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Some new criteria for judging $\mathcal{H}$-tensors and its application

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Abstract. In this paper, some new criteria which only depend on elements of the given tensors are proposed to judge $\mathcal{H}$-tensors. Moreover, based on these new criteria, some sufficient conditions of the positive definiteness for even-order real symmetric tensors are obtained. In addition, some numerical examples are presented to illustrate those new results.

1. Introduction

Let $n \geq 2$ and $m \geq 2$ be integers, $N = \{1, 2, \ldots, n\}$, and $\mathbb{C}(\mathbb{R})$ be the set of all complex(real) numbers. A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is called a complex (real) order m dimension n tensor, if $a_{i_1 i_2 \cdots i_m} \in \mathbb{C}(\mathbb{R})$, where $i_j = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, m$. Let $\mathbb{C}^{[m,n]}$ ($\mathbb{R}^{[m,n]}$) be the set of all complex (real) order m dimension n tensors. A tensor $\mathcal{I} = (\delta_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is called the unit tensor [1], if its elements satisfy

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, if there exists a complex number λ and a complex vector $x = (x_1, x_2, \ldots, x_n)^T \neq (0, 0, \ldots, 0)^T$ satisfy the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$ and x is its corresponding eigenvector [2–4], where $\mathcal{A}x^{m-1}$ and $\lambda x^{[m-1]}$ are vectors, and whose ith components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m \in N} |a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m},$$

and

$$x_i^{[m-1]} = x_i^{m-1},$$

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respectively, for all $i \in N$. In particular, if λ and x are real, then λ is called an H-eigenvalue of $\mathcal{A}$ and x is its corresponding H-eigenvector [2].

For an mth degree homogeneous polynomial of n variables $f(x)$ can be usually denoted as

$$f(x) = \sum_{i_1,i_2,\ldots,i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. The homogeneous polynomial $f(x)$ can be represented as the tensor product of a symmetric tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ and x^m denoted by

$$f(x) \equiv \mathcal{A}x^m = \sum_{i_1,i_2,\ldots,i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

where the tensor $\mathcal{A}$ is called symmetric if its elements are invariant under all permutation of indices $\{i_1, i_2, \ldots, x_m\}$ and $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ [2]. If m is even, and

$$f(x) > 0,\ \mathit{for\ all}\ x \in \mathbb{R}^n,\ x \neq 0,$$

then we call that $f(x)$ is positive definite.

The positive definiteness of homogeneous polynomial play a key role in automatic control [11, 12], magnetic resonance imaging [13] and so on. However, when $n > 3$, $m > 4$ and m is even, it is difficult to judge the positive definiteness of the homogeneous polynomial $f(x)$. In order to solve this problem, L.Q. Qi proposed in [2] that $f(x)$ is positive definite if and only if the real symmetric tensor $\mathcal{A}$ is positive definite, and L.Q. Qi gave a method to verify the positive definiteness of $\mathcal{A}$ by eigenvalue, that is,

Theorem 1. [2] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a symmetric tensor and m be even, then $\mathcal{A}$ is positive definite if and only if all of its H-eigenvalues are positive.

According to Theorem 1, one can verify the positive definiteness of an even-order symmetric tensor $\mathcal{A}$ by calculating the H-eigenvalues of $\mathcal{A}$. However, it is hard to compute all these H-eigenvalues of $\mathcal{A}$ if m and n are large. In order to solve this problem, a practical sufficient condition was provided for judging the positive definiteness of an even-order symmetric tensor as follow.

Theorem 2. [7] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ with $a_{kk\cdots k} > 0$ for all $k \in N$ and m be even. If $\mathcal{A}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is positive definite.

Based on the fact that the identification of $\mathcal{H}$-tensor is useful in checking the positive definiteness of homogeneous polynomials, some criteria for judging $\mathcal{H}$-tensor have been widely proposed, see [14–25]. In this paper, we still focus on judging of $\mathcal{H}$-tensors, and some new criteria which only depend on elements of the given tensors are proposed. As an application, for an even-order real symmetric tensor, some sufficient conditions of the positive definiteness are obtained. Moreover, some numerical examples are presented to illustrate those new results.

2. Some criteria for judging nonsingular $\mathcal{H}$-tensors

In this section, some new criteria for judging $\mathcal{H}$-tensors are proposed. Before that, some notations, definitions, lemmas and theorems are listed firstly. The calligraphy letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{H}$, $\cdots$ represent tensors; the capital letters A, B, $\cdots$ denote matrices; the lowercase letters x, y, $\cdots$ refer to vectors.

For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, we denote

$r_i(\mathcal{A}) = \sum\limits_{\substack{i_2 \cdots i_m \in N \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| = \sum\limits_{i_2 \cdots i_m \in N} |a_{i i_2 \cdots i_m}| - |a_{ii\cdots i}|$,

$N_1 = \{i \in N : 0 < |a_{ii\cdots i}| \leq r_i(\mathcal{A})\}$, $N_2 = \{i \in N : |a_{ii\cdots i}| > r_i(\mathcal{A})\}$,

$s_i = \frac{|a_{ii\cdots i}|}{r_i(\mathcal{A})}$, $t_i = \frac{r_i(\mathcal{A})}{|a_{ii\cdots i}|}$, $r = \max\{\max\limits_{i \in N_1} s_i, \max\limits_{i \in N_2} t_i\}$,

$r_i^S(\mathcal{A}) = \sum\limits_{\substack{i_2\cdots i_m \in S \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|,\ r_i^{S^*}(\mathcal{A}) = r_i(\mathcal{A}) - r_i^S(\mathcal{A}),$

