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On maps preserving C-symmetric triple Jordan product of pairs of operators

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Abstract. A bounded linear operator T, which operates on a complex separable Hilbert space H, is referred to as C-symmetric if $T = CT^*C$, where C represents a conjugate-linear isometric involution on H. In this paper, we thoroughly investigate mappings $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the algebra of all bounded linear operators on H, that fulfill the following condition:

$$ABA \text{ is } C\text{-symmetric} \iff \Phi(A)\Phi(B)\Phi(A) \text{ is } C\text{-symmetric}$$

for all $A, B \in \mathcal{B}(H)$ and conjugations C on H.

1. Introduction

In the present paper, we shall consistently adopt the symbol H to represent a separable Hilbert space, over the complex field $\mathbb{C}$, of dimension greater than two, endowed with the inner product $\langle \cdot, \cdot \rangle$. The notation $\mathcal{B}(H)$ will refer to the algebra of all bounded linear operators that act on H, and the identity operator is denoted by I.

A conjugate-linear map $C : H \to H$ is called a *conjugation on* H if it is isometric; i.e., $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in H$, and is involutive; i.e., $C^2 = I$.

One can readily observe that, for any orthonormal basis $\mathcal{E}$, the map K, defined by

$$Kx = \sum_{e\in\mathcal{E}} \overline{\langle x, e\rangle} e \quad \text{for every } x \in H,$$

is a conjugation on H. It is important to note that this specific conjugation satisfies $Ke = e$ for every $e \in \mathcal{E}$.

Conversely, it is proved in [11, Lemma 1], that for any conjugation C on H, there is a corresponding orthonormal basis $\mathcal{E}$ satisfying the condition:

$$Ce = e \quad \text{for every } e \in \mathcal{E}. \tag{1}$$

An operator $T \in \mathcal{B}(H)$ is referred to as *C-symmetric* if it fulfills the equality $T = CT^*C$, where T^* denotes the adjoint of T. We also use the term *complex symmetric* when there is no need to specify the conjugation.

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Notably, if T is C-symmetric, then T^* and λT are also C-symmetric for any $\lambda \in \mathbb{C}$. Moreover, it can be easily seen that the set of all C-symmetric operators, denoted by $\mathcal{L}_C$, is a closed subspace of $\mathcal{B}(H)$ and is stable under the triple Jordan product $A.B = ABA$.

It is well-known that an operator is complex symmetric if and only if, in some orthonormal basis, it can be represented by a symmetric (i.e., self-transpose) matrix (see [11]). In fact, for any operator $T \in \mathcal{B}(H)$ and an orthonormal basis $\mathcal{E}$ of H that satisfies (1), it can be verified that:

$$T \text{ is } C\text{-symmetric} \iff \langle Te, f\rangle = \langle Tf, e\rangle \text{ for all } e, f \in \mathcal{E}; \tag{2}$$

implying that T possesses a symmetric matrix relative to $\mathcal{E}$.

The pioneering investigation of such operators was conducted by Garcia, Putinar, and Wogen ([9–12, 14]). Subsequently, complex symmetry has captivated the attention of numerous researchers. The applications of complex symmetric operators extend across various disciplines, including complex analysis, matrix theory, differential equations, function theory, and even quantum mechanics ([9–12, 18]).

Given a subset $\mathcal{L} \subset \mathcal{B}(H)$, a map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is said to *preserve the triple Jordan product of* $\mathcal{L}$ if:

$$ABA \in \mathcal{L} \quad \Longrightarrow \quad \Phi(A)\Phi(B)\Phi(A) \in \mathcal{L} \quad \text{for all } A, B \in \mathcal{B}(H),$$

and it is referred to as *preserves the triple Jordan product of* $\mathcal{L}$ *in both directions* if:

$$ABA \in \mathcal{L} \quad \Longleftrightarrow \quad \Phi(A)\Phi(B)\Phi(A) \in \mathcal{L} \quad \text{for all } A, B \in \mathcal{B}(H).$$

In recent decades, significant attention has been drawn to the intriguing *Non-linear preserving problem*. This area revolves around describing maps Φ on matrix algebras, linear operator algebras, or more generally, Banach algebras, that retain specific subsets or properties under certain operations, without assuming linearity or additivity in Φ. Notably, researchers have extensively investigated the special case $\mathcal{L} = \{0\}$ (see for instance, [6–8]).

In a noteworthy result ([7, Theorem 2.2]), it was shown that if X is an infinite-dimensional Banach space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ is a surjective map that preserves the triple Jordan product of $\{0\}$ in both directions, then there exists $f : \mathcal{B}(X) \to \mathbb{F} \setminus \{0\}$ such that one of the following statements holds:

(i) There exists a bounded linear (or conjugate-linear) bijective operator $T : X \to X$ such that

$$\Phi(A) = f(A)TAT^{-1} \quad \text{for every } A \in \mathcal{B}(X).$$

(ii) X is reflexive and there exists a bounded linear (or conjugate-linear) bijective operator $T : X^* \to X$ such that

$$\Phi(A) = f(A)TA^*T^{-1} \quad \text{for every } A \in \mathcal{B}(X);$$

here $A^* : X^* \to X^*$ denotes the Banach space adjoint of A.

