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Some new class of ideals in semirings and their applications

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Abstract. In the present paper, we focus on semirings, which are positive cones of a class of lattice-ordered rings. We establish a lattice isomorphism between semiring l-ideals and ring l-ideals of cancellative l-semirings and its difference l-ring, and from this, we obtain a structure of semiring l-ideals via ring l-ideals in cancellative l-semirings. Smith in [26] defined f-semirings as a suitable class of l-semirings in which one can establish a structure theorem. A new class of f-semirings, namely $\mathcal{P}$-semirings, is defined to focus solely on positive cones of abundant function rings, e.g., $C(X)$. We bring notions of z-ideals and z°-ideals into commutative semirings. It is shown that these ideals are equally important in investigating $\mathcal{P}$-semirings like $C^+(X)$. The structure of z-ideals and z°-ideals are obtained in $\mathcal{P}$-semirings via the z-ideals and z°-ideals of its difference ring. We show that each k-ideal of a $\mathcal{P}$-semiring is a z-ideal if and only if it is a von Neumann regular semiring.

1. Introduction

One of the main themes of this paper is to study additively cancellative semirings, which are positive cones of partially ordered rings. The nature of semirings, which are positive cones of partially ordered rings, is noted in Chapter VI, Theorem 2 of [11]. An important example in this context is the semiring of continuous non-negative real-valued functions on a topological space X, denoted by $C^+(X)$. In the course of this paper, we characterize different classes (e.g., Bezout semiring and von Neumann regular semiring) of the semiring $C^+(X)$ via the topological properties of X and vice versa.

Hereinafter, by a semiring $(S,+,\cdot)$, we mean a commutative semiring with 0 (absorbing with respect to multiplication) and 1, where $1 \neq 0$. By $D(S)$, we always mean the ring of differences of a cancellative semiring S. Consequently, by any ideal of $D(S)$, we mean a ring ideal of $D(S)$; by any ideal of S, we mean a semiring ideal of S. A topological space X is always considered to be a Tychonoff space.

The class of *positive semirings* among others, has some better properties, namely every maximal ideal of a positive semiring is an l-ideal; see [26], [25]. Smith, in his seminal paper [26], has obtained a structure theory for f-semirings, which is a class of lattice-ordered semirings. It is noted in [26] that a positive

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lattice-ordered semiring S is an f-semiring, and a cancellative positive lattice-ordered semiring can be embedded in an f-ring by taking differences. Some authors use the term "positive semiring" for different kinds of semirings (see [22]), but here, the positivity of a cancellative l-semiring is closely related to bounded inversion property of an f-ring.

In Section 3, we have discussed a correlation between the lattice of l-ideals of a cancellative lattice-ordered semiring S and lattice l-ideals of its difference ring $D(S)$. As a result, we get a structure of l-ideals of cancellative lattice-ordered semiring via l-ideals of its difference ring. Next we revisit the characterization F-spaces via the f-semiring $C^+(X)$. This leads us to define convexity conditions on f-semirings like in f-rings (cf. [17]). We note that each prime ideal of $C^+(X)$ is an l-ideal, which follows from the 2^{nd}-convexity property of $C^+(X)$. Moreover, we obtain the structure of prime ideals in $C^+(X)$. Next, we define a new class of f-semirings and call them $\mathcal{P}$-semirings, which are positive cones of f-rings like $C(X)$, $\mathbb{R}^X$, a direct product of totally-ordered rings, etc. In the rest of the paper, we give the deserved attention to $\mathcal{P}$-semirings.

In Section 4, we introduce z-ideals in commutative semirings. Our primary focus here is to study $\mathcal{P}$-semirings through the lens of z-ideals. In function rings like $C(X)$, the topological notion of z-ideals is useful. It is well known that the ring $C(X)$ is a von Neumann regular ring if and only if each ideal of $C(X)$ is a z-ideal (cf. Exercise 4J, [12]). Mason has generalized this for commutative rings with unity; see [19]. The notion of algebraic z-ideals is indispensable for von Neumann regularity; a commutative ring R with unity is von Neumann regular if and only if each ideal of R is a z-ideal. We have achieved a similar characterization of $\mathcal{P}$-semirings via z-ideals. Moreover, we characterize the space X for which semiring ideals and z-ideals coincide entirely in $C^+(X)$.

In Section 5, we give some new necessary and sufficient conditions (related to annihilators) on an ideal to be a minimal prime ideal of a semiring, extending the work in [22]. Then, we define z°-ideals in commutative semirings. In literature, ring z°-ideals are sometimes referred to as *d-ideals*; see [20]. We give a new characterization of semisimple semirings via z-ideals and z°-ideals. Lastly, we characterize almost P-spaces via z°-ideals of $C^+(X)$.

2. Preliminaries

2.1. Lattice-ordered rings and f-rings

By a lattice-ordered ring or l-ring $(R, +, \cdot, \vee, \wedge)$, we mean a lattice-ordered group or an l-group (see [8]) that is also a ring in which product of two positive elements is again a positive element. A lattice-ordered ring R is said to be an f-ring, in the sense of Birkhoff and Pierce [9] if, for $a, b, c \in R, a \wedge b = 0$ and $c \geq 0$ imply that $ac \wedge b = ca \wedge b = 0$.

For any lattice-ordered ring R, the *positive cone* R^+ is defined as $R^+ = \{a \in R \colon a \geq 0\}$. For any $a \in R$, we define $a^+ = a \vee 0$, $a^- = a \wedge 0$ and $|a| = a \vee (-a)$. Then $a = a^+ + a^-$ and $|a| = a^+ - a^-$. As some special properties of f-rings, we note

$$a^2 \geq 0 \text{ and } |ab| = |a||b|,$$

for all $a, b \in R$. An f-ring R is said to have *bounded inversion* property if for any $a \in R$, $a > 1_R$ implies a has an inverse.

Some of the excellent references for l-groups, l-rings and f-rings are [5], [8], [9], [11] and [18].

An ideal I of an l-ring R is said to be an l-ideal if it is a ring ideal of R and for any $a \in I$, $b \in R$ and $|b| \leq |a|$ imply $b \in I$. In the case of the f-ring $C(X)$, l-ideals are called absolutely convex ideals; see [12].

2.2. Semirings

In [13], Golan defines *semiring* as an algebra $(S, +, \cdot, 0, 1)$, with $0 \neq 1$ such that the following conditions are satisfied.

1. $(S, +, 0)$ is a commutative monoid with identity element 0;
2. $(S, \cdot, 1)$ is a commutative monoid with identity element 1;
3. Multiplication distributes over addition from either side;

4. $0 \cdot s = 0 = s \cdot 0$ for all $s \in S$.

Definition 2.1. ([13], [14])

- A non-empty subset I of a semiring S is said to be an *ideal* if for all $a, b \in I$ and $s \in S$, $s(a+b) \in I$.
- An ideal P is said to be a *prime ideal* if for all $a, b \in S$, $ab \in P$ is equivalent to $a \in P$ or $b \in P$.
- An ideal I is said to be a *k-ideal* or *subtractive* if for all $a, b \in S$, $a, a+b \in I$ implies $b \in I$.
- An ideal I is said to be an *l-ideal* or *strong* if for all $a, b \in S$, $a+b \in I$ implies $a \in I$ and $b \in I$.

For an ideal I of S, the set $\bar{I} = \{x \in S \mid x + a = b \text{ for some } a, b \in I\}$ is called the *subtractive closure* or *k-closure* of I in S. Clearly, an ideal I is a k-ideal of S if and only if $\bar{I} = I$. The *radical of an ideal I*, denoted by $\sqrt{I}$, is defined as

$$\sqrt{I} = \{a \in S \colon a^n \in I \text{ for some } n \in \mathbb{N}\} \text{ (cf. Proposition 7.28, [13])}$$

A semiring ideal I is called a *radical ideal* if $I = \sqrt{I}$.