$N_1^{m-1} = \{i_2 i_3 \cdots i_m : i_j \in N_1, j = 2, 3, \ldots, m\},$

$N^{m-1} \backslash N_1^{m-1} = \{i_2 i_3 \cdots i_m : i_2 i_3 \cdots i_m \in N^{m-1}\ and\ i_2 i_3 \cdots i_m \notin N_1^{m-1}\}$

$R_j(A) = \sum\limits_{j_2\cdots j_m \in N^{m-1}\backslash N_2^{m-1}} |a_{jj_2\cdots j_m}| + \mu \sum\limits_{\substack{j_2\cdots j_m \in N_2^{m-1} \\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}|,\ j \in N_2,$

$\mu_j = \frac{\sum\limits_{j_2\cdots j_m \in N^{m-1}\backslash N_2^{m-1}} |a_{jj_2\cdots j_m}|}{|a_{jj\cdots j}| - \sum\limits_{\substack{j_2\cdots j_m \in N_2^{m-1} \\ \delta_{jj_2\cdots j_m}=0}} |a_{jj_2\cdots j_m}|}$ and $\mu = \max\{\mu_j\}, j \in N_2.$

Definition 1. [8] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If there is a positive vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ such that

$$|a_{ii\cdots i}| x_i^{m-1} > \sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| x_{i_2} \cdots x_{i_m},$$

where $|a|$ for the modulus of $a \in \mathbb{C}$, then $\mathcal{A}$ is called an $\mathcal{H}$-tensor.

Definition 2. [2] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

$$|a_{ii\cdots i}| > r_i(\mathcal{A}),\ i \in N,$$

then $\mathcal{A}$ is called an strictly diagonally dominant tensor.

Definition 3.[15] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ and $X = diag(x_1, x_2, \ldots, x_n)$. If

$$\mathcal{B} = (b_{i_1 i_2\cdots i_m}) = \mathcal{A} X^{m-1},$$

where

$$b_{i_1 i_2\cdots i_m} = a_{i_1 i_2\cdots i_m} x_{i_2} \ldots x_{i_m},\ i_j \in N,\ j = 2, 3, \ldots, m,$$

then $\mathcal{B}$ is called the product of the tensor $\mathcal{A}$ and the matrix X.

Definition 4.[6] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If there exists a $\emptyset \neq S \subset N$ such that $a_{i_1 i_2\cdots i_m} = 0$, $\forall i_1 \in S$ and $i_2, \ldots, i_m \notin S$, then $\mathcal{A}$ is called reducible. Otherwise, $\mathcal{A}$ is called irreducible.

Definition 5.[15] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$, for $i, j \in N$ $(i \neq j)$, if there exist indices $k_1, k_2, \ldots, k_l$ with

$$\sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{k_s i_2\cdots i_m}=0 \\ k_{s+1} \in \{i_2,\ldots,i_m\}}} |a_{k_s i_2\cdots i_m}| \neq 0,\ s = 0, 1, \ldots, l,$$

where $k_0 = i$, $k_{l+1} = j$, we call that there is a nonzero elements chain from i to j.

Lemma 1. [8] If $\mathcal{A}$ is an strictly diagonally dominant tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Lemma 2.[7] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. $\mathcal{A}$ is an $\mathcal{H}$-tensor, if
(i) $\mathcal{A}$ is irreducible,
(ii) $|a_{ii\cdots i}| \geq r_i(\mathcal{A})$, for each $i \in N$,
(iii) for inequality of (ii), strictly inequality holds for at least one i.

Lemma 3. [15] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. $\mathcal{A}$ is an $\mathcal{H}$-tensor, if

(i) $|a_{ii\cdots i}| \geq r_i(\mathcal{A}),\ i \in N$,

(ii) $N_2 = \{i \in N : |a_{ii\cdots i}| > r_i(\mathcal{A})\} \neq \emptyset$,

(iii) for any $i \in N_2$, there exists a nonzero elements chain from i to j such that $j \in N_1$.

Lemma 4.[15, 17] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If there exists a positive diagonal matrix X such that $\mathcal{A}X^{m-1}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

In the following, some new criteria are proposed to judge $\mathcal{H}$-tensors.

Theorem 3. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

$$|a_{ii\cdots i}|s_i > h_i,\ i \in N_1,$$

where

$$h_i = \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|(s_{i_2})^{\frac{1}{m-1}} \cdots (s_{i_m})^{\frac{1}{m-1}} + \sum_{i_2\cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{t_j\}|a_{ii_2\cdots i_m}| + r \sum_{i_2\cdots i_m \in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})} |a_{ii_2\cdots i_m}|, \tag{1}$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Let

$$M_i = \frac{|a_{ii\cdots i}|s_i - h_i}{\sum\limits_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}|},\ i \in N_1.$$

If $\sum\limits_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| = 0$, we define $M_i = +\propto$. Obviously, $M_i > 0$, $i \in N_1$. Hence, there exists $\varepsilon > 0$ such that

$$0 < \varepsilon < min\left\{\min_{i\in N_1} M_i, 1 - \max_{i\in N_2} t_i\right\}.$$

Construct diagonal matrix $X = diag\{x_1, x_2, \ldots, x_n\}$ and denote $\mathcal{B} = (b_{i_1 i_2\cdots i_m}) = \mathcal{A}X^{m-1}$, where

$$x_j = \begin{cases} (s_j)^{\frac{1}{m-1}}, & j \in N_1, \\ (\varepsilon + t_j)^{\frac{1}{m-1}}, & j \in N_2. \end{cases}$$

Obviously, X is a positive diagonal matrix.

Next, we will prove that $\mathcal{B}$ is an strictly diagonally dominant tensor, and divided it into two cases as follows.