The problem of characterizing maps on $\mathcal{B}(H)$, when H is finite-dimentional, that preserve the Jordan product of $\mathcal{L}_C$, in both directions, was considered in [1]. It is proved that those maps must satisfy

$$\Phi(A) = f(A)A \ \text{ for every } A \in \mathcal{B}(H) \quad \text{or} \quad \Phi(A) = f(A)A^* \ \text{ for every } A \in \mathcal{B}(H)$$

for some map $f : \mathcal{B}(H) \to \mathbb{C} \setminus \{0\}$.

In the context of linear preserving problems, the authors in [15] established an interesting result. They proved that if H is infinite-dimensional, C is a conjugation on H, and $\Phi : \mathcal{L}_C \to \mathcal{L}_C$ is an additive surjection preserving the usual product of $\{0\}$ in both directions, then there must exist $c \in \mathbb{C}$ and a linear (or conjugate-linear) bijection $T : H \to H$ satisfying $TCT^*C = I$ so that

$$\Phi(A) = cTACT^*C \quad \text{for every } A \in \mathcal{L}_C.$$

Additional findings related to preserving problems involving C-symmetric operators and other operators associated with conjugations have been achieved. For more details, the interested reader is encouraged to consult the references [2–4, 16].

The main objective of this paper is to characterize the maps on $\mathcal{B}(H)$ that preserve the triple Jordan product of $\mathcal{L}_C$ in both directions, for every conjugation C on H.

2. Main Result and Proof

For an orthonormal basis $\mathcal{E}$ of H, the matrix of an operator $T \in \mathcal{B}(H)$ with respect to $\mathcal{E}$ is denoted by $\mathcal{M}_{\mathcal{E}}(T)$. More precisely, if $\mathcal{E} = \{e_n\}_n$, then

$$\mathcal{M}_{\mathcal{E}}(T) = (\langle Te_m, e_n \rangle)_{n,m} .$$

The main result of this paper is presented in the following theorem.

Theorem 2.1. *Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a map. The following statements are equivalent:*

(i) *For all $A, B \in \mathcal{B}(H)$ and every orthonormal basis $\mathcal{E}$ of H,*

$$\mathcal{M}_{\mathcal{E}}(ABA) \text{ is symmetric} \quad \Longleftrightarrow \quad \mathcal{M}_{\mathcal{E}}\left(\Phi(A)\Phi(B)\Phi(A)\right) \text{ is symmetric.}$$

(ii) *For all $A, B \in \mathcal{B}(H)$ and every conjugation C on H,*

$$ABA \in \mathcal{L}_C \quad \Longleftrightarrow \quad \Phi(A)\Phi(B)\Phi(A) \in \mathcal{L}_C.$$

(iii) *There exists $f : \mathcal{B}(H) \to \mathbb{C} \setminus \{0\}$ such that*

$$\Phi(A) = f(A)A \text{ for every } A \in \mathcal{B}(H) \quad \text{or} \quad \Phi(A) = f(A)A^* \text{ for every } A \in \mathcal{B}(H).$$

The hypothesis that the map Φ preserves the triple Jordan product of $\mathcal{L}_C$ in one direction, for every conjugation C, is not adequate to fully characterize Φ. This inadequacy is illustrated through an example presented in [1] within the context of Jordan product preservers of $\mathcal{L}_C$. For the sake of completness, we will make minor modifications to that example to demonstrate the essential role of "both directions" in Theorem 2.1.

Example 2.2. *Consider the operator $A_t = B_t \oplus 0 \in \mathcal{B}(H)$, where B_t is defined, with respect to an orthonormal subset $\{e_1, e_2, e_3\}$ of H, as follows:*

$$B_t = \begin{bmatrix} 1 & t & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix} .$$

Define $\Omega = \{A_t : t > 0\}$, and let us show that any map Φ vanishing on $\mathcal{B}(H) \setminus \Omega$ must satisfy the direct implication of assertion (ii) of Theorem 2.1 for every conjugation.

If $T = A_t$ and $S = A_s$ where $t, s > 0$, we have $TST = A_{2t+s}$. Moreover, letting $r = 2t + s$, calculations show that the trace of the following operator

$$B_r^* B_r^2 B_r^{*2} B_r - B_r B_r^{*2} B_r^2 B_r^*$$

is equal to $r^2 > 0$. Hence, by [13, Proposition 2.5], the operator B_r is not J-symmetric for any conjugation J on $\operatorname{Span}\{e_1, e_2, e_3\}$. *Furthermore, it follows from [14, Lemma 1] that*

$$TST = B_r \oplus 0 \notin \mathcal{L}_C \text{ for any conjugation } C \text{ on } H.$$

On the other hand, if $T \notin \Omega$ or $S \notin \Omega$, then

$$\Phi(T)\Phi(S)\Phi(T) = 0 \in \mathcal{L}_C \text{ for all conjugations } C \text{ on } H.$$

Remark 2.3. *The equivalence between statements* (i) *and* (ii) *can easily be deduced from* (2). *Additionally, the implication* (iii)⇒(ii) *is obviously true. Thus, our focus now lies solely on proving the implication* (ii)⇒(iii).