Definition 2.2. An element x of S is said to be *additively cancellable* if $x + b = x + c$ implies $b = c$ for all $b, c \in S$.

If every element of S is additively cancellable, then we call S a *additively cancellative semiring* or just *cancellative semiring*. For a cancellative semiring $(S, +, \cdot)$ the set of differences $D(S) = \{a - b \mid a, b \in S\}$ forms a ring. We call $D(S)$ the *ring of differences* of S. It is the smallest ring where S is element-wise fixed; see Theorem 5.11, [14].

Definition 2.3. ([26], [25]) A commutative semiring S is called *positive* if $1 + s$ is a unit for every $s \in S$.

In literature, positive semirings are also referred to as *Gelfand semirings*; see [13]. In a positive semiring S, the class of maximal ideals is contained in the class of l-ideals. For proof, see Theorem 4 of [25].

Definition 2.4 (cf. Definition 1.6, [26]). *A semiring S is said to be* semisimple *if the intersection of all maximal ideals is* $\{0\}$.

2.3. Lattice of ideals of cancellative semirings

Unlike the lattice of ideals of a ring R, denoted by $L(R)$, the behavior of lattice of ideals of a semiring S, denoted by $L(S)$, is atypical; see [4]. Now, if S is a cancellative semiring, then we can make bridges between the lattices $L(S)$ and $L(D(S))$. We define α, β on $L(D(S))$ and $L(S)$ respectively by;

$$\alpha : L(D(S)) \to L(S) \text{ and } \beta : L(S) \to L(D(S)) \text{ such that}$$
$$\alpha(I) = I \cap S \text{ and } \beta(J) = J - J = \{x - y \mid x, y \in J\}$$

We shall use the following facts heavily in the rest of the paper. For proof, see Theorem 7.4 of [14].

1. For each ring ideal I of $(D(S), +, \cdot)$ the ideal $\alpha(I) = I \cap S$ is a k-ideal.
2. For each semiring ideal J of $(S, +, \cdot)$ there exists a unique smallest ring ideal $\beta(J) = J - J = \{x - y \mid x, y \in J\}$ of $(D(S), +, \cdot)$ which satisfies $J \subseteq \beta(J)$.
3. In this context $\beta(I \cap S) \subseteq I$ and $J \subseteq \beta(J) \cap S$ are always satisfied, and $\beta(J) \cap S = J$, that is, $\alpha(\beta(J)) = J$ if and only if J is a k-ideal.

2.4. Lattice-ordered semirings and f-semirings

A *natural quasi-order* " $\leq$ " can be defined on any semiring S. We say $a \leq b$ in S if and only if $a + x = b$ is solvable in S for some $x \in S$ (cf. [26]). This property is also known as *semisubtractive* (cf. p.17, [14]).

Definition 2.5. ([26]) A semiring S is said to be a *partially ordered semiring* if S is partially ordered under the natural quasi-order.

Definition 2.6. ([11]) A semiring S is called a *conic semiring* if no non-zero element has its additive inverse in S. Equivalently, a sum of elements of S is never 0 unless all terms are 0.

In literature, conic semiring also has the name *zerosumfree semiring*; see [13]. A characterization of semirings, which are positive cones of partially ordered rings, can be found in Theorem 2 of [11].

We recall from Chapter VI, Theorem 2 of [11], that a semiring is a positive cone of some partially ordered ring if only S is a conic semiring and the additive semigroup $(S, +)$ is commutative and cancellative.

Remark 2.7. Notably, for a commutative cancellative conic partially ordered semiring S, the partial order in $D(S)$ is induced from S.

An ideal I of a partially ordered semiring S is said to be *convex* if $a \leq b \leq c$ and $a, c \in I$ implies $b \in I$ or equivalently if $0 \leq a \leq b$ and $b \in I$ implies $a \in I$. The class of convex ideals and the class of l-ideals coincide in S (cf. Proposition 2.4, [26]).

Definition 2.8. ([26]) A semiring S will be called an l-semiring if S is a lattice under the natural quasi-order and both $a + (b \vee c) = (a + b) \vee (a + c)$ and $a + (b \wedge c) = (a + b) \wedge (a + c)$ hold for all $a, b, c \in S$.

Definition 2.9. ([26]) An l-semiring S will be called f-semiring if $a \wedge b = 0$ implies $a \wedge bc = 0$ for all $c \in S$.

Every positive l-semiring S is an f-semiring (cf. Theorem 5.12, [26]). Moreover if S is a positive cancellative l-semiring, then S can be embedded in an f-ring A such that $S = \{x \in A : x \geq 0\}$, that is, $A^+ = S$ (cf. Theorem 5.13, [26]).

We know that the ring of real-valued continuous functions, denoted by $C(X)$ and totally ordered division rings, are essential examples of f-rings. We shall now see an example of f-semirings.

Example 2.10. For any topological space X, consider the set $C^+(X)$ of all continuous non-negative real-valued continuous functions on X with pointwise addition and pointwise multiplication. Then $(C^+(X), +, \cdot)$ is a cancellative positive semiring. Indeed for any $f \in C^+(X)$, $\mathbf{1} + f$ is a unit in $C^+(X)$, where $\mathbf{1}(x) = 1$ for all $x \in X$. We can define a partial order on $C^+(X)$ pointwise, that is, for any $f, g \in C^+(X)$, we say $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. Under this partial order $C^+(X)$ forms a lattice-ordered semiring, that is, $(f \vee g)(x) = Max\{f(x), g(x)\}$ and $(f \wedge g)(x) = Min\{f(x), g(x)\}$ for all $x \in X$. Therefore $C^+(X)$ is an f-semiring. Moreover $C^+(X)$ is embedded in the f-ring $C(X)$ as the positive cone and for any $f \in C(X)$, $f = f^+ - (-f^-)$, with $f^+, -f^- \in C^+(X)$, affirming that $C(X)$ is the difference ring of $C^+(X)$.

3. l-ideals of l-semirings

The k-ideals of a cancellative semiring behave nicely under the maps α and β (cf. Theorem 7.4, [14]), but there are examples of ideals that are not k-ideals in cancellative semirings.

Example 3.1. A. Let S be the cancellative semiring of non-negative integers. Then the unique maximal ideal $I = S \setminus \{1\}$ is not a k-ideal.

B. We define $f, h \in C^+(\mathbb{R})$ such that,

$$f(x) = \begin{cases} |x|, & x \geq 0. \\ \frac{4}{3}|x|, & x < 0. \end{cases} \quad \text{and } h(x) = \begin{cases} 0, & x \geq 0. \\ \frac{|x|}{3}, & x < 0. \end{cases}$$

Let I be an ideal in $C^+(\mathbb{R})$ such that $I = (|i|, f)$, where $|i|(x) = |x|$. Then clearly $|i| + h = f \in I$ and $|i| \in I$. If I is a subtractive ideal, then $h \in I$ and so h can be written as $h = k_1|i| + k_2 f$ for some $k_1, k_2 \in C^+(\mathbb{R})$. But then $k_1(x)|x| + k_2(x)|x| = 0$ for $x \geq 0$ and $k_1(x)|x| + \frac{4}{3}k_2(x)|x| = \frac{|x|}{3}$ for $x \leq 0$. Since $k_1 + k_2 \in C^+(\mathbb{R})$, it follows that $k_1(x) + k_2(x) = 0$ when $x \geq 0$ and hence $k_1 = 0 = k_2$ for all $x \geq 0$. Again for $x < 0$ we get $k_1(x) + \frac{4}{3}k_2(x) = \frac{1}{3}$ which implies at least one of k_1, k_2 is nonzero at 0. This is a contradiction due to the former case. Hence $h \notin I$.

It is obvious from Definition 2.1 that every l-ideal is a k-ideal. The class of l-ideals often properly lies inside the class of k-ideals in f-semirings. This is evident from the following example.

