}{\sqrt{EG - F^2}} + \frac{\lambda_2(r)}{\sqrt{EG - F^2}}\{Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\}\} \\ &\quad \{\frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\}, \\ \Rightarrow \delta(r) \cdot N_1(r) &= \frac{\lambda_1(r)}{\kappa(r)\sqrt{EG - F^2}}\{x''(\sigma_x \times \sigma_y) \cdot \sigma_x + y''(\sigma_x \times \sigma_y) \cdot \sigma_y + x'^2(\sigma_x \times \sigma_y) \cdot \sigma_{xx} \\ &\quad + 2x'y'(\sigma_x \times \sigma_y) \cdot \sigma_{xy} + y'^2(\sigma_x \times \sigma_y) \cdot \sigma_{yy}\} + \frac{\lambda_1(r)}{\kappa(r)\sqrt{EG - F^2}}\{Ex'\sigma_y \\ &\quad + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\} \cdot \{x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}\}. \end{aligned} \tag{16}$$

After simplification and employing equations (13) and (14), we obtain

$$\begin{aligned} \delta(r) \cdot N_1(r) &= \frac{\lambda_1(r)}{\kappa(r)\sqrt{EG - F^2}}\{x'^2L + 2x'y'M + y'^2N\} + \frac{\lambda_2(r)}{\kappa(r)\sqrt{EG - F^2}}\{(y''x' - y'x'') \\ &\quad (EG - F^2) + x'^3\{E(F_x - \frac{E_y}{2}) - F\frac{E_x}{2}\} + 2x'^2y'\{E\frac{G_x}{2} - F\frac{E_y}{2}\} + x'y'^2\{E\frac{G_y}{2} \\ &\quad - F(F_y - \frac{G_x}{2})\} + x'^2y'\{F(F_x - \frac{E_y}{2}) - G\frac{E_x}{2}\} + 2x'y'^2\{F\frac{G_x}{2} - G\frac{E_y}{2}\} \\ &\quad + y'^3\{F\frac{G_y}{2} - G(F_y - \frac{G_x}{2})\}\}, \\ \Rightarrow \delta(r) \cdot N_1(r) &= \frac{\lambda_1(r)}{\kappa(r)}\kappa_n + \frac{\lambda_2(r)}{\kappa(r)}\zeta(E, F, G, E_x, E_y, F_x, F_y, G_x, G_y), \end{aligned} \tag{17}$$

where

$$\zeta(E,F,G,E_x,E_y,F_x,F_y,G_x,G_y) = \frac{1}{\sqrt{EG-F^2}}\{(y''x'-y'x'')(EG-F^2)+x'^3\{E(F_x-\frac{E_y}{2})-F\frac{E_x}{2}\} +2x'^2y'\{E\frac{G_x}{2}-F\frac{E_y}{2}\}+x'y'^2\{E\frac{G_y}{2}-F(F_y-\frac{G_x}{2})\} +x'^2y'\{F(F_x-\frac{E_y}{2})-G\frac{E_x}{2}\}+2x'y'^2\{F\frac{G_x}{2}-G\frac{E_y}{2}\} +y'^3\{F\frac{G_y}{2}-G(F_y-\frac{G_x}{2})\}\}. \quad (18)$$

Similarly, we can express

$$\tilde{\delta}(r)\cdot\tilde{N}_1(r) = \frac{\tilde{\lambda}_1(r)}{\tilde{\kappa}(r)}\tilde{\kappa}_n+\frac{\tilde{\lambda}_2(r)}{\tilde{\kappa}(r)}\zeta(E,F,G,E_x,E_y,F_x,F_y,G_x,G_y). \quad (19)$$

Since $\delta(r)$ and $\tilde{\delta}(r)$ are Darboux normal curves on the surfaces Q and $\tilde{Q}$ respectively, we can express $\tilde{\lambda}_1(r) = \lambda_1(r)$ and $\tilde{\lambda}_2(r) = \lambda_2(r)$.

By subtracting equation (17) from (19), we obtain

$$\tilde{\delta}(r)\cdot\tilde{N}_1(r)-\delta(r)\cdot N_1(r) = \lambda_1(r)\{\frac{\tilde{\kappa}_n}{\tilde{\kappa}(r)}-\frac{\kappa_n}{\kappa(r)}\}+\lambda_2(r)\{\frac{1}{\tilde{\kappa}(r)}-\frac{1}{\kappa(r)}\}\zeta(E,F,G,E_x,E_y,F_x,F_y,G_x,G_y).$$