For the remainder of this section, we denote by $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ a mapping that fulfills the second assertion of Theorem 2.1.

To simplify notation throughout this paper, we will use the symbol $T \sim S$ to denote that for every conjugation C, the following equivalence holds:

$$T \text{ is } C\text{-symmetric} \iff S \text{ is } C\text{-symmetric}.$$

Given an orthonormal basis $\mathcal{E}$ of H, we shall say that an operator $T \in \mathcal{B}(H)$ is $\mathcal{E}$-diagonal if, for every $e \in \mathcal{E}$, there exists some $\alpha_e \in \mathbb{C}$ such that $Te = \alpha_e e$.

Proposition 2.4. *Let $A, B \in \mathcal{B}(H)$ be such that $A = A_1 \oplus A_2$ with respect to an orthogonal decomposition $H = H_1 \oplus H_2$, where $H_2 \neq \{0\}$, A_1 is a C_1-symmetric operator, and A_2 is $\{e_i\}_{i \in I_2}$-diagonal. If $A \sim B$, then*

(i) *$B = B_1 \oplus B_2$ with respect to the decomposition $H = H_1 \oplus H_2$, where B_1 is C_1-symmetric and B_2 is $\{e_i\}_{i \in I_2}$-diagonal.*
(ii) *For $n, m \in I_2$, if $\langle A_2 e_n, e_n\rangle = \langle A_2 e_m, e_m\rangle$, then $\langle B_2 e_n, e_n\rangle = \langle B_2 e_m, e_m\rangle$.*

Proof. Let C_2 be the conjugate-linear map given by $C_2 \left(\sum \alpha_i e_i\right) = \sum \overline{\alpha_i} e_i$. It is easy to check that C_2 is a conjugation on H_2. Furthermore, for every $i \in I_2$,

$$C_2 A_2 C_2 e_i = C_2 A_2 e_i = C_2 \left(\langle A_2 e_i, e_i\rangle e_i\right) = \overline{\langle A_2 e_i, e_i\rangle} e_i = \langle A_2^* e_i, e_i\rangle e_i = A_2^* e_i,$$

and so $C_2 A_2 C_2 = A_2^*$ and A_2 is C_2-symmetric. According to [1, Lemma 2.4], we may write $B = B_1 \oplus B_2$ with respect to the decomposition $H = H_1 \oplus H_2$ with B_1 being C_1-symmetric. Moreover, by the same lemma, if A_2 is J-symmetric, for some conjugation J, then so is B_2. Therefore, by [1, Remark 2.3], B_2 must be $\{e_i\}_{i \in I_2}$-diagonal and $\langle B_2 e_n, e_n\rangle = \langle B_2 e_m, e_m\rangle$ whenever $\langle A_2 e_n, e_n\rangle = \langle A_2 e_m, e_m\rangle$. □

Remark 2.5. *Taking $H_1 = \{0\}$ in Proposition 2.4, we readily see that if two operators T and S are such that $T \sim S$ with T being diagonal, with respect to an orthonormal basis $\mathcal{E}$, then so is S with respect to the same orthonormal basis. In this case, we have*

$$\langle Te, e\rangle = \langle Tf, f\rangle \iff \langle Se, e\rangle = \langle Sf, f\rangle \text{ for all } e, f \in \mathcal{E}.$$

In particular, if $T = \alpha P + \beta(I - P)$ where P is an orthogonal projection and $\alpha, \beta \in \mathbb{C}$, then there exist $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}$ such that the following statements hold:

(i) *$S = \tilde{\alpha} P + \tilde{\beta}(I - P)$.*
(ii) *If $P \neq 0$ and $P \neq I$, then $\alpha = \beta$ if and only if $\tilde{\alpha} = \tilde{\beta}$.*

Proposition 2.6. *Let $A \in \mathcal{B}(H)$ be a non-zero operator such that $A^n = \lambda I$ for some positive integer n and $\lambda \in \mathbb{C}$. Then, there exist orthonormal vectors x and y in H so that $\langle Ax, x\rangle$ and $\langle Ay, y\rangle$ are both non-zero.*

Proof. Since $A \neq 0$, there exists a unit vector $x \in H$ such that $\langle Ax, x\rangle \neq 0$. Suppose for the sake of contradiction that $\langle Ay, y\rangle = 0$ for all $y \in \{x\}^\perp$.

First, we show that the rank of A is less than 3. Let P be the orthogonal projection onto $\{x\}^\perp$. Then, for every $y \in \{x\}^\perp$, we have

$$\langle PAPy, y\rangle = \langle APy, Py\rangle = \langle Ay, y\rangle = 0.$$

Furthermore, it is obvious that $\langle PAPx, x\rangle = 0$, which implies that $\langle PAPh, h\rangle = 0$ for all $h \in H$; that is, $PAP = 0$. Now, noting that $I - P$ is rank-one, we can see that

$$A = PA(I - P) + (I - P)A$$

has rank less than 3.

Note that if $\lambda \neq 0$, then A is invertible, and as $\dim H \geq 3$, the rank of A would be greater than 2; a contradiction.