Example 3.2. Suppose $I = (|i|)$, that is, the ideal generated by $|i|$ in $C^+(\mathbb{R})$. Every principal ideal of $C^+(\mathbb{R})$ is a k-ideal. Indeed, if $g, f + g \in I$, then $g = s|i|$ and $f + g = f + s|i| = t|i|$, for some $s, t \in C^+(\mathbb{R})$. Since $f \in C^+(\mathbb{R})$, we have $t|i| \geq s|i|$. Therefore $f = (t - s)|i| \in I$. Hence I is a k-ideal. We can replace $\mathbb{R}$ with any topological space X and $|i|$ with any function in $C^+(X)$ and we get that every principal ideal of $C^+(X)$ is a k-ideal. Now we define $p, q \in C^+(\mathbb{R})$ as follows;

$$p(x) = \begin{cases} \frac{|x|}{2}, & x > 0. \\ \frac{2|x|}{3}, & x \leq 0. \end{cases}, q(x) = \begin{cases} \frac{|x|}{2}, & x > 0. \\ \frac{|x|}{3}, & x \leq 0. \end{cases}$$

Then $p + q = |i| \in I$ but $p, q \notin I$. Hence, I is not an l-ideal.

In any l-ring, the sum and the intersection of finitely many l-ideals are again an l-ideal (cf. Theorem 1.24, [18]), and so the collection of l-ideals forms a lattice under the perceptible sense. Next, we show that we can correlate semiring l-ideals of a cancellative l-semiring S and ring l-ideals of its difference ring $D(S)$. It is worth noting that the arbitrary intersection of l-ideals in any semiring is again an l-ideal. Therefore, the collection of all l-ideals of a semiring forms a *complete lower semilattice* under the intersection operation.

Lemma 3.3. *Let S be a cancellative l-semiring embedded in the difference l-ring $D(S)$ such that $D(S)^+ = S$. Then, the following statements hold.*

1. *Let I be an l-ideal of the semiring S. Then, the difference ring ideal $\beta(I)$ of $D(S)$ is also an l-ideal.*
2. *Let J be an l-ideal of the ring $D(S)$. Then $\alpha(J) = J \cap S$ is an l-ideal of the semiring S. Moreover $\beta(J \cap S) = J$ if and only if J is an l-ideal of $D(S)$.*

Proof. (1) For any l-ideal I of S consider the difference ring ideal $\beta(I)$ of $D(S)$. Now if $|a| \leq |b|$ in $D(S)$ for some $b \in \beta(I)$. Then $b = c - d$, for some $c, d \in I$ and so $|a| \leq c + d$ (cf. Theorem 1.7, [18]). Therefore $|a| \in I$, since convex ideals and l-ideals coincide in S, and so I is an l-ideal in S. From the Definition of l-ideal we conclude that $a^+, -a^- \in I$ because $|a| = a^+ + (-a^-)$, whence $a = a^+ - (-a^-) \in \beta(I)$. Hence $\beta(I)$ is an l-ideal in $D(S)$.

(2) It is easy to observe that $\alpha(J)$ is an l-ideal of S for any l-ideal J of $D(S)$.

To show that $J \subseteq \beta(J \cap S)$ for any l-ideal J of $D(S)$. For any $a \in J$ we have $a^+, -a^- \in J$, because $|a^+| \leq |a|$ and $|-a^-| \leq |a|$. Hence $a^+, -a^- \in J \cap S$ as $|a| \in J \cap S$ and $J \cap S$ is an l-ideal in S. Therefore $a = a^+ - (-a^-) \in \beta(J \cap S)$, which implies $J \subseteq \beta(J \cap S)$. The converse inclusion is trivial. □

In case of $C^+(X)$ the equality $\beta(J \cap C^+(X)) = J$ may not hold when J is only a convex ideal of $C(X)$ which is not an l-ideal.

Example 3.4. Let J be the ideal generated by the identity function i in $C(\mathbb{R})$. By 2H1 of [12], J is of the form $J = \{f \in C(\mathbb{R})\colon f(0) = 0 \text{ and } f'(0) \text{ exists finitely}\}$. Then J is a convex ideal but not an l-ideal because $|i| \notin J$. It is easy to observe that $J \cap C^+(\mathbb{R})$ is an l-ideal in the semiring $C^+(\mathbb{R})$ and $\beta(J \cap C^+(\mathbb{R}))$ is an l-ideal of the ring $C(\mathbb{R})$, but $\beta(J \cap C^+(\mathbb{R})) \neq J$, since J is not an l-ideal of $C(\mathbb{R})$.

With the correlation of l-ideals (cf. Lemma 3.3) and the complete lower semilattice structure of the collection of l-ideals, we next show that we can make the collection of l-ideals of a cancellative l-semiring into a lattice under the perceptible sense.

Lemma 3.5. *Let I and J be two ideals of a cancellative semiring S. Then $\beta(I+J)=\beta(I)+\beta(J)$ in the difference ring $D(S)$.*

Proposition 3.6. *Let S be a cancellative l-semiring and $D(S)$ be the difference ring such that S is the positive cone of $D(S)$, that is, $D(S)^+=S$. Then the sum of two l-ideals in S is again an l-ideal in S.*

Proof. The difference ring $D(S)$ is evidently lattice-ordered (cf. Theorem 1.4 of [18]). Let I and J be two l-ideals of S. Then $\beta(I)$ and $\beta(J)$ are l-ideals of the difference ring $D(S)$ by Lemma 3.3, moreover $\beta(I)+\beta(J)=\beta(I+J)$ by Lemma 3.5. Now let $a \leq b+c$, where $a \in S, b \in I$ and $c \in J$. Then by *Riesz decomposition theorem* (cf. Proposition 1.1.4 of [5]), $a=s+t$ in $D(S)$, where $0 \leq s \leq b$ and $0 \leq t \leq c$ and hence $s \in \beta(I), t \in \beta(J)$. Therefore $s \in \beta(I) \cap S = I$ and $t \in \beta(J) \cap S = J$. We conclude that $a \in I+J$. Hence $I+J$ is an l-ideal. □

The following theorem shows that the lattice of l-ideals of a cancellative l-semiring, denoted by $\mathfrak{L}(S)$, is in lattice isomorphism with the lattice of l-ideals of the difference l-ring $D(S)$, denoted by $\mathfrak{L}(D(S))$.

Theorem 3.7. *Let S be a cancellative l-semiring and $D(S)$ be the difference ring of S with $D(S)^+=S$. Then $\mathfrak{L}(S)$ is lattice isomorphic to $\mathfrak{L}(D(S))$.*

Proof. The maps α and β are well-defined onto maps by Lemma 3.3. If $I \subseteq J$ in $\mathfrak{L}(D(S))$ then clearly $\alpha(I) \subseteq \alpha(J)$. Therefore, α is injective. Similarly, β is also injective. Moreover $\alpha\beta = 1_{\mathfrak{L}(S)}$ and $\beta\alpha = 1_{\mathfrak{L}(D(S))}$. We conclude that α and β are order-isomorphisms and, hence, lattice isomorphisms (cf. Proposition 2.19 of [10]). □

Therefore, each l-ideal of a cancellative l-semiring S is of the form $I \cap S$, for some unique l-ideal I of the difference ring $D(S)$.

Corollary 3.8. *The map $\alpha\colon \mathfrak{L}(C(X)) \to \mathfrak{L}(C^+(X))$ defined by $\alpha(I) = I \cap C^+(X)$, is a lattice isomorphism.*

It is well known that every ring ideal of $C(X)$ is an l-ideal if and only if X is an F-space (cf. Theorem 14.25, [12]). Similarly, one might wonder when semiring ideals and semiring l-ideals coincide entirely in $C^+(X)$, likewise when each ideal of $C^+(X)$ is a k-ideal. The following theorem gives another characterization of F-spaces via $C^+(X)$.

