Filomat 38:29 (2024), 10207–10221
https://doi.org/10.2298/FIL2429207D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hopf structures on closure spaces

Sibel Demiralp[a], Ozgur Ege[b,*]

[a]*Department of Mathematics, Kastamonu University, Kastamonu, Turkey*
[b]*Department of Mathematics, Ege University, Bornova, Izmir, 35100, Turkey*

Abstract. This study extends algebraic perspectives to non-topological closure spaces by introducing Hopf structures. We define closure Hopf spaces and groups, investigate their properties, and explore homotopy theory within this framework. Contravariant functors are established between the homotopy category of closure Hopf groups and the category of groups. We also introduce the concept of sub-CH groups for analyzing similar algebraic properties within CH group subsets. This research significantly advances our understanding of algebraic structures in closure spaces, and broadening the scope of mathematical exploration in this field.

1. Introduction

As a multidisciplinary field, algebraic topology uses algebraic techniques to transform topological problems into algebraic ones, providing practical tools for their solution. This mathematical discipline covers two main areas: homotopy theory and homology theory.

Homotopy theory relies on the notion of homotopy of mappings: Two maps $f, g : X \to Y$ are homotopic if and only if there exists a continuous function $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We use the notation $\simeq$ for homotopy of functions [22].

Within the framework of transitioning from topology to algebra, the Hopf group (H-group) concept emerges as an extension of the notion of a topological group, employing the machinery of homotopy theory. An H-group is characterized by a structure closely resembling a group, exhibiting an associative multiplication operation defined up to homotopy, along with homotopy inverses and a homotopy identity element. More precisely, a topological H-group is a pointed topological space (X, x_0) with a binary multiplication $m : X \times X \to X$ and the constant function $c : X \to X, c(x) = x_0$ such that:

i) $m(c, 1_X) \simeq 1_X \simeq m(1_X, c)$
ii) $m(m \times 1_X) \simeq m(1_X \times m)$
iii) $m(n, 1_X) \simeq c \simeq m(1_X, n)$, for a continuous function $n : X \to X$.

If only (i) holds, then (X, x_0) is called a Hopf space (H-space); see [2, 10, 19]. n is said to be a homotopy inverse.

2020 *Mathematics Subject Classification*. Primary 54A05; Secondary 55P45, 55P65
Keywords. Hopf space, Čech closure space, homotopy theory, closure Hopf space
Received: 22 February 2024; Revised: 28 August 2024; Accepted: 10 September 2024
Communicated by Ljubiša D. R. Kočinac
* Corresponding author: Ozgur Ege
Email addresses: sdemiralp@kastamonu.edu.tr (Sibel Demiralp), ozgur.ege@ege.edu.tr (Ozgur Ege)

H-space and H-group structures are topics of study across various mathematical areas. The digital counterparts of these concepts have been defined in [8, 12–16]. Investigating H-group structures extends to fuzzy topological spaces in [5, 6]. Serving as the dual concept to H-space, the notion of co-H-space has been examined in the context of digital images in [9, 15], fuzzy topological spaces in [7], and closure spaces in [4].

This work focuses on constructing H-space and H-group structures on closure spaces, also called pretopological spaces.

A closure space [3] is defined as a generalization of a topological space in the following manner: Given a space X and the operator $c : P(X) \to P(X)$, where $P(X)$ represents the power set of X. If the following conditions are satisfied:

i) $c(\emptyset) = \emptyset$
ii) $A \subseteq c(A)$ for all $A \in P(X)$
iii) $c(A \cup B) = c(A) \cup c(B)$

then the operator c is referred to as a closure operator on X, and the space equipped with this operator is termed a closure space. If the closure operator c is topological, i.e., it satisfies the axiom:

iv) $c(c(A)) = c(A)$, for all $A \in P(X)$,

then we arrive at the classical definition of a topological space. However, this additional axiom can introduce limitations in specific applications. For instance, in graph theory, when considering the closure of a vertex in a graph as the set of all adjacent points, it becomes evident that this operator is not a topological closure operator. Consequently, closure spaces find significance in fields like graph theory, where mathematical objects are employed for modeling purposes in both natural and social sciences.

This study aims to establish H-space and H-group structures in closure spaces by utilizing the homotopy definition provided in [18]. Consequently, in non-topological closure spaces, a group-like structure is established, transforming problems in closure spaces into algebraic problems.

The sections of this study are organized as follows. In Section 2, we provide essential information about closure spaces. Furthermore, we define certain operations between closure operators and offer illustrative examples. Section 3 is divided into two parts. In the first part, 3.1, we introduce the concept of H-space within closure spaces. We present an example of the H-space structure in non-topological closure spaces. Additionally, we demonstrate that the H-space concept remains preserved in closure spaces under Cartesian multiplication. We also define deformation retracts, retracts, and weak retracts in the context of closure spaces and investigate the relationships between these structures and the H-space concept. For instance, we establish that a weak retract of a commutative closure H-space retains its commutative H-space property. The second part of Chapter 3, 3.2, focuses on the study of H-groups in closure spaces. After defining the concept of Closure H-group, we prove that having the same homotopy type preserves the H-group structure within closure spaces. We also explore the transitions of retract species within Closure H-group structures. In Section 3, we establish a group structure on the set of homotopy classes of functions defined on pointed closure H-spaces. Consequently, we establish the existence of a contravariant functor between the category of closure Hopf groups and the category of groups. This result is achieved by demonstrating that the set comprising homotopy classes of functions between closure Hopf groups possesses the fundamental characteristics of a group. Therefore, we establish a significant link between these two categories. In Section 4, we define the subspace of the H-space concept within closure spaces and analyze some of its properties. Finally, Section 5 concludes the paper with summary.

Within this study, each non-original theorem, definition, example, etc., is presented with proper citations.

2. Closure space

Kuratowski introduced the concept of a closure space via a topological closure operator capable of constructing a topological space [11]. Subsequently, Cech introduced the Cech closure space, a more general form of the earlier closure space, which is also referred to as a pretopological space [3]. In Cech's closure space, the closure operator is not constrained to satisfy the idempotent property, which states that c(c(A)) = c(A).

In our study, our focus will be on Cech's closure spaces. For conciseness, we will refer to them as closure spaces instead of specifying them as Cech closure spaces. If a closure operator happens to be idempotent, we will designate that space as a topological space.

Definition 2.1. ([3]) Let (X, c) be a closure space. A subset $A \subset X$ is called a closed set if $c(A) = A$. If its complement is closed, i.e., $c(X - A) = X - A$, then A is called an open set. For all $A \subseteq X$, if $c(A) = A$, then c is called discrete. If $c(A) = X$, for all nonempty $A \subseteq X$, then c is called a trivial closure operator on X.

If multiple closure spaces are under consideration, we will employ the notation c_X to represent the closure operator associated with the space X.

Definition 2.2. ([3]) Let $Y \subseteq X$, then the subspace (Y, c_Y) of (X, c_X) is a closure space with the closure operator $c_Y(A) = c_X(A) \cap Y$, for all $A \subseteq Y$.

Definition 2.3. ([3]) Let (X, c_X) and (Y, c_Y) are closure spaces. A map $f : (X, c_X) \to (Y, c_Y)$ is said to be continuous iff $f(c_X(A)) \subseteq c_Y(f(A))$ for all $A \subseteq X$. Also f is called closed iff $f(c_X(A)) = c_Y(f(A))$.

Like Kuratowski's closure operator, an interior operator is also defined to establish a topological space.