Now if $\lambda = 0$, then A is nilpotent and finite-rank. Consequently, the trace of A, denoted as trace(A), exists and equals 0. However, choosing any orthonormal basis $\mathcal{E}$ of H containing x, we can observe that

$$\text{trace}(A) = \sum_{e\in\mathcal{E}} \langle Ae, e\rangle = \langle Ax, x\rangle + \sum_{e\in\mathcal{E}, e\neq x} \langle PAPe, e\rangle = \langle Ax, x\rangle \neq 0,$$

which leads to a contradiction. Therefore, there must exist $y \in \{x\}^\perp$ satisfying $\langle Ay, y\rangle \neq 0$. □

Let u and v be non-zero vectors in H. We use $u \otimes v$ to represent the rank-one operator defined as $(u \otimes v)(x) = \langle x, v\rangle u$ for all $x \in H$. It is a recognized fact that each rank-one operator acting on a H can be expressed in this form. Additionally, note that

$$u \otimes v \text{ is diagonal} \iff u \text{ and } v \text{ are linearly dependent}, \tag{3}$$

and that

$$u \otimes v \text{ is an orthogonal projection} \iff u \otimes v = x \otimes x \tag{4}$$

for some $x \in H$ satisfying $\|x\| = 1$.

Next, we provide the form of Φ when it is restricted to the set of multiples of orthogonal projections.

Lemma 2.7. *Let $P \in \mathcal{B}(H)$ be an orthogonal projection, and let $\lambda \in \mathbb{C}$ be non-zero. Then, there exists a non-zero $\alpha_{\lambda,P} \in \mathbb{C}$ such that $\Phi(\lambda P) = \alpha_{\lambda,P} P$.*

Proof. Since $\lambda^3 P = (\lambda P)^3 \sim \Phi(\lambda P)^3$, according to Remark 2.5, there exist $\gamma_{\lambda,P}, \beta_{\lambda,P} \in \mathbb{C}$, with $\gamma_{\lambda,P} \neq \beta_{\lambda,P}$ when $P \notin \{0, I\}$, such that

$$\Phi(\lambda P)^3 = \begin{bmatrix} \gamma_{\lambda,P} I & 0 \\ 0 & \beta_{\lambda,P} I \end{bmatrix} \begin{matrix} \text{Ran}(P) \\ \text{Ker}(P) \end{matrix}. \tag{5}$$

Moreover, noting that $\Phi(\lambda P)$ commutes with $\Phi(\lambda P)^3$, we can see that

$$\Phi(\lambda P) = \begin{bmatrix} A_{\lambda,P} & 0 \\ 0 & B_{\lambda,P} \end{bmatrix} \begin{matrix} \text{Ran}(P) \\ \text{Ker}(P) \end{matrix} \tag{6}$$

for some $A_{\lambda,P} \in \mathcal{B}(\text{Ran}(P))$ and $B_{\lambda,P} \in \mathcal{B}(\text{Ker}(P))$. Clearly, we need to show that $A_{\lambda,P}$ is a non-zero multiple of the identity and that $B_{\lambda,P} = 0$. The remainder of the proof is divided into three claims:

Claim 1. For all orthonormal vectors x and y in H, we have

$$\Phi(x \otimes x)\Phi(y \otimes y)\Phi(x \otimes x) = 0.$$

As $(x \otimes x)(y \otimes y)(x \otimes x) = (y \otimes y)(x \otimes x)(y \otimes y) = 0$, we obtain by Remark 2.5 that

$$\Phi(x \otimes x)\Phi(y \otimes y)\Phi(x \otimes x) = \gamma I \quad \text{and} \quad \Phi(y \otimes y)\Phi(x \otimes x)\Phi(y \otimes y) = \beta I$$

for some $\gamma, \beta \in \mathbb{C}$. We need to show that $\gamma = 0$. For the sake of contradiction, we assume that $\gamma \neq 0$. Then, we have

$$\begin{aligned} \Phi(y \otimes y) = \gamma^{-1}\Phi(y \otimes y)(\gamma I) &= \gamma^{-1}\Phi(y \otimes y)\Phi(x \otimes x)\Phi(y \otimes y)\Phi(x \otimes x) \\ &= \gamma^{-1}(\beta I)\Phi(x \otimes x) = \gamma^{-1}\beta\Phi(x \otimes x), \end{aligned}$$

and hence $\Phi(y \otimes y)^3 = \gamma^{-3}\beta^3\Phi(x \otimes x)^3$. Since, by (4), $x \otimes x$ is an orthogonal projection, it follows from (5) that, for every unit vector $z \in \{x, y\}^\perp \subseteq \text{Ker}(x \otimes x)$,

$$\langle \Phi(y \otimes y)^3 y, y\rangle = \langle \gamma^{-3}\beta^3\Phi(x \otimes x)^3 y, y\rangle = \langle \gamma^{-3}\beta^3\Phi(x \otimes x)^3 z, z\rangle = \langle \Phi(y \otimes y)^3 z, z\rangle.$$

Consequently, Remark 2.5 implies that

$$1 = \langle (y \otimes y)^3 y, y\rangle = \langle (y \otimes y)^3 z, z\rangle = 0;$$

a contradiction. Thus, $\gamma = 0$, and the claim is proved.