Case 1: For any $i \in N_1$, we obtain

$$\begin{aligned} r_i(\mathcal{B}) &= \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_2^{m-1}} |b_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})} |b_{ii_2\cdots i_m}| \\ &= \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|(s_{i_2})^{\frac{1}{m-1}} \cdots (s_{i_m})^{\frac{1}{m-1}} + \sum_{i_2\cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}|(\varepsilon + t_{i_2})^{\frac{1}{m-1}} \cdots (\varepsilon + t_{i_m})^{\frac{1}{m-1}} \\ &\quad + \sum_{i_2\cdots i_m \in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})} |a_{ii_2\cdots i_m}|x_{i_2} \cdots x_{i_m} \\ &\leq \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|(s_{i_2})^{\frac{1}{m-1}} \cdots (s_{i_m})^{\frac{1}{m-1}} + \sum_{i_2\cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}|(\varepsilon + \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{t_j\}) \\ &\quad + (r + \varepsilon) \sum_{i_2\cdots i_m \in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})} |a_{ii_2\cdots i_m}|. \end{aligned} \tag{2}$$

If $\sum\limits_{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}}|a_{ii_2\cdots i_m}|=0$, then from inequality (2) and condition that $|a_{ii\cdots i}|s_i>h_i$ for each $i\in N_1$, it is easy to obtain that

$$\begin{aligned}r_i(\mathcal{B}) &\le \sum_{\substack{i_2\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|(s_{i_2})^{\frac{1}{m-1}}\cdots(s_{i_m})^{\frac{1}{m-1}}\\ &< |a_{ii\cdots i}|s_i\\ &= |b_{ii\cdots i}|.\end{aligned}$$

If $\sum\limits_{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}}|a_{ii_2\cdots i_m}|\neq 0$, then from inequality (2) and condition that $|a_{ii\cdots i}|s_i>h_i$ for each $i\in N_1$, we obtain

$$\begin{aligned}r_i(\mathcal{B}) &\le h_i+\varepsilon\sum_{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}}|a_{ii_2\cdots i_m}|\\ &< h_i+M_i\sum_{i_2\cdots i_m\in N^{m-1}\setminus N_1^{m-1}}|a_{ii_2\cdots i_m}|\\ &= |a_{ii\cdots i}|s_i\\ &= |b_{ii\cdots i}|\end{aligned}$$

Case 2: For any $i\in N_2$, we obtain $|a_{ii\cdots i}|>r_i(\mathcal{A})\ge 0$, and from $0<x_{i_j}\le 1$ for $i_j\in N$ and $j=2,3,\ldots,m$, thus we get

$$\begin{aligned}|b_{ii\cdots i}|-r_i(\mathcal{B}) =&|a_{ii\cdots i}|(\varepsilon+t_i)-\sum_{i_2\cdots i_m\in N_1^{m-1}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}-\sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}\\ &-\sum_{i_2\cdots i_m\in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}\\ \ge&\varepsilon|a_{ii\cdots i}|+r_i(\mathcal{A})-\sum_{i_2\cdots i_m\in N_1^{m-1}}|a_{ii_2\cdots i_m}|-\sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|-\sum_{i_2\cdots i_m\in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})}|a_{ii_2\cdots i_m}|\\ =&\varepsilon|a_{ii\cdots i}|\\ >&0.\end{aligned}$$

From Cases 1 and 2, we obtain $|b_{ii\cdots i}|>r_i(\mathcal{B})$ for all $i\in N$, that is, $\mathcal{B}$ is an strictly diagonally dominant tensor, thus from Lemmas 1 and 4, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

From Theorem 3, we obtain the following corollary.

Corollary 1. Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{C}^{[m,n]}$. If

$$|a_{ii\cdots i}|s_i>p_i,\ i\in N_1,$$

where

$$p_i=\sum_{\substack{i_2\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\{s_j\}|a_{ii_2\cdots i_m}|+\sum_{i_2\cdots i_m\in N_2^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}|+r\sum_{i_2\cdots i_m\in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})}|a_{ii_2\cdots i_m}|, \tag{3}$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof From equalities (1) and (3), it is obvious to get $p_i\ge h_i$ for any $i\in N_1$, then from the condition that

$$|a_{ii\cdots i}|s_i>p_i,\ i\in N_1,$$

we obtain

$$|a_{ii\cdots i}|s_i > p_i \geq h_i,\ i \in N_1,$$

thus from Theorem 3, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

The following example is presented to illustrate Theorem 3.

Example 1. Let us consider tensor $\mathcal{A} = (a_{ijk}) = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{C}^{[3,3]}$, where

$$A(1,:,:) = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A(2,:,:) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 24 \end{pmatrix}\ and\ A(3,:,:) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 100 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 16,\ r_1(\mathcal{A}) = 25,\ |a_{222}| = 9,\ r_2(\mathcal{A}) = 25,\ |a_{333}| = 100\ and\ r_3(\mathcal{A}) = 10,$$

hence $N_1 = \{1, 2\}$, $N_2 = \{3\}$. By calculation, we obtain

$$s_1 = \frac{16}{25},\ s_2 = \frac{9}{25},\ t_3 = \frac{1}{10}\ and\ r = \frac{16}{25}.$$

Thus, we get

$$|a_{111}|s_1 = \frac{256}{25} > 9 = h_1 = \sum_{\substack{jk \in N_1^2 \\ \delta_{1jk}=0}} |a_{1jk}|(s_j)^{\frac{1}{2}}(s_k)^{\frac{1}{2}} + 0$$

and

$$|a_{222}|s_2 = \frac{81}{25} > \frac{76}{25} = h_2 = \sum_{\substack{jk \in N_1^2 \\ \delta_{2jk}=0}} |a_{2jk}|(s_j)^{\frac{1}{2}}(s_k)^{\frac{1}{2}} + \sum_{jk \in N_2^2} \max_{l \in \{j,k\}}\{t_l\}|a_{2jk}| + 0,$$

hence, from Theorem 3, $\mathcal{A}$ is an H-tensor.