Theorem 3.9. *The following are equivalent for a completely regular topological space X.*

1. *Each ideal of $C^+(X)$ is a k-ideal.*
2. *The lattice of ideals of $C^+(X)$, denoted by $L(C^+(X))$, is a modular lattice.*
3. *The lattice $L(C^+(X))$ is a distributive lattice.*
4. *X is an F-space.*
5. *The maps α and β are injective.*
6. *Each ideal of $C^+(X)$ is an l-ideal.*
7. *Each ring ideal of $C(X)$ is an l-ideal.*
8. *The lattices $L(C(X))$ and $L(C^+(X))$ are isomorphic.*
9. *For all $f,g \in C^+(X)$, $(f,g)=(f+g)$, that is, every finitely generated ideal of $C^+(X)$ is principal.*

Proof. The implications (1) through (5) and (8) have been shown for $C^+(X)$ in Theorem 2.1 of [23]. For the implication (4) $\iff$ (7) see the subsection 14.26 of [12].

(4) $\Rightarrow$ (9) Suppose X is an F-space. Equivalently, every cozero-set in X is $C^\star$-embedded. Now if $0 \leq f \leq f+g$ for some $f,g \in C^+(X)$. Then the function

$$h(x) = \frac{f(x)}{f(x)+g(x)}, \text{ where } x \in X \setminus Z(f+g)$$

belongs to $C^*(X \setminus Z(f+g))$. By the hypothesis, h has an extension to a function $k \in C(X)$ such that $f = k(f+g)$. This implies that $f = |k|(f+g)$ and $|k| \in C^+(X)$ as $k \in C(X)$. Similarly $g = s(f+g)$ for some $s \in C^+(X)$. Therefore, we get $(f, g) \subseteq (f + g)$, and the reverse inclusion is obvious. We conclude that $(f, g) = (f + g)$.

(9) ⇒ (6) Trivial.

(6) ⇒ (7) Let I be any ideal in $C(X)$. Then for any two $f, g \in I$, $h \in C(X)$ and $g \leq h \leq f$, we have $0 \leq h - g \leq f - g$. Clearly $h - g \in C^+(X)$ and $f - g \in I \cap C^+(X)$, where $I \cap C^+(X)$ is an l-ideal by our hypothesis. Then $h - g \in I \cap C^+(X)$. Since $g \in I$, we conclude that $h \in I$. Hence, I is a convex ideal in $C(X)$. Therefore, every ideal of $C(X)$ is convex.

□

In [17], Larson has defined n^{th}-convexity property for f-rings and has observed that $C(X)$ satisfies 1^{st}-convexity property if and only if X is an F-space. The definition of n^{th}-convexity solely relies on its positive cone. Also, in the above proof of Theorem 3.9, the implication (4) ⇒ (9) gives us an essence of 1^{st}-convexity property in $C^+(X)$. This observation is a clue to define n^{th}-convexity for commutative f-semirings.

Definition 3.10. An f-semiring S is said to satisfy n^{th}-convexity property if for any $u, v \in S$ with $u \leq v^n$, then there exists $w \in S$ such that $u = wv$.

It is easy to observe that an f-ring R with bounded inversion property has n^{th}-convexity property if and only if the positive cone R^+ has n^{th}-convexity property.

In arbitrary l-semirings, the class of prime ideals may not be properly contained in the class of l-ideals (cf. Example 3.1A). In the case of semiring $C^+(X)$, we can give a more precise structure of prime ideals. The following result is analogous to Exercise 1D.3 of [12].

Proposition 3.11. *If $f \leq g^r$ for some real $r > 1$, then f is a multiple of g. Moreover if $f \leq g$, then f^r is a multiple of g.*

In other words Proposition 3.11 infers that $C^+(X)$ satisfies n^{th}-convexity property for all $n \geq 2$. We give an alternate proof of Corollary 2.1 of [23] using the techniques used in Theorem 5.5 of [12].

Theorem 3.12. *Each prime ideal P of $C^+(X)$ is an l-ideal.*

Proof. Let P be a prime ideal of $C^+(X)$ with $f + g \in P$ for some $f, g \in C^+(X)$. Observe that $0 \leq f \leq f + g$. Now define

$$s(x) = \frac{f^2(x)}{(f+g)(x)} \text{ for } x \notin Z(f+g), \text{ and } s(x) = 0 \text{ for } x \in Z(f+g).$$

Clearly $h \in C^+(X)$ and $f^2 = s(f + g)$ for every $x \in X$, whence $f^2 \in P$ and by our hypothesis $f \in P$. Similarly, we have $g \in P$, and thus, we conclude that P is an l-ideal. □

We see that 2^{nd}-convexity property plays a vital role in Theorem 3.12. The following is a generalization of Theorem 3.12 and Theorem 5.5 of [12].

Theorem 3.13. *Each prime ideal of a nilpotent-free f-ring R with 2^{nd}convexity property is an l-ideal. Moreover, if R is the difference ring of its positive cone S, then each prime ideal of S is a semiring l-ideal and of the form $P \cap S$ for some prime ideal P of R.*

The second half of the statement follows from Theorem 3.7. As a direct consequence of Theorem 3.13, we conclude important facts about the prime and maximal ideals of the semiring $C^+(X)$.

Corollary 3.14. *For any topological space X, the following statements hold.*

1. *Each prime ideal of $C^+(X)$ is of the form $P \cap C^+(X)$ for some prime ideal P of $C(X)$.*
2. *Each maximal ideal of $C^+(X)$ is of the form $M \cap C^+(X)$ for some maximal ideal M of $C(X)$.*

From the above discussion, we find the motivation to define a new class of partially ordered semirings.

Definition 3.15. A partially ordered semiring S is said to be a P-convex if each prime ideal of S is an l-ideal.

Clearly, from Theorem 4 of [25] and Corollary 7.13 of [13], every P-convex semiring is a positive semiring. Something more can be said if we consider lattice-ordered cancellative conic P-semiring with 2^{nd}-convexity property (cf. Theorem 3.13). We define a new class of f-semirings.

Definition 3.16. A cancellative conic l-semiring S is said to be $\mathcal{P}$-semiring if it is P-convex.

Every $\mathcal{P}$-semiring S is positive and embedded in a difference f-ring $D(S)$ such that $D(S)^+ = S$ (cf. Theorem 5.13, [26]). By Lemma 3.3 and Theorem 3.7, each prime ideal of a $\mathcal{P}$-semiring S is of the form $Q \cap S$, where Q is an l-ideal of the f-ring $D(S)$. Moreover for any $a, b \in D(S)$ if $ab \in Q$ then $|ab| \in Q \cap S$ as Q is an l-ideal. From here, we get $|a||b| \in Q$. Indeed, $|ab| = |a||b|$ as $D(S)$ is an f-ring. Therefore either $|a| \in Q$ or $|b| \in Q$, and so either $a \in Q$ or $b \in Q$. Proving that Q is a prime ideal. From this observation, we arrive at the following proposition.

Proposition 3.17. *Each prime ideal of a $\mathcal{P}$-semiring S is of the form $P \cap S$ for some prime l-ideal P of $D(S)$ where $D(S)^+ = S$.*

Each prime ideal of a $\mathcal{P}$-semiring S is a k-ideal. Moreover, any maximal ideal of a $\mathcal{P}$-semiring S is of the form $M \cap S$ for some maximal ideal M of $D(S)$. Some natural examples of $\mathcal{P}$-semirings are as follows.

Example 3.18.
1. The semirings $C^+(X)$ and $\mathbb{R}^X_+$ (positive cone of the f-ring $\mathbb{R}^X$) are natural examples of $\mathcal{P}$-semirings. More generally the positive cones of f-rings with 2^{nd}-convexity property (cf. Theorem 3.13).
2. Any finite product of $\mathbb{R}^+$ and $\mathbb{Q}^+$. More generally any finite product of positive cones of totally ordered fields.