This proves the result. □

Corollary 3.6. *Let Q and $\tilde{Q}$ be two smooth surfaces and $J : Q \to \tilde{Q}$ be an isometry. Suppose $\delta(r)$ and $\tilde{\delta}(r)$ are Darboux normal curves on the surfaces Q and $\tilde{Q}$ respectively. Then under the isometry, the component of the position vector of the curve $\delta(r)$ in the direction of the normal vector to the curve at $\delta(r)$ is invariant, if $\frac{\tilde{\kappa}_n}{\tilde{\kappa}(r)} = \frac{\kappa_n}{\kappa(r)}$ and $\frac{1}{\tilde{\kappa}(r)} = \frac{1}{\kappa(r)}$, i.e., if*

$$\tilde{\delta}(r)\cdot\tilde{N}_1(r) = \delta(r)\cdot N_1(r), \text{ i.e., if } \frac{\tilde{\kappa}_n}{\tilde{\kappa}(r)} = \frac{\kappa_n}{\kappa(r)} \text{ and } \frac{1}{\tilde{\kappa}(r)} = \frac{1}{\kappa(r)} \text{ holds.}$$

Theorem 3.7. *Let Q and $\tilde{Q}$ be two smooth surfaces and $J : Q \to \tilde{Q}$ be an isometry. If $\delta(r)$ and $\tilde{\delta}(r)$ are Darboux normal curves on the surfaces Q and $\tilde{Q}$ respectively. Then under the isometry, for the component of the position vector of the curve $\delta(r)$ and $\tilde{\delta}(r)$ along the binormal vector $B_1(r)$ and $\tilde{B}_1(r)$, the following relation holds*

$$\tilde{\delta}(r)\cdot\tilde{B}_1(r)-\delta(r)\cdot B_1(r) = \lambda_1(r)\eta(E,F,G)\left\{\frac{1}{\tilde{\kappa}(r)}-\frac{1}{\kappa(r)}\right\}+\lambda_2(r)\left\{\frac{\tilde{\kappa}_n}{\tilde{\kappa}(r)}-\frac{\kappa_n}{\kappa(r)}\right\},$$

where $\eta(E,F,G) = \sqrt{EG-F^2}\left\{(y''x'-y'x'')+x'^3\Gamma^2_{11}+2x'^2y'\Gamma^2_{12}+x'y'^2\Gamma^2_{22}-x'^2y'\Gamma^1_{11}+2x'y'^2\Gamma^1_{12}-y'^3\Gamma^1_{22}\right\}$.

Proof. Let $J : Q \to \tilde{Q}$ be an isometry between the smooth surfaces Q and $\tilde{Q}$ and $\delta(r)$ be Darboux normal curve on the surface Q. From (5) and (9), it follows that

$$\delta(r)\cdot B_1(r) = \lambda_1(r)\frac{(\sigma_x\times\sigma_y)}{\sqrt{EG-F^2}}\cdot B_1(r)+\frac{\lambda_2(r)}{\sqrt{EG-F^2}}\left\{Ex'\sigma_y+F(y'\sigma_y-x'\sigma_x)-Gy'\sigma_x\right\}\cdot B_1(r).$$
$$= \frac{\lambda_1(r)}{\kappa(r)\sqrt{EG-F^2}}\{(y''x'-y'x'')U_1\cdot(\sigma_x\times\sigma_y)+x'^3(\sigma_x\times\sigma_y)\cdot(\sigma_x\times\sigma_{xx}) +2x'^2y'(\sigma_x\times\sigma_y)\cdot(\sigma_x\times\sigma_{xy})+x'y'^2(\sigma_x\times\sigma_y)\cdot(\sigma_x\times\sigma_{yy})+x'^2y'(\sigma_x\times\sigma_y)\cdot(\sigma_y\times\sigma_{xx}) +2x'y'^2(\sigma_x\times\sigma_y)\cdot(\sigma_y\times\sigma_{xy})+y'^3(\sigma_x\times\sigma_y)(\sigma_y\times\sigma_{yy})\}+\frac{\lambda_2(r)}{\kappa(r)\sqrt{EG-F^2}}\{Ex'\sigma_y +F(y'\sigma_y-x'\sigma_x)-Gy'\sigma_x\}\cdot\{(y''x'-y'x'')U_1+x'^3(\sigma_x\times\sigma_{xx})+2x'^2y'(\sigma_x\times\sigma_{xy}) +x'y'^2(\sigma_x\times\sigma_{yy})+x'^2y'(\sigma_y\times\sigma_{xx})+2x'y'^2(\sigma_y\times\sigma_{xy})+y'^3(\sigma_y\times\sigma_{yy})\}. \quad (20)$$