Theorem 4. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

(i) $\mathcal{A}$ is irreducible,

(ii) $|a_{ii\cdots i}|s_i \geq h_i$, for each $i \in N_1$, where h_i is defined as equality (1),

(iii) for inequality of (ii), strictly inequality holds for at least one $i \in N_1$,

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Firstly, let the diagonal matrix $X = diag\{x_1, x_2, \ldots, x_n\}$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$, where

$$x_j = \begin{cases} (s_j)^{\frac{1}{m-1}}, & j \in N_1, \\ (t_j)^{\frac{1}{m-1}}, & j \in N_2. \end{cases}$$

Since $\mathcal{A}$ is irreducible, we get $0 < t_j < 1$ for all $j \in N_2$, and from the definition of s_j, we obtain X is a positive diagonal matrix.

From the condition that $\mathcal{A}$ has at least one $i \in N_1$ such that $|a_{ii\cdots i}|s_i > h_i$, therefore, without less of generality, suppose $|a_{jj\cdots j}|s_j > h_j$, $j \in N_1$.

Secondly, similar to the proof of Theorem 3, we conclude that $|b_{ii\cdots i}| \geq r_i(\mathcal{B})$ for all $i \in N\backslash\{j\}$ and $|b_{jj\cdots j}| > r_j(\mathcal{B})$.

Finally, since $\mathcal{A}$ is irreducible and X is a positive diagonal matrix, $\mathcal{B}$ is also irreducible, thus $\mathcal{B}$ satisfies the conditions of Lemma 2. Therefore, by Lemmas 2 and 4, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

From Theorem 4, it is easy to get the following corollary.

Corollary 2. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If
(i) $\mathcal{A}$ is irreducible,
(ii) $|a_{ii\cdots i}|s_i \geq p_i$, for each $i \in N_1$, where p_i is defined as equality (3),
(iii) for inequality of (ii), strictly inequality holds for at least one $i \in N_1$,
then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Using the same technique as in the proof of Corollary 1, obviously, we obtain that $\mathcal{A}$ is an $\mathcal{H}$-tensor.

The example 2 is presented to illustrate Theorem 4.

Example 2. Let us consider irreducible tensor $\mathcal{A} = (a_{ijk}) = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{C}^{[3,3]}$, where

$$A(1,:,:) = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A(2,:,:) = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 12 \end{pmatrix} \ and\ A(3,:,:) = \begin{pmatrix} 20 & 5 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 192 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 16,\ r_1(\mathcal{A}) = 16,\ |a_{222}| = 9,\ r_2(\mathcal{A}) = 16,\ |a_{333}| = 192 \ and\ r_3(\mathcal{A}) = 33,$$

hence $N_1 = \{1, 2\}$, $N_2 = \{3\}$. By calculation, we obtain

$$s_1 = 1,\ s_2 = \frac{9}{16},\ t_3 = \frac{11}{64} \ and\ r = 1.$$

Thus, we get

$$|a_{111}|s_1 = 16 > 9 = h_1 = \sum_{\substack{jk\in N_1^2 \\ \delta_{1jk}=0}} |a_{1jk}|(s_j)^{\frac{1}{2}}(s_k)^{\frac{1}{2}} + 0$$

and

$$|a_{222}|s_2 = \frac{81}{16} = h_2 = \sum_{\substack{jk\in N_1^2 \\ \delta_{2jk}=0}} |a_{2jk}|(s_j)^{\frac{1}{2}}(s_k)^{\frac{1}{2}} + \sum_{jk\in N_2^2} \max_{l\in\{j,k\}}\{t_l\}|a_{2jk}| + 0,$$

hence, $\mathcal{A}$ satisfies the conditions of Theorem 4, that is, $\mathcal{A}$ is an $\mathcal{H}$-tensor by Theorem 4.

Theorem 5. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If
(i) $\mathcal{A}$ is irreducible,
(ii) $|a_{ii\cdots i}|s_i \geq h_i$, $i \in N_1$, where h_i is defined as equality (1),
(iii) at least one $a_{i_1 i_2 \cdots i_m} \neq 0$, $i_j \in N_2$, $j = 1, 2, \ldots, m$ and $i_1 \neq i_2 \neq \cdots \neq i_m$,
then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Firstly, let the diagonal matrix $X = diag\{x_1, x_2, \ldots, x_n\}$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$, where

$$x_j = \begin{cases} (s_j)^{\frac{1}{m-1}}, & j \in N_1, \\ (t_j)^{\frac{1}{m-1}}, & j \in N_2. \end{cases}$$

Similarly as in the proof of Theorem 4, we obtain X is a positive diagonal matrix.

From the condition that $\mathcal{A}$ has at least one $a_{i_1 i_2 \cdots i_m} \neq 0$, where $i_j \in N_2$, $j = 1, 2, \ldots, m$ and $i_1 \neq i_2 \neq \cdots \neq i_m$, therefore, without less of generality, suppose $a_{i_p i_2 \cdots i_m} \neq 0$, where $i_j \in N_2$, $j = p, 2, 3, \ldots, m$ and $i_p \neq i_2 \neq \cdots \neq i_m$.