Definition 2.4. ([3]) Let $P(X)$ denote the set of all subsets of a set X. The operator $int_c : P(X) \to P(X)$ defined as

$$int_c(u) = X - c(X - u)$$

is called an interior operator and $int_c(u)$ is called the interior of u. A set $v \subseteq X$ is called a neighbourhood of u iff $u \subseteq X - c(X - v)$. The set of all neighbourhood of u is called as neighborhood systems of u and denoted by $\mathcal{V}_u$.

Lemma 2.5. ([21]) *Let (X, c) be a closure space and $\alpha : X \to Y$ be an onto map. Then $c_\alpha : P(Y) \to P(Y)$, defined as $c_\alpha(B) = \alpha c \alpha^{-1}(B)$, is a closure operator on Y, named as quotient closure operator induced by c.*

Example 2.6. Let $X = \{1, 2, 3\}$ and define a closure operator c on X such that

$$c(\emptyset) = \emptyset, c(\{1\}) = c(\{2\}) = c(\{1, 2\}) = \{1, 2\},$$

$$c(\{3\}) = \{3\}, c(\{1, 3\}) = c(\{2, 3\}) = c(\{1, 2, 3\}) = \{1, 2, 3\}.$$

a) $Y = \{a, b, c\}$ and $\alpha : X \to Y$ be defined as $\alpha(1) = b$, $\alpha(2) = a$, $\alpha(3) = c$. The quotient closure operator c_α is defined as

$$c_\alpha(\emptyset) = \emptyset, c_\alpha(\{a\}) = c_\alpha(\{b\}) = c_\alpha(\{a, b\}) = \{a, b\},\ c_\alpha(\{c\}) = \{c\},$$

$$c_\alpha(\{a, c\}) = c_\alpha(\{b, c\}) = c_\alpha(\{a, b, c\}) = \{a, b, c\}.$$

b) $Z = \{a, b\}$ and $\beta : X \to Z$ be defined as $\beta(1) = \beta(2) = a, \beta(3) = b$. The quotient closure operator c_β is defined as

$$c_\beta(\emptyset) = \emptyset, c_\beta(\{a\}) = \{a\}, c_\beta(\{b\}) = \{b\}, c_\beta(\{a, b\}) = \{a, b\}.$$

c) $W = \{a, b, c\}$ and $\gamma : X \to W$ be defined as $\gamma(1) = \gamma(2) = a, \gamma(3) = b$. Then

$$c_\gamma(\{c\}) = \gamma c \gamma^{-1}(\{c\}) = \gamma c(\emptyset) = \gamma(\emptyset) = \emptyset.$$

Therefore, c_γ is not a closure operator on W, since γ is not an onto map.

Lemma 2.7. ([18]) *Let (X, c) be a closure space, $\alpha : X \to Y$ be an onto map and (Y, c_α) be the closure space induced by α. Then c_α is the finest closure operator on Y, which makes α continuous.*

In [3], a particular closure operator is defined uniquely via a neighborhood system on closure spaces. Before we get to that description, let us first identify the concept of a neighborhood base defined on closure spaces.

Definition 2.8. ([18]) Let (X, c) be a closure space and $A \subset X$. If a collection $\mathcal{B} \subset P(X)$ satisfies the following conditions, then $\mathcal{B}$ is called a base of the neighborhood system $\mathcal{V}_A$:

i) $B \in \mathcal{V}_A$, for all $B \in \mathcal{B}$,
ii) For all $U \in \mathcal{V}_A$ there exists $B \in \mathcal{B}$ such that $B \subset U$.

A subbase of the neighborhood system $\mathcal{V}_A$ is a collection $\mathcal{S}_A \subset P(X)$ such that the collection of all finite intersections of elements of $\mathcal{S}_A$ is a base of the neighborhood system $\mathcal{V}_A$.

Theorem 2.9. ([3]) *Let $(X_\alpha, c_\alpha)_{\alpha\in I}$ be a collection of closure spaces, let $\prod_{\alpha\in I} X_\alpha$ be the cartesian product of underlying sets, and let $\pi_\beta : \prod_{\alpha\in I} X_\alpha \to (X_\beta, c_\beta)$ be the projection mappings. For each $x \in \prod_{\alpha\in I} X_\alpha$, let*

$$\mathcal{V}_x = \{\pi_\beta^{-1}(V) : \beta \in I, V \subset X_\beta \text{ a neighborhood of } \pi_\beta(x) \in X_\beta\}.$$

Then there exist a unique closure structure c_Π on $\prod_{\alpha\in I} X_\alpha$ such that $\mathcal{V}_x$ is a subbase for each $x \in \prod_{\alpha\in I} X_\alpha$.

In certain instances, to prevent any potential confusion, we prefer to utilize the notation $c_{X\times Y}$ to represent the closure operator of $X \times Y$.

3. Hopf structures on closure spaces

This part defines Hopf space and Hopf group structures on closure spaces. Also, the subspace concept is defined for these structures.

In the subsequent part of the study, unless otherwise specified, we will consider (X, c), (X, c_X), (Y, c_Y), etc., as closure spaces.

3.1. Closure Hopf space

Following [18], continuous functions $f, g : (X, c_X) \to (Y, c_Y)$ are called homotopic, denoted by $f \simeq g$, if there exists a continuous map

$$H : (X \times I, c_\Pi) \to (Y, c_Y)$$

such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$, where $I = [0, 1]$ with topological closure structure T. Then H is called a homotopy between f and g.

The homotopy relation " $\simeq$ " is an equivalence relation. We use $[f]$ to denote the homotopy class of f, and $[(X, c_X), (Y, c_Y)]$ to denote the set of all homotopy classes of the functions from (X, c_X) to (Y, c_Y):

$$[(X, c_X); (Y, c_Y)] = \{\, [f] \mid f : (X, c_X) \to (Y, c_Y)\}$$

$$[f] = \{\, g \mid f \simeq g,\ g : (X, c_X) \to (Y, c_Y)\}$$

It is easy to see that if the continuous functions $f, g : (X, c_X) \to (Y, c_Y)$ are homotopic with the homotopy F, then $h \circ f \simeq h \circ g$ with the homotopy $H = h \circ F$ for any continuous function $h : (Y, c_Y) \to (Z, c_Z)$.

Definition 3.1. A continuous map $f : (Y, c_Y) \to (Z, c_Z)$ is called a monomorphism if, $f \circ g \simeq f \circ h$ implies that $g \simeq h$, for the continuous maps $g, h : (X, c_X) \to (Y, c_Y)$.

Let (X, c_X) be a closure space and $x_0 \in X$ be a point. Then (X, x_0, c_X) is called a pointed closure space and x_0 is called base point of (X, x_0, c_X). In this study, only functions from (X, x_0, c_X) to (Y, y_0, c_Y) that preserve the base point, that is, satisfy the condition $f(x_0) = y_0$, are considered. Also, homotopies that are relative to the base point are discussed, i.e., if $f \simeq g$ and F is the homotopy, then $F(x_0, t) = f(x_0) = g(x_0)$ for all $t \in [0, 1]$.

Definition 3.2. Let (X, x_0, c) be a pointed closure space, $m : X \times X \to X$ be a continuous multiplication, $\varsigma : X \to X$ be a constant function such that $\varsigma(x) = x_0$ for all $x \in X$, and 1_X be the identity function on X. Then (X, x_0, c) is called as a closure Hopf space (CH-space for short) if $m \circ (\varsigma, 1_X) \simeq 1_X \simeq m \circ (1_X, \varsigma)$. Also ς is called homotopy identity of (X, x_0, c).