In the next section, we introduce z-ideals in commutative semirings. We will obtain a characterization of von Neumann regularity of $\mathcal{P}$-semirings in terms of semiring z-ideals.

4. z-ideals of commutative semirings

In this section, we introduce z-ideals for commutative semirings with unity. First, we recall that every ideal I of a commutative semiring S with unity is contained in a maximal ideal M (cf. Proposition 6.59, [13]). Let us denote the set of all maximal ideals of S as $\mathcal{M}^+$. We define similar notations for semiring S as in [19]. For any $a \in S$

$$\mathcal{M}^+(a) = \{M \in \mathcal{M}^+ \mid a \in M\} \text{ and } \mathcal{M}^+_a = \textstyle\bigcap_{M \in \mathcal{M}^+(a)} M.$$

Definition 4.1. An ideal I of a commutative semiring S with unity is said to be a z-ideal if $\mathcal{M}^+(a) = \mathcal{M}^+(b)$ for any $a \in I$ and $b \in S$ imply $b \in I$.

Before proceeding, we give some examples of semiring z-ideals.

Example 4.2. (a) Every maximal ideal is a z-ideal.

(b) Intersection of z-ideals is a z-ideal.

(c) Let I and J be ideals of a semiring S. The *ideal quotient* $(J : I)$ is defined as $(J : I) = \{a \in S : aI \subset J\}$. If J is a z-ideal then so is $(J : I)$ for any I. Therefore $Ann_S(I) = \{a \in S : ax = 0 \text{ for all } x \in I\} = (I : \mathfrak{o})$ is a z-ideal, where $\mathfrak{o}$ is the zero ideal.

Another good source of examples is minimal prime ideals in certain kinds of semirings (see Proposition 4.3 below). We recall that in a nilpotent-free semiring S, an ideal P is a minimal prime ideal if and only if for any $x \in P$ there exists $y \notin P$ such that $xy = 0$. For the proof, see Corollary 3.6 of [22].

Proposition 4.3. *Every minimal prime ideal of a semisimple semiring S is a z-ideal.*

Proof. Let P be a minimal prime ideal in a semiring S with $\bigcap_{M\in\mathcal{M}^+} M = 0$. Let $a, b \in S$ with $\mathcal{M}^+(a) = \mathcal{M}^+(b)$ and $a \in P$. Since P is a minimal prime ideal, then there exists $c \notin P$ such that $ac = 0$ (cf. Corollary 3.6, [22]). To show $b \in P$, we claim that $bc = 0$. If not, then from the fact that $\bigcap_{M\in\mathcal{M}^+} M = 0$, we know that there is a maximal ideal M such that $bc \notin M$. Moreover by maximality of M, we have $(M, bc) = S$ and so there are elements $r \in S$ and $m \in M$ such that $m + rbc = 1$, which further implies $a = am \in M$. Again from $\mathcal{M}^+(a) = \mathcal{M}^+(b)$ we have $b \in M$ and hence $bc \in M$. A contradiction. Therefore $bc = 0 \in P$, and since P is prime and $c \notin P$, we conclude that $b \in P$. Hence P is a z-ideal. □

The following equivalent conditions for an ideal to be a z-ideal will be helpful in various contexts.

Theorem 4.4. *Let I be an ideal in a commutative semiring S with unity. Then the following statements are equivalent.*

1. *I is a z-ideal in S.*
2. *$\mathcal{M}^+(a) \supseteq \mathcal{M}^+(b)$ and $b \in I$ implies $a \in I$.*
3. *For every $a \in I$, $\mathcal{M}^+_a \subseteq I$.*

For every ideal I of S we define a *z-cover*, namely $I_z = \bigcap\{J\colon J \text{ is a } z\text{-ideal}, I \subseteq J\}$, that is, I_z is the smallest z-ideal containing I.

Proposition 4.5. *Let S be a semiring and I, J be ideals of S. Then, we have the following.*

1. $I \subseteq J \implies I_z \subseteq J_z$.
2. $(\sum I_\alpha)_z = (\sum (I_\alpha)_z)_z$.
3. $(I_z)_z = I_z$.
4. $I \subseteq \sqrt{I} \subseteq I_z$.
5. $(I^n)_z = I_z$. $(n \in \mathbb{Z}^+)$
6. *If I, J are z-ideals in a commutative l-semiring S with root property (every positive element in S has a square root) then IJ is a z-ideal if and only if $IJ = I \cap J$.*
7. *If I is a z-ideal, then $I = \sqrt{I} = I_z$.*

Proof. We only give the proof of (4). Observe that (7) follows directly from (4).

(4) We recall Krull's theorem (cf. Proposition 7.28 of [13]), that is, for any ideal I of a semiring S, the radical of I is $\sqrt{I} = \{a \in S\colon a^n \in I \text{ for some positive integer } n\}$. Then clearly $I \subseteq \sqrt{I}$. Now let $a \in \sqrt{I}$. Then there exist $n \in \mathbb{N}$ such that $a^n \in I$. If J is a z-ideal containing I, then $a^n \in I$. We observe that $\mathcal{M}^+(a) = \mathcal{M}^+(a^n)$. By the definition of z-ideals, we conclude that $a \in J$. Since J is an arbitrary z-ideal containing I, we have $a \in I_z$. Therefore $\sqrt{I} \subseteq I_z$. □

From (7) of Proposition 4.5 we deduce that semiring z-ideals are radical (semiprime) ideals. Hence, semiring z-ideals are an intersection of all prime ideals containing it (cf. Proposition 7.25, [13]).

The following result depicts the relationship between semiring z-ideals and minimal prime ideals of a commutative semiring S. The proof follows the same line of arguments as in the proof of Theorem 1.1 of [19].

Theorem 4.6. *If P is minimal in the class of prime ideals containing a z-ideal I of a semiring S, then P is a z-ideal in S.*

In the f-ring $C(X)$, the abstract definition of z-ideal coincides with the topological definition of z-ideal; see Exercise 4A.5 of [12]. The following proposition correlates abstract definition of z-ideals in $C^+(X)$ with the possible topological definition of z-ideals in $C^+(X)$, that is, an ideal I of $C^+(X)$ is a z-ideal if and only if $Z(f) \subseteq Z(g)$ and $f \in I$ imply $g \in I$.

Proposition 4.7. *For any $f \in C^+(X)$, $\mathcal{M}_f^+ = \{g \in C^+(X) \mid Z(f) \subseteq Z(g)\}$.*

Proof. Suppose $g \in \mathcal{M}_f^+$ and let $x \in Z(f)$. Then $f(x) = 0$ and thus $f \in M_x \cap C^+(X) = \{h \in C^+(X) \mid h(x) = 0\}$. By definition of $\mathcal{M}_f^+$ we have $g \in M_x \cap C^+(X)$ but then $g(x) = 0$ implies $x \in Z(g)$. Hence $Z(f) \subseteq Z(g)$.
Conversely let $h \in \{g \in C^+(X) \mid Z(f) \subseteq Z(g)\}$. Then $Z(f) \subseteq Z(h)$. By Gelfand-Kolmogoroff theorem, we know that any maximal ideal of $C(X)$ is of the form M^p (cf. Theorem 7.3 of [12]). Therefore any maximal ideal of $C^+(X)$ is of the form $M_+^p = M^p \cap C^+(X) = \{g \in C^+(X) \mid p \in Cl_{\beta X}Z(g)\}$ (cf. Corollary 3.14), then $Z(f) \subseteq Z(h)$ implies $Cl_{\beta X}Z(f) \subseteq Cl_{\beta X}Z(h) \Rightarrow h \in \mathcal{M}_f^+ \Rightarrow \{g \in C^+(X) \mid Z(f) \subseteq Z(g)\} \subseteq \mathcal{M}_f^+$. Hence, the statement follows. □

There are examples of semiring z-ideals that are not k-ideals. For example the unique maximal ideal of $(\mathbb{N} \cup \{0\}, +, \cdot)$ is a z-ideal but not a k-ideal. We recall that each maximal ideal of a positive semiring is an l-ideal (cf. Theorem 4, [25]). We have the following theorem.