After simplifying and applying equations (16) and (17), we obtain

$$\begin{aligned}
\delta(r)\cdot B_1(r) &= \frac{\lambda_1(r)}{\kappa(r)\sqrt{EG-F^2}}\{(y''x'-y'x'')(EG-F^2)+x'^3\{E(F_x-\frac{E_y}{2})-F\frac{E_x}{2}\}\\
&+2x'^2y'\{E\frac{G_x}{2}-F\frac{E_y}{2}\}+x'y'^2\{E\frac{G_y}{2}-F(F_y-\frac{G_x}{2})\}+x'^2y'\{F(F_x-\frac{E_y}{2})-G\frac{E_x}{2}\}\\
&+2x'y'^2\{F\frac{G_x}{2}-G\frac{E_y}{2}\}+y'^3\{F\frac{G_y}{2}-G(F_y-\frac{G_x}{2})\}\}+\frac{\lambda_2(r)}{\kappa(r)\sqrt{EG-F^2}}\{(Ex'^4\\
&+2Fx'^3y'+Gx'^2y'^2)\sigma_x\cdot(\sigma_x\times\sigma_{xx})+(2Ex'^3y'+4Fx'^2y'^2+2Gx'y'^3)\sigma_x\cdot(\sigma_x\times\sigma_{xy})\\
&+(Ex'^2y'^2+2Fx'y'^3+Gy'^4)\sigma_y\cdot(\sigma_x\times\sigma_{yy})\}.\\
\Rightarrow \delta(r)\cdot B_1(r) &= \frac{\lambda_1(r)}{\kappa(r)}\sqrt{EG-F^2}\{(y''x'-y'x'')+x'^3\Gamma^2_{11}+2x'^2y'\Gamma^2_{12}+x'y'^2\Gamma^2_{22}-x'^2y'\Gamma^1_{11}\\
&+2x'y'^2\Gamma^1_{12}-y'^3\Gamma^1_{22}\}+\frac{\lambda_1(r)}{\kappa(r)}\{x'^2\sigma_{xx}\cdot U_1+2x'y'\sigma_{xy}\cdot U_1+y'^2\sigma_{yy}\cdot U_1\}, \qquad (21)
\end{aligned}$$

where Γ^w_{uv} (where $u, v, w = 1, 2$) represents the Christoffel symbols of the second kind, and are defined as follows

$$\begin{aligned}
\Gamma^1_{11} &= \tfrac{1}{2H^2}\{GE_x+F[E_y-2F_x]\}, \quad \Gamma^2_{11} = \tfrac{1}{2H^2}\{E[2F_x-E_y]-FE_y\},\\
\Gamma^2_{22} &= \tfrac{1}{2H^2}\{EG_y+F[G_y-2F_y]\}, \quad \Gamma^1_{22} = \tfrac{1}{2H^2}\{G[2F_y-G_x]-FG_y\},\\
\Gamma^2_{12} &= \tfrac{1}{2H^2}\{EG_x-FE_y\} = \Gamma^2_{21}, \quad \Gamma^1_{21} = \tfrac{1}{2H^2}\{GE_y-FG_x\} = \Gamma^1_{12},
\end{aligned}$$

where $H^2 = (EG - F^2)$. Now equation (21), implies that

$$\begin{aligned}
\delta(r)\cdot B_1(r) &= \frac{\lambda_1(r)}{\kappa(r)}\sqrt{EG-F^2}\{(y''x'-y'x'')+x'^3\Gamma^2_{11}+2x'^2y'\Gamma^2_{12}+x'y'^2\Gamma^2_{22}-x'^2y'\Gamma^1_{11}\\
&+2x'y'^2\Gamma^1_{12}-y'^3\Gamma^1_{22}\}+\frac{\lambda_1(r)}{\kappa(r)}\{x'^2L+2x'y'M+y'^2N\},\\
\Rightarrow \delta(r)\cdot B_1(r) &= \frac{\lambda_1(r)}{\kappa(r)}\sqrt{EG-F^2}\{(y''x'-y'x'')+x'^3\Gamma^2_{11}+2x'^2y'\Gamma^2_{12}+x'y'^2\Gamma^2_{22}-x'^2y'\Gamma^1_{11}\\
&+2x'y'^2\Gamma^1_{12}-y'^3\Gamma^1_{22}\}+\frac{\lambda_1(r)}{\kappa(r)}\kappa_n. \qquad (22)
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
\tilde{\delta}(r)\cdot\tilde{B}_1(r) &= \frac{\tilde{\lambda}_1(r)}{\tilde{\kappa}(r)}\sqrt{EG-F^2}\{(y''x'-y'x'')+x'^3\tilde{\Gamma}^2_{11}+2x'^2y'\tilde{\Gamma}^2_{12}+x'y'^2\tilde{\Gamma}^2_{22}-x'^2y'\tilde{\Gamma}^1_{11}\\
&+2x'y'^2\tilde{\Gamma}^1_{12}-y'^3\tilde{\Gamma}^1_{22}\}+\frac{\tilde{\lambda}_2(r)}{\tilde{\kappa}(r)}\tilde{\kappa}_n. \qquad (23)
\end{aligned}$$