Consequently, if (X, x_0, c) is a CH-space, then the following diagram is homotopy commutative:

$$\begin{array}{ccccc} X & \xrightarrow{(\varsigma,1_X)} & X \times X & \xleftarrow{(1_X,\varsigma)} & X \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle 1_X} & \\ & & X & & \end{array}$$

In the case of more than one CH-space, we use the notations m_X and ς_X for the continuous multiplication and homotopy identity of the CH-space (X, x_0) to avoid confusion.

An example of a non-topological CH-space is provided below.

Example 3.3. Consider the graph represented as $(\mathbb{Z}, E)$, where $\mathbb{Z}$ is the set of all integers as the vertex set, and the set of edges is defined as $E = \{\{x, x+2\} \mid x \in \mathbb{Z}\}$. Then $(\mathbb{Z}, E)$ is a closure space with the closure operator defined as $c(A) = \bigcup_{x \in A}\{c(x)\}$ where $c(x) = \{x-2, x, x+2\}$. Since the closure operator c is not idempotent, it is not topological.

Now let us examine the pointed closure space $(G, 0, c)$ where $G = (\mathbb{Z}, E)$. Define $\varsigma : \mathbb{Z} \to \mathbb{Z}$ as $\varsigma(x) = 0$ for all $x \in \mathbb{Z}$. Let $m : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined such that $m(x, y) = x + y$ for all $x, y \in \mathbb{Z}$. Then, we can observe the following:

$$(m \circ (\varsigma, 1_{\mathbb{Z}}))(x) = m(0, x) = x$$

$$(m \circ (1_{\mathbb{Z}}, \varsigma))(x) = m(x, 0) = x$$

Therefore, $(G, 0, c)$ is a CH-space.

Now, we prove that the product of CH-spaces is a CH-space.

Theorem 3.4. *Let (X, x_0, c_X) and (Y, y_0, c_Y) be CH-spaces. Then $X \times Y$ is a CH-space.*

Proof. Define $m_{X \times Y} = (m_X \circ (\pi_1 \times \pi_1), m_Y \circ (\pi_2 \times \pi_2))$

$$\begin{array}{ccccc} (X \times Y) \times (X \times Y) & \xrightarrow{\pi_1 \times \pi_1} & X \times X & \xrightarrow{m_X} & X \\ & {\scriptstyle \pi_2 \times \pi_2}\searrow & & & \\ & & Y \times Y & \xrightarrow{m_Y} & Y \end{array}$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the first and second projection maps, respectively. Let $\varsigma_{X \times Y} = (\varsigma_X \times \varsigma_Y)$ for the homotopy identities ς_X and ς_Y of (X, x_0, c_X) and (Y, y_0, c_Y), respectively. Then

$$\begin{array}{rcl} m_{X\times Y} \circ (1_{X\times Y}, \varsigma_{X\times Y}) & = & (m_X \circ (\pi_1 \times \pi_1), m_Y \circ (\pi_2 \times \pi_2)) \circ (1_X \times 1_Y, \varsigma_X \times \varsigma_Y) \\ & = & (m_X \circ (1_X, \varsigma_X), m_Y \circ (1_Y, \varsigma_Y)) \\ & \simeq & 1_X \times 1_Y = 1_{X\times Y}. \end{array}$$

Similarly, we have

$$m_{X\times Y} \circ (\zeta_{X\times Y}, 1_{X\times Y}) \simeq 1_{X\times Y}.$$

□

Definition 3.5. Let (X, x_0, c_X) and (Y, y_0, c_Y) be CH-spaces. A function

$$h : (X, x_0, c_X) \to (Y, y_0, c_Y)$$

is called an H-homomorphism if $h \circ m_X \simeq m_Y \circ (h \times h)$.

Theorem 3.6. *Let $g : (X, x_0, c_X) \to (Y, y_0, c_Y)$ and $h : (Y, y_0, c_Y) \to (Z, z_0, c_Z)$ be H-homomorphisms. Then $h \circ g$ is an H-homomorphism.*

Proof. Since g and h are H- homomorphisms

$$g \circ m_X \simeq m_Y \circ (g \times g) \quad \text{and} \quad h \circ m_Y \simeq m_Z \circ (h \times h).$$

Therefore, $h \circ g \circ m_X \simeq h \circ m_Y \circ (g \times g) \simeq m_Z \circ (h \times h) \circ (g \times g) = m_Z \circ (h \circ g \times h \circ g)$. □

The concept of retract plays a fundamental role in understanding and studying topological spaces. Retracts help us understand how subspaces of a topological space can be retracted within the main space. This aids in comprehending the topological properties of a space. Also retracts closely related to the homotopy theory. Retracts are used to describe homotopy equivalence relationships. Deformation retracts, a specific subset of retracts, demonstrates how a space can be deformed within itself. This concept is beneficial for examining and understanding the topological properties of a space. Deformation retracts show that a space is homotopically equivalent to another space; see [20] for further details on related topics.

Let us introduce and delve into these concepts within closure Hopf spaces.

Definition 3.7. A subspace (Z, c_Z) of a closure space (X, c_X) is called a retract of (X, c_X) if there exists a map $r : (X, c_X) \to (Z, c_Z)$ such that $r(x) = x$, for all $x \in Z$. This means $r \circ i = 1_Z$ for the inclusion map $i : (Z, c_Z) \hookrightarrow (X, c_X)$. In this case, r is called a retraction. If $r \circ i \simeq 1_Z$, then (Z, c_Z) is called a weak retract of (X, c_X).

It is clear that every retract of a closure space is a weak retract of it.

Theorem 3.8. *A weak retract of a CH-space is also a CH-space.*

Proof. Let (Z, z_0, c_Z) be a weak retract of a CH-space (X, x_0, c_X) and r be the retraction. Let $m_Z = r \circ m_X \circ (i \times i)$. Then m_Z is a continuous multiplication of (Z, z_0, c_Z). Let ς be the homotopy identity of (X, x_0, c_X) and $\varsigma|_Z$ be the restriction of ς to Z: $\varsigma|_z(z) = \varsigma(z) = x_0$, for all $z \in Z$. Therefore,

$$m_Z \circ (1_Z, \varsigma|_z) = (r \circ m_X \circ (i \times i)) \circ (1_Z, \varsigma|_z) = r \circ (m_X \circ (1_X, \varsigma)) \circ i \simeq r \circ 1_X \circ i = r \circ i \simeq 1_Z.$$

By the same way, $m_Z \circ (\varsigma|_z, 1_Z) \simeq 1_Z$. Consequently, (Z, z_0, c_Z) is a CH-space. □

Definition 3.9. A subspace (Z, z_0, c_Z) of a closure space (X, x_0, c_X) is called a deformation retract if there exists a homotopy such that $i \circ r \simeq 1_X$ for the inclusion map i and the retraction r.