Theorem 4.8. *Each z-ideal of a positive semiring is an l-ideal.*

As a direct corollary, we get that each z-ideal of $C^+(X)$ is an l-ideal. In Theorem 3.9, we have recorded the structure of l-ideals in certain kinds of semirings, namely each l-ideal J of a cancellative conic l-semiring S is of the form $J = I \cap S$ for some ring l-ideal I of $D(S)$. This motivates us to find a similar structure of z-ideals in certain kinds of semirings, viz., in $\mathcal{P}$-semirings. We need the following lemma.

Lemma 4.9. *Let S be a $\mathcal{P}$-semiring. Then the difference ring $D(S)$ is an f-ring with bounded inversion.*

Theorem 4.10. *Let S be a $\mathcal{P}$-semiring with difference ring $D(S)$ such that $D(S)^+ = S$. Then the following holds.*

1. *Let I be a z-ideal of the semiring S. Then the difference ring ideal $\beta(I)$ of $D(S)$ is also a z-ideal.*
2. *Let J be a z-ideal of the ring $D(S)$. Then $J \cap S$ is a z-ideal of the semiring S. Moreover $\beta(J \cap S) = J$.*

Proof. We only give the proof of (1).
To show that $\mathcal{M}_a \subseteq \beta(I)$ for every $a \in \beta(I)$, where $\mathcal{M}_a = \bigcap_{M \in \mathcal{M}(a)} M$ (cf. [19]). Let $b \in \mathcal{M}_a$ for some $a \in \beta(I)$. Then by Lemma 4.9 every maximal ideal of $D(S)$ is an l-ideal (cf. Lemma 1.1, [15]) and hence we have $b \in \mathcal{M}_{|a|}$. Further we get $|b| \in \mathcal{M}_{|a|}$ and $|b| \in \bigcap_{M \in \mathcal{M}(|a|)}(M \cap S)$. Hence $b^+, -b^- \in M \cap S$. Indeed, because $|b| = b^+ + (-b^-)$ and $M \cap S$ is an l-ideal for all maximal ideal M. Since I is a z-ideal in S, then $b^+, -b^- \in I$ as $b^+, -b^- \in \mathcal{M}_{|a|}^+$ and $|a| \in I$. Therefore $b = b^+ + b^- \in \beta(I)$ and so $\mathcal{M}_a \in I$. □

By $ZId(S)$ we mean the lattice of z-ideals of a semiring S partially ordered by inclusion and $I \vee J = (I + J)_z$ and $I \wedge J = I \cap J$ in $(ZId(S), \vee, \wedge)$. The following result about z-ideals is analogous to Theorem 3.7.

Theorem 4.11. *Let S be a $\mathcal{P}$-semiring with difference ring $D(S)$. Then $ZId(S)$ is lattice-isomorphic to $ZId(D(S))$.*

The proof easily follows from the fact that in a $\mathcal{P}$-semiring every semiring z-ideal is a semiring l-ideal, and in the difference f-ring (with bounded inversion), every z-ideal is an l-ideal. We deduce the following.

Corollary 4.12. *Any z-ideal in a $\mathcal{P}$-semiring S is of the form $I \cap S$ for some ring z-ideal I of the difference ring $D(S)$.*

As a direct consequence of Corollary 4.12 and Theorem 2.7 of [19], we have the following important result.

Theorem 4.13. *The following are equivalent for a z-ideal in $\mathcal{P}$-semiring S with difference ring $D(S)$ such that $D(S)^+ = S$.*

1. *I is a prime ideal in S.*
2. *I contains a prime ideal of S.*
3. *$\beta(I)$ is a prime ideal $D(S)$.*
4. *$\beta(I)$ contains a prime ideal in $D(S)$.*

Our main goal in this section is to examine how the von Neumann regularity of a $\mathcal{P}$-semiring and the von Neumann regularity of its ring of differences are correlated. We recall two types of regularity in a semiring.

Definition 4.14. ([13], [3]) A semiring S is said to be von Neumann regular if for any $a \in S$ there exists $s \in S$ such that $a = asa$. A semiring S is said to be Bourne regular or k-regular if for each $a \in S$ there exists x and y in S such that $a + axa = aya$.

For a positive semiring S, these two kinds of regularities coincide. We recall the following important result from [3].

Lemma 4.15. *(cf. Lemma A.4.1, [3]) Let S be an additively commutative and cancellative semiring and $R = D(S)$ its difference ring. If R is regular, then S is k-regular. The converse holds if S is semisubtractive, but not in general.*

Theorem 4.16. *The following are equivalent for a $\mathcal{P}$-semiring S and its difference ring $D(S)$ with $D(S)^+ = S$.*

1. *$D(S)$ is a von Neumann regular ring.*
2. *Every ideal of $D(S)$ is a z-ideal.*
3. *Every k-ideal of S is a z-ideal.*
4. *Every k-ideal of S is a semiprime ideal.*
5. *S is a von Neumann regular semiring.*

Proof. The implication (1) $\iff$ (2) is from Theorem 1.2 of [19]. The implication (1) $\iff$ (5) follows from Lemma 4.15 and the fact that von Neumann regularity and k-regularity coincide in positive semirings.

(2) $\Rightarrow$ (3) Every k-ideal I can be written as $\beta(I) \cap S$. Here, $\beta(I)$ is a z-ideal of $D(S)$ by our hypothesis. Hence I is a z-ideal by Theorem 4.10.

Every z-ideal is a semiprime ideal (cf. Proposition 4.5). Therefore (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (5) If very k-ideal is a semiprime ideal, then S is a k-regular semiring (cf. Theorem 3.2, [21]). □

Corollary 4.17. *The following are equivalent for a topological space X.*

1. *X is a P-space.*
2. *Every ideal of $C^+(X)$ is a z-ideal.*
3. *$C^+(X)$ is a von Neumann regular semiring.*

Proof. (1) $\Rightarrow$ (2) If X is a P-space, then it is also an F-space. Therefore, every ideal of $C^+(X)$ is an l-ideal by Theorem 3.9 and so every ideal is of the form $I \cap C^+(X)$ for l-ideals I of $C(X)$. We know that X is a P-space if and only if every ideal of $C(X)$ is a z-ideal if and only if $C(X)$ is a von Neumann regular ring(cf. Exercise 4J, [12]). Then by Theorem 4.10, every ideal of $C^+(X)$ is a z-ideal.

(2) $\Rightarrow$ (3) and (3) $\iff$ (1) Follows from Theorem 4.16. □

It is important to note that the ideals of $C^+(X)$ when X is a P-space are of a special kind. Indeed, because they entirely consist of zero-divisors. In the next section, we focus on such ideals in a more general setting.

5. z°-ideals of semirings

We need the following notations for the rest of the section.

- For any $s \in S$, the *double annihilator* is an ideal of S defined by, $Ann_S^2(s) = \{x \in S \colon xy = 0 \text{ for all } y \in Ann_S(s)\}$.
- The collection of all minimal prime ideals of S is denoted by $Min^+(S)$.
- For any $a \in S, \mathcal{P}_a^+ = \bigcap\{Q \in Min^+(S) \colon a \in Q\}$.
- For any subset $A \subseteq S$, $V^+(A) = \{P \in Min^+(S) \colon P \subseteq B\}$ and for an element $x \in S$, $V^+(x) = \{P \in Min^+(S) \colon x \in P\}$.

Definition 5.1. ([13]) A nonempty subset A of a semiring S is said to be an m-system if and only if $a, b \in A$ implies that there exists an element $s \in S$ such that $asb \in S$.