Since $\delta(r)$ and $\tilde{\delta}(r)$ are Darboux normal curves on the surfaces Q and $\tilde{Q}$ respectively, and J is an isometry between the smooth surfaces Q and $\tilde{Q}$, we have $\tilde{\lambda}_1(r) = \lambda_1(r)$, $\tilde{\lambda}_2(r) = \lambda_2(r)$ and $\tilde{\Gamma}^w_{uv} = \Gamma^w_{uv}$, for $u, v, w = \{1, 2\}$. Then equation (23) becomes

$$\begin{aligned}
\tilde{\delta}(r)\cdot\tilde{B}_1(r) &= \frac{\lambda_1(r)}{\tilde{\kappa}(r)}\sqrt{EG-F^2}\{(y''x'-y'x'')+x'^3\Gamma^2_{11}+2x'^2y'\Gamma^2_{12}+x'y'^2\Gamma^2_{22}-x'^2y'\Gamma^1_{11}\\
&+2x'y'^2\Gamma^1_{12}-y'^3\Gamma^1_{22}\}+\frac{\lambda_2(r)}{\tilde{\kappa}(r)}\tilde{\kappa}_n. \qquad (24)
\end{aligned}$$

The difference of (22) and (24) entails

$$\begin{aligned}\tilde{\delta}(r)\cdot\tilde{B}_1(r)-\delta(r)\cdot B_1(r) &= \lambda_1(r)\sqrt{EG-F^2}\{(y''x'-y'x'')+x'^3\Gamma^2_{11}+2x'^2y'\Gamma^2_{12}+x'y'^2\Gamma^2_{22}\\ &\quad -x'^2y'\Gamma^1_{11}+2x'y'^2\Gamma^1_{12}-y'^3\Gamma^1_{22}\}\{\frac{1}{\tilde{\kappa}(r)}-\frac{1}{\kappa(r)}\}+\lambda_1(r)\{\frac{\tilde{\kappa}_n}{\tilde{\kappa}(r)}-\frac{\kappa_n}{\kappa(r)}\},\\ \Rightarrow \tilde{\delta}(r)\cdot\tilde{B}_1(r)-\delta(r)\cdot B_1(r) &= \lambda_1(r)\eta(E,F,G)\{\frac{1}{\tilde{\kappa}(r)}-\frac{1}{\kappa(r)}\}+\lambda_2(r)\{\frac{\tilde{\kappa}_n}{\tilde{\kappa}(r)}-\frac{\kappa_n}{\kappa(r)}\},\end{aligned}$$

where

$$\eta(E,F,G)=\sqrt{EG-F^2}\{(y''x'-y'x'')+x'^3\Gamma^2_{11}+2x'^2y'\Gamma^2_{12}+x'y'^2\Gamma^2_{22}-x'^2y'\Gamma^1_{11}+2x'y'^2\Gamma^1_{12}-y'^3\Gamma^1_{22}\}.$$

This proves the result. □

Corollary 3.8. *Let Q and $\tilde{Q}$ be two smooth surfaces and $J : Q \to \tilde{Q}$ be an isometry between them. Suppose $\delta(r)$ and $\tilde{\delta}(r)$ are Darboux normal curves on the surfaces Q and $\tilde{Q}$ respectively. Then under the isometric transformation, the component of the position vector of the curve $\delta(r)$ along the binormal vector $B_1(r)$ to the curve at $\delta(r)$ remains unchanged, if $\tilde{\kappa}(r)=\kappa(r)$ and $\frac{\tilde{\kappa}_n}{\tilde{\kappa}(r)}=\frac{\kappa_n}{\kappa(r)}$ holds.*

4. Conclusion

In this study, we examined the matrix representation of the Frenet frame and Darboux frame, establishing the relationship between them. Additionally, we explored the invariant sufficient conditions for the isometric image of Darboux normal curves on smooth surfaces immersed in Euclidean space. Finally, we computed the deviations along the normal and tangential components of the position vector of the Darboux normal curve under surface isometry and obtained the conditions under which these components remain invariant during isometric transformation.

For further research, one could introduce the concepts of Darboux rectifying and Darboux osculating curves under conformal and isometric transformations on smooth surfaces in Euclidean 3-space, using the Darboux frame. Additionally, these findings could be extended to Euclidean 4-space by incorporating the concepts of both the Darboux frame and the Frenet frame.

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Data availability
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Conflict of interest
The authors affirm that there are no conflicts of interest.

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