We have the following transitions between these concepts:

$$\text{Deformation Retract} \longrightarrow \text{Retract} \longrightarrow \text{Weak Retract}$$

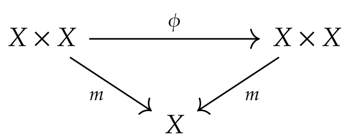
Therefore, we obtain the following corollary:

Corollary 3.10. *A retract (deformation retract) of a CH-space is itself a CH-space.*

Definition 3.11. Let (X, x_0, c_X) be a CH-space. The multiplication m is called homotopy abelian and (X, x_0, c_X) is called an abelian CH-space if there exists a map

$$\phi : X \times X \to X \times X, \ \phi(a, b) = (b, a)$$

such that $m \circ \phi \simeq m$, that is the following diagram homotopy commutative:



Theorem 3.12. *A weak retract of an abelian CH-space is also an abelian CH-space.*

Proof. Let (X, x_0, c_X) be an abelian CH-space and (Y, y_0, c_Y) be a weak retract of it. Then $m_X \circ \phi \simeq m_X$ for a map $\phi : X \times X \to X \times X$, $\phi(a, b) = (b, a)$. Then (Y, y_0, c_Y) is a CH-space with the multiplication $m_Y = r \circ m_X \circ (i \times i)$, by Theorem 3.8. Let $\phi' = \phi|_{Y \times Y}$. Therefore,

$$m_Y \circ \phi' = r \circ m_X \circ (i \times i) \circ \phi' = r \circ m_X \circ \phi \circ (i \times i) \simeq r \circ m_X \circ (i \times i) = m_Y.$$

So (Y, y_0, c_Y) is an abelian CH-space. □

Theorem 3.13. *Let (X, x_0, c_X) be a CH-space and (Z, z_0, c_Z) be a deformation retract of (X, x_0, c_X). Then, the inclusion map i and the retraction r are H-homomorphisms.*

Proof. Let $m_Z = r \circ m_X \circ (i \times i)$. Then

$$i \circ m_Z = i \circ (r \circ m_X \circ (i \times i)) \simeq 1_X \circ m_X \circ (i \times i) = m_X \circ (i \times i).$$

So the inclusion map i is an H-homomorphism. Also

$$m_Z \circ (r \times r) = r \circ m_X \circ (i \times i) \circ (r \times r) \simeq r \circ m_X \circ 1_X \times 1_X = r \circ m_X.$$

Therefore, the retraction r is an H-homomorphism. □

Theorem 3.14. *Let (X, x_0, c_X) be a CH-space and (Y, y_0, c_Y) has the same homotopy type with (X, x_0, c_X). Then (Y, y_0, c_Y) is a CH-space.*

Proof. Let ς be the homotopy identity of (X, x_0, c_X) and $f : X \to Y$ be a homotopy equivalence with a homotopy inverse $g : Y \to X$. Let $m_Y = f \circ m_X \circ (g \times g)$ and $\varsigma'(y) = y_0$ for all $y \in Y$. Then

$$m_Y \circ (1_Y, \varsigma') = (f \circ m_X \circ (g \times g)) \circ (1_Y, \varsigma') = f \circ (m_X \circ (1_X, \varsigma)) \circ g \simeq f \circ 1_X \circ g = f \circ g \simeq 1_Y,$$

$$m_Y \circ (\varsigma', 1_Y) = (f \circ m_X \circ (g \times g)) \circ (\varsigma', 1_Y) = f \circ (m_X \circ (\varsigma, 1_X)) \circ g \simeq f \circ 1_X \circ g = f \circ g \simeq 1_Y.$$

□

Now, we construct a Hopf structure on a set with the help of the quotient closure operator.

Theorem 3.15. *Let (X, x_0, c) be a CH-space with the homotopy identiti ς, (Y, y_0) be a pointed space and α be a surjective mapping from (X, x_0, c) to (Y, y_0). Then (Y, y_0) is a CH-space.*

Proof. (Y, y_0) is a closure space with the quotient closure operator $c_\alpha = \alpha \circ c \circ \alpha^{-1}$.
Define $m_Y = \alpha \circ m_X \circ (\alpha^{-1} \times \alpha^{-1})$ and $\varsigma'(y) = y_0$ for all $y \in Y$. We have

$$\begin{aligned} m_Y \circ (1_Y, \varsigma') &= (\alpha \circ m_X \circ (\alpha^{-1} \times \alpha^{-1})) \circ (1_Y, \varsigma') \\ &= \alpha \circ (m_X \circ (1_X, \varsigma)) \circ \alpha^{-1} \simeq \alpha \circ 1_X \circ \alpha^{-1} \\ &= \alpha \circ \alpha^{-1} = 1_Y, \end{aligned}$$

$$\begin{aligned} m_Y \circ (\varsigma', 1_Y) &= (\alpha \circ m_X \circ (\alpha^{-1} \times \alpha^{-1})) \circ (\varsigma', 1_Y) \\ &= \alpha \circ (m_X \circ (\varsigma, 1_X)) \circ \alpha^{-1} \simeq \alpha \circ 1_X \circ \alpha^{-1} \\ &= \alpha \circ \alpha^{-1} = 1_Y. \end{aligned}$$

Therefore, (Y, y_0, c_α) is a CH-space. □

3.2. *Closure Hopf group*

In closure spaces, it is possible to establish a Hopf group structure, a group-like algebraic structure. This section presents the concept of a closure Hopf group (a CH-group for shorts).

Definition 3.16. Let (X, x_0, c) be a CH-space with the homotopy identity ς.

1. $m : X \times X \to X$ is called homotopy associative if $m \circ (m \times 1_X) \simeq m \circ (1_X \times m)$, i.e., the following diagram is homotopy commutative:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times 1_X} & X \times X \\ {\scriptstyle 1_X \times m}\downarrow & & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow{\quad m \quad} & X \end{array}$$

2. A continuous function $\eta : X \to X$ such that $m \circ (\eta, 1_X) \simeq c \simeq m \circ (1_X, \eta)$, is called a homotopy inverse.

A CH-group is a CH-space that has a homotopy associative multiplication and a homotopy inverse.

Theorem 3.17. *Let (X, x_0, c_X) be a CH-group and (Y, y_0, c_Y) has the same homotopy type with (X, x_0, c_X). Then (Y, y_0, c_Y) is a CH-group.*

Proof. Let $g : X \to Y$ be a homotopy equivalence with a homotopy inverse $h : Y \to X$ and $m_Y = g \circ m_X \circ (h \times h)$ be continuous multiplication of (Y, y_0, c_Y). Then (Y, y_0, c_Y) is a CH-space by Theorem 3.14.
Now let us show that m_Y is homotopy associative:

$$\begin{aligned} m_Y \circ (m_Y \times 1_Y) &= \big(g \circ m_X \circ (h \times h)\big) \circ \big((g \circ m_X \circ (h \times h)) \times 1_Y\big) \\ &= (g \circ m_X) \circ (h \times h) \circ (g \times g) \circ (m_X \times 1_X) \circ (h \times h \times h) \\ &\simeq (g \circ m_X) \circ 1_{X \times X} \circ (m_X \times 1_X) \circ (h \times h \times h) \\ &= g \circ (m_X \circ (m_X \times 1_X)) \circ (h \times h \times h) \\ &\simeq g \circ (m_X \circ (1_X \times m_X)) \circ (h \times h \times h) \\ &\simeq g \circ m_X \circ (h \times h) \circ (g \times g) \circ (1_X \times m_X) \circ (h \times h \times h) \\ &= \big(g \circ m_X \circ (h \times h)\big) \circ \big(1_Y \times (g \circ m_X \circ (h \times h))\big) \\ &= m_Y \circ (1_Y \times m_Y). \end{aligned}$$

Now let us show that (Y, y_0, c_Y) has a homotopy inverse:
Let η_X be the homotopy inverse of (X, x_0, c_X) and $\eta_Y = g \circ \eta_X \circ h$. Then

$$\begin{aligned} m_Y \circ (1_Y, \eta_Y) &= (g \circ m_X \circ (h \times h)) \circ (1_Y, g \circ \eta_X \circ h) \\ &= (g \circ m_X) \circ (h, h \circ g \circ \eta_X \circ h) \simeq (g \circ m_X) \circ (h, \eta_X \circ h) \\ &= g \circ (m_X \circ (1_X, \eta_X)) \circ h \simeq g \circ (m_X \circ (\eta_X, 1_X)) \circ h \\ &= (g \circ m_X) \circ (\eta_X \circ h, h) \simeq (g \circ m_X) \circ (h \circ g \circ \eta_X \circ h, h) \\ &= (g \circ m_X \circ (h \times h)) \circ (g \circ \eta_X \circ h, 1_Y) = m_Y \circ (\eta_Y, 1_Y). \end{aligned}$$