The following theorem is an easy consequence of Proposition 7.15 of [13].

Theorem 5.2. *Let I be an ideal of a semiring S and $\mathfrak{M}$ be an m-system of S with $\mathfrak{M} \cap I = \emptyset$. Then $\mathfrak{M}$ is contained in a maximal m-system $\mathfrak{N}$ with $\mathfrak{N} \cap I = \emptyset$.*

Lemma 5.3. *The following are equivalent for a nilpotent-free semiring S.*

1. *$\mathfrak{M}$ is a maximal m-system of S.*
2. *For any non-zero $a \notin \mathfrak{M}$, there exists $b \in \mathfrak{M}$ such that $ab = 0$.*

Proof. (1) ⇒ (2) Let $\mathfrak{M}$ is a maximal m-system. Then $S \setminus \mathfrak{M}$ is a prime ideal. Since every prime ideal contains a minimal prime ideal (cf. Proposition 7.14, [13]), by maximality of $\mathfrak{M}$ it follows that $S \setminus \mathfrak{M}$ is a minimal prime ideal. Then (2) follows from Corollary 3.6 of [22].

(2) ⇒ (1) Suppose $\mathfrak{M}$ is an m-system satisfying (2) such that $\mathfrak{M}$ is not a maximal m-system. Then by Theorem 5.2, $\mathfrak{M}$ is contained in a maximal m-system $\mathfrak{N}$. Choose $a \in \mathfrak{N} \setminus \mathfrak{M}$, then by (2) there exists $b \in \mathfrak{M}$ such that $ab = 0$. Since S is nilpotent-free, then $(a)(b) = 0$. This is a contradiction to the fact that $S \setminus \mathfrak{N}$ is a prime ideal. Therefore $\mathfrak{M}$ must be a maximal m-system.
□

Proposition 5.4. *A prime ideal P of a nilpotent-free semiring S is minimal if and only if $Ann_S(x) \setminus P \neq \emptyset$ for any $x \in P$.*

Proof. Let P be a minimal prime ideal in S. Then $S \setminus P$ is a maximal m-system in S and there exists $y \in S \setminus P$ such that $xy = 0$ by Lemma 5.3. Thus $y \in Ann_S(x) \setminus P$. The converse follows from the definition. □

The following theorem is another easy consequence of Lemma 5.3.

Theorem 5.5. *A prime ideal P is minimal in a semiring S if and only if exactly one of (x), $Ann_S(x)$ is contained in P*

Proof. Let P be a minimal prime ideal of S and $x \in P$. Then by Proposition 5.4 $Ann_S(x) \nsubseteq P$. Moreover, if $x \notin P$, then $Ann_S(x) \subseteq P$ by Lemma 5.3.

The converse statement follows easily from Lemma 5.3 and Theorem 3.5 of [22]. □

The following lemmas are analogous to Lemma 1.2 and Lemma 1.3 of [20].

Lemma 5.6. *Following holds in a nilpotent-free semiring S.*

1. *For an ideal I of S, $Ann_S(I) = \bigcap\{P \in Min^+(S) \colon I \nsubseteq P\}$. Moreover $Ann_S^2(I) = \bigcap\{P \in Min^+(S) \colon Ann_S(I) \nsubseteq P\}$.*
2. *$Ann_S^2(I) = \bigcap\{P \in Min^+(S) \colon Ann_S^2(I) \subseteq P\}$. Moreover $Ann_S^2(x) = \mathcal{P}_x^+$ for any $x \in S$.*

3. $V^+(a) = V^+(Ann_S^2(a))$ *for any* $a \in S$.

Proof. (1) ⇒ For any ideal I of S, $I \cdot Ann_S(I) = (0)$. Therefore, either I or $Ann_S(I)$ is contained in a prime ideal P. Moreover if $P \in Min^+(S)$, then $I \notin P$ implies $Ann_S(I) \in P$. Therefore we have $Ann_S(I) \subseteq \bigcap\{P \in Min^+(S)\colon I \nsubseteq P\}$. Now let $a \in \bigcap\{P \in Min^+(S)\colon I \nsubseteq P\} \setminus I$. Then there exists $b \in I$ such that $ab \neq 0$. Consider $\mathfrak{M} = \{(xy)^n\colon n = 1, 2, \cdots\}$, which is an m-system. Then by Theorem 5.2, $\mathfrak{M}$ is contained in a maximal m-system $\mathfrak{N}$ in S. Hence $xy \notin S \setminus \mathfrak{N}$, so $x \notin S \setminus \mathfrak{N}$ and $y \notin S \setminus \mathfrak{N}$. Therefore $I \notin S \setminus \mathfrak{N}$ and so $\bigcap\{P \in Min^+(S)\colon I \nsubseteq P\} \subseteq S \setminus \mathfrak{N}$, which implies $x \in S \setminus \mathfrak{N}$. A contradiction.

(2) ⇒ Clear from (1).

(3) ⇒ Clearly $V^+(a) \supseteq V^+(Ann_S^2(a))$ as $a \in Ann_S^2(a)$. For the reverse inclusion take any $P \in V^+(a)$, then there exists $x \in Ann_S(a) \setminus P$ by Proposition 5.4. Then for any $y \in Ann_S^2(a)$, we have $xy = 0 \in P$ and thus $y \in P$. Hence $V^+(a) \subseteq V^+(Ann_S^2(a))$.

□

We abide by the set-theoretic convention that the intersection of an empty set of ideals is the whole set.

Definition 5.7. An ideal I of a semiring S will be called z°-ideal if $a \in I$ implies $\mathcal{P}_a^+ \subseteq I$.

From the definition, it easily follows that a z°-ideal consists entirely of zero-divisors. We give examples of z°-ideal in semirings.

Example 5.8.

1. For any zero-divisor a in a semiring S, the ideal $\mathcal{P}_a^+$ is a z°-ideal. The ideal $\mathcal{P}_a^+$ for any zero-divisor a will be called a *basic z°-ideal*.
2. For any nonzero ideal I of a nilpotent-free semiring S, the ideal $Ann_S(I)$ is a z°-ideal. Indeed, because $Ann(I) = \bigcap\{P \in Min(S)\colon I \nsubseteq P\}$. Moreover the ideal $(I : A) = \{a \in S\colon aS \subseteq I\}$, where $A \nsubseteq I$, is a z°-ideal of S.
3. Maximal ideals of $\mathbb{R}^+ \times \underbrace{\mathbb{R}^+ \cdots \times \mathbb{R}^+}_{n \text{ times}}$ are z°-ideals. Moreover, for any finite X, every maximal ideal of $C^+(X)$ is a z°-ideal.
4. Any maximal of $C^+(X)$ consisting of zero-divisors is a z°-ideal.

Theorem 5.9. *Let S be a nilpotent-free semiring. Then for any $a \in S$ we have* $\mathcal{P}_a^+ = \{b \in S \mid Ann_S(a) \subseteq Ann_S(b)\}$.

Proof. Let $b \in \mathcal{P}_a^+$ and $c \in Ann(a)$. Then $ac = 0$. We shall show that $bc = 0$ means that $c \in Ann_S(b)$. Let P be any minimal prime ideal in S. If $a \in P$, then $b \in \mathcal{P}_a^+$ implies that $b \in P$ and $bc \in P$. On the other hand if $a \notin P$, then $ac = 0$ implies $c \in P$ and so $bc \in P$. Thus $Ann_S(a) \subseteq Ann_S(b)$.
Conversely, let $Ann_S(a) \subseteq Ann_S(b)$. Let P be a minimal prime ideal in S containing a. Then, by Corollary 5 of [22], there exists $c \in S \setminus P$ such that $ac = 0$. So $c \in Ann_S(a) \subseteq Ann_S(b)$, and hence $bc = 0 \in P$ which implies that $b \in P$ as P is a prime ideal and $c \notin P$. We conclude $b \in \mathcal{P}_a^+$. This completes the proof. □

For any $f \in C^+(X)$, $\mathcal{P}_f^+$ has one additional structure analogous to Lemma 2.1 of [6].