Claim 2. For each unit vector $u \in H$, there is a non-zero $\alpha_u \in \mathbb{C}$ satisfying $\Phi(u \otimes u) = \alpha_u u \otimes u$.

It follows from (6) that $\Phi(u \otimes u)$ can be expressed as

$$\Phi(u \otimes u) = \begin{bmatrix} \alpha_u I & 0 \\ 0 & A_u \end{bmatrix} \begin{matrix} \mathrm{Span}\{u\} \\ \{u\}^\perp \end{matrix},$$

where $A_u^3 = \beta_u I$ and $\alpha_u^3 \neq \beta_u$.

Suppose that $A_u \neq 0$. According to Proposition 2.6, there are orthonormal vectors e_1 and e_2 in $\{u\}^\perp$ satisfying $\langle A_u e_1, e_1\rangle \neq 0$ and $\langle A_u e_2, e_2\rangle \neq 0$. Furthermore, by Claim 1, we have $\Phi(e_1 \otimes e_1)\Phi(u \otimes u)\Phi(e_1 \otimes e_1) = 0$, and in particular, we also have $\Phi(e_1 \otimes e_1)^3\Phi(u \otimes u)\Phi(e_1 \otimes e_1)^3 = 0$. So, we obtain

$$\begin{aligned} \alpha_{e_1}^2 \langle A_u e_1, e_1\rangle &= \langle \Phi(u \otimes u)\alpha_{e_1} e_1, \overline{\alpha_{e_1}} e_1\rangle \\ &= \langle \Phi(u \otimes u)\Phi(e_1 \otimes e_1)e_1, \Phi(e_1 \otimes e_1)^* e_1\rangle \\ &= \langle \Phi(e_1 \otimes e_1)\Phi(u \otimes u)\Phi(e_1 \otimes e_1)e_1, e_1\rangle = 0, \end{aligned}$$

and similarly, we get

$$\beta_{e_1}^2 \langle A_u e_2, e_2\rangle = \langle \Phi(e_1 \otimes e_1)^3\Phi(u \otimes u)\Phi(e_1 \otimes e_1)^3 e_2, e_2\rangle = 0.$$

Since $\alpha_u^3 \neq \beta_u$, at least one of α_{e_1} and β_{e_1} is non-zero; this contradicts the above equalities. Consequently, $A_u = 0$ and $\alpha_u^3 \neq \beta_u = 0$. Thus, $\Phi(u \otimes u) = \alpha_u u \otimes u$ with $\alpha_u \neq 0$.

Claim 3: $\Phi(\lambda P)$ has the the desired form.

The reader will see that we can assume, without loss of generality, that $\mathrm{Ran}(P) \neq \{0\}$ and $\mathrm{Ker}(P) \neq \{0\}$. Let $u \in \mathrm{Ran}(P)$ and $v \in \mathrm{Ker}(P)$ be unit vectors. As $(\lambda P)(u \otimes u)(\lambda P) = \lambda^2 u \otimes u$ and $(v \otimes v)(\lambda P)(v \otimes v) = 0$, Remark 2.5 and Claim 2 imply that there exist $\alpha_0, \beta_0, \gamma_0 \in \mathbb{C}$ such that

$$\begin{aligned} (\Phi(\lambda P)u) \otimes (\Phi(\lambda P)^* u) &= \Phi(\lambda P)(u \otimes u)\Phi(\lambda P) = \alpha_u^{-1}\Phi(\lambda P)\Phi(u \otimes u)\Phi(\lambda P) \\ &= \begin{bmatrix} \alpha_0 I & 0 \\ 0 & \beta_0 I \end{bmatrix} \begin{matrix} \mathrm{Span}\{u\} \\ \{u\}^\perp \end{matrix} \end{aligned}$$

with $\alpha_0 \neq \beta_0$, and

$$\langle \Phi(\lambda P)v, v\rangle v \otimes v = (v \otimes v)\Phi(\lambda P)(v \otimes v) = \alpha_v^{-2}\Phi(v \otimes v)\Phi(\lambda P)\Phi(v \otimes v) = \gamma_0 I.$$

As $\dim H > 2$, we must have $\beta_0 = \gamma_0 = 0$. Consequently, $\alpha_0 \neq 0 = \beta_0$,

$$0 \neq A_{\lambda,P} u = \Phi(\lambda P)u \in \mathrm{Span}\{u\},$$

and

$$\langle B_{\lambda,P} v, v\rangle = \langle \Phi(\lambda P)v, v\rangle = 0.$$

Since u and v were arbitrary, we get that $A_{\lambda,P} = \alpha_{\lambda,P} I$ for some non-zero $\alpha_{\lambda,P} \in \mathbb{C}$, and that $B_{\lambda,P} = 0$. This proves the last claim and ends the proof of the lemma. □

Remark 2.8. *Using elementary linear algebra, one can easily show that every rank-one operator $x \otimes y$ in $\mathcal{B}(H)$ can be presented as*

$$x \otimes y = (au + bv) \otimes v = \begin{bmatrix} 0 & a & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} u \\ v \\ \{u, v\}^\perp \end{matrix} \tag{7}$$

where u and v are orthonormal vectors and $a, b \in \mathbb{C}$. Moreover, according to [11, Example 6], the operator

$$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \begin{matrix} u \\ v \end{matrix}$$

is J-symmetric for some conjugation J on $\mathrm{Span}\{u, v\}$.