Secondly, similarly as in the proof of Theorem 3, we conclude that $|b_{ii\cdots i}| \geq r_i(\mathcal{B})$ for all $i \in N\backslash\{i_p\}$ and $|b_{i_p i_p \cdots i_p}| > r_{i_p}(\mathcal{B})$.

Finally, since $\mathcal{A}$ is irreducible and X is a positive diagonal matrix, $\mathcal{B}$ is also irreducible, thus $\mathcal{B}$ satisfies the conditions of Lemma 2. Therefore, by Lemmas 2 and 4, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

From Theorem 5, we get the following corollary.

Corollary 3. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

(i) $\mathcal{A}$ is irreducible,

(ii) $|a_{ii\cdots i}|s_i \geq p_i$, $i \in N_1$, where p_i is defined as equality (3),

(iii) at least one $a_{i_1 i_2 \cdots i_m} \neq 0$, $i_j \in N_2$, $j = 1, 2, \ldots, m$ and $i_1 \neq i_2 \neq \cdots \neq i_m$,

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Using the same technique as in the proof of Corollary 1, it is easy to get that $\mathcal{A}$ is an $\mathcal{H}$-tensor.

The example 3 is presented to illustrate Theorem 5.

Example 3. Let us consider irreducible tensor $\mathcal{A} = (a_{ijk}) = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{C}^{[3,3]}$, where

$$A(1,:,:) = \begin{pmatrix} 16 & 0 & 5 \\ 4 & 16 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A(2,:,:) = \begin{pmatrix} 20 & 2 & 2 \\ 0 & 100 & 0 \\ 0 & 0 & 4 \end{pmatrix} \ and\ A(3,:,:) = \begin{pmatrix} 2 & 5 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 100 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 16,\ r_1(\mathcal{A}) = 25,\ |a_{222}| = 100,\ r_2(\mathcal{A}) = 28,\ |a_{333}| = 100 \ and\ r_3(\mathcal{A}) = 15,$$

hence $N_1 = \{1\}$, $N_2 = \{2, 3\}$. By calculation, we obtain

$$s_1 = \frac{16}{25},\ t_2 = \frac{7}{25},\ t_3 = \frac{3}{20} \ and\ r = \frac{16}{25}.$$

Since $\mathcal{A}$ is irreducible, $N_1 = \{1\}$, $a_{233} = 4$ and

$$|a_{111}|s_1 = \frac{256}{25} = h_1 = \sum_{jk \in N_2^2} \max_{l \in \{j,k\}}\{t_l\}|a_{1jk}| + r \sum_{jk \in N^2 \backslash (N_1^2 \cup N_2^2)} |a_{1jk}| + 0,$$

$\mathcal{A}$ satisfies the conditions of Theorem 5, thus from Theorem 5, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Theorems 4 and 5 are given to judge whether an irreducible tensor is $\mathcal{H}$-tensor. Obviously, Example 2 does not satisfy the conditions of Theorem 5 since $N_2 = \{3\}$, Example 3 does not satisfy the conditions of Theorem 4 since $N_1 = \{1\}$ and $|a_{111}|s_1 = \frac{256}{25} = h_1$. Therefore, Examples 2 and 3 show that Theorem 4 and Theorem 5 are mutually included.

Theorem 6. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

(i) $|a_{ii\cdots i}|s_i \geq h_i$ for each $i \in N_1$, where h_i is defined as equality (1),

(ii) $G_1(\mathcal{A}) \cup G_2(\mathcal{A}) \neq \emptyset$, where $G_1(\mathcal{A}) = \{i : |a_{ii\cdots i}|s_i > h_i, i \in N_1\}$ and $G_2(\mathcal{A}) = \{i : |a_{ii\cdots i}|t_i > h_i, i \in N_2\}$,

(iii) for any $i \in (N_1 \backslash G_1(\mathcal{A}) \cup G_2 \backslash G_2(\mathcal{A}))$, there exists a nonzero elements chain from i to j such that $j \in (G_1(\mathcal{A}) \cup G_2(\mathcal{A}))$,

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Firstly, construct diagonal matrix $X = diag\{x_1, x_2, \ldots, x_n\}$ and denote $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$, where

$$x_j = \begin{cases} (s_j)^{\frac{1}{m-1}}, & j \in N_1, \\ (t_j)^{\frac{1}{m-1}}, & j \in N_2. \end{cases}$$

From the condition that for any $i \in (N_1 \backslash G_1(\mathcal{A}) \cup G_2 \backslash G_2(\mathcal{A}))$, $\mathcal{A}$ exists a nonzero elements chain from i to j such that $j \in (G_1(\mathcal{A}) \cup G_2(\mathcal{A}))$ and set $G_2(\mathcal{A})$ and Definition 5, it is easy to get $r_i(\mathcal{A}) \neq 0$ for any $i \in N_2$. Thus, from the definitions of s_j and t_j, we obtain X is a positive diagonal matrix.

Secondly, similar to the proof of Theorem 3, we conclude that $|b_{ii\cdots i}| \geq r_i(\mathcal{B})$ for all $i \in N$. From the condition $G_1(\mathcal{A}) \cup G_2(\mathcal{A}) \neq \emptyset$, we obtain that there exists at least an $i_p \in N$ such that $|b_{i_p i_p \cdots i_p}| > r_{i_p}(\mathcal{B})$.

On the other hand, if $|b_{ii\cdots i}| = r_i(\mathcal{B})$, then $i \in (N_1\backslash G_1(\mathcal{A}) \cup N_2\backslash G_2(\mathcal{A}))$, and from the condition that for any $i \in (N_1\backslash G_1(\mathcal{A}) \cup N_2\backslash G_2(\mathcal{A}))$, $\mathcal{A}$ exists a nonzero elements chain from i to j such that $j \in (G_1(\mathcal{A}) \cup G_2(\mathcal{A}))$, we obtain that $\mathcal{B}$ exists a nonzero elements chain from i to j with $|b_{jj\cdots j}| > r_j(\mathcal{B})$.