Proposition 5.10. *For any* $f \in C^+(X)$, $\mathcal{P}_f^+ = \{g \in C^+(X)\colon intZ(f) \subseteq intZ(g)\}$.

Proof. It is sufficient to show that

$$intZ(f) \subseteq intZ(g) \iff Ann_{C^+(X)}(f) \subseteq Ann_{C^+(X)}(g), \text{ for any } g \in C^+(X).$$

First, let $intZ(f) \subseteq intZ(g)$ and choose $k \in Ann_{C^+(X)}(f)$. Then $kf = \mathbf{0}$ implies that $Z(k) \cup Z(f) = X$, so $X \setminus Z(k) \subseteq intZ(f) \subseteq intZ(g) \subseteq Z(g)$. Therefore $Z(k) \cup Z(g) = X$ and $kg = \mathbf{0}$. Thus $k \in Ann_{C^+(X)}(g)$.

Conversely, let $Ann_{C^+(X)}(g) \subseteq Ann_{C^+(X)}(f)$. It is sufficient to prove that $intZ(f) \subseteq Z(g)$. If possible, let $s \in intZ(f) \setminus Z(g)$. Then by complete regularity of X, there exists $k \in C(X)$ such that $k(X \setminus intZ(f)) = \{0\}$ and $k(s) = 1$. Clearly $kf = \mathbf{0}$ and so $|k|f = \mathbf{0}$, where $|k| \in C^+(X)$. Therefore $|k| \in Ann_{C^+(X)}(f)$ but $|k| \notin Ann_{C^+(X)}(g)$, as $|k|(s)g(s) = g(s)$. A contradiction to our hypothesis, and we are through. □

In the following, we list some equivalent conditions for an ideal I to be a z°-ideal in a nilpotent-free semiring S.

Proposition 5.11. *Let S be a nilpotent-free semiring. Then the following are equivalent.*

1. *I is a z°-ideal of S.*
2. *For $a, b \in S$, $\mathcal{P}_a^+ = \mathcal{P}_b^+$ and $b \in I$, imply that $a \in I$.*
3. *For $a, b \in S$, $V^+(a) = V^+(b)$ and $a \in I$, imply $b \in I$.*
4. *For any $a \in I$, $Ann_S^2(a) \subseteq I$.*

Proof. (1) ⇒ (2) Since $a \in \mathcal{P}_a^+$ and $\mathcal{P}_a^+ = \mathcal{P}_b^+ \subseteq I$ as $b \in I$. We have $a \in I$.

(2) ⇒ (3) Suppose $V^+(a) = V^+(b)$ and $b \in I$. Then $a \in \mathcal{P}_a^+ = \bigcap V^+(a) = \bigcap V^+(b) = \mathcal{P}_b^+ \subseteq I$ as $b \in I$. Hence $a \in I$.

(3) ⇒ (4) This is clear from the fact that for any $a \in I$, $V^+(a) = V^+(Ann_S^2(a))$ by Lemma 5.6.

(4) $\implies$ (1) By Lemma 5.6, $Ann_S^2(a) = \mathcal{P}_a^+$. Hence for any $a \in I$ $\mathcal{P}_a^+ \subseteq I$, so I is a z°-ideal. □

In the following, we give an interesting characterization of semisimple semirings.

Theorem 5.12. *A semiring S is semisimple if and only if every z°-ideal is a z-ideal.*

Proof. Let I be a z°-ideal with $\mathcal{M}^+(a) = \mathcal{M}^+(b)$, such that $b \in I$. Then $a \in Ann_S^2(b) \subseteq I$. Indeed, if $s \in Ann(b)$ and $as \neq 0$, then there is a maximal ideal M such that $as \notin M$, and so by $sb = 0$, it follows that $b \in M$ and hence $a \in M$ by the above condition. A contradiction.

Conversely, suppose $s \neq 0$ and $s \in \bigcap_{M \in \mathcal{M}^+} M$. If P is a minimal prime ideal of S not containing s, then $\mathcal{M}^+(s) = \mathcal{M}^+(0)$ and $s \notin P$. Hence P is a z°-ideal which is not a z-ideal. □

Corollary 5.13. *Each z°-ideal of $C^+(X)$ is a z-ideal.*

The following example shows that not all z-ideal in a semiring is a z°-ideal.

Example 5.14. Consider the polynomial semiring $\mathbb{Q}^+[x]$, which is a cancellative semiring. Evidently $\mathbb{Q}[x]$ is the difference ring of $\mathbb{Q}^+[x]$. We claim that a maximal ideal M of $\mathbb{Q}^+[x]$ cannot consist of zero-divisors entirely. Assuming the contrary, we have $x \notin M$. Then $(M, x) = \mathbb{Q}^+[x]$, and so $1 = xf + g$ for some $f \in \mathbb{Q}^+[x]$ and $g \in M$. As g is a zero-divisor then there exists $s \in \mathbb{Q}^+[x]$ such that $gs = 0$ and so $s = xfs$, which forces x to be a unit in $\mathbb{Q}^+[x]$. A contradiction.

Proposition 5.15. *Each z°-ideal of a conic semiring is an l-ideal.*

Proof. Let I be a z°-ideal of a conic semiring S and $a + b \in I$ for some $a, b \in S$. Then $a + b$ is a zero-divisor in S, that is, there exists $s \in S$ such that $(a + b)s = 0$. As S is a conic semiring, we have $as = 0 = bs$. Therefore $V^+(a) \supseteq V^+(a + b)$. By taking intersections we have $\mathcal{P}_a^+ \subseteq \mathcal{P}_{a+b}^+ \subseteq I$, so $a \in I$. Similarly, we have $b \in I$. □

Corollary 5.16. *Each z°-ideal of a $\mathcal{P}$-semiring is an l-ideal.*

Remark 5.17. As each z°-ideal of a semisimple semiring is a z-ideal, then by Corollary 4.12, each z°-ideal of a semisimple $\mathcal{P}$-semiring S is of the form $I \cap S$, where I is a z-ideal of $D(S)$. Moreover, I consists entirely of zero-divisors.

Like z-ideals of a $\mathcal{P}$-semiring S, we can correlate between z°-ideals of S and z° ideals of its difference ring $D(S)$. The proof follows the same line of arguments as in Theorem 4.10.

Theorem 5.18. *Let S be a $\mathcal{P}$-semiring with difference ring $D(S)$ such that $D(S)^+ = S$. Then the following statements hold.*

1. *Let I be a z°-ideal of the semiring S. Then the difference ring ideal $\beta(I)$ of $D(S)$ is also a z°-ideal.*
2. *Let J be a z°-ideal of the ring $D(S)$. Then $J \cap S$ is a z°-ideal of the semiring S. Moreover $\beta(J \cap S) = J$.*

Corollary 5.19. *Each z°-ideal of $C^+(X)$ is of the form $I \cap C^+(X)$ for some z°-ideal I of $C(X)$.*

A topological space X is said to be *almost P-space* if every non-empty zero-set of X has a non-empty interior. Equivalently, if every element of $C^+(X)$ is either a unit or zero-divisor. As a direct consequence of Corollary 5.19, we have the following characterization of *almost P-spaces* via $C(X)$ (cf. Theorem 2.14, [6]) and $C^+(X)$.

Theorem 5.20. *The following are equivalent for a topological space X.*

1. *X is an almost P-space.*
2. *Every maximal ideal of $C^+(X)$ consists entirely of zero-divisors.*
3. *Every maximal ideal of $C(X)$ consists entirely of zero-divisors.*
4. *Every maximal ideal of $C^+(X)$ is a z°-ideal.*

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