Consequently, (Y, y_0, c_Y) is a CH-group. □

Theorem 3.18. *Let (X, x_0, c_X) and (Y, y_0, c_Y) have the same homotopy type. If (X, x_0, c_X) is an abelian CH-group, then (Y, y_0, c_Y) also an abelian CH-group.*

Proof. Let $f : X \to Y$ be a homotopy equivalence with a homotopy inverse $g : Y \to X$. By Theorem 3.17, (Y, y_0, c_Y) is a CH-group with the multiplication $m_Y = f \circ m_X \circ (g \times g)$. Since m_X is homotopy commutative, then there exists a map $\phi : X \times X \to X \times X, \phi(a,b) = (b,a)$ such that $m_X \circ \phi \simeq m_X$. Let $\phi' : Y \times Y \to Y \times Y$ be defined as $\phi'(a',b') = (b',a')$ for all $a',b' \in Y$. Then

$$m_Y \circ \phi' = (f \circ m_X \circ (g \times g)) \circ \phi' = f \circ m_X \circ \phi \circ (g \times g) \simeq f \circ m_X \circ (g \times g) = m_Y.$$

So m_Y is homotopy commutative. □

Theorem 3.19. *A CH-space (X, x_0, c_X) is a CH-group if and only if the map*

$$\alpha : X \times X \to X \times X$$

defined as $\alpha(x,y) = (x, m(x,y))$ is a homotopy equivalence, where $m : X \times X \to X$ is a continuous multiplication.

Proof. Let (X, x_0, c_X) be a CH-group with homotopy inverse η. Define $\beta : X \times X \to X \times X$ such that $\beta(x,y) = (x, m(\eta(x), y))$, for all $x, y \in X$. Then

$$(\alpha \circ \beta)(x,y) = \alpha(x, m(\eta(x), y)) = (x, m(x, m(\eta(x), y)))$$

implies that $\alpha \circ \beta \simeq 1_{X\times X}$, since m is homotopy associative and η is homotopy inverse. Similarly $\beta \circ \alpha \simeq 1_{X\times X}$. Therefore, α is homotopy equivalence.
Conversely, let α be a homotopy equivalence and $\gamma : X \times X \to X \times X$ be a homotopy inverse of α such that $\alpha \circ \gamma \simeq 1_{X\times X} \simeq \gamma \circ \alpha$. Let show that (X, x_0, c_X) is a CH-group.
Let π_1 and π_2 be projections and define $\Phi = \pi_2 \circ \gamma \circ f$ where $f : X \to X \times X, f(x) = (x, x_0)$ for all $x \in X$. Then $\pi_1 \circ \alpha = \pi_1$ and $\pi_2 \circ \alpha = m$, so we have:

$$\pi_1 \simeq \pi_1 \circ \alpha \circ \gamma = \pi_1 \circ \gamma,$$

$$\pi_2 \simeq \pi_2 \circ \alpha \circ \gamma = m \circ \gamma.$$

Since $\pi_1 \circ \gamma \circ f \simeq \pi_1 \circ f = 1_X$, where $c : X \to X, c(x) = x_0$ is the constant map, we can further show that:

$$\begin{aligned} m \circ (1_X, \Phi) &\simeq m \circ (\pi_1 \circ \gamma \circ f, \pi_2 \circ \gamma \circ f) \\ &= m \circ (\pi_1 \circ \pi_2) \circ (\gamma \circ f) \\ &= m \circ \gamma \circ f \simeq \pi_2 \circ f = c. \end{aligned}$$

Using the same reasoning, we can also show that $m \circ (\Phi, 1_X) \simeq c$. Consequently, (X, x_0, c_X) is a CH-group. □

Theorem 3.20. *A deformation retract of a CH-group is also a CH-group.*

Proof. In Theorem 3.17, take $h = i$ and $g = r$, then the proof is obvious. □

Corollary 3.21. *A deformation retract of an abelian CH-group is also an abelian CH-group.*

Theorem 3.22. *Let (X, x_0, c_X) be a CH-group. Then for every pointed closure space (Y, y_0, c_Y), the set $[(Y, y_0, c_Y); (X, x_0, c_X)]$ is a group of homotopy classes. Also $[(Y, y_0, c_Y); (X, x_0, c_X)]$ is abelian if m_X is abelian.*

Proof. Let us define a binary operation M on $[(Y, y_0, c_Y); (X, x_0, c_X)]$ such that, for all $[f], [g] \in [(Y, y_0, c_Y); (X, x_0, c_X)]$, $M([f], [g]) = [m_X \circ (f, g)]$.

Let $([f_1], [f_2]) = ([g_1], [g_2])$. Then $f_1 \simeq g_1$ and $f_2 \simeq g_2$ for homotopies

$$F, G : (Y \times I, c_{(Y\times I)}) \to (X, c_X).$$

Let $H : (Y \times I, c_{(Y\times I)}) \to (X, c_X)$ be defined as $H = m_X \circ (F, G)$. Then

$$H(y, 0) = \big(m_X \circ (F, G)\big)(y, 0) = m_X(F(y,0), G(y,0)) = m_X(f_1(y), f_2(y))$$

$$H(y,1) = \Big(m_X \circ (F,G)\Big)(y,1) = m_X(F(y,1), G(y,1)) = m_X(g_1(y), g_2(y)).$$

Therefore, $m_X \circ (f_1, f_2) \simeq m_X \circ (g_1, g_2)$. So

$$M([f_1],[f_2]) = [m_X \circ (f_1, f_2)] = [m_X \circ (g_1, g_2)] = M([g_1],[g_2]).$$

Then M is well defined. Let $e : Y \to X, e(y) = x_0$, for all $y \in Y$. Then for any $[f] \in [(Y, y_0, c_Y); (X, x_0, c_X)]$,

$$M([f],[e]) = [m_X \circ (f, e)] = [m_X \circ (1_X, c) \circ f] = [1_X \circ f] = [f].$$

Similarly, $M([e],[f]) = [f]$. So $[e]$ is the unit element of $[(Y, y_0, c_Y); (X, x_0, c_X)]$ for M.
Let $1_{[\,]}$ be the unit function of $[(Y, y_0, c_Y); (X, x_0, c_X)]$. Let us show M is associative:

$$\begin{aligned}
\Big(M \circ (1_{[\,]} \times M)\Big)([f_1],([f_2],[f_3])) &= M\Big([f_1], M([f_2],[f_3])\Big) = M\Big([f_1], m_X \circ (f_2, f_3)]\Big) \\
&= [m_X \circ (f_1, m_X \circ (f_2, f_3)] \\
&= [m_X \circ (1_X \times m_X) \circ (f_1, (f_2, f_3))] \\
&= [m_X \circ (m_X \times 1_X) \circ (f_1, (f_2, f_3))] \\
&= [m_X((m_X \circ (f_1, f_2)), f_3)] \\
&= M\Big([m_X \circ (f_1, f_2)], [f_3]\Big) = M\Big(M([f_1],[f_2]), [f_3]\Big) \\
&= \Big(M \circ (M \times 1_{[\,]})\Big)(([f_1],[f_2]),[f_3]).
\end{aligned}$$

Therefore, M is associative.

Let η be the homotopy inverse of (X, x_0, c_X). For any $[f] \in [(Y, y_0, c_Y); (X, x_0, c_X)]$,

$$M([f],[\eta \circ f)]) = [m_X \circ ((f, \eta \circ f)] = [m_X \circ (1_X, \eta) \circ f] = [c \circ f] = [e].$$

Similarly, $M([\eta \circ f],[f]) = [e]$. Therefore, $[\eta \circ f]$ is the homotopy inverse of $[f]$.