Finally, based on above analysis, we draw a conclusion that $\mathcal{B}$ satisfies the conditions of Lemma 3, hence by Lemmas 3 and 4, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

From Theorem 6, we obtain the following corollary.

Corollary 4. Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

(i) $|a_{ii\cdots i}|s_i \geq p_i$, for each $i \in N_1$, where p_i is defined as equality (3),

(ii) $K_1(\mathcal{A}) \cup K_2(\mathcal{A}) \neq \emptyset$, where $K_1(\mathcal{A}) = \{i : |a_{ii\cdots i}|s_i > p_i, i \in N_1\}$ and $K_2(\mathcal{A}) = \{i : |a_{ii\cdots i}|t_i > p_i, i \in N_2\}$,

(iii) for any $i \in (N_1\backslash K_1(\mathcal{A}) \cup N_2\backslash K_2(\mathcal{A}))$, there exists a nonzero elements chain from i to j such that $j \in (K_1(\mathcal{A}) \cup K_2(\mathcal{A}))$,

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Using the same technique as in the proof of Corollary 1, obviously, we get that $\mathcal{A}$ is an $\mathcal{H}$-tensor.

The following example is presented to illustrate Theorem 6.

Example 4. Let us consider tensor $\mathcal{A} = (a_{ijk}) = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{C}^{[3,3]}$, where

$$A(1,:,:) = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A(2,:,:) = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 16 & 0 \\ 0 & 0 & 27 \end{pmatrix}\ and\ A(3,:,:) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 27 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 16,\ r_1(\mathcal{A}) = 20,\ |a_{222}| = 16,\ r_2(\mathcal{A}) = 32,\ |a_{333}| = 27\ and\ r_3(\mathcal{A}) = 4,$$

hence $N_1 = \{1, 2\}$, $N_2 = \{3\}$. By calculation, we obtain

$$s_1 = \frac{4}{5},\ s_2 = \frac{1}{2},\ t_3 = \frac{4}{27}\ and\ r = \frac{4}{5}.$$

Since $J_1(\mathcal{A}) = \{1\}$, $J_2(\mathcal{A}) = \emptyset$, $a_{213} = 5$, $a_{312} = 1$,

$$|a_{111}|s_1 = \frac{64}{5} > 10 = h_1 = \sum_{\substack{jk \in N_1^2 \\ \delta_{1jk}=0}} |a_{1jk}|(s_j)^{\frac{1}{2}}(s_k)^{\frac{1}{2}} + 0,$$

and

$$|a_{222}|s_2 = 8 = h_2 = \sum_{jk \in N_2^2} \max_{l \in \{j,k\}}\{t_l\}|a_{2jk}| + r \sum_{jk \in N^2\backslash(N_1^2 \cup N_2^2)} |a_{2jk}| + 0,$$

thus from Theorem 6, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Theorem 7. Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

$$|a_{ii\cdots i}|s_i > \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|(s_{i_2})^{\frac{1}{m-1}} \cdots (s_{i_m})^{\frac{1}{m-1}} + r \sum_{i_2\cdots i_m \in N^{m-1}\backslash(N_1^{m-1} \cup N_2^{m-1})} |a_{ii_2\cdots i_m}|,\ i \in N_1, \tag{4}$$

and

$$\sum_{i_2\cdots i_m \in N^{m-1}\backslash N_2^{m-1}} |a_{ii_2\cdots i_m}| = 0,\ i \in N_2,$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof From inequality (4), we obtain that for each $i \in N_1$, there exists a positive number $K > 1$ such that for any $i \in N_1$,

$$|a_{ii\cdots i}|s_i > \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|(s_{i_2})^{\frac{1}{m-1}} \cdots (s_{i_m})^{\frac{1}{m-1}} + r \sum_{i_2\cdots i_m \in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})} |a_{ii_2\cdots i_m}| + \frac{1}{K} \sum_{i_2\cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{t_j\}|a_{ii_2\cdots i_m}|.$$

For any $i \in N_1$, denote

$$T_i = \frac{|a_{ii\cdots i}|s_i - q_i}{\sum\limits_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}|},\ i \in N_1,$$

where

$$q_i = \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|(s_{i_2})^{\frac{1}{m-1}} \cdots (s_{i_m})^{\frac{1}{m-1}} + \frac{1}{K} \sum_{i_2\cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{t_j\}|a_{ii_2\cdots i_m}| + r \sum_{i_2\cdots i_m \in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})} |a_{ii_2\cdots i_m}|.$$

If $\sum\limits_{i_2\cdots i_m \in N^{m-1}\setminus N_1^{m-1}} |a_{ii_2\cdots i_m}| = 0$, we define $T_i = +\infty$. Obviously, $T_i > 0$, $i \in N_1$. Hence, there exists $\varepsilon > 0$ such that

$$0 < \varepsilon < min\left\{\min_{i\in N_1} T_i, 1 - \max_{i\in N_2} \frac{t_i}{K}\right\}.$$

Construct diagonal matrix $X = diag\{x_1, x_2, \ldots, x_n\}$, and denote $\mathcal{B} = (b_{i_1 i_2\cdots i_m}) = \mathcal{A}X^{m-1}$, where

$$x_j = \begin{cases} (s_j)^{\frac{1}{m-1}}, & j \in N_1, \\ (\varepsilon + \frac{t_j}{K})^{\frac{1}{m-1}}, & j \in N_2. \end{cases}$$

Obviously, X is a positive diagonal matrix.