The next lemma reveals Φ on the set of rank-one operators. Denote such set by $\mathcal{F}_1(H)$.

Lemma 2.9. *For every $R \in \mathcal{F}_1(H)$ there exists a non-zero $\alpha_R \in \mathbb{C}$ such that either $\Phi(R) = \alpha_R R$ or $\Phi(R) = \alpha_R R^*$.*

Proof. Let $R \in \mathcal{F}_1(H)$. According to Lemma 2.7, we may assume that R is not a multiple of an orthogonal projection. In this case, by (7) and (3), we can write

$$R = \begin{bmatrix} 0 & a & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} u \\ v \\ \{u, v\}^\perp \end{matrix}$$

where u and v are orthonormal vectors in H and $a, b \in \mathbb{C}$ with $a \neq 0$. By Lemma 2.7, we have $\Phi(I) = \alpha_I I$ with $\alpha_I \neq 0$; therefore

$$R = IRI \sim \Phi(I)\Phi(R)\Phi(I) \sim \alpha_I^2 \Phi(R) \sim \Phi(R).$$

Then, Remark 2.8 and Proposition 2.4 imply that

$$\Phi(R) = \begin{bmatrix} \eta & \alpha & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \lambda I \end{bmatrix} \begin{matrix} u \\ v \\ \{u, v\}^\perp \end{matrix} \tag{8}$$

for some $\alpha, \beta, \eta, \gamma, \lambda \in \mathbb{C}$. Let P be the orthogonal projection onto $\{u, v\}^\perp$. Since $(u \otimes u)R(u \otimes u) = PRP = 0$, by Remark 2.5, the operators $\Phi(u \otimes u)\Phi(R)\Phi(u \otimes u)$ and $\Phi(P)\Phi(R)\Phi(P)$ are both multiples of the identity, and so, by Lemma 2.7, we get

$$\eta u \otimes u = (u \otimes u)\Phi(R)(u \otimes u) \in \mathbb{C}I \quad \text{and} \quad \lambda P = P\Phi(R)P \in \mathbb{C}I,$$

and consequently, as $u \otimes u \neq I$ and $P \neq I$, we must have $\eta = \lambda = 0$.

Using a similar argument, one can show that

$$b = 0 \quad \Longleftrightarrow \quad \gamma = 0. \tag{9}$$

Note that if α and β are both zero, then $(I - P)\Phi(R)(I - P)$ is a non-zero multiple of $v \otimes v$. It follows by Lemma 2.7 that

$$v \otimes v \sim (I - P)\Phi(R)(I - P) \sim \Phi(I - P)\Phi(R)\Phi(I - P) \sim (I - P)R(I - P).$$

Hence, using Remark 2.5, we infer that $(I - P)R(I - P) \in \mathbb{C}v \otimes v + \mathbb{C}(I - v \otimes v)$, and consequently,

$$au \otimes v + bv \otimes v = (I - P)R(I - P) \in \mathbb{C}v \otimes v,$$

which is impossible because $a \neq 0$. Therefore, $\alpha \neq 0$ or $\beta \neq 0$. Now, noting that

$$R\left((\bar{b}u - \bar{a}v) \otimes (\bar{b}u - \bar{a}v)\right)R = 0,$$

then, using similar arguments as in the preceding, we get

$$\Phi(R)\left((\bar{b}u - \bar{a}v) \otimes (\bar{b}u - \bar{a}v)\right)\Phi(R) = 0;$$

which implies that

$$\Phi(R)(\overline{a}u - \overline{b}v) = 0 \quad \text{or} \quad \Phi(R)^*(\overline{a}u - \overline{b}v) = 0. \tag{10}$$

Calculations show that $\alpha = 0$ or $\beta = 0$. Consequently,

$$\text{either} \quad \alpha \neq 0 \quad \text{or} \quad \beta \neq 0. \tag{11}$$

If $b = 0$, the proof of the lemma is completed by combining (9) and (11). Suppose that $b \neq 0$. From (8) and (10), one can easily see that

$$\begin{cases} \Phi(R)^*(\overline{a}u - \overline{b}v) = 0 & \text{if } \alpha \neq 0; \\ \Phi(R)(\overline{a}u - \overline{b}v) = 0 & \text{if } \beta \neq 0, \end{cases}$$

and so, calculations show that

$$\begin{cases} \Phi(R) = a^{-1}\alpha R & \text{if } \alpha \neq 0; \\ \Phi(R) = \overline{a^{-1}}\beta R^* & \text{if } \beta \neq 0. \end{cases}$$

The proof is now completed. □

Now, we shall improve the result in the previous lemma.

Lemma 2.10. *We have either*

$$\Phi(R) = \alpha_R R \quad \textit{for every } R \in \mathcal{F}_1(H) \tag{12}$$

or

$$\Phi(R) = \alpha_R R^* \quad \textit{for every } R \in \mathcal{F}_1(H), \tag{13}$$

where α_R is a non-zero complex number that depends on R.