Now let m_X be abelian. Then

$$M([f],[g]) = [m_X \circ (f, g)] = [m_X \circ (g, f)] = M([g],[f]).$$

Therefore, $[(Y, y_0, c_Y); (X, x_0, c_X)]$ is abelian. □

4. Homotopy category of closure Hopf spaces

In this section, we introduce the concept of the homotopy category of CH-spaces, wherein the objects are CH-spaces, and the morphisms are the homotopy classes of the base point preserving continuous functions on CH-spaces. To provide context, we begin by revisiting the fundamental definition of a mathematical category.

Definition 4.1. ([1]) A category $\mathcal{F}$ includes;

C_1) a collection of objects
C_2) a set $hom(U, V) = \{f | f : U \to V\})$ of morphisms, for every object U and V,
C_3) composition of morphisms such that the following axioms hold:
  i) $f \circ (g \circ h) = (f \circ g) \circ h$
  ii) for every object A, there exists a unique morphism $1_A \in hom(A, A)$ called the identity morphism for A, such that all morphisms $f \in hom(B, A), g \in hom(A, B)$ satisfy $1_A \circ f = f$ and $g \circ 1_A = g$.

We define the homotopy category of CH-spaces, denoted by $\mathcal{CH}$, where the objects in this category are CH-spaces, and the set of morphisms is the set of homotopy classes of the continuous functions on CH-spaces

$$hom((X, x_0, c_X), (Y, y_0, c_Y)) = [(X, x_0, c_X), (Y, y_0, c_Y)].$$

The composition of morphisms in this category is defined as the operation M, as introduced in Theorem 3.22.

Definition 4.2. ([1]) A contravariant functor C from the category $\mathcal{F}$ to the category $\mathcal{G}$ is a function, which maps each object A of $\mathcal{F}$ to an object $C(A)$ of $\mathcal{G}$ and each morphism $f \in hom(A, B)$ of $\mathcal{F}$ to a morphism $C(f) : C(B) \to C(A)$, such that

i) $C(1_A) = 1_{C(A)}$
ii) $C(g \circ f) = C(f) \circ C(g)$.

Theorem 4.3. *Let (Y, y_0, c_Y) be a CH-group. Then there exists a contravariant functor from $\mathcal{CH}$ to the category of groups and homomorphisms, denoted by $\mathcal{G}$.*

Proof. Let define a map Γ^Y from $\mathcal{CH}$ to the category of sets and functions, denoted by $\mathcal{S}$ such that associates to an object (X, x_0, c_X) the set

$$\Gamma^Y((X, x_0, c_X)) = [(X, x_0, c_X), (Y, y_0, c_Y)]$$

and to a morphism $[g]$ the function

$$\Gamma^Y([g]) = g^* : [(Z, z_0, c_Z), (Y, y_0, c_Y)] \to [(X, x_0, c_X), (Y, y_0, c_Y)], g^*([f]) = [f \circ g]$$

where $[g] \in [(X, x_0, c_X), (Z, z_0, c_Z)]$. Let $[f], [h] \in [(Z, z_0, c_Z), (Y, y_0, c_Y)]$.

$$\begin{aligned} g^*(M([f], [h])) &= g^*([m_Y \circ (f, h)]) \\ &= [(m_Y \circ (f, h)) \circ g] \\ &= [m_Y \circ (f \circ g, h \circ g)] \\ &= M([f \circ g], [h \circ g]) \\ &= M(g^*([f]), g^*([h])). \end{aligned}$$

Therefore, g^* is a homomorphism. Also $\Gamma^Y(X, x_0, c_X) = [(X, x_0, c_X), (Y, y_0, c_Y)]$ is a group with the binary operation M by the Theorem 3.22. Therefore, objects and morphisms of the category that range of the Γ^Y are groups and homomorphisms, respectively. Now let us show that Γ^Y is a contravariant functor.

Let $[1_X] \in [(X, x_0, c_X), (X, x_0, c_X)]$ be the unit morphism of $\mathcal{CH}$. Then

$$\Gamma^Y([1_X] = 1_X^* : [(X, x_0, c_X), (Y, y_0, c_Y)] \to [(X, x_0, c_X), (Y, y_0, c_Y)]$$

and for any morphism $[f] \in [(X, x_0, c_X), (Y, y_0, c_Y)]$, $1_X^*([f]) = [f \circ 1_X] = [f]$. So $\Gamma^Y([1_X])$ is the unit morphism of $\mathcal{G}$.

Let $[f] \in [(X, x_0, c_X), (Z, z_0, c_Z)]$, $[g] \in [(Z, z_0, c_Z), (W, w_0, c_W)]$. For any morphism $[h] \in [(W, w_0, c_W), (Y, y_0, c_Y)]$,

$$\begin{aligned} \Gamma^Y([g \circ f])([h]) &= [h \circ (g \circ f)] = [(h \circ g) \circ f] \\ &= \Gamma^Y([f])([h \circ g]) \\ &= \Gamma^Y([f])(\Gamma^Y([g])([h])) \\ &= (\Gamma^Y([f]) \circ \Gamma^Y([g]))([h]). \end{aligned}$$

Then $\Gamma^Y([g \circ f]) = \Gamma^Y([f]) \circ \Gamma^Y([g])$. Therefore, Γ^Y is a contravariant functor since it preserves the composition and the identity. □

Corollary 4.4. *Let (Y, y_0, c_Y) be an abelian CH-group. Then there exists a contravariant functor from $\mathcal{CH}$ to the category of abelian groups and homomorphisms.*

The following theorem shows that the converse of the Theorem 4.3 is valid.

Theorem 4.5. *Let (X, x_0, c_X) be a pointed closure space and Γ^X takes values in $\mathcal{G}$, then (X, x_0, c_X) is a CH-group.*

Proof. Let $m : X \times X \to X$ be a map such that $[m] = M([\pi_1], [\pi_2])$ where M is the binary operation on the group $[X \times X, (x_0, x_0), c_\Pi), (X, x_0, c_X)]$.
For any map $h, h' : (Y, y_0, c_Y) \to (X, x_0, c_X)$, for a pointed closure space (Y, y_0, c_Y),
$\Gamma^X([h, h']) = (h, h')^* : [(X \times X, (x_0, x_0), c_\Pi), (X, x_0, c_X)] \to [(Y, y_0, c_Y), (X, x_0, c_X)]$ is a homomorphism.

$$\begin{aligned}
[m \circ (h, h')] = (h, h')^*([m]) &= (h, h')^*(M([\pi_1], [\pi_2])) \\
&= M((h, h')^*([\pi_1], [\pi_2])) \\
&= M([\pi_1 \circ (h, h')], [\pi_2 \circ (h, h')]) \\
&= M([h], [h']).
\end{aligned}$$