Next, similarly as in the proof of Theorem 3, we obtain that $\mathcal{B}$ is an strictly diagonally tensor. Thus, from Lemmas 1 and 4, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

From Theorem 7, a corollary is obtained as follows.

Corollary 5. Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

$$|a_{ii\cdots i}|s_i > \sum_{\substack{i_2\cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{s_j\}|a_{ii_2\cdots i_m}| + r \sum_{i_2\cdots i_m \in N^{m-1}\setminus(N_1^{m-1}\cup N_2^{m-1})} |a_{ii_2\cdots i_m}|,\ i \in N_1,$$

and

$$\sum_{i_2\cdots i_m \in N^{m-1}\setminus N_2^{m-1}} |a_{ii_2\cdots i_m}| = 0,\ i \in N_2,$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof Using the same technique as in the proof of Corollary 1, obviously, it is easy to obtain that $\mathcal{A}$ is an $\mathcal{H}$-tensor.

The following example is presented to illustrate Theorem 7.

Example 5. Let us consider tensor $\mathcal{A} = (a_{ijk}) = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{C}^{[3,3]}$, where

$$A(1,:,:) = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{pmatrix},\ A(2,:,:) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 9 & 1 \\ 0 & 1 & 22 \end{pmatrix}\ \textit{and}\ A(3,:,:) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 10 & 2 \\ 0 & 2 & 100 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 16,\ r_1(\mathcal{A}) = 25,\ |a_{222}| = 9,\ r_2(\mathcal{A}) = 25,\ |a_{333}| = 100\ \textit{and}\ r_3(\mathcal{A}) = 14,$$

hence $N_1 = \{1, 2\}$, $N_2 = \{3\}$. By calculation, we obtain

$$s_1 = \frac{16}{25},\ s_2 = \frac{9}{25},\ t_3 = \frac{7}{50}\ \textit{and}\ r = \frac{16}{25}.$$

Thus, we obtain

$$|a_{111}|s_1 = \frac{256}{25} > \frac{36}{5} = \sum_{\substack{jk\in N_1^2\\ \delta_{1jk}=0}} |a_{1jk}|(s_j)^{\frac{1}{2}}(s_k)^{\frac{1}{2}} + r \sum_{jk\in N^2\setminus(N_1^2\cup N_2^2)} |a_{1jk}|,$$

and

$$|a_{222}|s_2 = \frac{81}{25} > \frac{48}{25} = \sum_{\substack{jk\in N_1^2\\ \delta_{2jk}=0}} |a_{2jk}|(s_j)^{\frac{1}{2}}(s_k)^{\frac{1}{2}} + r \sum_{jk\in N^2\setminus(N_1^2\cup N_2^2)} |a_{2jk}|,$$

hence $\mathcal{A}$ satisfies the conditions of Theorem 7, thus from Theorem 7, $\mathcal{A}$ is an $\mathcal{H}$-tensor.

At the end of this section, we give a numerical example, which shows that some criteria of existing $\mathcal{H}$-tensor can not be used to determine whether it is an $\mathcal{H}$-tensor, but we can use new criterion which is proposed by us to judge that it is an $\mathcal{H}$-tensor. Before that, some criteria for judging $\mathcal{H}$-tensor are recalled.

Theorem 8. [15] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

$$|a_{ii\cdots i}|s_i > r \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m\in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}|,\ \forall i \in N_1$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Theorem 9. [16] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If

$$|a_{ii\cdots i}| > \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m\in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}|,\ \forall i \in N_1$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Theorem 10. [17] Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$. If for all $i \in N_1$, $j \in N_2$

$$\left(R_j(A) - \sum_{\substack{j_2\cdots j_m\in N_2^{m-1}\\ \delta_{jj_2\cdots j_m}=0}} \max_{j\in\{j_2,j_3,\ldots,j_m\}}\{t_j\}|a_{jj_2\cdots j_m}|\right)\left(|a_{ii\cdots i}| - \sum_{\substack{i_2\cdots i_m\in N^{m-1}\setminus N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|\right)$$
$$> \sum_{j_2\cdots j_m\in N^{m-1}\setminus N_2^{m-1}} |a_{jj_2\cdots j_m}| \sum_{i_2\cdots i_m\in N_1^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}}\{t_j\}|a_{ii_2\cdots i_m}|,$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Theorem 11. [19] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If there exists a nonempty subset S of n such that

(i) $|a_{ii\cdots i}| > r_i(\mathcal{A}),\ \forall i \in S,$

(ii) $|a_{ii\cdots i}|(|a_{jj\cdots j}| - r_j^{S^*}(\mathcal{A})) > r_i(\mathcal{A}) r_j^{S}(\mathcal{A}),\ \forall i \in S, j \in \overline{S}$, where $S \cup \overline{S} = n$,

then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Example 6. We still consider the tensor $\mathcal{A}$ in Example 1. By calculation, we obtain

$$|a_{111}| = 16,\ r_1(\mathcal{A}) = 25,\ |a_{222}| = 9,\ r_2(\mathcal{A}) = 25,\ |a_{333}| = 100,\ r_3(\mathcal{A}) = 10,$$
$$N = \{1,2,3\},\ N_1 = \{1,2\},\ N_2 = \{3\},\ s_1 = \frac{16}{25},\ s_2 = \frac{9}{25},\ t_3 = \frac{1}{10} \text{ and } r = \frac{16}{25}.$$