Proof. The proof is divided into two claims:
Claim 1: If $a, b \in \mathbb{C}$, with $a \neq 0$, and u and v are orthonormal vectors, then

$$\Phi(u \otimes v) \in \mathbb{C}u \otimes v \iff \Phi\left((au + bv) \otimes v\right) \in \mathbb{C}(au + bv) \otimes v. \tag{14}$$

Note first that, by the previous lemma, $\Phi(u \otimes v)$ and $\Phi\,(au + bv) \otimes v)$ are non-zero, and we have

$$\text{“}\,\Phi(u \otimes v) \in \mathbb{C}u \otimes v \quad \text{or} \quad \Phi(u \otimes v) \in \mathbb{C}v \otimes u\,\text{”}$$

and

$$\text{“}\,\Phi\left((au + bv) \otimes v\right) \in \mathbb{C}(au + bv) \otimes v \quad \text{or} \quad \Phi\left((au + bv) \otimes v\right) \in \mathbb{C}v \otimes (au + bv)\,\text{”}.$$

Since $(u \otimes v)\left((au + bv) \otimes v\right)(u \otimes v) = 0$, Remark 2.5 implies that

$$\Phi(u \otimes v)\Phi\left((au + bv) \otimes v\right)\Phi(u \otimes v) \in \mathbb{C}I.$$

But, $\Phi(u \otimes v)$ is a rank-one operator; hence

$$\Phi(u \otimes v)\Phi\left((au + bv) \otimes v\right)\Phi(u \otimes v) = 0. \tag{15}$$

Now, a simple calculation shows that if

$$\Phi(u \otimes v) \in \mathbb{C}u \otimes v \quad \text{and} \quad \Phi\left((au + bv) \otimes v\right) \notin \mathbb{C}(au + bv) \otimes v,$$

then (15) does not hold; a contradiction. This proves the direct implication in (14). The reverse implication can be shown by the same argument.

Claim 2: Either property (12) or property (13) is satisfied.

Combining (7), Lemma 2.9 and Claim 1, it is clear that we need only show that, for all pairs of orthonormal sets $\{u, v\}$ and $\{x, y\}$, we have

$$\Phi(u \otimes v) \in \mathbb{C}u \otimes v \implies \Phi(x \otimes y) \in \mathbb{C}x \otimes y.$$

To this end, we first consider the case $v = x$. Arbitrarily choose $\epsilon \in \mathbb{C}$ so that $\langle u+\epsilon y, u\rangle \neq 0$ and $\langle u+\epsilon y, y\rangle \neq 0$. Then, we have

$$(u \otimes v)\,(v \otimes (u + \epsilon y))\,(u \otimes v) \neq 0$$

and

$$(v \otimes (u + \epsilon y))\,(x \otimes y)\,(v \otimes (u + \epsilon y)) = (v \otimes (u + \epsilon y))\,(x \otimes y)\,(x \otimes (u + \epsilon y)) = 0.$$

Hence, by Claim 1 and Remark 2.5, we must have

$$\Phi(u \otimes v)\Phi\,(v \otimes (u + \epsilon y))\,\Phi(u \otimes v) \neq 0$$

and

$$\Phi\,(v \otimes (u + \epsilon y))\,\Phi(x \otimes y)\Phi\,(v \otimes (u + \epsilon y)) = 0.$$

Consequently, it is not hard to see that

$$\begin{aligned} \Phi(u \otimes v) \in \mathbb{C}u \otimes v &\implies \Phi\,(v \otimes (u + \epsilon y)) \in \mathbb{C}v \otimes (u + \epsilon y) \\ &\implies \Phi(x \otimes y) \in \mathbb{C}x \otimes y. \end{aligned}$$

Now, let us consider the general case. Choose arbitrarily a unit vector $z \in \{v, x\}^{\perp}$. Then, by the special case, we have

$$\begin{aligned} \Phi(u \otimes v) \in \mathbb{C}u \otimes v \implies \Phi(v \otimes z) \in \mathbb{C}v \otimes z &\implies \Phi(z \otimes x) \in \mathbb{C}z \otimes x \\ &\implies \Phi(x \otimes y) \in \mathbb{C}x \otimes y. \end{aligned}$$

□

Lemma 2.11. *Assume that* Φ *satisfies* (12). *Then, for every* $A \in \mathcal{B}(H)$, *we have* $\mathrm{Ker}(\Phi(A)) = \mathrm{Ker}(A)$.

Proof. Let $A \in \mathcal{B}(H)$. Then,

$$\begin{aligned} x \in \mathrm{Ker}(A) &\iff Ax \otimes A^*y = 0, \forall y \in H \\ &\iff A(x \otimes y)A = 0, \forall y \in H \\ &\iff \Phi(A)(x \otimes y)\Phi(A) = 0, \forall y \in H \;\; \text{(By Remark 2.5 and (12))} \\ &\iff \Phi(A)x \otimes \Phi(A)^*y = 0, \forall y \in H \\ &\iff x \in \mathrm{Ker}(\Phi(A)). \end{aligned}$$

□

With these results at hand, we are ready to prove the main result of this paper.