So it turns out that M is induced by m.
Let $[c]$ be the identity for $[(X, x_0, c_X), (X, x_0, c_X)]$ where $c : (X, x_0, c_X) \to (X, x_0, c_X)$ is the constant map, defined as $c(x) = x_0$, for all $x \in X$. Then

$$[m \circ (1_X, c)] = M([1_X], [c]) = [1_X].$$

Similarly $[m \circ (1_X, c)] = [1_X]$. Therefore, (X, x_0, c_X) is a CH-space. Let $\rho_1, \rho_2, \rho_3 : X \times X \times X \to X$ be projections. Then

$$\begin{aligned}
[m \circ (1_X \times m)] &= (1_X \times m)^*([m]) \\
&= M((1_X \times m)^*([\pi_1]), (1_X \times m)^*([\pi_2])) \\
&= M([\pi_1 \circ (1_X \times m)], [\pi_2 \circ (1_X \times m)]) \\
&= M([\pi_1 \circ (1_X \times M([\pi_1], [\pi_2]))], [\pi_2 \circ (1_X \times M([\pi_1], [\pi_2]))]) \\
&= M([\rho_1], M([\rho_2], [\rho_3])) \\
&= M(M([\rho_1], [\rho_2]), [\rho_3]) \\
&= M([\pi_1 \circ (M([\pi_1], [\pi_2]) \times (1_X)], [\pi_2 \circ M([\pi_1], [\pi_2]) \times (1_X)]) \\
&= M([\pi_1 \circ (m \times 1_X)], [\pi_2 \circ (m \times 1_X)]) \\
&= (m, 1_X)^*([m]) = [m \circ (m \times 1_X)].
\end{aligned}$$

Therefore, $m \circ (1_X \times m) \simeq m \circ (m \times 1_X)$. So m is homotopy associative. Let $[\eta]$ be the inverse of $[1_X]$, for the map $\eta : X \to X$. Then

$$[m \circ (1_X, \eta)] = M([1_X], [\eta]) = [c].$$

Likewise, $[m \circ (\eta, 1_X)] = [c]$. Consequently, (X, x_0, c_X) is a CH-group with the homotopy inverse η. □

The following theorem shows that the commutative feature is preserved for the Theorem 4.5.

Theorem 4.6. *Let (X, x_0, c_X) be a pointed closure space and Γ^X takes values in the category of abelian groups and homomorphisms, then (X, x_0, c_X) is an abelian CH-group.*

Proof. (X, x_0, c_X) is a CH-group with the multiplication m, defined as Theorem 4.5. Let $\phi : X \times X \to X \times X, \phi(x, y) = (y, x)$. Since M is commutative,

$$\begin{aligned}
[m \circ \phi] &= \phi^*([m]) = \phi^*(M([\pi_1], [\pi_2])) = M([\pi_1 \circ \phi], (\pi_2 \circ \phi]) \\
&= M(\pi_1], [\pi_2]) = M([\pi_2], [\pi_1]) = [m].
\end{aligned}$$

Therefore, m is homotopy commutative. □

Theorem 4.7. *Let (X, x_0, c_X) and (Y, y_0, c_Y) be CH-spaces and*

$$f : (X, x_0, c_X) \to (Y, y_0, c_Y)$$

be an H-homomorphism. Then there exists a homomorphism from $[(Z, z_0, c_Z), (X, x_0, c_X)]$ to $[(Z, z_0, c_Z), (Y, y_0, c_Y)]$ for any pointed closure space (Z, z_0, c_Z).

Proof. Let $[h], [h'] \in [(Z, z_0, c_Z), (X, x_0, c_X)]$ and define

$$f_* : [(Z, z_0, c_Z), (X, x_0, c_X)] \to [(Z, z_0, c_Z), (Y, y_0, c_Y)]$$

such that $f_*([h]) = [f \circ h]$. Then

$$f_*(M([h], [h'])) = f_*([m_X \circ (h, h')]) = [f \circ m_X \circ (h, h')].$$

Also $m_Y \circ (f \times f) \simeq f \circ m_X$, since f is H-homomorphism. Therefore

$$\begin{aligned} [f \circ m_X \circ (h, h')] &= [m_Y \circ (f \times f) \circ (h, h')] \\ &= M([f \circ h], [f \circ h']) \\ &= M(f_*([h]), f_*([h'])). \end{aligned}$$

Consequently, f_* is a homomorphism. □

The following theorem is a result of Theorem 4.3,4.7 and 4.5.

Theorem 4.8. *Let σ be a map between CH-groups (Y, y_0, c_Y) and $(Y', y'_0, c_{Y'})$. Then there exists a natural transformation from Γ^Y to $\Gamma^{Y'}$ in the category $\mathcal{G}$ if and only if σ is an H-homomorphism.*

Proof. Define $\sigma_* : \Gamma^Y \to \Gamma^{Y'}$ as a map assigns every object (X, x_0, c_X) to a morphism

$$\sigma_*^X : [(X, x_0, c_X), (Y, y_0, c_Y)] \to [(X, x_0, c_X), (Y', y'_0, c_{Y'})]$$

such that $\sigma_*^X([g]) = [\sigma \circ g]$. It is easy to see that the following diagram commutes:

$$\begin{array}{ccc} [(X, x_0, c_X), (Y, y_0, c_Y))] & \xrightarrow{\Gamma^Y([h])} & [(X', x'_0, c_{X'}), (Y, y_0, c_Y)] \\ \Big\downarrow{\scriptstyle \sigma_*^X} & & \Big\downarrow{\scriptstyle \sigma_*^{X'}} \\ [(X, x_0, c_X), (Y', y'_0, c_{Y'})] & \xrightarrow[\Gamma^{Y'}([h])]{} & [(X', x'_0, c_{X'}), (Y', y'_0, c_{Y'})] \end{array}$$

for the map $h : (X, x_0, c_X) \to (X', x'_0, c_{X'})$. Therefore, σ_* is a natural transformation between Γ_Y and $\Gamma_{Y'}$ □

5. Subspace of closure Hopf groups

Within the framework of closure spaces, it is reasonable to extend the concept of the sub-H group, as initially defined in [17]. In this context, this section introduces the notion of a sub-CH group and investigates the conditions that establish when a subset within a CH group merits classification as a sub-CH group.

Definition 5.1. Let (Y, x_0, c_X) be a pointed closure subspace of (X, x_0, c_X) which is a CH-group with homotopy inverse η and homotopy identity c. If (Y, x_0, c_X) is a CH-group with the multiplication $m_Y = m_X|_{Y \times Y}$, homotopy inverse $\eta' = \eta|_Y$ and homotopy identity $c' = c|_Y$ such that the inclusion map $i : Y \hookrightarrow X$ is an H-homomorphism, then (Y, x_0, c_X) is called a sub-CH-group of (X, x_0, c_X).

It is clear that if (Y, x_0, c_X) is a sub-CH-group of (X, x_0, c_X), then there exists a continuous function $\phi : Y \to Y$ such that $i \circ m_Y \simeq m_X \circ (i \times i)$.

We get the following result from the definition of the sub-CH-group and Theorem 4.8.

Corollary 5.2. *Let the pointed closure space (Y, x_0, c_Y) be a subspace of (X, x_0, c_X). Then (Y, x_0, c_Y) is a sub-CH-group of (X, x_0, c_X) if and only if i_* is a natural transformation, defined as Theorem 4.8, from Γ^Y to Γ^X for the inclusion map $i : Y \hookrightarrow X$.*

Theorem 5.3. *Let (Y, y_0, c_Y) be a pointed closure subspace of a CH-group (X, x_0, c_X). If the following conditions are satisfied:*

i) there exists a continuous function $m_Y : Y \times Y \to Y$ such that $i \circ m_Y \simeq m_X \circ (i \times i)$,
ii) for the constant map $c' : Y \to Y, c'(y) = y_0$, where c is the homotopy identity of (X, x_0, c_X),
iii) there exists a continuous map $\phi' : Y \to Y$ such that $i \circ \phi' \simeq \phi \circ i$,
iv) the inclusion map $i : Y \hookrightarrow X$ is a monomorphism,

then (Y, y_0, c_Y) is a sub-CH-group of (X, x_0, c_X).