Thus, we get

$$|a_{111}|s_1 = \frac{256}{25} < 16 = r \sum_{\substack{jk\in N^2\setminus N_2^2 \\ \delta_{1jk}=0}} |a_{1jk}| + 0 \text{ and } |a_{111}| = 16 < 25 = \sum_{\substack{jk\in N^2\setminus N_2^2 \\ \delta_{1jk}=0}} |a_{1jk}| + 0,$$

hence we can not judge whether the tensor $\mathcal{A}$ is an $\mathcal{H}$-tensor or not by Theorems 8, 9 and 10. Since $N_2 = \{3\}$, we can only take $S = \{3\}$ and $\overline{S} = \{1,2\}$ from Theorem 11, thus we obtain $|a_{111}| - r_1^{S^*}(\mathcal{A}) = -9$, it is obvious that the $\mathcal{A}$ does not satisfy the conditions of Theorem 11, hence we can not use Theorem 11 to determine whether it is an $\mathcal{H}$-tensor. However, in fact, $\mathcal{A}$ is an $\mathcal{H}$-tensor from Example 1.

3. An application

In this section, based on new criteria for judging $\mathcal{H}$-tensors in section 2, some new criteria for identifying the positive definiteness of an even-order real symmetric tensor are presented.

From Theorems 3, 4, 5, 6 and 7, we get the following result.

Theorem 12. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}(m, n \geq 2)$. If m is even, $a_{pp\cdots p} > 0$ for all $p \in N$, $\mathcal{A}$ is symmetric and satisfies one of the following conditions, then $\mathcal{A}$ is positive definite:

(i) all the conditions of Theorem 3;

(ii) all the conditions of Theorem 4;

(iii) all the conditions of Theorem 5;

(iv) all the conditions of Theorem 6;

(v) all the conditions of Theorem 7.

From Theorem 8 and Corollaries 1, 2, 3, 4 and 5, it is easy to obtain the following corollary.

Corollary 6. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. If m is even, $a_{pp\cdots p} > 0$ for all $p \in N$, $\mathcal{A}$ is symmetric and satisfies one of the following conditions, then $\mathcal{A}$ is positive definite:

(i) all the conditions of Corollary 1;

(ii) all the conditions of Corollary 2;

(iii) all the conditions of Corollary 3;

(iv) all the conditions of Corollary 4,

(v) all the conditions of Corollary 5.

The following example is given to show this result.

Example 7. Consider the following 4th-degree homogeneous polynomial

$$f(x) = 598000x_1^4 + 64x_2^4 + 27x_3^4 + 320x_1^3x_2 + 452x_1^3x_3 + 4x_1x_3^3 + 36x_1x_2^3 + 18x_1^2x_2^2 + 18x_2^2x_3^2,$$

where $x = (x_1, x_2, x_3)^T$. Then we can obtain a symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$, where

$$A(1,1,:,:) = \begin{pmatrix} 598000 & 80 & 113 \\ 80 & 3 & 0 \\ 113 & 0 & 0 \end{pmatrix}, A(1,2,:,:) = \begin{pmatrix} 80 & 3 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A(1,3,:,:) = \begin{pmatrix} 113 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$A(2,1,:,:) = \begin{pmatrix} 80 & 3 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A(2,2,:,:) = \begin{pmatrix} 3 & 9 & 0 \\ 9 & 64 & 0 \\ 0 & 0 & 3 \end{pmatrix}, A(2,3,:,:) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix},$$
$$A(3,1,:,:) = \begin{pmatrix} 113 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A(3,2,:,:) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}, A(3,3,:,:) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 27 \end{pmatrix}.$$

Obviously,

$$|a_{1111}| = 598000,\ r_1(\mathcal{A}) = 598,\ |a_{2222}| = 64,\ r_2(\mathcal{A}) = 125,\ |a_{3333}| = 27 \textit{ and } r_3(\mathcal{A}) = 125,$$

so $N_1 = \{2,3\}, N_2 = \{1\}$. By simple calculation, we obtain

$$t_1 = \frac{1}{1000},\ s_2 = \frac{64}{125},\ s_3 = \frac{27}{125} \textit{ and } r = \frac{64}{125}.$$

Thus, we get

$$|a_{2222}|s_2 = \frac{4096}{125} > \frac{2638}{125} = h_2 = \sum_{\substack{jkl \in N_1^3 \\ \delta_{2jkl}=0}} |a_{2jkl}|(s_j)^{\frac{1}{3}}(s_k)^{\frac{1}{3}}(s_l)^{\frac{1}{3}} + \sum_{jkl \in N_2^3} \max_{j \in \{j,k,l\}} \{t_j\}|a_{2jkl}| + r \sum_{jkl \in N^3 \setminus (N_1^3 \cup N_2^3)} |a_{2jkl}|,$$

and

$$|a_{3333}|s_3 = \frac{729}{125} > \frac{1021}{200} = h_3 = \sum_{\substack{jkl \in N_1^3 \\ \delta_{3jkl}=0}} |a_{3jkl}|(s_j)^{\frac{1}{3}}(s_k)^{\frac{1}{3}}(s_l)^{\frac{1}{3}} + \sum_{jkl \in N_2^3} \max_{j \in \{j,k,l\}} \{t_j\}|a_{3jkl}| + r \sum_{jkl \in N^3 \setminus (N_1^3 \cup N_2^3)} |a_{3jkl}|,$$

which means that $\mathcal{A}$ satisfies the conditions of Theorem 3, and $m = 4$, hence, $f(x)$ is positive definite by Theorem 8.

4. Conclusions

In this paper, some new criteria are proposed for judging $\mathcal{H}$-tensors, which is easy to verify since they only depend on elements of the given tensors. As an application, some sufficient conditions of the positive definiteness for even-order real symmetric tensors are obtained. In addition, some numerical examples are presented to illustrate those new results.

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