Proof. [Proof of Theorem 2.1] (ii)⇒(iii). Suppose first that Φ satisfies (12), and let us show that Φ has the first form in Theorem 2.1 (iii). By the previous lemma, we have $\Phi(0) = 0$, and by Lemma 2.10, Φ maps every rank-one operator R to a non-zero multiple of R. Hence, it suffices to show that for every operator $A \in \mathcal{B}(H)$ whose rank is at least 2, we have $\Phi(A) = \alpha A$ for some non-zero $\alpha \in \mathbb{C}$. Moreover, given that $\mathrm{Ker}(\Phi(A)) = \mathrm{Ker}(A)$, it suffices to prove that for each $h \in \mathrm{Ker}(A)^{\perp}$, there exists some $\alpha_h \in \mathbb{C}$ such that $\Phi(A)h = \alpha_h Ah$. Let $h \in \mathrm{Ker}(A)^{\perp}$ be a non-zero vector. Two situations are distinguished:

Case 1. $Ah \in \operatorname{Ran}(A^*)$. Let $k \in H$ be such that $Ah = A^*k$. Then,

$$A(h \otimes k)A = (Ah) \otimes (Ah) = \|Ah\|^2 \left(\|Ah\|^{-1}Ah\right) \otimes \left(\|Ah\|^{-1}Ah\right)$$

is a non-zero multiple of an orthogonal projection. Applying Remark 2.5 and (12), we get that

$$(\Phi(A)h) \otimes (\Phi(A)^*k) = \Phi(A)(h \otimes k)\Phi(A) \in \mathbb{C}\left(\|Ah\|^{-1}Ah\right) \otimes \left(\|Ah\|^{-1}Ah\right).$$

Whence, $\Phi(A)h \in \operatorname{Span}\{Ah\}$.

Case 2. $Ah \notin \operatorname{Ran}(A^*)$. Since A has rank ≥ 2, then so has A^*. So, we can choose linearly independent vectors A^*u_1 and A^*u_2 in $\operatorname{Ran}(A^*)$. Noting that, for every $i \in \{1, 2\}$, the rank-one operator $A(h \otimes u_i)A$ can be expressed as

$$A(h \otimes u_i)A = \begin{bmatrix} F_i & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \operatorname{Span}\{Ah, A^*u_i\} \\ \{Ah, A^*u_i\}^\perp \end{matrix}$$

for some complex symmetric operator F_i, Proposition 2.4 and (12) imply that, for every $i \in \{1, 2\}$, we have

$$(\Phi(A)h) \otimes (\Phi(A)^*u_i) = \Phi(A)(h \otimes u_i)\Phi(A) = \begin{bmatrix} R_i & 0 \\ 0 & \lambda_i I \end{bmatrix} \begin{matrix} \operatorname{Span}\{Ah, A^*u_i\} \\ \{Ah, A^*u_i\}^\perp \end{matrix}$$

for some operator R_i and $\lambda_i \in \mathbb{C}$. Observe that if there is $i \in \{1, 2\}$ such that $\lambda_i \neq 0$, then we must have $R_i = 0$ because the rank of $(\Phi(A)h) \otimes (\Phi(A)^*u_i)$ must not exceed 1. Combining this fact with Remark 2.5, we get that

$$Ah \otimes A^*u_i = A(h \otimes u_i)A = \begin{bmatrix} \alpha_i I & 0 \\ 0 & \beta_i I \end{bmatrix} \begin{matrix} \operatorname{Span}\{Ah, A^*u_i\} \\ \{Ah, A^*u_i\}^\perp \end{matrix},$$

for some $\alpha_i, \beta_i \in \mathbb{C}$. Then, by (3), Ah and A^*u_i should be linearly dependent; a contradiction because $Ah \notin \operatorname{Ran}(A^*)$. Thus, λ_1 and λ_2 are both zero, and $\Phi(A)h \in \operatorname{Span}\{Ah, A^*u_i\}$ for every $i \in \{1, 2\}$. Consequently, since $Ah \notin \operatorname{Span}\{A^*u_1, A^*u_2\}$ and $\{A^*u_1, A^*u_2\}$ is a linearly independent set, elementary linear algebra can show that

$$\Phi(A)h \in \operatorname{Span}\{Ah, A^*u_1\} \cap \operatorname{Span}\{Ah, A^*u_2\} = \operatorname{Span}\{Ah\};$$

the desired property.

Now Assume that Φ satisfies (13). Define a map $\Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ by $\Psi(A) = \Phi(A^*)$ for every $A \in \mathcal{B}(H)$. Then, it is easy to see that Ψ satisfies both Theorem 2.1 (ii) and (12). Hence, there must exist a map $f : \mathcal{B}(H) \to \mathbb{C} \setminus \{0\}$ such that $\Psi(A) = f(A)A$ for every $A \in \mathcal{B}(H)$. Therefore, for every $A \in \mathcal{B}(H)$, we have

$$\Phi(A) = \Phi((A^*)^*) = \Psi(A^*) = f(A)A^*.$$

This shows that Φ has the second form in Theorem 2.1 (iii) and concludes the proof. □

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