Proof. By i) and ii),

$$\begin{aligned} i \circ m_Y \circ (1_Y, c') &\simeq m_X \circ (i \times i) \circ (1_Y, c') = m_X \circ (i \circ 1_Y, i \circ c') \\ &= m_X \circ (1_X \circ i, c \circ i) = m_X \circ (1_X, c) \circ i \\ &\simeq 1_X \circ i = i \circ 1_Y. \end{aligned}$$

So $i \circ m_Y \circ (1_Y, c') \simeq i \circ 1_Y$. Then $m_Y \circ (1_Y, c') \simeq 1_Y$, since i is monomorphism. Similarly $m_Y \circ (c', 1_Y) \simeq 1_Y$. Hence, c' is a homotopy identity for m_Y.
Following i),

$$\begin{aligned} i \circ m_Y \circ (m_Y \times 1_Y) &\simeq m_X \circ (i \times i) \circ (m_Y \times 1_Y) = m_X \circ [(i \circ m_Y) \times (i \circ 1_Y)] \\ &\simeq m_X \circ [(m_X \circ (i \times i)) \times (1_X \circ i)] = m_X \circ (m_X \times 1_X) \circ (i \times i \times i) \\ &\simeq m_X \circ (1_X \times m_X) \circ (i \times i \times i) \\ &= m_X \circ [(1_X \circ i) \times (m_X \circ (m_X \circ (i \times i))] \\ &= m_X \circ (i \times i) \circ (1_Y \times m_Y) \simeq i \circ m_Y \circ (1_Y \times m_Y). \end{aligned}$$

Therefore, $m_Y \circ (m_Y \times 1_Y) \simeq m_Y \circ (1_Y \times m_Y)$, since i is monomorphism. So

$$\begin{aligned} i \circ c' &= c \circ i \simeq m_X \circ (1_X, \phi) \circ i \\ &= m_X \circ (1_X \circ i, \phi \circ i) \simeq m_X \circ (i \circ 1_Y, i \circ \phi') \\ &\simeq i \circ m_Y \circ (1_Y, \phi'). \end{aligned}$$

Since m_Y is homotopy associative,

$$c' \simeq m_Y \circ (1_Y, \phi')$$

and by the same way $c' \simeq m_Y \circ (\phi', 1_Y)$. Hence ϕ' is a homotopy inverse for m_Y. So (Y, y_0, c_Y) is a CH-group. By ii), $(i \circ c')(y_0) = x_0 = (c \circ i)(y_0)$, and by i), i is an H-homomorphism. Consequently, (Y, y_0, c_Y) is a sub-CH-group of (X, x_0, c_X). □

The following theorem provides a characterization for the deformation retract of a CH-group.

Theorem 5.4. *A deformation retract with the same base point of a CH-group is a sub-CH-group.*

Proof. Let (X, x_0, c_X) be a CH-group with the homotopy identity c_X and homotopy inverse η_X and (Y, x_0, c_Y) be a subspace of (X, x_0, c_X).
Define $m_Y = r \circ m_X \circ (i \times i), c_Y = r \circ c_X \circ i$ and $\eta_Y = r \circ \eta_X \circ i$, for the inclusion i and retraction r. Then

i) $i \circ m_Y = i \circ (r \circ m_X \circ (i \times i) \simeq_1 X \circ (m_X \circ (i \times i)$,
ii) $i \circ c_Y = i \circ r \circ c_X \circ i \simeq 1_X \circ c_X \circ i = c_X \circ i$ and since the base point of Y is x_0, $i \circ c_Y = c_X \circ i$,
iii) $i \circ \eta_Y = i \circ r \circ \eta_X \circ i \simeq 1_X \circ \eta_X \circ i = \eta_X \circ i$.

Therefore, (Y, x_0, c_Y) is a sub-CH-group of (X, x_0, c_X) . □

6. Conclusion

In conclusion, this study explored the concept of Hopf structures in closure spaces, extending the algebraic perspective to non-topological closure spaces. We defined closure Hopf spaces and closure Hopf groups and investigated their properties. Additionally, we delved into the homotopy theory within this framework and demonstrated the relationships between structures, such as retracts and deformation retracts. Furthermore, the study established the existence of contravariant functors between the category of closure Hopf groups and the category of groups. It also introduced the homotopy category of closure Hopf spaces and demonstrated the existence of contravariant functors from this category to the category of groups and, under certain conditions, to the category of abelian groups.

Moreover, we introduced the concept of sub-CH groups, providing a framework for analyzing subsets of CH groups that exhibit similar algebraic properties. Overall, this study contributed to the understanding of algebraic structures within closure spaces, broadening the scope of mathematical research in this area.

Acknowledgements

The authors thank Prof. Ljubiša D. R. Kočinac and the anonymous referees for their very useful and detailed comments which greatly improve the quality of the manuscript.

References

[1] M. R. Adhikari, *Basic Algebraic Topology and Its Applications*, New Delhi: Springer, 2016.
[2] M. Arkowitz, *Introduction to Homotopy Theory: H-spaces and Co-H-spaces*, Springer, 2011.
[3] E. Cech, *Topological Spaces*, Czechoslovak Academy of Sciences, Prague, 1966.
[4] S. Demiralp, *Co-Hopf space structure on closure spaces*, Turkish J. Math. Comput. Sci. **14** (2022), 321–330.
[5] S. Demiralp, H. H. Essam, E. Guner, *Some properties of fuzzy H-spaces*, JP J. Algebra, Number Theory Appl. **40** (2018), 429–448.
[6] S. Demiralp, E. Guner, *Some characterizations of Hopf group on fuzzy topological spaces*, Iran. J. Fuzzy Syst. **11** (2014), 111–121.
[7] O. Ege, *Fuzzy co-Hopf spaces*, ICMME-2016, Firat University, Elazig, 12-14 May, 2016.
[8] O. Ege, I. Karaca, *Some properties of digital H-spaces*, Turkish J. Elec. Eng. Comp. Sci. **24** (2016), 1930–1941.
[9] O. Ege, I. Karaca, *Digital co-Hopf spaces*, Filomat **34** (2020), 2705–2711.
[10] I. M. James, *On H-spaces and their homotopy groups*, Q. J. Math. **11** (1960), 161–179.
[11] K. Kuratowski, *Topology: Vol. I, Vol. 1*, Elsevier, 2014.
[12] D. W. Lee, *On the digitally quasi comultiplications of digital images*, Filomat **31** (2017), 1875–1892.
[13] D. W. Lee, *Near-rings on digital Hopf groups*, Appl. Algebra Engrg. Comm. Comput. **29** (2018), 261–282.
[14] D. W. Lee, *Digital H-spaces and actions in the pointed digital homotopy category*, Appl. Algebra Engrg. Comm. Comput. **31** (2020), 149–169.
[15] D. W. Lee, *Digital Hopf spaces and their duals*, J. Math. (2022), Art. I 8435007.
[16] D. W. Lee, *On the digital Hopf images and monoid isomorphisms*, Publ. Math. Debrecen **103** (2023), 13–39.
[17] K. Park, *On sub-H-groups of an H group and their duals*, J. Korean Math. Soc. **6** (1969), 41–46.
[18] A. Rieser, *Cech closure spaces: A unified framework for discrete and continuous homotopy*, Topology Appl. **296** (2021), Art. ID 107613.
[19] J. P. Serre, *Homologie singulière des espaces fibrés*, Ann. of Math. **54** (1951), 425–505.
[20] E. H. Spanier, *Algebraic Topology: Homotopy and the Fundamental Group*, Springer, 1966.
[21] U. M. Swamy, R. S. Rao, *Algebraic topological closure operators*, Southeast Asian Bull. Math. **26** (2003), 669–678.
[22] R. M. Switzer, *Algebraic Topology: Homotopy and Homology*, Springer, 2017.